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Global bifurcation and stability of steady states for a reaction-diffusion-chemotaxis model with volume-filling effect

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Abstract

This paper is devoted to studying a reaction-diffusion-chemotaxis model with a volume-filling effect in a bounded domain with Neumann boundary conditions. We first establish the global existence of classical solutions bounded uniformly in time. Then applying the asymptotic analysis and bifurcation theory, we obtain both the local and global structure of steady states bifurcating from the homogeneous steady states in one dimension by treating the chemotactic coefficient as a bifurcation parameter. Moveover we find the stability criterion of the bifurcating steady states and give a sufficient condition for the stability of steady states with small amplitude. The pattern formation of the model is numerically shown and the stability criterion is verified by our numerical simulations.

Keywords: steady states, bifurcation theory, stability
Mathematics Subject Classification: 35K55, 35K57, 35K45, 35K50, 92C15, 92C17

(Some figures may appear in colour only in the online journal)

1. Introduction

The mathematical modeling of chemotaxis was started from the pioneering works of Patlak in 1953 [19] and Keller and Segel in 1970 [14, 13]. Since then, various chemotaxis models based on the Keller-Segel model have been proposed to describe the chemotactic aggregation process. Many of these works treat the cells as point masses and hence the formation of cell aggregation was interpreted as a finite-time blow-up of cell density [9, 10]. To take into account cell sizes, a so-called volume-filling chemotaxis was proposed in [18] so that arbitrarily high cell densities can be precluded by setting an impassable threshold value for cell density. This idea was further developed in [25, 24] for generic cell types. A generalized form of volume-filling chemotaxis model of [18, 25] reads

$$\begin{cases} u_t = \nabla\cdot(d(u)\nabla u - \chi h(u)\nabla v) + \mu u(1-u/u_c), & x\in\Omega,\ t>0,\\ v_t = \Delta v - v + u, & x\in\Omega,\ t>0,\\ \nabla u\cdot\nu = \nabla v\cdot\nu = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where the cell density-dependent diffusion coefficient $d(u)$ and the chemotactic sensitivity function $h(u)$ are of the form

$$d(u) = D(1-u)^{-\alpha}, \quad h(u) = u(1-u)^{\beta}, \qquad u\in[0,1), \tag{1.2}$$

with constants $D>0, \alpha, \beta\in\mathbb{R}$, and ν denotes the outward normal vector of $\partial\Omega$. The number 1 in (1.2) is defined as *crowding capacity*, the maximal cell numbers that can be accommodated in an unit volume of space. The cell has a logistic growth with growth rate $\mu>0$ and carrying capacity u_c with $0<u_c<1$ (see details in [25]), and $\chi>0$ is called the chemotactic coefficient. The detailed derivation of (1.1)–(1.2) can be found in [24, 27]. The striking and interesting feature of the model (1.1)–(1.2) is the possible singularity or degeneracy at where u attains the threshold value 1 in either the diffusion coefficient or the chemotactic sensitivity or both. Therefore whether the solution u attains 1 is the foremost theoretical question. When the cell growth is neglected (i.e. $\mu=0$), the results of [27, 26] showed that if $\alpha+\beta>1$ or $\alpha=0, \beta=1$, the solution u satisfies $0<u<1$ for any $(x,t)\in\mathbb{R}\times(0,\infty)$ with initial data (u_0, v_0) satisfying

$$(u_0, v_0)\in[W^{1,\infty}(\Omega)]^2 \text{ and } 0\leqslant u_0(x)<1,\ v_0(x)\geqslant 0,\ x\in\overline{\Omega}. \tag{1.3}$$

In other regimes of parameters α and β, the singularity or degeneracy (meaning u attains 1) may happen in either finite or infinite time, except for a borderline case $\alpha>0, \alpha+\beta=1$ which still remains unknown (see [27]).

The existence of non-constant steady states of (1.1) with $\mu>0$ for $\alpha+\beta>1$ has been established by the authors in [17] by the degree theory. The purpose of this paper is to use the global bifurcation theorem to find the local and global structures of non-constant steady states of (1.1) with $\mu>0$ bifurcating from the constant steady state, and then find the stability criterion of the bifurcating steady states. Since the singularity may occur when $\alpha+\beta\leqslant 1$ (except for $\alpha=0, \beta=1$), we restrict our attention in this paper to the case

$$\alpha+\beta>1, \alpha, \beta\in\mathbb{R}. \tag{1.4}$$

While for the case $\alpha=0, \beta=1$, many results are available. First without cell growth ($\mu=0$), the existence of steady states of chemotaxis system (1.1) was rigorously established in [23] in one dimension via the global bifurcation theorem in [22] while a detailed local bifurcation analysis was performed previously in [20]. The global existence of classical solutions have been obtained in [28, 29] for $\mu\geqslant 0$ and the convergence of solutions to equilibria with $\mu=0$ was studied in [12]. The pattern formation of (1.1) was numerically investigated in [18, 25] for both $\mu=0$ and $\mu>0$. It was found that the model (1.1) with cell growth ($\mu>0$) appears to typically exhibit merging and emerging chaotic patterns in contrast to single merging aggregation patterns for $\mu=0$. Then an important question arise as whether or not the volume-filling chemotaxis model for $\mu>0$ can develop stationary patterns. This question was first confirmed analytically in [16] by the degree theory, which differs from the bifurcation approach used in [23, 20] where $\mu = 0$ and the cell mass conservation is essentially used. More delicate analysis of local dynamics of the aggregation patterns was performed recently in [15].

The paper is organized as follows. In section 2, we establish the global existence of classical solutions to (1.1)–(1.3) and show that the solution is bounded below and above by constants. In section 3, the local and global bifurcation analysis will be performed to examine the structure of bifurcating steady states. The analysis on the stability of bifurcating steady states with small amplitude will be given in section 4. Finally in section 5, we shall show numerical simulations of bifurcating patterns and verify our analytical results numerically.

2. Existence of global solutions

We first introduce some notations used in the paper for readers' convenience. In the sequel, we denote the measure of the set A by $|A|$; let $W^{m,p}(\Omega, \mathbb{R}^N)$ for $m \geqslant 1,\ 1 < p < +\infty$ be Sobolev space of $\mathbb{R}^N$-valued functions with norm $\|\cdot\|_{m,p}$. When $p = 2$, $W^{m,2}(\Omega, \mathbb{R}^N)$ is written as $H^m(\Omega)$. Let $L^p(\Omega)(1 \leqslant p \leqslant \infty)$ denote the usual Lebesgue space in a bounded domain $\Omega \subset \mathbb{R}^n$ with norm $\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \mathrm{d}x\right)^{1/p}$ for $1 \leqslant p < \infty$ and $\|f\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_{x\in\Omega} |f(x)|$. When $p \in (n, +\infty)$, $W^{1,p}(\Omega, \mathbb{R}^2) \hookrightarrow C(\Omega, \mathbb{R}^2)$ which is the space of $\mathbb{R}^2$-valued continuous functions.

In this section, we shall show that the parabolic system (1.1)–(1.3) has an invariant region

$$\mathcal{X} = \{(u, v) : 0 \leqslant u < 1, 0 \leqslant v < 1\} \tag{2.1}$$

for any initial values u_0 and v_0 fulfilling (1.3). This is a consequence of the following theorem.

Theorem 2.1. *Let (u_0, v_0) fulfill (1.3), and α and β satisfy (1.4). Then the problem (1.1)–(1.3) with (1.4) has a global classical solution (u, v). Moreover, there exists a constant $\delta > 0$ such that*

$$0 \leqslant u(x,t) \leqslant 1-\delta,\quad 0 \leqslant v(x,t) \leqslant 1-\delta,\quad \text{for all}\ \ (x,t) \in \Omega \times (0, \infty). \tag{2.2}$$

For the case of no cell growth ($\mu = 0$), the existence of the global in-time solutions was proved in [27, 26]. In this section we study the case for $\mu > 0$. In order to prove theorem 2.1, we shall first show the existence of local solutions, then verify (u, v) is uniformly bounded in t. In particular, one needs to show that $u(x, t)$ is separated from 1 for any $t > 0$.

Lemma 2.2 (Local existence). *Suppose that (u_0, v_0) satisfies (1.3). There exists a positive constant T_0 depending on initial data (u_0, v_0) such that the initial-boundary problem (1.1) with (1.2) has a unique maximal solution (u, v) defined on $\Omega \times [0, T_0)$ satisfying*

$$(u, v) \in C(\overline{\Omega} \times [0, T_0); \mathbb{R}^2) \cap C^{2,1}(\overline{\Omega} \times (0, T_0); \mathbb{R}^2)$$

with $0 \leqslant u, v < 1$. Furthermore if $T_0 < \infty$, then

$$\lim_{t \nearrow T_0} \sup \|u(x,t)\|_{L^\infty(\Omega)} = 1. \tag{2.3}$$

Proof. We shall apply the abstract theory developed by Amann [4] to prove this lemma. Let $\omega = (u, v) \in \mathbb{R}^2$. Then the system (1.1) with (1.2) can be reformulated as

$$\begin{cases} \omega_t = \nabla \cdot (\mathbb{A}(\omega)\nabla\omega) + \mathbb{G}(\omega), & x \in \Omega \\ \dfrac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \\ \omega(0, \cdot) = (u_0, v_0), & x \in \overline{\Omega}, \end{cases} \tag{2.4}$$

where

$$\mathbb{A}(\omega) = \begin{pmatrix} D(1-u)^{-\alpha} & -\chi u(1-u)^{\beta} \\ 0 & 1 \end{pmatrix}, \ \mathbb{G}(\omega) = \begin{pmatrix} \mu u(1-u/u_c) \\ u-v \end{pmatrix}.$$

