Analysis and Applications, Vol. 15, No. 1 (2017) 83–106
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DOI: 10.1142/S0219530515500220

Patterns in a generalized volume-filling chemotaxis model with cell proliferation

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Received 29 October 2014
Accepted 27 May 2015
Published 30 September 2015

In this paper, we consider the following system

$$\begin{cases} -\nabla\cdot\left(\dfrac{D}{(1-u)^{\alpha}}\nabla u-\chi u(1-u)^{\beta}\nabla v\right)=\mu u\left(1-\dfrac{u}{u_c}\right), & x\in\Omega,\\ -\Delta v=u-v, & x\in\Omega,\\ \dfrac{\partial u}{\partial\nu}=\dfrac{\partial v}{\partial\nu}=0, & x\in\partial\Omega, \end{cases}$$

which corresponds to the stationary system of a generalized volume-filling chemotaxis model with logistic source in a bounded domain in $\mathbb{R}^N$ $(N\geq 1)$ with zero Neumann boundary conditions. Here the parameters D,χ,μ,u_c are positive and $\alpha,\beta\in\mathbb{R}$, and ν denotes the outward unit normal vector of $\partial\Omega$. With the *priori* positive lower- and upper-bound solutions derived by the Moser iteration technique and maximum principle, we apply the degree index theory in an annulus to show that if the chemotactic coefficient χ is suitably large, the system with $\alpha+\beta>1$ admits pattern solutions under certain conditions. Numerical simulations of the pattern formation are shown to illustrate the theoretical results and predict the interesting phenomenon for further studies.

Keywords: Chemotaxis; volume-filling effect; nonconstant steady state; degree index; pattern formation.

Mathematics Subject Classification 2010: 35K55, 35K57, 35K45, 35K50, 92C15, 92C17, 92B99

1. Introduction

Chemotaxis is a process directing the motion of species up or down a chemical concentration gradient. It has been recognized as an important mechanism, in addition to the Turing mechanism, for many biological pattern formations such as the propagation of traveling band of bacterial toward the oxygen [1, 2], the outward propagation of concentric ring waves by *E. coli* [5–7], the spiral wave patterns during the aggregation of *Dictyostelium discoideum* [10, 16]. The mathematical modeling of chemotaxis was dated to the pioneering works of Patlak in 1953 [22] and Keller and Segel in 1970 [14, 15]. Since then, a number of particular chemotaxis models have been proposed to model the aggregation phase of chemotaxis. Most of these works treat the cells as point masses and hence the formation of cell aggregation was interpreted as a finite-time blow-up of cell density [11, 12]. The ideas of taking into account cell sizes have been developed in the past two decades so that arbitrarily high cell densities can be precluded by setting an impassable threshold value for cell density. Among these is a so-called volume-filling effect, which was first introduced by Painter and Hillen [20] and further developed in [27] for generic cell types. The general form of volume-filling chemotaxis model in [20, 27] reads

$$\begin{cases} u_t = \nabla\cdot(d(u)\nabla u - \chi u\phi(u)\nabla v) + \mu u(1-u/u_c), & x\in\Omega, \quad t>0,\\ v_t = \Delta v - v + u, & x\in\Omega, \quad t>0,\\ \nabla u\cdot\nu = \nabla v\cdot\nu = 0, & x\in\partial\Omega, \ t>0,\\ u(x,0)=u_0(x), v(x,0)=v_0(x), & \qquad x\in\Omega, \end{cases} \tag{1.1}$$

where the cell density-dependent diffusion coefficient $d(u)$ and the chemotactic sensitivity function $\phi(u)$ are of the form

$$d(u) = D(q(u) - uq'(u)), \quad \phi(u) = q(u),$$

with $D, \chi > 0$ being constants and $q(u)$ is a probability function that cells make a jump to their neighboring locations, which satisfies the following general properties: there is a maximal cell number $\tilde{u}$, called *crowding capacity*, that can be accommodated in a unit volume (or area) of space such that

$$q(\tilde{u}) = 0, \quad \text{and} \quad 0 < q(u) \le 1, q'(u) < 0 \quad \text{for all } 0 \le u < \tilde{u}.$$

Moreover the last term in the first equation of (1.1) accounts for the cell growth with the rate $\mu \ge 0$ and the carrying capacity $u_c > 0$ which is smaller than $\tilde{u}$ in general [27], and ν is the outward unit normal vector of $\partial\Omega$ where Ω is a bounded domain in $\mathbb{R}^N$ ($N \ge 1$) with smooth boundary. The precise form of the probability function $q(u)$ is generally unknown and not directly accessible to experiments. By assuming $\tilde{u} = 1$ for convenience, a natural choice based on the above properties of $q(u)$ is (see [20, 26, 27])

$$q(u) = \begin{cases} (1-u)^r, & 0 \le u \le 1,\\ 0, & u > 1, \end{cases}$$

where $r > 0$ is a parameter. Consequently for any $0 \le u < 1$ it holds that

$$d(u) = D(1-u)^{r-1}[1 - u(1-r)], \quad \phi(u) = (1-u)^r.$$

Then for all $u \in [0,1)$, one can easily check that $1 \le \frac{d(u)}{D(1-u)^{r-1}} \le 1+r$ if $r>1$, and $d(u)=D$ if $r=1$ as well as $r \le \frac{d(u)}{D(1-u)^{r-1}} \le 1$ if $r<1$. Therefore

$$d(u) \sim \begin{cases} (1-u)^{r-1}, & r \neq 1, \\ 1, & r=1, \end{cases}$$

from which we see the nonlinear diffusion $d(u)$ is degenerate if $r>1$ and singular if $r<1$, and is a constant if $r=1$. Hence we are motivated to consider the following generalized $d(u)$ and $\phi(u)$ in (1.1)

$$d(u) = D(1-u)^{-\alpha}, \quad \phi(u) = (1-u)^{\beta}, \quad u \in [0,1), \tag{1.2}$$

where α and β are real numbers. Then system (1.1) with (1.2) leads to a class of parabolic systems with singularity or degeneracy as u approaches the threshold value 1 in either the diffusion coefficient or the chemotactic sensitivity or both. Hence whether the solution u attains 1 is the foremost question. When the cell growth is neglected (i.e. $\mu=0$), the results of [28, 29, 32] showed that if $\alpha+\beta>1$ or $\alpha=0, \beta=1$, the solution u is strictly less than 1 for any initial data (u_0, v_0) satisfying

$$(u_0, v_0) \in [W^{1,\infty}(\Omega)]^2 \quad \text{and} \quad 0 \le u_0(x) < 1, \;\; v_0(x) \ge 0, \;\; x \in \Omega. \tag{1.3}$$

In other regimes of parameters α and β, the singularity or degeneracy (meaning u attains 1) may happen in either finite or infinite time, except for a borderline case $\alpha>0, \alpha+\beta=1$ which still remains unknown (see [29]).

When $\alpha=0, \beta=1$, the pattern formation (i.e. steady states) of (1.1) was numerically investigated first in [20] and further elaborated in [27] for both zero and non-zero cell growth. It was observed in [20, 27] that the volume-filling chemotaxis model with cell growth ($\mu>0$) will typically exhibit merging and emerging chaotic patterns in contrast to stationary aggregation patterns in the case of zero cell growth ($\mu=0$). Then an important question arises as whether or not the volume-filling chemotaxis model with cell growth can develop stationary patterns. This question was first confirmed analytically in a recent work [17] for the case $\alpha=0, \beta=1$. The purpose of this paper is to develop the results of [17] into a more general parameter regime

$$\alpha+\beta>1. \tag{1.4}$$

But it should be emphasized that our present work is not a simple extension but an essential development of work [17] in the following two senses. First in this paper, we apply degree index theory in an annulus instead of in a positive cone in [17], which enables us to remove one essential condition of [17, Theorem 3.10] for the existence of nonconstant steady states (i.e. stationary patterns). Second, when $\alpha=0, \beta=1$ as considered in [17], the maximum principle applies directly to obtain the bound of u. However the maximum principle no longer holds in the case (1.4) due to the possible singularity/degenaracy. Here we apply the Moser iteration to derive *a priori* bound for u, which is much more technical than the previous case $\alpha = 0, \beta = 1$. It is also worthwhile to note that since the singularity may occur when $\alpha + \beta \leq 1$ except for $\alpha = 0, \beta = 1$ (see details in [29]), the parameter regime (1.4) together with $\alpha = 0, \beta = 1$ may constitute the largest parameter regime allowing the stationary patterns for the volume-filling chemotaxis model (1.1)–(1.2). The existence of global classical solutions of (1.1)–(1.2) with initial data (1.3) and (1.4) is established in a separate paper [18]. In the present paper, we focus on the stationary solutions of (1.1)–(1.2) only.