Since the given initial conditions (1.3) satisfy $0 \leqslant u_0 < 1-\delta_0$ for some $0 < \delta_0 < 1$, it is clear that the matrix $\mathbb{A}(\omega)$ are positively definite at $t=0$. Hence the system (2.4) is normally parabolic and local existence of solution follows from [3, theorem 7.3], i.e. there exists a $T_0 > 0$ such that the unique solution $(u, v) \in C(\overline{\Omega} \times [0, T_0); \mathbb{R}^2) \cap C^{1,2}(\overline{\Omega} \times (0, T_0); \mathbb{R}^2)$ with $u < 1-\delta_0$ exists. Next we apply the maximum principle to prove that $u, \ v \geqslant 0$. To this end, we write the first equation of (1.1) as

$$\begin{aligned} u_t = {} & D(1-u)^{-\alpha}\Delta u + [D\alpha(1-u)^{-1-\alpha}\nabla u - \chi(1-u)^{\beta}\nabla v \\ & +\chi\beta u(1-u)^{\beta-1}\nabla v]\nabla u - \chi u(1-u)^{\beta}\Delta v + \mu u((1-u/u_c). \end{aligned} \tag{2.5}$$

Then the strong maximum principle applied to (2.5) with the Neumann boundary condition asserts that $u > 0$ for all $(x, t) \in \Omega \times (0, T_0)$ due to $u_0 \not\equiv 0$. Similarly we have $v > 0$ for any $(x, t) \in \Omega \times (0, T_0)$ by the strong maximum principle applied to the second equation of (1.1). Next we prove that $\overline{v} = 1-\delta_0$ is an upper solution of the v equation. Define an operator $\mathbb{T}$ by $\mathbb{T}v = \frac{\partial v}{\partial t} - \Delta v - u + v$. Then $\mathbb{T}\overline{v} = -u + 1 - \delta_0 \geqslant 0$. By the comparison principle, we have $0 \leqslant v(x, t) \leqslant \overline{v} \leqslant 1-\delta_0$ for all $(x, t) \in \Omega \times [0, T)$. Finally the assertion (2.3) follows from [2, theorem 5.2] since $\mathbb{A}(w)$ is an upper triangular matrix. The proof of lemma 2.2 is completed. □

To extend the local solutions to global ones in time, a *priori* estimates $u(x, t) < 1$ for all $t > 0$ is needed. Due to the possible singularity/degeneracy in the diffusion, maximum principle can not be used to achieve such a goal. Here we adopt the idea of [27, 7] by deriving that $\|\frac{1}{1-u}\|_{L^\infty} < \infty$ for all $t > 0$ using the Moser iteration. To this end, we need to establish the L^p estimates for $\frac{1}{1-u}$ with $p > 1$. Before embarking on this, we first present a result on the cell mass $M = \int_\Omega u(x, t)\mathrm{d}x$, which enables us to employ the procedure of [27] to derive the L^∞ estimate of $\frac{1}{1-u}$.

Lemma 2.3. *Let $(u, v)(x, t)$ be a non-negative solution of (1.1)–(1.2) with $0 < u_c < 1$. Then it holds that*

$$M = \|u(\cdot, t)\|_{L^1(\Omega)} < |\Omega| \text{ for all } t > 0.$$

Proof. Integrating the first equation of (1.1) and using the Neumann boundary condition, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\mathrm{d}x = \mu\int_\Omega u(1-u/u_c)\mathrm{d}x.$$

Noticing that $u(1-u/u_c) = \frac{1}{u_c}u(u_c-u)$, one can readily derives that $u(1-u/u_c) \leqslant -u+u_c$ for all $u \geqslant 0$, which leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\mathrm{d}x + \mu\int_\Omega u\mathrm{d}x \leqslant \mu u_c|\Omega|.$$

Then integration of above inequality yields

$$\int_\Omega u\mathrm{d}x \leqslant \mathrm{e}^{-\mu t}\left(\int_\Omega u_0\mathrm{d}x - u_c|\Omega|\right) + u_c|\Omega|.$$

Then we have

$$\int_\Omega u\mathrm{d}x \leqslant \begin{cases} u_c|\Omega|, & \text{if } \int_\Omega u_0\mathrm{d}x \leqslant u_c|\Omega| \\ \int_\Omega u_0\mathrm{d}x, & \text{if } \int_\Omega u_0\mathrm{d}x > u_c|\Omega|. \end{cases}$$

That is $\int_\Omega u\mathrm{d}x \leqslant \max\{\|u_0\|_{L^1(\Omega)}, u_c|\Omega|\}$. Since $0 \leqslant u_0 < 1$, we have $\int_\Omega u_0\mathrm{d}x < |\Omega|$. Moreover $u_c|\Omega| < |\Omega|$ due to $0 < u_c < 1$. Then the proof is completed. □

Lemma 2.4. *Assume α and β satisfy (1.4) and (u_0, v_0) satisfies (1.3). Let (u, v) be a solution of (1.1)–(1.2) such that $0 \leqslant u < 1$ in $\Omega \times (0, T)$ and $v \in L^\infty((0, T); C^1(\Omega))$ satisfies $\frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega$ and $|\nabla v| \leqslant K$ in $\Omega \times (0, T)$ with some constant $K > 0$. Then, for any $p > 1$, there exists a constant $C(K, p) > 0$ such that*

$$\int_\Omega (1-u)^{-p}(x,t)\mathrm{d}x \leqslant C(K,p), \qquad \text{for all } t \in (0,T). \tag{2.6}$$

Proof. Let $w(x,t) = 1-u(x,t)$ and then multiple the first equation of (1.1) by $w^{-p-1} (p > 1)$ and integrate it over Ω. Applying Green's formula and the Neumann boundary condition, we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^{-p}\mathrm{d}x + (p+1)D\int_\Omega w^{-p-2-\alpha}|\nabla w|^2\mathrm{d}x \\ &\quad = -(p+1)\chi\int_\Omega uw^{-p-2+\beta}\nabla v\nabla w\mathrm{d}x + \mu\int_\Omega u(1-u/u_c)w^{-p-1}\mathrm{d}x \\ &\quad \leqslant (p+1)\chi\int_\Omega uw^{-p-2+\beta}|\nabla v\nabla w|\mathrm{d}x + \mu\int_\Omega u(1-u/u_c)w^{-p-1}\mathrm{d}x. \end{aligned} \tag{2.7}$$

Let $\epsilon = \frac{\chi}{D}$, $a = w^{\frac{-p-\alpha-2}{2}}|\nabla w|$, $b = w^{\frac{-p+\alpha+2\beta-2}{2}}u|\nabla v|$. Then Young's inequality $ab \leqslant \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$ and the fact $0 \leqslant u < 1$ yield

$$\begin{aligned} &(p+1)\chi\int_\Omega uw^{-p-2+\beta}|\nabla v\nabla w|\mathrm{d}x \\ &\quad \leqslant \frac{(p+1)D}{2}\int_\Omega w^{-p-2-\alpha}|\nabla w|^2\mathrm{d}x + \frac{(p+1)\chi^2K^2}{2D}\int_\Omega w^{-p+\alpha+2\beta-2}\mathrm{d}x. \end{aligned} \tag{2.8}$$

From $p > 1$ it follows that $p \leqslant p+1 \leqslant 2p$. By (2.7) and (2.8), we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^{-p}\mathrm{d}x + \frac{pD}{2}\int_\Omega w^{-p-2-\alpha}|\nabla w|^2\mathrm{d}x \\ &\quad \leqslant \frac{p\chi^2K^2}{D}\int_\Omega w^{-p+\alpha+2\beta-2}\mathrm{d}x + \mu\int_\Omega u(1-u/u_c)w^{-p-1}\mathrm{d}x. \end{aligned} \tag{2.9}$$

Since $\int_\Omega w^{-p-2-\alpha}|\nabla w|^2\mathrm{d}x = \frac{4}{(p+\alpha)^2}\int_\Omega |\nabla w^{-\frac{p+\alpha}{2}}|^2\mathrm{d}x$, the inequality (2.9) is equivalent to

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^{-p}\mathrm{d}x + \frac{4p^2D}{(p+\alpha)^2}\int_\Omega |\nabla w^{-\frac{p+\alpha}{2}}|^2\mathrm{d}x \\ &\quad \leqslant \frac{p^2\chi^2K^2}{D}\int_\Omega w^{-p+\alpha+2\beta-2}\mathrm{d}x + \mu p\int_\Omega u(1-u/u_c)w^{-p-1}\mathrm{d}x \end{aligned} \tag{2.10}$$

for all $t \in (0,T)$. To proceed, we let $p > 1$ to be sufficiently large to fulfill

$$\frac{p-\alpha-2\beta+2}{p+\alpha} \geqslant \frac{1}{2}, \text{ for } p > |\alpha| \text{ and } p > -\frac{n\alpha}{2}. \tag{2.11}$$

By the lemma 2.3, we have $M = \int_\Omega u\mathrm{d}x < |\Omega|$. Hence, for any $a \in (1, \frac{|\Omega|}{M})$, we have $|\{u(\cdot,t) > \frac{aM}{|\Omega|}\}| \leqslant \frac{|\Omega|}{a}$ and hence

$$|\{u(\cdot,t) \leqslant aM/|\Omega|\}| \geqslant \frac{a-1}{a}|\Omega|, \quad \text{for all } \ t \in (0,T). \tag{2.12}$$

By (2.12) and (2.11), it is easy to check that

$$\left|\left\{(1-u)^{-\frac{p+\alpha}{2}} \leqslant (1-au_c)^{-\frac{p+\alpha}{2}}\right\}\right| \geqslant \frac{a-1}{a}|\Omega|, \quad \text{for all } \ t \in (0,T).$$

Then we can follow the proof of [27, lemma 5.1] with (2.11) to find constants $c_3, c_4 > 0$ such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^{-p}\mathrm{d}x \leqslant c_3 - \frac{p^2D}{(p+\alpha)^2}\left(\frac{1}{c_4}\int_\Omega w^{-p}\mathrm{d}x - 1\right)^{\frac{p+\alpha}{p}} \\ + \mu\int_\Omega u(1-u/u_c)w^{-p-1}\mathrm{d}x\end{aligned} \tag{2.13}$$

for all $t \in (0,T)$. For brevity, we omit the details here. The last term in (2.13) can be estimated as

$$\begin{aligned}\mu\int_\Omega u(1-u/u_c)w^{-p-1}\mathrm{d}x &\leqslant \mu\int_{|\{u\leqslant u_c\}|} u(1-u/u_c)w^{-p-1}\mathrm{d}x \\ &\leqslant \mu u_c(1-u_c)^{-p-1}|\Omega|.\end{aligned} \tag{2.14}$$

Therefore there exists a constant $c_5 > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^{-p} \leqslant c_3 - \frac{p^2D}{(p+\alpha)^2}\left(\frac{1}{c_4}\int_\Omega w^{-p}\mathrm{d}x - 1\right)^{\frac{p+\alpha}{p}}. \tag{2.15}$$

By the comparison argument of ordinary differential equations, it follows from (2.15) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^{-p} \leqslant \max\left\{\int_\Omega (1-u_0)^{-p},\ c_4\left[\left(\frac{(p+\alpha)^2c_5}{p^2D}\right)^{\frac{p}{p+\alpha}} + 1\right]\right\} \tag{2.16}$$

for $t \in (0,T)$. The proof is completed. □

We continue to carry out the L^∞ estimate of w^{-1} by applying a variant of the Moser-Alikakos iterative developed in [1].

Lemma 2.5. *Let the assumptions in lemma 2.4 hold. Then there exists a constant $C(K)$ such that*

$$\frac{1}{1-u} \leqslant C(K). \tag{2.17}$$

Proof. The proof of this lemma is similar to lemma 4.2 in [27]. Here we just sketch the different parts and leave out the similar parts for brevity. We first construct a recursive sequence $\{p_k\}_{k\in\mathbb{N}}$ by fixing $p_0 > 1$ such that $p_0 > \frac{n|\alpha|}{2}$ and $p_0 > 4(\alpha+\beta-1)-\alpha$. Set $p_k = 2p_{k-1} - \alpha,\ k \geqslant 1$. It is clear that $\{p_k\}_{k\in\mathbb{N}}$ is strictly increasing and there are constants σ_1 and σ_2 such that $\sigma_1 2^k \leqslant p_k \leqslant \sigma_2 2^k$, for all $k \geqslant 0$. Let

$$q_k = \frac{2(p_k - \alpha - 2\beta + 2)}{p_k + \alpha} \equiv 2 - \frac{4(\alpha+\beta-1)}{p_k+\alpha},\ k \geqslant 1.$$

Then by the monotonicity of $\{p_k\}_{k\in\mathbb{N}}$, we have $1 < q_k \leqslant 2$, $k \geqslant 1$. Furthermore, set $\tilde{q}_k = \frac{2p_k}{p_k+\alpha}$, $k \geqslant 1$, we have $1 < \tilde{q}_k \leqslant \frac{2p_k}{p_k-|\alpha|} \leqslant \frac{2p_0}{p_0-|\alpha|}$, $k \geqslant 1$, and it follows that $\overline{q} = \frac{2p_0}{p_0-|\alpha|} < \frac{2n}{(n-2)_+}$. Our goal is to derive upper bounds for

$$A_k = \max\left\{1, \int_\Omega w^{-p_k}(x,t)\mathrm{d}x\right\}, \quad k \geqslant 0,$$

where $w(x,t) = 1 - u(x,t)$. To this end, we recall (2.10), which results in

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^{-p_k}\mathrm{d}x + b_1\int_\Omega |\nabla w^{\frac{p_k+\alpha}{2}}|^2\mathrm{d}x\\ &\quad\leqslant b_2 p_k^2\int_\Omega w^{-p_k+\alpha+2\beta-2}\mathrm{d}x + \mu\int_\Omega u(1-u/u_c)w^{-p-1}\mathrm{d}x\end{aligned} \tag{2.18}$$

for $b_1 \in (0,1]$, $b_2 > 0$ and all $t \in (0,T)$, b_1 and b_2, and b_3, b_4 used later, may depend on K but not on t, T or k. We note that (2.19) corresponds to (4.21) in [27]. Henceforward using the same technique as in [27], we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^{-p_k}\mathrm{d}x &\leqslant -b_3\left(\int_\Omega w^{-p_k}\mathrm{d}x\right)^{\frac{p_k+\alpha}{p_k}} + b_4 2^{(n+2)k}A_{k-1}^2\\ &\quad+\mu\int_\Omega u(1-u/u_c)w^{-p_k-1}\mathrm{d}x\end{aligned} \tag{2.19}$$

for b_3, $b_4 > 0$ and all $t \in (0,T)$. Note that the last term in (2.19) is bounded. Since $A_{k-1} \geqslant 1$, then exists a constant $b_5 > 0$ such that the last term in (2.19) can be bounded by $b_5 A_{k-1}^2$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^{-p_k}\mathrm{d}x \leqslant -b_3\left(\int_\Omega w^{-p_k}\mathrm{d}x\right)^{\frac{p_k+\alpha}{p_k}} + [b_5 + b_4 2^{(n+2)k}]A_{k-1}^2. \tag{2.20}$$

Then integrating (2.20) leads to

$$\int_\Omega w^{-p_k}\mathrm{d}x \leqslant \max\left\{\int_\Omega (1-u_0)^{-p_k}\mathrm{d}x, \left(\frac{b_4 2^{(n+2)k} + b_5}{b_3}A_{k-1}^2\right)^{\frac{p_k}{p_k+\alpha}}\right\}, \ t \in (0,T)$$

by which, we have, for all $k \geqslant 1$,

$$A_k \leqslant \max\left\{1, \int_\Omega (1-u_0)^{-p_k}\mathrm{d}x, b^k A_{k-1}^{2(1+\delta_k)}\right\}$$

with $\delta_k = |\frac{-\alpha}{p_k+\alpha}|$ and some constant $b > 1$ independent of k. Whereafter, the same argument in the proof of [27, lemma 4.2] yields (2.17) by noting that A_0 is finite from lemma 2.4. The proof is completed. □

We are now in a position to prove theorem 2.1.

Proof of theorem 2.1. Since $0 \leqslant u \leqslant 1$ and ∇v_0 is bounded, the parabolic regularity applied to the second equation of (1.1) asserts that there exists $K > 0$ such that $|\nabla v| \leqslant K$ in $\Omega \times (0, T_{\max})$ (see [30, proposition 1] for details). By lemma 2.5, we have

$$\sup_{T\in(0,T_{\max})} A_0(T) < \infty.$$

Then the results in theorem 2.1 immediately follow from lemmas 2.2 and 2.5. □

3. Bifurcation analysis

System (1.1)–(1.2) has two constant steady states $\mathcal{O} = (0, 0)$ and $\tilde{\omega} = (u_c, u_c)$. By a routine linearized stability analysis, we find that $\mathcal{O}$ is always unstable, and $\tilde{\omega}$ is globally stable provided the parameter χ satisfies

$$\chi \leqslant \frac{\mu + d(u_c) + 2\sqrt{\mu d(u_c)}}{h(u_c)} \overset{\text{def}}{=} \chi_c. \tag{3.1}$$

Furthermore, when $\chi > \chi_c$, $\tilde{\omega}$ is locally unstable. The existence of non-constant steady state system of (1.1)–(1.2) with (1.4) is established under certain conditions in [17]. In this section, we shall choose χ as a bifurcation parameter and fix the rest of parameters to explore the structure of non-constant steady states of (1.1) bifurcating from the constant steady state $\tilde{\omega}$ in one dimensional. Before proceeding, we present some properties about the negative Laplace operator $-\Delta$. Let

$$0 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots \tag{3.2}$$

be the eigenvalues of the operator $-\Delta$ on Ω with the homogeneous Neumann boundary condition, $E(\lambda_i)$ be the eigenspace corresponding to λ_i in $H^1(\Omega, \mathbb{R}^2)$, $\{\varphi_{ij} : j = 1, \cdots, \dim E(\lambda_i)\}$ be an orthonormal basis of $E(\lambda_i)$, and $X_{ij} = \{\mathbf{c}\varphi_{ij} : \mathbf{c} \in \mathbb{R}^2\}$. Let $\mathbf{X} = H^1(\Omega, \mathbb{R}^2)$. Then we have

$$\mathbf{X}_i = \bigoplus_{j=1}^{\dim E(\lambda_i)} X_{ij}, \qquad \mathbf{X} = \bigoplus_{i=1}^{\infty} \mathbf{X}_i. \tag{3.3}$$

Moreover, it is well-known that the following eigenvalue problem

$$\begin{cases} -\varphi''(x) = \lambda\varphi(x), & x \in (0, l), \\ \varphi'(x) = 0, & x = 0, l. \end{cases} \tag{3.4}$$

has a sequence of simple eigenvalues with corresponding eigenfunctions given by

$$\lambda_j = (\pi j/l)^2, \ \varphi_j(x) = \begin{cases} 1, & j = 0, \\ \cos(\pi j x/l), & j > 0 \end{cases} \tag{3.5}$$

where $j = 0, 1, 2, \cdots$. Clearly this set of eigenfunctions constitutes an orthonormal basis in $L^2(0, l)$.

In one dimension, we let $\Omega = (0, l)$ with $l > 0$. Then a steady state of (1.1) is a positive solution $(u(x), v(x))$ to the elliptic system:

$$\begin{cases} -(d(u)u')' + \chi(h(u)v')' = \mu u(1 - u/u_c), \ x \in (0, l), \\ -v'' = u - v, \ x \in (0, l), \\ u'(0) = u'(l) = 0, \ v'(0) = v'(l) = 0, \end{cases} \tag{3.6}$$

where $' = \frac{\mathrm{d}}{\mathrm{d}x}$. Next we shall use χ as the bifurcation parameter and apply the bifurcation theory to study the local and global structures of solutions to (3.6) in $\mathcal{X}$ define in (2.1). Finally based on the solution structure, we shall explore the stability of bifurcating steady states and find the stability conditions. Hereafter, (1.4) is assumed without mention anymore.