Finally we mention some recent works related to the model (1.1). When $\mu = 0$, the existence of steady states of chemotaxis system (1.1) for $\alpha = 0, \beta = 1$ was rigorously established in [25] in one dimension via the global bifurcation theorem in [24], whereas the local bifurcation analysis was performed previously in [23] for the same case. The analysis of works [23, 25] essentially relies on the cell mass conservation which is apparently not true for the case $\mu > 0$ as considered in the present paper. When $\alpha = 0, \beta = 1$, the global existence of classical solutions has been obtained in [30, 31] for $\mu \geq 0$ and the convergence of solutions to equilibria with $\mu = 0$ was studied in [13].

Notation. Throughout this paper, we denote the measure of the set A by $|A|$; let $W^{m,p}(\Omega, \mathbb{R}^N)$ for $m \geq 1, 1 < p < +\infty$ be Sobolev space of $\mathbb{R}^N$-valued functions with norm $\|\cdot\|_{m,p}$. When $p = 2$, $W^{m,2}(\Omega, \mathbb{R}^N)$ is written as $H^m(\Omega)$. Let $L^p(\Omega)(1 \leq p \leq \infty)$ denote the usual Lebesgue space in a bounded domain $\Omega \subset \mathbb{R}^n$ with norm $\|f\|_{L^p(\Omega)} = (\int_\Omega |f(x)|^p dx)^{1/p}$ for $1 \leq p < \infty$ and $\|f\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_{x\in\Omega} |f(x)|$. When $p \in (n, +\infty)$, $W^{1,p}(\Omega, \mathbb{R}^2) \hookrightarrow C(\Omega, \mathbb{R}^2)$ which is the space of $\mathbb{R}^2$-valued continuous functions.

2. *Priori* Estimates for Steady States

In this section, we shall show that the steady state solutions of (1.1)–(1.3) have *priori* lower and upper bounds and $u(x)$ cannot attain the threshold value 1. It is difficult to prove that $u(x) < 1$ for all $x \in \overline{\Omega}$ directly. Here we achieve it by deriving that $\|\frac{1}{1-u}\|_{L^\infty}$ is bounded for all $x \in \overline{\Omega}$. Such an idea was first developed in [8]. The corresponding steady state problem of (1.1)–(1.3) is

$$\begin{cases} -\nabla \cdot (D(1-u)^{-\alpha}\nabla u - \chi u(1-u)^\beta \nabla v) = \mu u(1 - u/u_c), \quad x \in \Omega, \\ -\Delta v = u - v, \quad x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega. \end{cases} \tag{2.1}$$

It should be noted that $0 < u_c < 1$ since the carrying capacity u_c is smaller than the crowding capacity 1 (see the introduction). We stress that we are only interested in the solution $0 \leq u(x) \leq 1$ in the sequel due to 1 is the cell *crowding capacity* in the volume-filling model. The lemma below asserts that the solution $(u, v)(x)$ has *a priori* positive lower bound. It can be readily derived by the method of upper and lower solutions.

Lemma 2.1. *Let* $(u(x), v(x))$ *be a nonconstant non-negative classical solution to the elliptic boundary value problem* (2.1) *with* $0 \le u(x) < 1$. *Then we have the following*:

(i) $u(x) > 0, v(x) > 0$ *for all* $x \in \overline{\Omega}$;
(ii) *there exists a constant* $K > 0$ *such that* $|\nabla v| \le K$ *for all* $x \in \overline{\Omega}$.

Proof. We first show that $u(x) > 0$ for all $x \in \overline{\Omega}$. To this end, we write the first equation of (2.1) as

$$\begin{cases} D\Delta u + D\alpha(1-u)^{-1}|\nabla u|^2 - \chi(1-u)^{\alpha+\beta}\nabla u \nabla v + \chi\beta u(1-u)^{\alpha+\beta-1}\nabla u \nabla v \\ \qquad - \chi u(1-u)^{\alpha+\beta}\Delta v = -\mu u(1-u)^{\alpha}(1-u/u_c), \quad x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, \quad x \in \partial\Omega. \end{cases} \tag{2.2}$$

We assume, by contradiction, that $u(x_0) = \min_{x\in\overline{\Omega}} u(x) = 0$. By the Hopf's lemma with zero Neumann boundary, u cannot be zero on $\partial\Omega$ and hence $x_0 \in \Omega$. On the other hand, if $x_0 \in \Omega$, then by the strong maximum principle, we see from Eq. (2.2) that $u(x) \equiv 0$ for all $x \in \Omega$. This contradicts that $u(x)$ is nonconstant. Therefore $u(x) > 0$ on $\overline{\Omega}$.

Next we show that $v(x) > 0$ for all $x \in \overline{\Omega}$. By contradiction, we assume that $v(x_0) = \min_{x\in\overline{\Omega}} v(x) = 0$. Then the Hopf's lemma applied to the problem

$$\begin{cases} -\Delta v + v = u, \quad x \in \Omega, \\ \dfrac{\partial v}{\partial \nu} = 0, \qquad\quad x \in \partial\Omega \end{cases} \tag{2.3}$$

entails that $x_0 \notin \partial\Omega$. Hence $x_0 \in \Omega$. Then the strong maximum principle to the above problem asserts that $v(x) \equiv 0$ for all $x \in \Omega$ and hence $u \equiv 0$ in Ω. This contradicts the fact that $u(x) > 0$ proved above, which shows that $v(x) > 0$ for all $x \in \overline{\Omega}$ by contradiction.

We proceed to show the boundedness of $|\nabla v|$. Indeed for (2.3), by the Agmon–Douglis–Nirenberg estimates [3] on linear elliptic equations with the (zero) Neumann boundary condition, we have

$$\|v(\cdot)\|_{W^{2,p}(\Omega)} \le C\|u\|_{L^p(\Omega)} < \infty.$$

This, along with the Sobolev embedding (see [9, (7.30), p. 158]): $W^{2,p}(\Omega) \hookrightarrow C^1_B(\Omega) := \{f \in C^1(\Omega) | \nabla f \in L^\infty(\Omega)\}$ if $p > n$, yields a constant $K > 0$ such that

$$\|\nabla v(\cdot)\|_{L^\infty(\Omega)} \le K \quad \text{for all } x \in \overline{\Omega}.$$

The proof is completed. □

It is necessary to prove that $u(x) < 1$ for all $x \in \overline{\Omega}$ to make the problem well-posed. Since the maximum principle is not valid here, we shall derive it by the method of Moser–Alikakos iteration as shown in [29]. First we shall use the following lemma extracted from [29], which presents a Poincaré–Sobolev type inequality for functions remaining suitably small in sets with appropriate large measure.

Lemma 2.2 ([29]). (i) *Let* $\varepsilon > 0$, $\kappa > 0$ *and* $q \geq 1$ *satisfying* $q \leq \frac{2n}{(n-2)_+}$. *If* $z \in W^{1,2}(\Omega)$ *is non-negative with* $|\{z \leq \varepsilon\}| \geq \kappa$. *Then there exists* $c = c(\kappa) > 0$ *such that*

$$\int_\Omega z^q \leq c \cdot \left\{ 1 + \left(\int_\Omega |\nabla z|^2 \right)^{\frac{q}{2}} \right\}.$$

(ii) *Let* $\overline{q} \in (1, \frac{2n}{(n-2)_+})$. *Then there exists a constant* $C > 0$ *such that*

$$\|z\|_{L^q(\Omega)} \leq C \|\nabla z\|^a_{L^2(\Omega)} \cdot \|\nabla z\|^{1-a}_{L^1(\Omega)} + C\|z\|_{L^1(\Omega)} \tag{2.4}$$

holds for all $z \in W^{1,2}(\Omega)$, $q \in [1, \overline{q}]$ *and* $a = \frac{2n(q-1)}{(n+2)q}$.

We first derive the L^p-estimates.

Lemma 2.3. *Assume* α *and* β *satisfy* (1.4) *and* (u_0, v_0) *satisfies* (1.3). *Let* $(u, v)(x)$ *be a solution of* (2.1) *such that* $0 \leq u < 1$ *in* Ω. *Then, for any* $p > 1$, *there exists a constant* $C(K, p) > 0$ *such that*

$$\int_\Omega (1 - u(x))^{-p} dx \leq C(K, p), \tag{2.5}$$

where K *is given in Lemma* 2.1.