3.1. Local bifurcation

The standard bifurcation technique and asymptotic analysis method will be used to obtain bifurcation points and a precise description for the structure of positive solutions of (3.6) near the bifurcation points. We first define a Banach space X by

$$X = \{(u, v) : u, v \in C^2([0, l]), u' = v' = 0 \text{ at } x = 0, \ l\} \tag{3.7}$$

equipped with the usual C^2-norm, and Hilbert space Y by $Y = L^2(0,l) \times L^2(0,l)$ with the inner product $(\omega_1, \omega_2)_Y = (u_1, u_2)_{L^2(0,l)} + (v_1, v_2)_{L^2(0,l)}$ for $\omega_1 = (u_1, v_1) \in Y$, $\omega_2 = (u_2, v_2) \in Y$. Define the map $H\colon (0,\infty) \times X \longrightarrow Y$ by

$$H(\chi, \omega) = \begin{pmatrix} (d(u)u')' + \chi(h(u)v')' + \mu u(1 - u/u_c) \\ v'' + u - v. \end{pmatrix}.$$

Then the solutions of (3.6) are just zeros of this map. Therefore, (3.6) is equivalent to

$$H(\chi, \tilde{\omega}) = 0 \text{ for all } \chi > 0.$$

Next we shall perform the asymptotic analysis for the non-constant solution $\omega^*(x) = (u^*(x), v^*(x))$ of (3.6) bifurcating from $\tilde{\omega}$ with small amplitude. For this purpose, we assume

$$\chi = \chi_0 + \sum_{k=1}^{\infty} \varepsilon^k \chi_k, \tag{3.8}$$

where $0 < \varepsilon \ll 1$. Then let $u^*(x)$ and $v^*(x)$ have expansions as power series in ε as follows:

$$\begin{cases} u^* = u_c + \sum_{k=1}^{\infty} \varepsilon^k u_k \\ v^* = u_c + \sum_{k=1}^{\infty} \varepsilon^k v_k. \end{cases} \tag{3.9}$$

Substituting (3.8) and (3.9) into (3.6) and equating the $O(\varepsilon)$ and $O(\varepsilon^2)$ terms, respectively, we get two systems

$$\begin{cases} d(u_c)u_1'' + (\chi_0 h(u_c) - \mu)u_1 - \chi_0 h(u_c)v_1 = 0, & x \in (0,l), \\ v_1'' + u_1 - v_1 = 0, & x \in (0,l), \\ u_1'(0) = u_1'(l) = 0, \\ v_1'(0) = v_1'(l) = 0 \end{cases} \tag{3.10}$$

and

$$\begin{cases} d(u_c)u_2'' + (\chi_0 h(u_c) - \mu)u_2 - \chi_0 h(u_c)v_2 = F_1, & x \in (0,l), \\ v_2'' + u_2 - v_2 = 0, & x \in (0,l), \\ u_2'(0) = u_2'(l) = 0, \\ v_2'(0) = v_2'(l) = 0, \end{cases} \tag{3.11}$$

where

$$F_1 = -d'(u_c)(u_1 u_1')' + \chi_0 h'(u_c)(u_1 v_1')' + \chi_1 h(u_c) v_1'' + \frac{\mu}{u_c} u_1^2. \tag{3.12}$$

Then we substitute the second equation of (3.10) into the first one, and obtain

$$\begin{cases} d(u_c)u_1'' - \mu u_1 - \chi_0 h(u_c)v_1'' = 0, & x \in (0,l), \\ v_1'' + u_1 - v_1 = 0, & x \in (0,l), \\ u_1'(0) = u_1'(l) = 0, \\ v_1'(0) = v_1'(l) = 0, \end{cases} \tag{3.13}$$

which has a matrix form of

$$\begin{pmatrix} -\mu + d(u_c)\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} & -\chi_0 h(u_c)\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} \\ 1 & -1 + \dfrac{\mathrm{d}^2}{\mathrm{d}x^2} \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = 0. \tag{3.14}$$

We can solve (3.14) directly and get non-constant solutions as

$$\begin{cases} u_1 = c_1(j)\varphi_j, \ c_1(j) = 1 + \lambda_j > 0, \\ v_1 = \varphi_j, \end{cases} \tag{3.15}$$

as long as χ_0 is given by

$$\chi_0 = \frac{\left(\mu + d(u_c)\lambda_j\right)(1 + \lambda_j)}{\lambda_j h(u_c)} \overset{\text{def}}{=} \chi_0^j, \ j = 1, 2, \cdots \tag{3.16}$$

where (λ_j, φ_j) is given by (3.5). Here we note that the solution given in (3.15) is unique up to a constant multiple for any positive integer j, and this constant can be absorbed into ε in (3.9). In addition, the uniqueness of solution indicates that $\chi_0^j \neq \chi_0^k$ for any integer $k \neq j$.

Let us set

$$\chi_{\min} = \min_{j\in\mathbb{Z}^+} \chi_0^j = \min_j \left\{ \frac{\left(\mu + d(u_c)\lambda_j\right)(1 + \lambda_j)}{\lambda_j h(u_c)}, \ j = 1, 2, \cdots \right\} = \chi_0^{j_0}$$

for a positive integer j_0. That is, j_0 is the wave mode minimizing χ_0^j such that $\chi_0^{j_0}$ is the bifurcation value. For convenience, we shall call j_0 the *principle wave mode*. Hence the first bifurcation will occur when the parameter χ crosses the bifurcation value $\chi_{\min}$. If the bifurcation is stable, it will then yield the pattern solution (u^*, v^*) given in (3.9) and (3.15). Then we have the following local bifurcation theorem which can also be proved by verifying the conditions given in [8].

Theorem 3.1. *If $\chi_0^j \neq \chi_0^k$ for any positive integer $k \neq j$, then χ_0^j is a bifurcation value of the equation $H(\chi, \omega) = 0$ with respect to the curve $(\chi, \widetilde{\omega})$, $\chi > 0$. Furthermore, there is a one-parameter family of non-trivial solutions $\Gamma(\varepsilon) = (\chi(\varepsilon), u^*(\varepsilon), v^*(\varepsilon))$ of (3.6) for $|\varepsilon|$ sufficiently small, where $\chi(\varepsilon)$, $u(\varepsilon)$ and $v(\varepsilon)$ are continuous functions such that $\chi(0) = \chi_0^j$ and*

$$u^*(\varepsilon) = u_c + \varepsilon c_1(j)\varphi_j + o(\varepsilon), \ \ v^*(\varepsilon) = u_c + \varepsilon\varphi_j + o(\varepsilon).$$

The zero-point set of $H(\chi, \omega)$ constitutes two curves $(\chi, \widetilde{\omega})$ and $\Gamma(\varepsilon)$ in a neighborhood of the bifurcation point $(\chi_0^j, \widetilde{\omega})$.

Remark 3.1. The bifurcation value χ_0^j has the following properties.

(i) When $\mu = 0$ (i.e. no cell growth), then

$$\chi_0^j = \frac{d(u_c)}{h(u_c)}(1 + \lambda_j) = \frac{D}{u_c(1 - u_c)^{\alpha+\beta}}(1 + \lambda_j), \quad \text{for each } j = 1, 2, \cdots$$

which attains the minimum at $j = j_0 = 1$. That is, the bifurcation occurs at the first wave mode $j = j_0 = 1$.

(ii) If the interval length l is sufficiently small, then λ_j will be large and χ_0^j attains the minimal value also at $j = j_0 = 1$. This can be seen from the fact

$$\chi_0^j = \frac{\left(d(u_c)\lambda_j + \mu\right)\left(1 + \lambda_j\right)}{h(u_c)\lambda_j} \sim \frac{d(u_c)}{h(u_c)}\lambda_j, \quad \text{for each } j = 1, 2, \cdots. \tag{3.17}$$

(iii) By a simple computation, we have, for some positive integer j_0,

$$\chi_{\min} \geqslant \chi_c,$$

where '=' holds if and only if

$$j_0 = \left[\frac{\mu}{d(u_c)}\right]^{\frac{1}{4}} \frac{l}{\pi} \quad \text{and is an integer.} \tag{3.18}$$

Here χ_c is defined in (3.1). Therefore we find that the first bifurcation value $\chi_{\min}$ is greater than the critical value χ_c. They coincide only if the mode j_0 at which χ_0^j attains minimum is given by (3.18).

Theorem 3.1 implies that $(\chi_0^j, \widetilde{\omega})$ is a bifurcation point with respect to the trivial branch $(\chi, \widetilde{\omega})$, and there are infinity such bifurcation points. Let C denote the closure of the non-constant solution set of $H(\chi, \omega) = 0$, and Γ_j the connected component of $C \cup \{(\chi_0^j, \widetilde{\omega})\}$ to which $\{(\chi_0^j, \widetilde{\omega})\}$ belongs. In a neighborhood of the bifurcation point $(\chi_0^j, \widetilde{\omega})$ the curve Γ_j is formulated by the eigenfunction φ_j. Obviously φ_j has j zero points in the interval $(0, l)$. Thus the non-constant solutions in Γ_j are called mode j steady states.

Theorem 3.1 provides the detail information of the bifurcating curve Γ_j near the bifurcation point. In order to understand its global structure, the global bifurcation theory and the Leray-Schauder degree for compact operators will be applied.

3.2. Global bifurcation

This subsection is to investigating the global nature of the curve of non-constant solutions $\Gamma(\varepsilon)$ in the $\chi - (u, v)$ plane. We first introduce the standard abstract bifurcation theorem from [5,6] for readers' convenience. Let $\mathbb{X}$ be a Banach space and let $T : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ be a compact, continuously differentiable operator such that $T(a, 0) = 0$. Assume that T can be written as

$$T(a, U) = K(a)U + W(a, U), \tag{3.19}$$

where $K(a)$ is a linear compact operator and the Fréchet derivative $W_U(a, 0) = 0$. Regarding a as a bifurcation parameter, we will undertake a global bifurcation analysis for the equation

$$U = T(a, U). \tag{3.20}$$

We suppose that $I - K : \mathbb{X} \to \mathbb{X}$ is a bijection. Then the Leray-Schauder degree $\deg(I - K, \hat{B}, 0) = (-1)^p$, where $\hat{B}$ is a ball centered at 0 in $\mathbb{X}$ and p is the sum of the algebraic multiplicities of the eigenvalues of K that are larger than 1. If x_0 is an isolated fixed point of the operator T and B is a ball centered at x_0 such that x_0 is the unique fixed point of T in B, the index of T at x_0 is defined as

$$\mathrm{index}(T, x_0) = \deg(I - T, B, x_0).$$

Moreover, if x_0 is a fixed point of T and $I - T'(x_0)$ is invertible, then x_0 is an isolated fixed point of T and

$$\mathrm{index}(T, x_0) = \deg(I - T, B, x_0) = \deg(I - T'(x_0), \hat{B}, 0),$$

where B and $\hat{B}$ are sufficiently small.

We now state the result on the global bifurcation for the operator T defined by (3.19).

Lemma 3.2. *(Theorem 3.2 in [5]) Let a_0 be such that $I - K(a)$ is invertible if $0 < |a - a_0| < \epsilon$ for $\epsilon > 0$. Assume that* $\mathrm{index}(T(a, \cdot), 0)$ *is constant on $(a_0 - \epsilon, a_0)$ and on $(a_0, a_0 + \epsilon)$; moreover, if $a_0 - \epsilon < a_1 < a_0 < a_2 < a_0 + \epsilon$, then* $\mathrm{index}(T(a_1, \cdot), 0) \neq \mathrm{index}(T(a_2, \cdot), 0)$. *Then there exists a continuum $\mathcal{C}$ in the a-U plane of solutions of (3.20) such that one of the following alternatives is true*

(i) $\mathcal{C}$ joins $(a_0, 0)$ to $(\hat{a}, 0)$ where $I - K(\hat{a})$ is not invertible.

(ii) $\mathcal{C}$ joins $(a_0, 0)$ to ∞ in $\mathbb{R} \times \mathbb{X}$.

This theorem is essentially the same as [21, theorem 1.3]. In order to conveniently apply lemma 3.2 to the system (3.6), we first rewrite (3.6) as

$$\begin{cases} -u'' = f(u, v), & x \in (0, l) \\ -v'' = g(u, v), & x \in (0, l) \\ u' = v' = 0, & x = 0, l, \end{cases} \tag{3.21}$$

where $g(u,v) = u - v$ and

$$f(u,v) = \alpha(1-u)^{-1}|u'|^2 + \frac{1}{D}(1-u)^{\alpha}[\chi u(1-u)^{\beta}(u-v) - \chi(1-u)^{\beta-1}(1-u-\beta u)u'v' + \mu u(1-u/u_c)].$$

Then, let $\tilde{u} = u - u_c$, $\tilde{v} = v - u_c$. (3.21) becomes

$$\begin{cases} -\tilde{u}'' = f_0\tilde{u} + f_1\tilde{v} + f_2(\tilde{u},\tilde{v}), & x \in (0,l) \\ -\tilde{v}'' = \tilde{u} - \tilde{v}, & x \in (0,l) \\ \tilde{u}' = \tilde{v}' = 0, & x = 0,\ l, \end{cases} \tag{3.22}$$

where f_2 is higher-order terms of $\tilde{u}$ and $\tilde{v}$, and

$$f_0 = f_u'(u_c,u_c) = \frac{\chi h(u_c) - \mu}{d(u_c)}, \quad f_1 = f_v'(u_c,u_c) = -\frac{\chi h(u_c)}{d(u_c)}. \tag{3.23}$$

The constant solution (u_c, u_c) of (3.6) is transformed to the zero solution $\mathcal{O} = (0,0)$ of (3.22). Let G_χ and G denote the inverse of the operators $f_0 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ and $1 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with homogeneous Neumann boundary condition, respectively. Set

$$U = (\tilde{u},\tilde{v}),\ K(\chi)U = \Big(2f_0G_\chi(\tilde{u}) + f_1G_\chi(\tilde{v}), G(\tilde{u})\Big), \text{ and } W(\chi,U) = \big(G_\chi(f_2(\tilde{u},\tilde{v})), 0\big).$$

Then the boundary value problem (3.22) is equivalent to the equation

$$U = K(\chi)U + W(\chi,U) \stackrel{\text{def}}{=} T(\chi,U), \quad K(\chi) = \begin{pmatrix} 2f_0G_\chi & f_1G_\chi \\ G & 0 \end{pmatrix} \tag{3.24}$$

in X defined in (3.7). Clearly $K(\chi)$ is a compact linear operator on X for any given $\chi > 0$, and $W(\chi,U) = o(\|U\|)$ for U near zero uniformly on closed χ sub-intervals of $(0,\infty)$ and is also a compact operator on X.

For a fixed integer j, the matrix $K(\chi_0^j)$ has the following property which plays a crucial role in the proof of the results of the global bifurcation:

Lemma 3.3. *Suppose that (1.4) holds. If $\chi_0^j \neq \chi_0^k$ for any integer $k \neq j$. Then 1 is an eigenvalue of $K(\chi_0^j)$ with algebraic multiplicity one.*

Proof. Assume that $\Psi = (\varphi,\psi)$, and $\varphi = \sum_{i=0}^{\infty} a_i\varphi_i$, $\psi = \sum_{i=0}^{\infty} b_i\varphi_i$. Let

$$\Big(K(\chi_0^j) - I\Big)\Psi = 0,$$

which leads to

$$\begin{pmatrix} -\mu + d(u_c)\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} & -\chi_0^j h(u_c)\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} \\ 1 & -1 + \dfrac{\mathrm{d}^2}{\mathrm{d}x^2} \end{pmatrix}\Psi = 0.$$

This system is same as (3.14) with $\chi_0 = \chi_0^j$. Hence from (3.15)–(3.16) and the paragraph following (3.16), we know that 1 is an eigenvalue of $K(\chi_0^j)$ with the unique eigenfunction $\Psi = \binom{1+\lambda_j}{1}\varphi_j$ which indicates $\dim\ker(K(\chi_0^j) - I) = 1$. Next we shall show the eigenvalue 1 is simple. Since the algebraic multiplicity of the eigenvalue 1 is the dimension of the generalized null space $\bigcup_{i=1}^{\infty}\ker(K(\chi_0^j) - I)^i$, it is sufficient to verify $\ker(K(\chi_0^j) - I) \cap R(K(\chi_0^j) - I) = \{0\}$.

Denote $K(\chi_0^j)$ by K. Let K^* be the adjoint of K. We now compute $\ker(K^* - I)$. Assume $(\varphi,\psi) \in \ker(K^* - I)$, we have from (3.24) that

$$\begin{cases} 2f_0^jG_\chi(\varphi) + G(\psi) = \varphi, \\ f_1^jG_\chi(\varphi) = \psi, \end{cases} \tag{3.25}$$

where $f_0^j = \frac{\mu}{\lambda_j d(u_c)} + 1 + \lambda_j$ and $f_1^j = -(1 + \frac{1}{\lambda_j})\frac{\mu}{d(u_c)} - 1 - \lambda_j$. According to the definition of G_χ and G, (3.25) can be rewritten in the form of

$$\begin{cases} -f_1^j \varphi'' = f_\varphi \varphi + f_\psi \psi, \\ -\psi'' = f_1^j \varphi - f_0^j \psi, \end{cases} \tag{3.26}$$

where

$$f_\varphi = 2 f_0^j f_1^j - f_1^j, \;\; f_\psi = f_1^j - 2((f_0^j)^2 - f_0^j).$$

Again set $\varphi = \sum_{i=0}^{\infty} a_i \varphi_i$, $\psi = \sum_{i=0}^{\infty} b_i \varphi_i$. By (3.26), we have

$$\sum_{i=0}^{\infty} L_i^* \begin{pmatrix} a_i \\ b_i \end{pmatrix} \varphi_i = 0, \;\; L_i^* = \begin{pmatrix} f_\varphi - f_1^j \lambda_i & f_\psi \\ f_1^j & -\lambda_i - f_0^j \end{pmatrix}.$$

By a straightforward calculation, we find $\det L_j^* = 0$ if and only if $i = j$ and

$$L_i^* = \begin{pmatrix} 0 & 0 \\ f_1^j & -\lambda_j - f_0^j \end{pmatrix}.$$

Therefore, $\ker(K^* - I)$ is spanned by $\begin{pmatrix} f_0^j + \lambda_j \\ f_1^j \end{pmatrix} \varphi_j$. Thus it is easy to check that the unique element Ψ in $\ker(K - I)$ does not belong to $(\ker(K^* - I))^\perp = R(K - I)$, and hence $\ker(K - I) \cap R(K - I) = \{0\}$. Therefore the algebraic multiplicity of the eigenvalue 1 is one. Thus, the lemma is proved. □

We are now in a position to present the result on global bifurcation for the system (3.6).

Theorem 3.4. *If $\chi_0^j \neq \chi_0^k$ for any integer $k \neq j$. Then the projection of the bifurcation curve Γ_j onto the χ-axis is the infinity interval (χ_0^j, ∞). Moreover, if $\chi > \chi_{\min}$ and $\chi \neq \chi_0^k$ for any positive integer k, then (3.6) has at least one non-constant positive solution.*

Proof. We only need to verify all the conditions of lemma 3.2. By lemma 3.3 and its proof, we know that, if $0 < \chi \neq \chi_0^j$ and χ lies in a small neighborhood of χ_0^j, then the linear operator $I - K(\chi) : X \to X$ is a bijection and thus $\mathcal{O}$ is an isolated solution of (3.24) for this fixed χ. We now compute index $(T(\chi, .), \mathcal{O})$ so that we can apply lemma 3.2. The index of this isolated zero fixed point of $T(\chi, .)$ is given by

$$\text{index}\,(T(\chi, .), \mathcal{O}) = \deg\,(I - K(\chi), \mathcal{B}, \mathcal{O}) = (-1)^\gamma,$$

where $\mathcal{B}$ is a sufficiently small ball centered at $\mathcal{O}$, and γ is the sum of the algebraic multiplicities of the eigenvalues of $K(\chi)$ that are large than 1. We shall verify that

$$\text{index}\left(T(\chi_0^j - \epsilon), \mathcal{O}\right) \neq \text{index}\left(T(\chi_0^j + \epsilon), \mathcal{O}\right) \tag{3.27}$$

for $\epsilon > 0$ sufficiently small. Indeed, if τ is an eigenvalue of $K(\chi)$ with eigenfunction (φ, ψ), then we have

$$\begin{cases} -\tau\varphi'' = (2 - \tau) f_0 \varphi + f_1 \psi, \\ -\tau\psi'' = \varphi - \tau\psi. \end{cases}$$

Applying $\varphi = \sum_{i=0}^{\infty} a_i \varphi_i$ and $\psi = \sum_{i=0}^{\infty} b_i \varphi_i$, we have

$$\sum_{i=0}^{\infty} \begin{pmatrix} (2 - \tau) f_0 - \tau\lambda_i & f_1 \\ 1 & -(1 + \lambda_i)\tau \end{pmatrix} \begin{pmatrix} a_i \\ b_i \end{pmatrix} \varphi_i = 0.$$

It is obvious that the set of eigenvalues of $K(\chi)$ is composed of all $\tau's$ that satisfy the characteristic equation

$$(f_0+\lambda_i)(1+\lambda_i)\tau^2 - 2f_0(1+\lambda_i)\tau - f_1 = 0, i = 0, 1, 2, \cdots. \tag{3.28}$$

Taking $\chi = \chi_0^j$, if $\tau = 1$ solves (3.28), then by (3.28) and (3.23) we have

$$\chi_0^j = \frac{(\mu + d(u_c)\lambda_i)\,(1+\lambda_i)}{h(u_c)\lambda_i} = \chi_0^i.$$

It follows from the assumption in theorem 3.4 that $j = i$. Therefore, without counting the eigenvalues corresponding to $i = j$ in (3.28), $K(\chi)$ has the same number of eigenvalues large than 1 for all χ close to χ_0^j, and they have the same multiplicities. For $i = j$, (3.28) has the following two roots:

$$\tau_1(\chi_0^j) = 1 \text{ and } \tau_2(\chi_0^j) = \frac{f_0^j - \lambda_j}{f_0^j + \lambda_j} < 1.$$

It is evident that $\tau_2(\chi) < 1$ always holds for χ sufficiently close to χ_0^j. So the change of $\tau_1(\chi)$ with the variable χ plays a critical role in our results.