Proof. Let $w(x) = 1 - u(x)$ and then multiple the first equation of (2.1) by w^{-p-1} with $p > 1$ and integrate it over Ω. Applying Green's formula and the Neumann boundary condition, we have

$$\begin{aligned}
&(p+1)D \int_\Omega w^{-p-2-\alpha} |\nabla w|^2 dx \\
&\quad = -(p+1)\chi \int_\Omega u w^{-p-2+\beta} \nabla v \nabla w dx + \mu \int_\Omega u(1 - u/u_c) w^{-p-1} dx.
\end{aligned} \tag{2.6}$$

Let $\epsilon = \frac{\chi}{D}$, $a = w^{\frac{-p-\alpha-2}{2}} |\nabla w|$, $b = w^{\frac{-p+\alpha+2\beta-2}{2}} u |\nabla v|$. Then Young's inequality $ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$ and the fact $0 \leq u < 1$ yield

$$\begin{aligned}
&(p+1)\chi \int_\Omega u w^{-p-2+\beta} |\nabla v \nabla w| dx \\
&\quad \leq \frac{(p+1)D}{2} \int_\Omega w^{-p-2-\alpha} |\nabla w|^2 dx + \frac{(p+1)\chi^2 K^2}{2D} \int_\Omega w^{-p+\alpha+2\beta-2} dx.
\end{aligned} \tag{2.7}$$

From $p > 1$ it follows that $p \leq p + 1 \leq 2p$. By (2.6) and (2.7), we have

$$\begin{aligned}
&\frac{pD}{2} \int_\Omega w^{-p-2-\alpha} |\nabla w|^2 dx \\
&\quad \leq \frac{p\chi^2 K^2}{D} \int_\Omega w^{-p+\alpha+2\beta-2} dx + \mu \int_\Omega u(1 - u/u_c) w^{-p-1} dx.
\end{aligned} \tag{2.8}$$

Since $\int_\Omega w^{-p-2-\alpha}|\nabla w|^2dx = \frac{4}{(p+\alpha)^2}\int_\Omega |\nabla w^{-\frac{p+\alpha}{2}}|^2dx$, the inequality (2.8) is equivalent to

$$\begin{aligned}&\frac{4p^2D}{(p+\alpha)^2}\int_\Omega |\nabla w^{-\frac{p+\alpha}{2}}|^2dx\\ &\quad\le \frac{p^2\chi^2K^2}{D}\int_\Omega w^{-p+\alpha+2\beta-2}dx + \mu p\int_\Omega u(1-u/u_c)w^{-p-1}dx. \qquad (2.9)\end{aligned}$$

To proceed, we let $p>1$ to be sufficiently large to fulfill

$$\frac{p-\alpha-2\beta+2}{p+\alpha}\ge\frac{1}{2},\quad \text{for } p>|\alpha| \quad \text{and} \quad p>-\frac{n\alpha}{2}. \qquad (2.10)$$

By the boundary condition, the first equation of (1.1) upon integration over Ω leads to $\int_\Omega udx = \frac{1}{u_c}\int_\Omega u^2dx$. By applying Hölder inequality to $\int_\Omega udx$, we have $\int_\Omega udx \le (\int_\Omega u^2dx)^{\frac{1}{2}}|\Omega|^{\frac{1}{2}}$ from which, it follows that $(\int_\Omega u^2dx)^{1/2}\le u_c\sqrt{|\Omega|}$, and then $\int_\Omega udx\le u_c|\Omega|$. Hence, for any $a>1$, we have $|\{u(x)>au_c\}|\le\frac{|\Omega|}{a}$ and hence

$$|\{u(x)\le au_c\}|\ge\frac{a-1}{a}|\Omega|. \qquad (2.11)$$

Noticing that $0<u_c<1$, we now take $a\in(1,\frac{1}{u_c})$ so that $au_c<1$. By (2.11) and (2.10), it is easy to check that

$$|\{(1-u)^{-\frac{p+\alpha}{2}}\le(1-au_c)^{-\frac{p+\alpha}{2}}\}|\ge\frac{a-1}{a}|\Omega|.$$

Applying Lemma 2.2 with $q=\frac{2(p-\alpha-2\beta+2)}{p+\alpha}$, and noting that $w=1-u$, there exists some constant $c_1>0$ possibly depending on K and p such that

$$\begin{aligned}\frac{p\chi^2K^2}{D}\int_\Omega w^{-p+\alpha+2\beta-2}dx &= \frac{p\chi^2K^2}{D}\int_\Omega (w^{-\frac{p+\alpha}{2}})^{\frac{2(p-\alpha-2\beta+2)}{p+\alpha}}dx\\ &\le c_1\cdot\left\{1+\left(\int_\Omega|\nabla w^{-\frac{p+\alpha}{2}}|^2dx\right)^{\frac{p-\alpha-2\beta+2}{p+\alpha}}\right\}. \qquad (2.12)\end{aligned}$$

Note that (2.10) and the condition $\alpha+\beta>1$ lead to $q\in[1,\frac{2n}{(n-2)_+}]$. Since $\frac{p-\alpha-2\beta+2}{p+\alpha}<1$, we can find $c_2=c_2(K,p)>0$, by employing Young's inequality, such that

$$c_1\left(\int_\Omega|\nabla w^{-\frac{p+\alpha}{2}}|^2dx\right)^{\frac{p-\alpha-2\beta+2}{p+\alpha}}\le\frac{pD}{(p+\alpha)^2}\int_\Omega|\nabla w^{-\frac{p+\alpha}{2}}|^2dx+c_2. \qquad (2.13)$$

Collecting (2.9), (2.12) and (2.13), we obtain that

$$\frac{pD}{(p+\alpha)^2}\int_\Omega|\nabla w^{-\frac{p+\alpha}{2}}|^2dx\le c_1+c_2+\mu\int_\Omega u(1-u/u_c)w^{-p-1}dx. \qquad (2.14)$$

Moreover, taking $\widetilde{q}=\frac{2p}{p+\alpha}$, by means of (2.10) again we have $\widetilde{q}$ satisfies $1\le\widetilde{q}\le\frac{2n}{(n-2)_+}$. Therefore, for some constant $c_3=c_3(K,p)>0$, applying Lemma 2.2 to $\widetilde{q}$, we have

$$\int_\Omega w^{-p}dx = \int_\Omega (w^{-\frac{p+\alpha}{2}})^{\frac{2p}{p+\alpha}}dx \le c_3\left\{1+\left(\int_\Omega |\nabla w^{-\frac{p+\alpha}{2}}|^2dx\right)^{\frac{p}{p+\alpha}}\right\}.$$

Thus, from (2.14) it follows that

$$\frac{pD}{(p+\alpha)^2}\left(\frac{1}{c_3}\int_\Omega w^{-p}dx-1\right)^{\frac{p+\alpha}{p}} \le c_1+c_2+\mu\int_\Omega u(1-u/u_c)w^{-p-1}dx. \tag{2.15}$$

The last term in (2.15) can be estimated as

$$\begin{aligned}\mu\int_\Omega & u(1-u/u_c)w^{-p-1}dx\\ &\le \mu\int_{|\{u\le u_c\}|} u(1-u/u_c)w^{-p-1}dx \le \mu u_c(1-u_c)^{-p-1}|\Omega|.\end{aligned} \tag{2.16}$$

Therefore there exists a constant $c_4 = c_4(p) > 0$ such that $\frac{pD}{(p+\alpha)^2}(\frac{1}{c_3}\int_\Omega w^{-p}dx - 1)^{\frac{p+\alpha}{p}} \le c_4$ which yields that

$$\int_\Omega w^{-p}dx \le c_3\left[\left(\frac{(p+\alpha)^2c_4}{p^2D}\right)^{\frac{p}{p+\alpha}}+1\right].$$

The proof is completed. □

We continue to carry out the L^∞ estimate of w^{-1} by applying a variant of the Moser-Alikakos iterative developed in [4].

Lemma 2.4. *Let the assumptions in Lemma* 2.3 *hold. Then there exists a constant* $C(K)$, *where* K *is given in Lemma* 2.1, *such that*

$$\frac{1}{1-u} \le C(K). \tag{2.17}$$

Proof. The proof of this lemma is tedious and similar to the proof of [29, Lemma 4.2]. So we present the proof in the appendix for completeness. □

Then the following result is apparent.

Proposition 2.5. *Let* $(u(x), v(x))$ *be a non-negative classical solution of* (2.1) *with* $0 \le u(x) \le 1$, *and* D, χ, μ *and* u_c *be fixed. If* α *and* β *satisfy* (1.4), *then*

$$0 < u(x) < 1, \quad 0 < v(x) < 1 \quad \textit{for all } x \in \overline{\Omega}. \tag{2.18}$$

Proof. The inequalities $0 < u(x) < 1$ and $v(x) > 0$ follow from Lemmas 2.1 and 2.4. Then by using the comparison principle, we have $v(x) < 1$. □

3. Nonexistence of Nonconstant Steady States

In this section we shall verify that the elliptic boundary value problem (2.1) has no nonconstant solution when the chemotactic parameter χ is sufficiently small. To proceed, we first present the decomposition in function space based on the elliptic operator $-\Delta$ subject to the Neumann boundary condition on Ω. Let

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots \tag{3.1}$$

be the sequence of eigenvalues for the elliptic operator $-\Delta$ subject to the homogeneous Neumann boundary condition on Ω and each λ_i have multiplicity $m_i \geq 1$. Let $\phi_{ij}, i \geq 0, 1 \leq j \leq m_i$, be the normalized eigenfunctions corresponding to λ_i. Let $E(\lambda_i)$ be the eigenspace associated with λ_i in $H^1(\Omega; \mathbb{R}^2)$. Then the set $\{\phi_{ij}, i \geq 0, j = 1, 2, \ldots, \dim E(\lambda_i)\}$ forms a complete orthogonal basis in $L^2(\Omega)$. Let $X = [H^1(\Omega)]^2$ and

$$X_{ij} = \{c\phi_{ij} : 1 \leq j \leq m_i, c \in R^2\}. \tag{3.2}$$

Then

$$X = \bigoplus_{i=1}^{\infty} X_i, \quad X_i = \bigoplus_{j=1}^{\dim E(\lambda_i)} X_{ij}, \tag{3.3}$$

where $\bigoplus$ denotes the direct sum of subspaces. So (3.3) demonstrates the direct sum decomposition of X and X_i.