By the quadratic equation (3.28), we can readily show that $\tau_1(\chi)$ is an increasing function of χ and crosses 1 as χ passes through χ_0^j. Thus we have

$$\tau_1(\chi_0^j - \epsilon) < 1 \text{ and } \tau_1(\chi_0^j + \epsilon) > 1.$$

Then the matrix $K(\chi_0^j + \epsilon)$ has exactly one more eigenvalue that is larger than 1 than $K(\chi_0^j - \epsilon)$ does. Using a similar argument as that in lemma 3.3, we can show that this eigenvalue has algebraic multiplicity one. So (3.27) is satisfied.

Now lemma 3.2 applied to $T(\chi,\cdot)$ asserts that Γ_j either meets ∞ in $R\times X$ or meets $(\chi_0^k,\tilde{\omega})$ for some $k \neq j$, $\chi_0^k > 0$. Since the solutions (u, v) of (3.6) are bounded by constants independent of χ which was verified in [17], we can prove the first alternative must occur by applying a reflective and periodic extension method exactly same as in [11] and any solution on the curve Γ_j must be positive. The proof of theorem 3.4 is completed. □

The theorem above shows that the bifurcation curve Γ_j emanating from $(\chi_0^j,\tilde{\omega})$ must join with ∞; however, we do not know whether it is possible that Γ_j meets some bifurcation points and then reaches ∞; Moreover the existence of non-constant solutions for $\chi = \chi_0^k$ is still not provided by our theorem.

4. Stability of bifurcation steady states

This section is devoted to studying the stability of the steady state (u_j^*, v_j^*) bifurcating from $(\chi_0^j,\tilde{\omega})$ by analyzing the sign of the principal eigenvalue of linearized operator around (u_j^*, v_j^*). To this end, we need to obtain the formula for χ_1 in (3.8). We first consider the adjoint system of the homogeneous system associated with (3.11):

$$\begin{cases} d(u_c)\overline{u}_2'' + (\chi_0 h(u_c) - \mu)\,\overline{u}_2 + \overline{v}_2 = 0,\\ \overline{v}_2'' - \chi_0 h(u_c)\overline{u}_2 - \overline{v}_2 = 0,\\ \overline{u}_2'(0) = \overline{u}_2'(l) = 0,\\ \overline{v}_2'(0) = \overline{v}_2'(l) = 0. \end{cases} \tag{4.1}$$

Solving (4.1), we obtain a nonzero solution

$$\begin{cases} \overline{u}_2 = c_2(j)\varphi_j, \quad c_2(j) = -\frac{1+\lambda_j}{\chi_0 h(u_c)} < 0,\\ \overline{v}_2 = \varphi_j, \end{cases} \tag{4.2}$$

where φ_j is defined as in (3.5) for $j = 1, 2, \cdots$. Then the solvability condition for (3.11) yields the solvability equation for χ_1 as follows:

$$\int_0^l \overline{u}_2 F_1 \mathrm{d}x = 0.$$

Substituting (3.12) into the above equation gives rise to

$$\chi_1 = \chi_1^j = 0, \quad \text{for all } j = 1, 2, \cdots.$$

In view of $\chi_1 = 0$ for each j, F_1 in (3.12) can be simplified to

$$F_1 = \frac{\mu}{2u_c} c_1^2 + \left(d'(u_c) c_1 \lambda_j - \chi_0^j h'(u_c) \lambda_j + \frac{\mu}{2u_c} c_1 \right) c_1 \cos(2\sqrt{\lambda_j} x).$$

By this, we can find a particular solution (u_2, v_2) of (3.11) in the form of

$$\begin{cases} u_2 = a_1(j) + a_2(j) \cos(2\sqrt{\lambda_j} x), \\ v_2 = a_3(j) + a_4(j) \cos(2\sqrt{\lambda_j} x), \end{cases} \tag{4.3}$$

where

$$a_1(j) = a_3(j) = -c_1^2/2u_c; \quad a_2(j) = (1 + 4\lambda_j) a_4(j);$$

$$a_4(j) = \frac{\left(d'(u_c)\lambda_j + \frac{\mu}{2u_c} \right) c_1^2 - \chi_0^j h'(u_c) c_1 \lambda_j}{4h(u_c)\chi_0^j \lambda_j - \left(\mu + 4d(u_c)\lambda_j \right) \left(1 + 4\lambda_j \right)}. \tag{4.4}$$

Due to $\chi_1 = 0$, we proceed to compute χ_2. Again we substitute (3.8) and (3.9) into (3.6) and equate the $O(\varepsilon^3)$ terms. Then we obtain

$$\begin{cases} d(u_c) u_3'' + (\chi_0 h(u_c) - \mu) u_3 - \chi_0 h(u_c) v_3 = F_2, \\ v_3'' + u_3 - v_3 = 0, \\ u_3'(0) = u_3'(l) = 0, \\ v_3'(0) = v_3'(l) = 0, \end{cases} \tag{4.5}$$

where

$$\begin{aligned} F_2 = & -\left[d'(u_c)(u_1 u_2' + u_2 u_1') + d''(u_c)(\frac{1}{2} u_1^2 u_1') \right]' \\ & + \chi_0 \left[h'(u_c)(u_1 v_2' + u_2 v_1') + h''(u_c)(\frac{1}{2} u_1^2 v_1') \right]' \\ & + \chi_1 \left[h(u_c) v_2' + h'(u_c)(u_1 v_1') \right]' \\ & + \chi_2 h(u_c) v_1'' + \frac{2\mu}{u_c} u_1 u_2. \end{aligned} \tag{4.6}$$

Similarly, by the solvability condition for (4.5), we have $\int_0^l \overline{u}_2 F_2 \mathrm{d}x = 0$ which gives

$$\int_0^l F_2 \varphi_j \mathrm{d}x = 0. \tag{4.7}$$

Then substituting (4.6) into (4.7) leads to

$$\begin{aligned} \chi_2^j = & \frac{-\chi_0^j}{h(u_c)} \left[h'(u_c) \left(a_1(j) - \frac{a_2(j)}{2} + c_1(j) a_4(j) \right) + \frac{1}{8} h''(u_c) c_1^2(j) \right] \\ & + \frac{1}{h(u_c)} \left[\frac{1}{\lambda_{j_0}} \frac{\mu c_1(j)}{u_c} (2a_1(j) + a_2(j)) + \frac{1}{8} d''(u_c) c_1^3(j) \right]. \\ & + \frac{1}{h(u_c)} \left[\frac{c_1(j)}{2} d'(u_c) \left(2a_1(j) + a_2(j) \right) \right]. \end{aligned}$$

From the above computation results, we know that when the parameter χ given by (3.8) is in the neighborhood of χ_0^j for each $j = 1, 2, \cdots$, the corresponding bifurcating solution (u^*, v^*) has a formula (3.9) with (u_1, v_1) and (u_2, v_2) given in (3.15) and (4.3), respectively. In order to show the relationship between the solution (u^*, v^*) and its bifurcation location χ_0^j, we relabel (u^*, v^*) as (u_j^*, v_j^*), i.e.

$$\begin{cases} u_j^* = u_c + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots, \\ v_j^* = u_c + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots, \end{cases}$$

Now we give some basic terminologies, $\tilde{\omega} = (u_c, u_c)$ is called the *base* term of the non-constant steady state (u_j^*, v_j^*) whose shape and amplitude are primarily determined by the leading term (u_1, v_1) when ε is small, i.e. $\|u_j^* - u_c\| \approx \varepsilon$ is the amplitude of response which is the change in the cell density from the base term. Note that this leading term has the wave mode j, see (3.15) and (3.5). Therefore we also refer to j as *the principal wave mode* of the solution (u_j^*, v_j^*).