The system (1.1) has two homogeneous steady states $(0, 0)$ and (u_c, u_c). We then employ the linear stability analysis to find the condition for the stability/instability of the homogeneous steady states. To this end, we linearize (1.1) at $(0, 0)$ and (u_c, u_c) and then look for solutions proportional to $e^{i\mathbf{k}\cdot\mathbf{x}+\sigma t}$, where $\mathbf{k}$ is the wave vector with magnitude $k = |\mathbf{k}|$ and σ is the temporal growth rate depending on k^2. Then the instability is expected when $\mathrm{Re}(\sigma) > 0$ for some $k > 0$. Following this standard procedure, after some elementary algebraic operations, we can show that a necessary condition for the pattern formation of the model (1.1) is

$$\begin{aligned}\chi > \chi_c =: &\frac{\mu + d(u_c) + 2\sqrt{\mu d(u_c)}}{u_c \phi(u_c)} \\ = &\frac{\mu + D(1-u_c)^{-\alpha} + 2\sqrt{\mu D(1-u_c)^{-\alpha}}}{u_c(1-u_c)^{\beta}},\end{aligned} \tag{3.4}$$

and allowable wave number k satisfies

$$\begin{aligned}k_1^2 = &\frac{1}{2D}(\eta - \sqrt{\eta^2 - 4\mu D(1-u_c)^{\alpha}}) < k^2 < k_2^2 \\ = &\frac{1}{2D}(\eta + \sqrt{\eta^2 - 4\mu D(1-u_c)^{\alpha}}),\end{aligned} \tag{3.5}$$

where $\eta = \chi u_c(1-u_c)^{\alpha+\beta} - \mu(1-u_c)^{\alpha} - D > 0$. We should remark that the condition (3.5) is necessary because allowable wave numbers are discrete in a finite domain and hence interval (k_1, k_2) does not necessarily contain the desired discrete number, for example $k = \frac{n\pi}{l}$ for $n = 1, 2, \ldots$ if $\Omega = (0, l)$. Then the following result is obvious.

Lemma 3.1. *Let* χ_c *be defined in* (3.4). *Then the steady state* $0 = (0, 0)$ *is always unstable, and the steady state* $\widetilde{\omega} = (u_c, u_c)$ *is asymptotically stable if* $\chi \le \chi_c$, *and is unstable if and only if* (3.4)–(3.5) *are satisfied.*

We now verify the following result.

Lemma 3.2. *If* $\chi = 0$, *namely, there is no chemotaxis, then system* (2.1) *does not have nonconstant steady states.*

Proof. Let $\chi = 0$; then v is decoupled and the system (2.1) reduces to the following two problems

$$\begin{cases} -\Delta u - \alpha(1-u)^{-1}|\nabla u|^2 = \dfrac{\mu}{D} u(1-u)^{\alpha}\left(1 - \dfrac{u}{u_c}\right), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, \quad x \in \partial\Omega, \end{cases} \tag{3.6}$$

and

$$\begin{cases} -\Delta v + v = u, & x \in \Omega, \\ \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.7}$$

We first show that there is no nonconstant solution to (3.6). By contradiction, we assume that (3.6) possesses a nonconstant classical solution $\tilde{u}(x)$. Denote $\tilde{u}(u_*) = \min_{x \in \overline{\Omega}} \tilde{u}(x)$. By Hopf's lemma, $x_* \notin \partial\Omega$ and hence $x_* \in \Omega$. Therefore $\nabla \tilde{u}(x_*) = 0, \Delta \tilde{u}(x_*) \ge 0$. Then the strong maximum principle applied to (3.6) asserts that $\tilde{u}(x_*) > 0$. Now we claim that

$$\tilde{u}(x_*) = \min_{x \in \overline{\Omega}} \tilde{u}(x) \ge u_c. \tag{3.8}$$

Indeed, if $0 < \tilde{u}(x_*) < u_c$, then $\Delta \tilde{u}(x_*) = -\mu \tilde{u}(x_*)(1 - \tilde{u}(x_*))^{\alpha}(1 - \tilde{u}(x_*)/u_c) < 0$, which is a contradiction and hence the claim (3.8) holds. We further assume that $\tilde{u}(u^*) = \max_{x \in \overline{\Omega}} \tilde{u}(x)$. The Hopf's lemma applied to (3.6) entails that $x^* \in \Omega$ and hence $\nabla \tilde{u}(x^*) = 0, \Delta \tilde{u}(x_*) \le 0$. Due to (3.8) and assumption that $\tilde{u}$ is nonconstant, we have $u(x^*) > u_c$. On the other hand, by Eq. (3.6), we have $\Delta \tilde{u}(x^*) = -\mu \tilde{u}(x^*)(1 - \tilde{u}(x^*))^{\alpha}(1 - \tilde{u}(x^*)/u_c) > 0$. This is a contradiction and hence (3.6) does not have nonconstant solution.

Next we show that (3.7) does not have nonconstant solution. From the proof above, we know u is a constant, denoted by $\bar{u}$. By contradiction, if we assume (3.7) has a nonconstant solution $\tilde{v}$ in $\overline{\Omega}$ with $\tilde{v}(v_*) = \min_{x \in \overline{\Omega}} \tilde{v}(x)$ and $\tilde{v}(v^*) = \max_{x \in \overline{\Omega}} \tilde{v}(x)$, then by the similar argument shown above, we can show that $\tilde{v}(v_*) \ge \bar{u}$ and $\tilde{v}(v^*) \le \bar{u}$. This indicates that $\tilde{v}(x) \equiv \bar{u}$, which is a constant and hence the proof is completed. $\square$

By Proposition 2.5 and the standard elliptic regularity arguments, we have the following result which plays an important role in proving our main result of this section.

Lemma 3.3. *Let D, χ, μ and u_c be fixed. Assume that the inequality* (1.4) *holds. Then any solution* $(u(x), v(x))$ *of* (2.1) *satisfies*

$$\|u(x)\|_{C^2(\overline{\Omega})} < 1, \quad \|v(x)\|_{C^2(\overline{\Omega})} < 1.$$

Now Lemmas 3.2 and 3.3 are employed to prove the lemma below.

Lemma 3.4. *Assume that D, μ and u_c be fixed. Let $\chi_n > 0, n = 1, 2, \ldots$ such that $\chi_n \to 0$ as $n \to \infty$. If (u_n, v_n) is a positive solution of* (2.1) *with $\chi = \chi_n$, then*

$$(u_n, v_n) \to (u_c, u_c) \quad in\ C^2(\overline{\Omega}) \times C^2(\overline{\Omega}) \quad as\ n \to \infty.$$

Proof. Lemmas 3.2 and 3.3 imply that there exists a non-negative constant solution $(\overline{u}, \overline{v})$ of (3.6) such that $(u_n, v_n) \to (\overline{u}, \overline{v})$ as $n \to \infty$, passing to a subsequence if necessary, in $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$. We will prove that $(\overline{u}, \overline{u}) = (u_c, u_c)$. To this end, integrating the first equation of (2.1) with u replaced by u_n over Ω and using integration by parts, we obtain

$$\int_\Omega u_n(1 - u_n/u_c)dx = 0. \tag{3.9}$$

Assume $\overline{u} < u_c$. Then $1 - u_n/u_c > 0$ when n is large enough since $u_n \to \overline{u}$. Noting that u_n is positive, by (3.9), this assumption is impossible. Similarly, $\overline{u} > u_c$ is impossible. Thus, we have that $\overline{u} = u_c$. From the second equation of (2.1) it immediately follows that $\overline{v} = u_c$. The proof is finished. □

Now we state the main result of this section.

Theorem 3.5. *Let* (1.4) *hold, and let D, μ and u_c be fixed. There exists $\chi_0 = \chi_0(D, \mu, u_c) > 0$ such that the elliptic boundary value problem* (2.1) *has no nonconstant solution for $\chi \in [0, \chi_0]$.*

Proof. We introduce an operator $\mathcal{P} : \mathbb{R} \times (H^2(\Omega) \times H^2(\Omega)) \to L^2(\Omega) \times L^2(\Omega)$ as

$$\mathcal{P}(\chi, u, v) = \begin{pmatrix} \nabla \cdot (d(u)\nabla u - \chi u\phi(u)\nabla v) + \mu u(1 - u/u_c) \\ \Delta v + u - v \end{pmatrix}.$$

By Lemma 3.2, $\mathcal{P}(0, u, v) = \mathbf{0}$ has the unique positive solution $(u, v) = (u_c, u_c)$. Let $D_{(u,v)}(0, u_c, u_c) : (H^2(\Omega) \times H^2(\Omega)) \to L^2(\Omega) \times L^2(\Omega)$ denote Jacobian matrix of $\mathcal{P}(0, u, v)$ at (u_c, u_c) with respect to (u, v). Then through a straightforward computation we have

$$D_{(u,v)}\mathcal{P}(0, u_c, u_c) = \begin{pmatrix} d(u_c)\Delta - \mu & 0 \\ 1 & \Delta - 1 \end{pmatrix}.$$

From (3.2), we know that $D_{(u,v)}\mathcal{P}(0, u_c, u_c)$ is invertible. Thus from the Implicit Function Theorem it follows that there exists $\chi_0, r > 0$ such that there is a unique solution of

$$\mathcal{P}(\chi, u, v) = \mathbf{0} \quad \text{in } [0, \chi_0] \times B_r(u_c, u_c),$$

where $B_r(u_c, u_c)$ denotes the open ball in $H^2(\Omega) \times H^2(\Omega)$ centered at (u_c, u_c) with the radius r. We know that this unique solution is the positive constant solution (u_c, u_c). Assume now that $\{\chi_n\}_{n\geq 0}$ is a sequence of positive real numbers such that $\chi_n \to 0$ as $n \to \infty$ and let (u_n, v_n) be an arbitrary solution of (2.1) for $\chi = \chi_n$, that is,

$$\mathcal{P}(\chi_n, u_n, v_n) = \mathbf{0}.$$

By Lemma 3.4, we have

$$(u_n, v_n) \to (u_c, u_c) \quad \text{in } C^2(\overline{\Omega}) \times C^2(\overline{\Omega}) \quad \text{as } n \to \infty.$$

It is seen that when n is large enough, (χ_n, u_n, v_n) remains in $(0, \chi_0) \times B_r(u_c, u_c)$. Then, from the results obtained above we see that $(u_n, v_n) = (u_c, u_c)$. Therefore, for $\chi = \chi_n$ small enough, (2.1) has only the positive constant solution (u_c, u_c). The proof is completed. □

4. Existence of Nonconstant Steady States

It has been shown in Sec. 3 that the system (1.1)–(1.3) has only two constant steady states $\mathbf{0} = (0, 0)$ and $\widetilde{\omega} = (u_c, u_c)$ for sufficiently small $\chi > 0$. In this section, we shall establish the existence conditions for nonconstant positive steady states of the volume-filling chemotaxis system (1.1)–(1.3) by applying the degree index theory, which is equivalent to find the nonconstant solutions to the elliptic boundary value problem (2.1). We begin with the following lemma, which enables us to apply the degree index theory in an annulus.

Lemma 4.1. *There exists a positive constant ϵ such that every possible positive solution (u, v) of* (2.1) *satisfies*

$$\min_{\overline{\Omega}} u, \min_{\overline{\Omega}} v > \epsilon. \tag{4.1}$$

Proof. By contradiction, we assume that the result is not true. Then there is a sequence of positive solutions $(u_n, v_n), n = 1, 2, \ldots$ with

$$\min\left\{\min_{\overline{\Omega}} u_n, \min_{\overline{\Omega}} v_n\right\} \to 0 \quad \text{as } n \to \infty.$$

In view of Lemma 3.3, there exists a non-negative solution (u, v) of (2.1) such that $(u_n, v_n) \to (u, v)$, passing to a subsequence if necessary, in $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ as $n \to \infty$. Our assumption implies that (u, v) satisfies $\min_{\overline{\Omega}} u = 0$ or $\min_{\overline{\Omega}} v = 0$. Then using the same method as in Lemma 2.1, we can prove that $(u, v) \equiv (0, 0)$. Then integrating the first equation of (2.1) with u replaced by u_n over Ω and using the Green's formula, we have $\int_\Omega u_n(1 - u_n/u_c)dx = 0$ for all positive integers n. But when n is large enough, it is true that $0 < u_n < u_c$ for $x \in \Omega$, and then we have $\int_\Omega u_n(1-u_n/u_c)dx \neq 0$ for sufficiently large n. This contradiction implies that $u \neq 0$. By this and the second equation of (2.1), we also have $v \neq 0$. Therefore, the assumption does not hold, and the proof is completed. □

From Proposition 2.5 and Lemma 4.1, it follows that all the possible positive solutions of (2.1) are located in the following set

$$\mathbb{B}(\epsilon) = \{\omega \in X : \epsilon < u, v < 1 \text{ on } \Omega\}. \tag{4.2}$$

Next we shall find the conditions under which the elliptic boundary value problem (2.1) possesses nonconstant positive solutions. To this end, we now rewrite the elliptic boundary value problem (2.1) in the vectorial form

$$\begin{cases} -\nabla \cdot (\mathbb{A}(\omega)\nabla\omega) = \mathbb{G}(\omega), & x \in \Omega, \\ \dfrac{\partial \omega}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{4.3}$$

where

$$\mathbb{A}(\omega) = \begin{pmatrix} d(u) & -\chi u\phi(u) \\ 0 & 1 \end{pmatrix}, \quad \mathbb{G}(\omega) = \begin{pmatrix} \mu u(1-u/u_c) \\ u - v \end{pmatrix}.$$

Denoting by Γ the inverse operator of $I - \nabla \cdot (\mathbb{A} \circ \nabla)$, we define the operator $\mathbb{J}(\chi, \omega)$ as

$$\mathbb{J}(\chi, \omega) = \omega - \Gamma(\omega + \mathbb{G}(\omega)). \tag{4.4}$$

Then ω is a positive solution to (4.3) if and only if ω satisfies

$$\mathbb{J}(\chi, \omega) = 0, \quad \omega \in \mathbb{B}(\epsilon) \tag{4.5}$$

with the homogeneous Neumann boundary condition. Since $\mathbb{J}(\chi, \cdot)$ is a compact perturbation of the identity operator for $\mathbb{B} = \mathbb{B}(\epsilon)$, and, by Proposition 2.5 and Lemma 4.1, $\mathbb{J}(\chi, \cdot) \neq 0$ on $\partial\mathbb{B}$, the Leray–Schauder degree $\deg(\mathbb{J}(\chi, \cdot), 0, \mathbb{B})$ is well defined. Next we need to compute the degree index of the positive zero-point $\widetilde{\omega}$ of $\mathbb{J}(\chi, \cdot)$. Recall that if $D_\omega\mathbb{J}(\chi, \widetilde{\omega})$ is invertible, the index of $\mathbb{J}(\chi, \cdot)$ at the zero point $\widetilde{\omega}$ is defined as

$$\operatorname{index}(\mathbb{J}(\chi, \cdot), \widetilde{\omega}) = (-1)^\gamma,$$

where γ is the sum of algebraic multiplicities of the negative eigenvalues of $D_\omega\mathbb{J}(\chi, \widetilde{\omega})$ (see [19, Theorem 2.8.1]).

By (4.4), we obtain

$$\begin{aligned} D_\omega\mathbb{J}(\chi, \widetilde{\omega}) &= I - D_\omega(\Gamma \circ (I + \mathbb{G}))(\widetilde{\omega}) = I - \Gamma_\omega(\widetilde{\omega}) \circ (I + \mathbb{G}_\omega)(\widetilde{\omega}) \\ &= I - (I - \nabla \cdot (\mathbb{A}_\omega(\widetilde{\omega}) \circ \nabla))^{-1} \circ (I + \mathbb{G}_\omega)(\widetilde{\omega}) \\ &= I - \begin{pmatrix} -d(u_c)\Delta + 1 & \chi u_c\phi(u_c)\Delta \\ 0 & -\Delta + 1 \end{pmatrix}^{-1} \begin{pmatrix} 1-\mu & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{4.6}$$

Furthermore, the right-hand side of (4.6) can be rewritten as

$$\begin{pmatrix} -d(u_c)\Delta+1 & \chi u_c\phi(u_c)\Delta \\ 0 & -\Delta+1 \end{pmatrix}^{-1} \begin{pmatrix} -d(u_c)\Delta+\mu & \chi u_c\phi(u_c)\Delta \\ -1 & -\Delta+1 \end{pmatrix}. \tag{4.7}$$