Now applying the conventional perturbation method to linearize equation (3.6), we set

$$\begin{cases} u = u_j^* + \phi e^{\rho t}, \\ v = v_j^* + \eta e^{\rho t} \end{cases}$$

and obtain the linearized equation of (3.6) as follows:

$$\begin{cases} d(u_j^*)\phi'' - \chi h(u_j^*)\eta'' - \chi h'(u_j^*)u_j^{*\prime}\eta' + R_1\phi' + R_2\phi = \rho\phi, \quad x \in (0, l), \\ \eta'' + \phi - \eta = \rho\eta, \quad x \in (0, l), \\ \phi' = \phi'(l) = 0, \\ \eta'(0) = \eta'(l) = 0, \end{cases} \tag{4.8}$$

where

$$R_1 = 2d'(u_j^*)u_j^{*\prime} - \chi h'(u_c)(u_j^*)v_j^{*\prime}$$

and

$$R_2 = d''(u_j^*)u_j^{*\prime 2} + d'(u_j^*)u_j^{*\prime\prime} - \chi h''(u_j^*)u_j^{*\prime}v_j^{*\prime} - \chi h'(u_j^*)v_j^{*\prime\prime} + \mu(1 - 2u_j^*/u_c).$$

Set

$$\begin{cases} \rho = \rho_0 + \varepsilon\rho_1 + \varepsilon^2\rho_2 + \cdots, \\ \phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots, \\ \eta = \eta_0 + \varepsilon\eta_1 + \varepsilon^2\eta_2 + \cdots, \end{cases}$$

and

$$\begin{cases} u_j^* = u_c + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots, \\ v_j^* = u_c + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots, \\ \chi = \chi_0^j + \varepsilon\chi_1^j + \varepsilon^2\chi_2^j + \cdots. \end{cases}$$

Substituting them into (4.8), we obtain a system by equating the $O(1)$ terms as follows:

$$\begin{cases} d(u_c)\phi_0'' + (\chi_0^j h(u_c) - \mu)\phi_0 - \chi_0^j h(u_c)\eta_0 = \rho_0\phi_0 + \rho_0\chi_0^j h(u_c)\eta_0, \quad x \in (0, l), \\ \eta_0'' + \phi_0 - \eta_0 = \rho_0\eta_0, \quad x \in (0, l), \\ \phi_0'(0) = \phi_0'(l) = 0, \\ \eta_0'(0) = \eta_0'(l) = 0. \end{cases} \tag{4.9}$$

It is well known that the sign of ρ_0 determines the stability of the stationary solution (u_j^*, v_j^*). Observing the space decomposition (3.3), we use $-\lambda_m(\phi_0, \eta_0)$ to replace (ϕ_0'', η_0'') in (4.9) and the existence of non-zero solution (ϕ_0'', η_0''), we get an equation for ρ_0 :

$$\rho_0^2 + (d(u_c)\lambda_m + \lambda_m + \mu + 1)\,\rho_0 + A = 0, \tag{4.10}$$

where

$$A = (d(u_c)\lambda_m + \mu)(1 + \lambda_m) - \chi_0^j h(u_c)\lambda_m = h(u_c)\lambda_m(\chi_0^m - \chi_0^j)$$

and χ_0^j is given by (3.16). Obviously, when $j \neq j_0$ in the above formula, there exists a positive integer $m = j_0$ such that $A < 0$. As such, equation (4.10) has a positive root $\rho_0 > 0$. Therefore, we can conclude an important result on the stability of the solution (u_j^*, v_j^*).

Proposition 4.1 (Stability criterion). *When the wave mode $j \neq j_0$, the steady state (u_j^*, v_j^*) in (3.9) is unstable. In other words, if (u_j^*, v_j^*) is stable, then $j = j_0$.*

The above result gives a necessary condition for the stability of the non-constant steady states. Next we shall derive a sufficient condition for the stability of the steady state $(u_{j_0}^*, v_{j_0}^*)$ with *principle wave mode* j_0. It is straightforward to check that the principal eigenvalue for (4.9) is $\rho_0 = 0$ with eigenvector

$$(\phi_0, \eta_0) = ((1 + \lambda_{j_0})\varphi_{j_0}, \varphi_{j_0}).$$

To determine the stability of $(u_{j_0}^*, v_{j_0}^*)$, we need further to find ρ_1. Thus, by the similar computation of obtaining equation (4.9), but now equating the $O(\varepsilon)$ terms gives a system of ρ_1:

$$\begin{cases} d(u_c)\phi_1'' + \left(\chi_0^{j_0} h(u_c) - \mu\right)\phi_1 - \chi_0^{j_0} h(u_c)\eta_1 = \rho_1\phi_0 + \rho_1\chi_0^{j_0} h(u_c)\eta_0 + G_1, \\ \eta_1'' + \phi_1 - \eta_1 = \rho_1\eta_0, \\ \phi_1'(0) = \phi_1'(l) = 0, \\ \eta_1'(0) = \eta_1'(l) = 0, \end{cases} \tag{4.11}$$

where

$$G_1 = \chi_0^{j_0} h'(u_c)\left[u_1\eta_0' + v_1'\phi_0\right]' - d'(u_c)\left[u_1\phi_0' + u_1'\phi_0\right]' + \frac{2\mu}{u_c}u_1\phi_0.$$

It follows from the solvability condition for the equation (4.11) that

$$\int_0^l [\rho_1\phi_0 + \rho_1\chi_0^{j_0} h(u_c)\eta_0 + G_1]\overline{u}_2 \mathrm{d}x + \int_0^l \rho_1\eta_0\overline{v}_2 \mathrm{d}x = 0,$$

where $(\overline{u}_2, \overline{v}_2)$ is given in (4.2) with $j = j_0$. By this, we have

$$\rho_1 = -\frac{\int_0^l G_1\overline{u}_2 \mathrm{d}x}{\int_0^l (\phi_0\overline{u}_2 + \chi_0^{j_0} h(u_c)\eta_0\overline{u}_2 + \eta_0\overline{v}_2)\mathrm{d}x}.$$

Through a direct computation, we obtain the values of the denominator and the numerator for ρ_1, respectively, as follows:

$$\begin{aligned} \int_0^l (\phi_0\overline{u}_2 + \chi_0^{j_0} h_2(u_c)\eta_0\overline{u}_2 + \eta_0\overline{v}_2)\mathrm{d}x &= \frac{1}{2}l\left(c_1(j_0)c_2(j_0) + \chi_0^{j_0} h_2(u_c)c_2(j_0) + 1\right) \\ &= -\frac{l}{2}\left(\frac{c_1^2(j_0)}{\chi_0^{j_0} h_2(u_c)} + \lambda_{j_0}\right) < 0 \end{aligned} \tag{4.12}$$

and

$$\int_0^l G_1\overline{u}_2 \mathrm{d}x = 0.$$

Hence $\rho_1 = 0$. For our purpose, we need to compute ρ_2. Firstly simplify G_1 as

$$G_1 = \frac{\mu}{u_c}c_1^2(j_0) + \left[2d'(u_c)c_1^2(j_0)\lambda_{j_0} - 2\chi_0^{j_0} h'(u_c)c_1(j_0)\lambda_{j_0} + \frac{\mu}{u_c}c_1^2(j_0)\right]\cos\left(2\sqrt{\lambda_{j_0}}x\right).$$

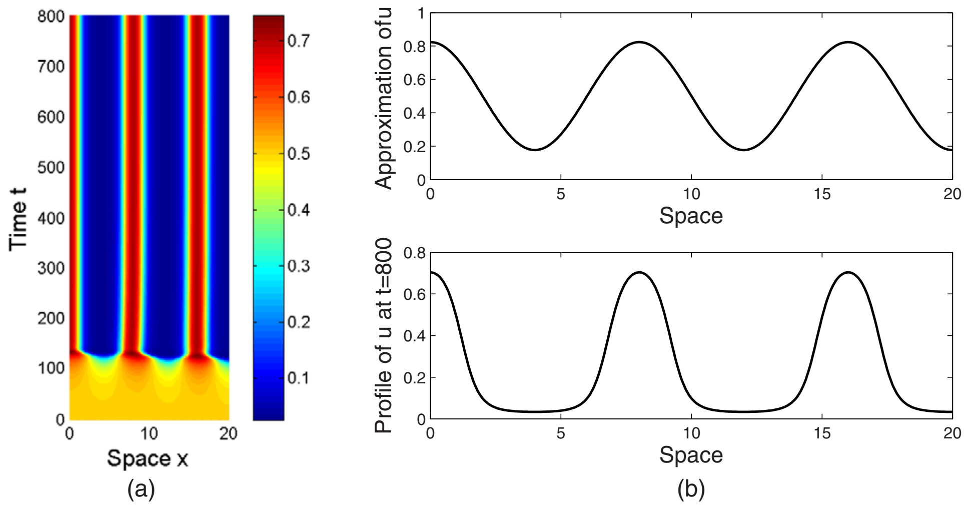


Figure 1. Numerical simulation of the bifurcating solution component u of the chemotaxis system (1.1) in a one dimensional interval $l = (0, 20)$ with parameter values: $D = 1$, $\mu = u_c = 0.5$, $\alpha = -0.2$, $\beta = 1.3$, $\chi = 14(j_0 = 5)$; the initial value is set as: $u_0 = v_0 = u_c + r$ where r is a 1% random small perturbation of the homogeneous steady state (u_c, u_c). The red color represents the high value of u and other colors (blue and yellow) represents the low value of u.

Then, we can find a particular solution (ϕ_1, η_1) for (4.11) as

$$\begin{cases} \phi_1 = \bar{a}_1 + \bar{a}_2 \cos(2\sqrt{\lambda_{j_0}}x), \\ \eta_1 = \bar{a}_3 + \bar{a}_4 \cos(2\sqrt{\lambda_{j_0}}x), \end{cases}$$

where

$$\bar{a}_i = 2a_i(j_0), \ i = 1, 2, 3, 4,$$

with $a_i(j_0), \ i = 1, 2, 3, 4$, as defined in (4.4). We again need to use the same computation as obtaining equation (4.9), but now equating the $O(\varepsilon^2)$ terms to create a system for ρ_2:

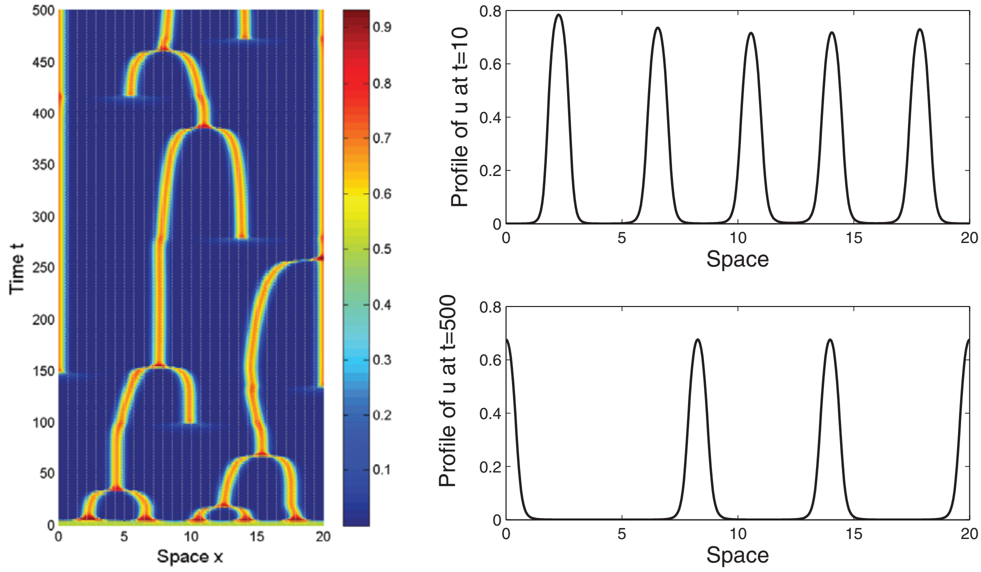
$$\begin{cases} d(u_c)\phi_2'' + \left(\chi_0^{j_0}h(u_c) - \mu\right)\phi_2 - \chi_0^{j_0}h(u_c)\eta_2 = \rho_2\phi_0 + \rho_2\chi_0^{j_0}h(u_c)\eta_0 + G_2, \\ \eta_2'' + \phi_2 - \eta_2 = \rho_2\eta_0, \\ \phi_1'(0) = \phi_1'(l) = 0, \\ \eta_1'(0) = \eta_1'(l) = 0, \end{cases} \tag{4.13}$$

where

$$\begin{aligned} G_2 = {} & \chi_0^{j_0}h'(u_c)\left[v_1'\phi_1 + v_2'\phi_0 + u_1\eta_1' + u_2\eta_0'\right]' - d'(u_c)\left[u_1'\phi_1 + u_2'\phi_0 + u_1\phi_1' + u_2\phi_0'\right]' \\ & + \chi_0^{j_0}h''(u_c)\left[u_1v_1'\phi_0 + \frac{1}{2}u_1^2\eta_0'\right]' - d''(u_c)\left[u_1u_1'\phi_0 + \frac{1}{2}u_1^2\phi_0'\right]' \\ & + \chi_2^{j_0}h(u_c)\eta_0'' + \frac{2\mu}{u_c}(u_1\phi_1 + u_2\phi_0). \end{aligned}$$

By applying the solvability condition of (4.13), we have

$$\rho_2 = -\frac{\int_0^l G_2\bar{u}_2\mathrm{d}x}{\int_0^l(\phi_0\bar{u}_2 + \chi_0^{j_0}h(u_c)\eta_0\bar{u}_2 + \eta_0\bar{v}_2)\mathrm{d}x}. \tag{4.14}$$



(a) (b)

Figure 2. Illustration of evolutionarily unstable bifurcating solution component u of the chemotaxis system (1.1) in $l = (0, 20)$, where $\chi = 48(j_0 = 18)$ and other parameters are same as those in figure 1. The initial value is set as: $u_0 = v_0 = u_c + r$ where r is a 1% random small perturbation of the homogeneous steady state (u_c, u_c). The red color represents the high value of u and other colors represent the lower value of u as shown in the color bar.

A tedious computation leads to

$$\int_0^l G_2\bar{u}_2\mathrm{d}x = -\frac{3}{2}lc_2\lambda_{j_0}\Theta$$

with

$$\Theta = \chi_0^{j_0}\left[h'(u_c)\left(a_1 - \frac{a_2}{2} + c_1a_4\right) + \frac{1}{8}h''(u_c)c_1^2\right] - \left[\frac{1}{\lambda_{j_0}}\frac{\mu c_1}{u_c}(2a_1 + a_2) + \frac{1}{8}d''(u_c)c_1^3\right] + \left[\frac{1}{3}h(u_c)\chi_2^{j_0} - \frac{3}{2}d'(u_c)c_1a_2\right].$$

Noting the expression (4.8) of $\chi_2^{j_0}$, we can simplify Θ as

$$\Theta = c_1(a_1 - a_2)d'(u_c) - \frac{2}{3}h(u_c)\chi_2^{j_0}.$$

By (4.13), the denominator of (4.14) is negative. Therefore, the stability of non-constant steady state $(u^*_{j_0}, v^*_{j_0})$ actually depends on the sign of Θ. In particular, if $\Theta < 0$, then the numerator of (4.14) is negative since $c_2 < 0$ and hence $\rho_2 < 0$. This implies the stability of small-amplitude steady state $(u^*_{j_0}, v^*_{j_0})$. Therefore, we can draw the following conclusion.

Theorem 4.2 (Stability). *Let $j_0 > 0$ be a positive integer such that $\chi_0^{j_0} = \chi_{\min}$. Then, the small-amplitude steady-state $(u^*_{j_0}, v^*_{j_0})$ is stable provided that*

$$\Theta < 0. \tag{4.15}$$

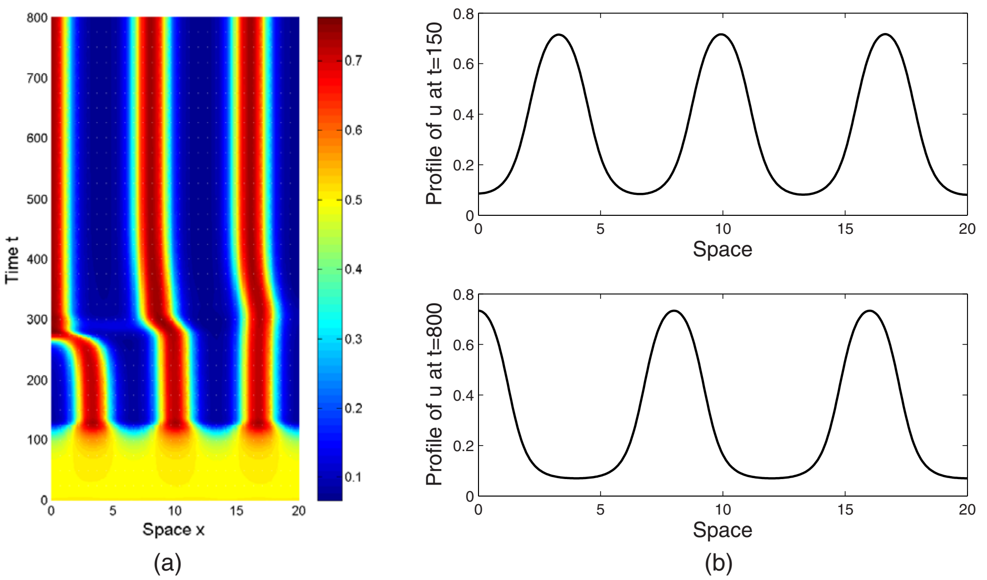


Figure 3. Numerical simulation of transition of unstable bifurcating solutions to stable solutions for the chemotaxis system (1.1) in $l = (0, 20)$, where parameter values are $D = 1, \mu = u_c = 0.5, \alpha = 0.2, \beta = 1, \chi = 12.6714(j_0 = 5)$; the initial value is set as: $u_0 = v_0 = u_c + r$ where r is a 1% random small perturbation of the homogeneous steady state $(u_c, u_c) = (0.5, 0.5)$. The red color represents the high value of u and other colors represent the lower value of u as shown in the color bar.

We should note that it is unclear whether the condition (4.15) is necessary for the stability of small-amplitude steady state $(u^*_{j_0}, v^*_{j_0})$. This leave an open question for the future study.

5. Simulation and conclusion

In this paper, we establish the global existence of classical solutions of the volume-filling chemotaxis system (1.1)–(1.3) subject to (1.4). Furthermore, we investigate the local and global structure of steady states bifurcating from the homogeneous steady state (u_c, u_c) via the asymptotic analysis and global bifurcation theory. Based on the local structure of solutions, we find a stability criterion and a sufficient condition for the stability of bifurcating steady states with small amplitude.

Next we shall show the numerical simulation in an interval $l = [0, 20]$ to illustrate the possible bifurcating patterns for the system (1.1)–(1.3) and verify the stability criterion given in proposition 4.1. The model is solved with MATLAB pde solver based on the finite difference scheme. For brevity, we only show the numerical result for the solution component u. We start from the case where the wave mode of bifurcating pattern is exactly the *principle wave mode* j_0. With the parameter values chosen in figure 1 where the condition (1.4) is satisfied, we can calculate from (3.16) that the *principle wave mode* $j_0 = 5$. By theorem 3.1, we know that the analytical approximation of the non-constant solution component u bifurcating from the homogenous steady states $(u_c, u_c) = (0.5, 0.5)$ is given by

$$u^*(\varepsilon) = 0.5 + \varepsilon(1 + \lambda_5)\varphi_5 + o(\varepsilon) = 0.5 + \varepsilon\left(1 + \left(\frac{\pi}{4}\right)^2\right)\cos\left(\frac{\pi x}{4}\right) + o(\varepsilon)$$

for some small $\varepsilon > 0$, whose graph has 5 zeros with the horizontal line $u = 0.5$ in the x-u plane, see a plot in figure 1(b) with $\varepsilon = 0.3$. By proposition 4.1, if the bifurcating solution of (1.1) is stable, then its wave mode must be $j = j_0 = 5$. This is perfectly verified by the numerical simulation shown in figure 1, where we see the stable steady state bifurcating from the homogeneous field (yellow area in figure 1(a)) has exactly five interactions with the basal line $u = u_c = 0.5$ (see the plot of profile of u at $t = 800$ in figure 1(b)). The stability criterion in proposition 4.1 is further confirmed by the simulation in figure 2 for a different scenario, where χ is chosen such that *principle wave mode* $j_0 = 18$ and other parameter values are same as those in figure 1. The result of proposition 4.1 asserts that only the bifurcating solution with wave mode 18 could be stable. From the numerical simulation, we can see that the bifurcation solution from the homogenous field has only 10 wave mode (see the plot of upper panel of figure 2(b)) and hence it is unstable, as seen in figure 2(a) where the aggregation pattern with mode $j = 10$ is bifurcated again followed with more bifurcations. In figure 3, we show another scenario where the bifurcating solution is initially unstable but evolutionarily stable. With the parameter values chosen, we have the *principle wave mode* $j_0 = 5$. The numerical simulation shows that the initial bifurcating solution has wave mode $j = 6$ (see the plot of profile of u at time $t = 150$) which is unstable by proposition 4.1. As we see from figure 3(a), this unstable solution transits to a stable one with $j = j_0 = 5$ at time $t = 280$ approximately by merging half of a peak into the boundary (see the plot of profile of u in figure 3(b)). This again supports the stability criterion given in proposition 4.1.

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