We shall use the decomposition (3.3) to analyze the eigenvalues of $D_\omega\mathbb{J}(\chi,\widetilde{\omega})$. It is well known that, for each integer $i \geq 1$ and $1 \leq j \leq \dim E(\lambda_i)$, X_{ij} defined in (3.2) is invariant under $D_\omega\mathbb{J}(\chi,\widetilde{\omega})$. By (4.7), we have that τ is an eigenvalue of $D_\omega\mathbb{J}(\chi,\widetilde{\omega})$ on X_{ij} if and only if, for some $i \geq 1$, it is an eigenvalue of the matrix

$$M_i \stackrel{def}{=} \begin{pmatrix} d(u_c)\lambda_i+1 & -\lambda_i\chi u_c\phi(u_c) \\ 0 & \lambda_i+1 \end{pmatrix}^{-1} \begin{pmatrix} d(u_c)\lambda_i+\mu & -\lambda_i\chi u_c\phi(u_c) \\ -1 & \lambda_i+1 \end{pmatrix}$$

and $D_\omega\mathbb{J}(\chi,\widetilde{\omega})$ is invertible if and only if the matrix M_i is nonsingular for all $i \geq 1$. In terms of this property and the definition of the degree index of an operator at its zero point, we have the following formula

$$\gamma(D_\omega\mathbb{J}(\chi,\widetilde{\omega})) = \sum_{i=1}^{\infty} \dim E(\lambda_i) \times \gamma(M_i), \tag{4.8}$$

where $\gamma(D_\omega\mathbb{J}(\chi,\widetilde{\omega}))$ and $\gamma(M_i)$ are the sum of the algebraic multiplicities of the negative eigenvalues of $D_\omega\mathbb{J}(\chi,\widetilde{\omega})$ and M_i, respectively (see [17] for details). In order to compute $\gamma(M_i)$, we need to use the values of $\det(M_i)$ and $\mathrm{trace}(M_i)$. As such, we first define a function $H(\lambda)$ as follows

$$\begin{aligned} H(\lambda) &= H(\lambda,\widetilde{\omega}) \\ &= \det\begin{pmatrix} d(u_c)\lambda+1 & -\lambda\chi u_c\phi(u_c) \\ 0 & \lambda+1 \end{pmatrix}^{-1} \det\begin{pmatrix} d(u_c)\lambda+\mu & -\lambda\chi u_c\phi(u_c) \\ -1 & \lambda+1 \end{pmatrix}. \end{aligned} \tag{4.9}$$

It is easy to see that $H(\lambda_i) = \det(M_i)$. Evidently, if $H(\lambda_i) \neq 0$, then $\gamma(M_i) = 1$ if and only if $H(\lambda_i) < 0$. Whereas if $H(\lambda_i) > 0$, $\gamma(M_i)$ is either 0 or 2. Thus, it follows from (4.8) that

$$\gamma(D_\omega\mathbb{J}(\chi,\widetilde{\omega})) = \sum_{i\geq 1, H(\lambda_i)<0} \dim E(\lambda_i) \pmod 2.$$

Therefore, we have the following lemma.

Lemma 4.2. *Supposed that $H(\lambda_i) \neq 0$ for all $i \geq 1$. Then*

$$\mathrm{index}(\mathbb{J},\widetilde{\omega}) = (-1)^\gamma, \quad \textit{where } \gamma = \sum_{i\geq 1, H(\lambda_i)<0} \dim E(\lambda_i).$$

To determine the value of γ in the above formula, we will discuss the sign of $H(\lambda)$. Since the first factor of $H(\lambda)$ is positive, we only need to analyze the sign of the second factor which is denoted as

$$h(\lambda) = \det \begin{pmatrix} d(u_c)\lambda + \mu & -\lambda\chi u_c \phi(u_c) \\ -1 & \lambda + 1 \end{pmatrix}.$$

By a simple calculation, we have

$$h(\lambda) = d(u_c)\lambda^2 + [\mu + d(u_c) - \chi u_c \phi(u_c)]\lambda + \mu. \tag{4.10}$$

To make $h(\lambda) < 0$, it is required that the discriminant of the quadratic function in (4.10) to be positive, that is, $[\mu + d(u_c) - \chi u_c \phi(u_c)]^2 - 4\mu d(u_c) > 0$. Through a direct computation, we find that when (3.4) is satisfied, the equation $h(\lambda) = 0$ has two positive roots $\lambda_\pm(\chi)$ satisfying $k_2^2 = \lambda_+(\chi) > \lambda_-(\chi) = k_1^2 > 0$, where $k_i^2\,(i = 1, 2)$ are defined in (3.5). It is easy to check that

$$\begin{cases} H(\lambda) > 0, & \lambda \in (-\infty, \lambda_-(\chi)) \cup (\lambda_+(\chi), +\infty), \\ H(\lambda) < 0, & \lambda \in (\lambda_-(\chi), \lambda_+(\chi)). \end{cases} \tag{4.11}$$

It is helpful to note that the results derived here show that k^2 corresponds to the eigenvalue λ of the negative Laplace operator, namely $\lambda = k^2$, $k_1^2 = \lambda_-(\chi), k_2^2 = \lambda_+(\chi)$. Based on Lemma 4.2 and (4.11), we have the result below.

Lemma 4.3. *Let D, μ and u_c be fixed parameters, and α, β satisfy* (1.4). *Let χ be large such that* (3.4) *holds. If there exist integers n_1, n_2 satisfying $n_2 > n_1 \geq 1$ such that*

$$\lambda_-(\chi) \in (\lambda_{n_1}, \lambda_{n_1+1}), \quad \lambda_+(\chi) \in (\lambda_{n_2}, \lambda_{n_2+1}) \tag{4.12}$$

and

$$\sum_{i=n_1+1}^{n_2} \dim E(\lambda_i) \quad \text{is odd}, \tag{4.13}$$

then

$$\operatorname{index}(\mathbb{J}(\chi, \cdot), \widetilde{\omega}) = -1. \tag{4.14}$$

Proof. From (3.1) and (4.12), it follows that

$$\begin{cases} \lambda_i < \lambda_-(\chi), & 1 \leq i \leq n_1, \\ \lambda_-(\chi) < \lambda_i < \lambda_+(\chi), & n_1 + 1 \leq i \leq n_2, \\ \lambda_i > \lambda_+(\chi), & i \geq n_2 + 1. \end{cases}$$

Then, by (4.11), we have

$$\begin{cases} H(\lambda_i) > 0, & 1 \leq i \leq n_1, \\ H(\lambda_i) < 0, & n_1 + 1 \leq i \leq n_2, \\ H(\lambda_i) > 0, & i \geq n_2 + 1. \end{cases}$$

Obviously, the matrix M_i for all $i \geq 1$ has no zero eigenvalue. Therefore, $D_\omega \mathbb{J}(\chi, \widetilde{\omega})$ is invertible, and then Lemma 4.2 and (4.13) lead to the desired result. □

The primary result of this paper is the following theorem.

Theorem 4.4. *Let λ_i $(i \geq 1)$ be eigenvalues of the Laplace operator $-\Delta$ under the homogeneous Neumann boundary condition. Let D, μ and u_c be fixed parameters and α, β satisfy* (1.4). *Then system* (2.1) *has at least one nonconstant positive solution provided that the following conditions hold*:

(i) *χ is large enough such that $\chi > \chi_c$, i.e. $\chi > \frac{D(1-u_c)^{-\alpha}+\mu+2\sqrt{\mu D(1-u_c)^{-\alpha}}}{u_c(1-u_c)^{\beta}}$*;

(ii) *there exist integers n_1 and n_2 satisfying $n_2 > n_1 \geq 1$ such that*

$$\lambda_-(\chi) \in (\lambda_{n_1}, \lambda_{n_1+1}), \quad \lambda_+(\chi) \in (\lambda_{n_2}, \lambda_{n_2+1}),$$

where $\lambda_-(\chi) = k_1^2 > 0, \lambda_+(\chi) = k_2^2$ and k_i^2 $(i = 1, 2)$ are defined in (3.5);

(iii) $\sum_{i=n_1+1}^{n_2} \dim E(\lambda_i)$ *is odd.*

Proof. For $\tau \in [0, 1]$, we define the operator

$$\mathbb{A}(\tau; \omega) = \begin{pmatrix} d(u) & -\tau\chi u\phi(u) \\ 0 & 1 \end{pmatrix}$$

and consider the following boundary problem

$$\begin{cases} -\nabla \cdot (\mathbb{A}(\tau; \omega)\nabla\omega) = \mathbb{G}(\omega), & x \in \Omega, \\ \dfrac{\partial\omega}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{4.15}$$

It is easily seen that ω is a positive nonconstant solution of (4.3) if and only if it is a positive nonconstant solution of (4.15) with $\tau = 1$. Obviously, $\widetilde{\omega}$ is the unique positive constant solution of (4.15) for any $0 \leq \tau \leq 1$; moreover, for any $0 \leq \tau \leq 1$, ω is a positive solution of (4.15) if and only if

$$\mathbb{J}(\tau; \omega) = \omega - (\omega - \nabla(\mathbb{A}(\tau; \omega)\nabla\omega)^{-1}(\omega + \mathbb{G}(\omega)) = 0 \quad \text{in } \mathbb{B}(\epsilon). \tag{4.16}$$

Then we have $\mathbb{J}(1; \omega) = \mathbb{J}(\chi, \omega)$, and $\mathbb{J}(0; \omega) = 0$ just corresponds to the steady state system (3.6). Thus, we have that $D_\omega\mathbb{J}(0, \widetilde{\omega}) = -D_{(u,v)}\mathcal{P}(0, u_c, u_c)$. Hence it is simple to verify that

$$\text{index}(\mathbb{J}(0; \cdot), \widetilde{\omega}) = (-1)^0 = 1. \tag{4.17}$$

By (i)–(iii) and Lemma 4.3, it has that

$$\text{index}(\mathbb{J}(1; \cdot), \widetilde{\omega}) = (-1)^\gamma = -1. \tag{4.18}$$

Using the same argument as in Lemma 2.4 and Proposition 2.5, we know that (4.16) has no solution on the boundary of $\mathbb{B}(\epsilon)$ for any $0 \leq \tau \leq 1$. Thus the Leray–Schauder degree $\deg(\mathbb{J}(\tau; \omega), 0, \mathbb{B}(\epsilon))$ is well defined, and according to the homotopy invariance of the topological degree, it is a constant for all $\tau \in [0, 1]$. Thus we have

$$\deg(\mathbb{J}(1; \omega), 0, \mathbb{B}(\epsilon)) = \deg(\mathbb{J}(0; \omega), 0, \mathbb{B}(\epsilon)). \tag{4.19}$$

Now suppose by contradiction that system (2.1) has no nonconstant solution in $\mathbb{B}(\epsilon)$ when all conditions in Theorem 4.4 are satisfied; moreover, Lemma 3.2 implies both $\mathbb{J}(0;\omega) = 0$ and $\mathbb{J}(1;\omega) = 0$ have only the positive solution $\widetilde{\omega}$ in $\mathbb{B}(\epsilon)$. Hence from (4.17) and (4.18) it follows that

$$\deg(\mathbb{J}(0;\omega), 0, \mathbb{B}(\epsilon)) = \text{index}(\mathbb{J}(0;\cdot), \widetilde{\omega}) = 1,$$

$$\deg(\mathbb{J}(1;\omega), 0, \mathbb{B}(\epsilon)) = \text{index}(\mathbb{J}(1;\cdot), \widetilde{\omega}) = -1,$$

which contradicts (4.19) and the proof is completed. □

Remark 4.5. The results of Theorem 4.4 apply to the model with $\alpha = 0, \beta = 1$ as considered in paper [17] where the existence of stationary patterns was established in [17, Theorem 3.10] under four conditions: three conditions same as those in Theorem 4.4 plus one additional condition $\lambda_i \neq \frac{\mu}{D}$ $(i \geq 1)$. The results of the present paper apparently improve the results of [17, Theorem 3.10] by removing the condition $\lambda_i \neq \frac{\mu}{D}$ $(i \geq 1)$.

5. Example

Now we present a special case of Theorem 4.4 where the space dimension is 1. Without loss of generality, we set $\Omega = (0, l)$. Then the system (2.1) becomes

$$\begin{cases} -(d(u))u')' + \chi(u\phi(u)v')' = \mu u(1 - u/u_c), & x \in (0, l), \\ -v'' = u - v, & x \in (0, l), \\ u'(0) = u'(l) = 0, & v'(0) = v'(l) = 0, \end{cases} \tag{5.1}$$

where $' = \frac{d}{dx}$. Applying Theorem 4.4, we have the following more explicit results.

Corollary 5.1. *Let D, μ and u_c be fixed parameters and α, β satisfy* (1.4). *Let $\lambda_i = \frac{i^2\pi^2}{l^2}, i = 0, 1, 2, 3, \ldots$. The system* (5.1) *has at least one nonconstant positive solution if the following conditions are fulfilled*:

(i) *parameters $D, \mu, u_c, \alpha, \beta$ and χ satisfy $\chi > \chi_c = \frac{\mu + d(u_c) + 2\sqrt{\mu d(u_c)}}{u_c \phi(u_c)}$*;
(ii) *there exist integers n_1, n_2 satisfying $n_2 > n_1 \geq 1$ such that $\lambda_-(\chi) \in (\lambda_{n_1}, \lambda_{n_1+1}), \lambda_+(\chi) \in (\lambda_{n_2}, \lambda_{n_2+1})$, where $\lambda_-(\chi) = k_1^2 > 0, \lambda_+(\chi) = k_2^2$ and $k_i^2 (i = 1, 2)$ are defined in* (3.5);
(iii) $\sum_{i=n_1+1}^{n_2} \dim E(\lambda_i) = n_2 - n_1$ *is odd.*

Proof. Notice that when $\Omega = (0, l)$, the eigenvalue problem associated with (5.1) is

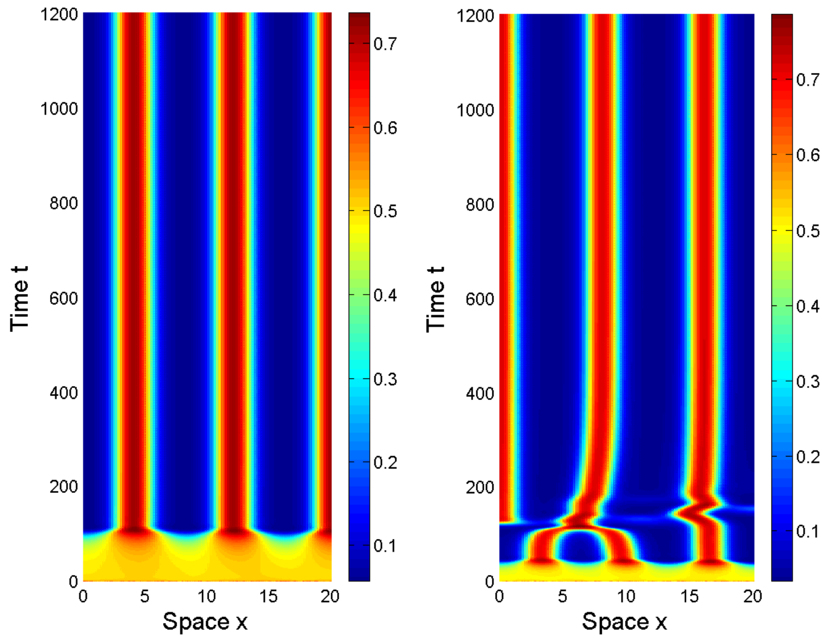
$$-\omega'' = \lambda\omega, \quad \omega'(0) = \omega'(l) = 0$$

which has countably many eigenvalues $\lambda_i = \frac{i^2\pi^2}{l^2}, i = 0, 1, 2, \ldots$, and the eigenvector corresponding to each eigenvalue λ_i is $\cos\frac{i\pi}{l}x$ which spans the eigenspace $E(\lambda_i)$, then $\dim E(\lambda_i) = 1$. Thus $\sum_{i=n_1+1}^{n_2} \dim E(\lambda_i) = n_2 - n_1$. Therefore, the application of Theorem 4.4 completes the proof. □

Next, we shall perform numerical simulations to illustrate the pattern formation of the model (1.1) under the condition (1.4). The numerical computation is implemented with the Matlab PDE computing package PDEPE based on the finite difference scheme. In general, chemotaxis model with logistic cell growth will generate chaotic patterns (see [21, 27]) and stationary patterns are not the typical ones. In this paper, Theorem 4.4 gives sufficient conditions for the existence of stationary patterns and also provides a numerical way to select the parameter values to generate stationary patterns. We numerically verify the Theorem 4.4 in one dimension in Fig. 1. By the parameter values chosen in Fig. 1(a), we can make simple calculations and obtain

$$\chi_c = 12.6577, \quad \lambda_-(\chi) = 0.3918, \quad \lambda_+(\chi) = 1.1111.$$

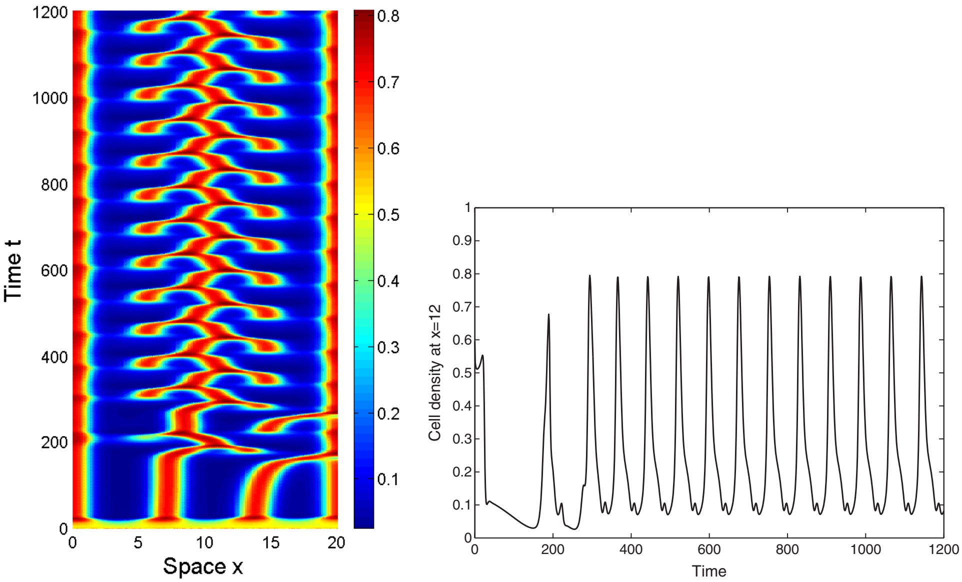
Since $\chi = 13.5 > \chi_c$, the first condition of Corollary 5.1 is satisfied. Furthermore simple computations entail that we can find $6 = n_2 > n_1 = 3$ such that $\lambda_3 = 0.2221, \lambda_4 = 0.3948, \lambda_6 = 0.8883, \lambda_7 = 1.2090$. Hence $\lambda_-(\chi) \in (\lambda_3, \lambda_4)$ and $\lambda_+(\chi) \in (\lambda_6, \lambda_7)$ with $\sum_{i=n_1+1}^{n_2} \dim E(\lambda_i) = n_2 - n_1 = 3$ which is odd. Hence the second condition of Corollary 5.1 is satisfied. Therefore by Corollary 5.1, a stationary pattern should be expected, as we see in Fig. 1(a). From Lemma 3.1, we know that the first two conditions in Theorem 4.4 are necessary. But we have to remark that it is unclear whether the third condition of Theorem 4.4 is necessary. The degree index



(a) (b)

Fig. 1. Stationary pattern formation of the chemotaxis model (1.1)–(1.4) in an interval $(0, 20)$, where the initial value $u_0 = v_0 = u_c + r$ with r being a 1% random spatial perturbation of the homogeneous steady state (u_c, u_c). The parameter values are: $\mu = 0.5, u_c = 0.5, D = 1, \alpha = 0.2, \beta = 1$ and the value of χ is: (a) $\chi = 13.5$; (b) $\chi = 15$.

theory cannot provide necessary and sufficient conditions in general for the existence of stationary solutions. Here we make a numerical investigation. We set the value of $\chi = 15$ as in Fig. 1(b), which gives $\lambda_-(\chi) = 0.2831, \lambda_+(\chi) = 1.5481$. Then we can find that $\lambda_-(\chi) \in (\lambda_3, \lambda_4)$ and $\lambda_+(\chi) \in (\lambda_7, \lambda_8) = (1.2090, 1.5791)$, which verifies that $\sum_{i=n_1+1}^{n_2} \dim E(\lambda_i) = n_2 - n_1 = 7 - 3 = 4$ which is even. However we still numerically obtain the stationary pattern. This indicates that the third condition in Theorem 4.4 might not be necessary. We suspect that the considered system may have meta-stability as for the case $\mu = 0$ [23], and the stationary patterns shown in Fig. 1 may make a transition after sufficiently long time. This leaves us an interesting problem to explore in the future. Another profound pattern found in our numerical simulation is the time-periodic patterns as shown in Fig. 2. It is well known (cf. [30]) that the volume-filling chemotaxis model without cell growth ($\mu = 0$) has the time monotone (decreasing) Lyapunov functional which excludes the existence of time-periodic orbits. However once the cell growth is included, the Lyapunov functional no longer exists and the time-periodic pattern may arise as shown in Fig. 2. However, if we perform the linear stability analysis, one can easily find that there is no Hopf bifurcation from the homogeneous steady state. So mechanism of generating the time-periodic pattern for chemotaxis models with logistic growth is still mysterious.



(a) (b)

Fig. 2. Time-periodic patterns of the chemotaxis model (1.1)–(1.4) in an interval $(0, 20)$, where $\chi = 16$. Other parameter values and the initial value are the same as those in Fig. 1. Left panel: the spatio-temporal pattern formation; right panel: the time evolution of the solution component u (cell density) at the fixed spatial point $x = 12$.

Acknowledgments

We are grateful to the referee for carefully reading the manuscript and comments which improve the exposition of the paper. The research of Z. A. Wang was supported in part by the Central Research Grant No. G-YBCS and a startup grant No. A-PL15 from the Hong Kong Polytechnic University, and the Hong Kong RGC ECS grant No. 509113. M. Ma gratefully acknowledges support from the provincial Natural Science Foundation of Zhejiang (LY15A010017) and the National Natural Science Foundation of China (No. 11271342).

Appendix

Proof of Lemma 2.4. The use of Moser iterative procedure to derive L^∞ estimates of the solution $u(x,t)$ for the time-dependent problem (1.1) with $\mu = 0$ has been exposed in [29, proof of Lemma 4.2]. Here we treat the stationary problem of (1.1) for $\mu > 0$ using slightly simplified procedure which is, however, essentially similar to [29]. For completeness, we present the detailed proofs below with some repeated arguments as in [29, proof of Lemma 4.2].

We first fix $p_0 > 1$ such that

$$p_0 > n|\alpha| \tag{A.1}$$

and

$$p_0 > 4(\alpha + \beta - 1) - \alpha. \tag{A.2}$$

Then let us define a recursive sequence $\{p_k\}_{k\in\mathbb{N}}$ by

$$p_k = 2p_{k-1} - \alpha, \quad k \ge 1. \tag{A.3}$$

It is easy to check that $\{p_k\}_{k\in\mathbb{N}}$ is strictly increasing and there are constants σ_1 and σ_2 such that

$$\sigma_1 2^k \le p_k \le \sigma_2 2^k, \quad \text{for all } k \ge 0. \tag{A.4}$$

Set

$$q_k = \frac{2(p_k - \alpha - 2\beta + 2)}{p_k + \alpha} \equiv 2 - \frac{4(\alpha + \beta - 1)}{p_k + \alpha}, \quad k \ge 1.$$

Then by (A.2) and the monotonicity of $\{p_k\}_{k\in\mathbb{N}}$, we have

$$1 < q_k \le 2, \quad k \ge 1. \tag{A.5}$$

Furthermore, setting

$$\widetilde{q}_k = \frac{2p_k}{p_k + \alpha}, \quad k \ge 1, \tag{A.6}$$

we have

$$1 < \widetilde{q}_k \le \frac{2p_k}{p_k - |\alpha|} \le \frac{2p_0}{p_0 - |\alpha|}, \quad k \ge 1, \tag{A.7}$$

and from (A.1) it follows that $\overline{q} = \frac{2p_0}{p_0-|\alpha|} < \frac{2n}{(n-2)_+}$. Our goal is to derive upper bounds for

$$A_k = \max\left\{1, \int_\Omega w^{-p_k}(x)dx\right\}, \quad k \geq 0,$$

where $w(x) = 1 - u(x)$. In view of (2.10), there exist constants $b_1 \in (0,1]$ and $b_2 > 0$ which, like constants $b_3, b_4, \dots$ used afterwards, possibly depend on K but not on k such that

$$b_1 \int_\Omega |\nabla w^{-\frac{p_k+\alpha}{2}}|^2 dx \leq b_2 p_k^2 \int_\Omega w^{-p_k+\alpha+2\beta-2} dx$$
$$+\mu \int_\Omega u(1-u/u_c) w^{-p_k-1} dx, \quad k \geq 1. \tag{A.8}$$

Noting (A.5), we can use Lemma 2.2(ii) to obtain

$$\begin{aligned} &b_2 p_k^2 \int_\Omega w^{-p_k+\alpha+2\beta-2} dx \\ &= b_2 p_k^2 \left\| w^{-\frac{p_k+\alpha}{2}} \right\|_{L^{q_k}(\Omega)}^{q_k} \\ &\leq b_3 p_k^2 \left\| \nabla w^{-\frac{p_k+\alpha}{2}} \right\|_{L^2(\Omega)}^{\frac{2n(q_k-1)}{n+2}} \cdot \left\| w^{-\frac{p_k+\alpha}{2}} \right\|_{L^1(\Omega)}^{\frac{2n-(n-2)q_k}{n+2}} + b_3 p_k^2 \left\| w^{-\frac{p_k+\alpha}{2}} \right\|_{L^1(\Omega)}^{q_k} \end{aligned} \tag{A.9}$$

with $b_3 > 0$. By (A.3), we have

$$\|w^{-\frac{p_k+\alpha}{2}}\|_{L^1(\Omega)} = \int_\Omega w^{-p_{k-1}}(x) dx \leq A_{k-1}, \tag{A.10}$$

and by (A.5), we have $\frac{2n(q_k-1)}{n+2} \leq \frac{2n}{n+2} < 2$. Thus, we can apply the Young's inequality to the last second term of (A.9) (see the derivation of (4.27) in [29] for details) and obtain

$$\begin{aligned} &b_2 p_k^2 \int_\Omega w^{-p_k+\alpha+2\beta-2} dx \\ &\leq \frac{n(q_k-1)}{n+2} b_1 \left\| \nabla w^{-\frac{p_k+\alpha}{2}} \right\|_{L^2(\Omega)}^2 \\ &\quad + \frac{2n+2-nq_k}{n+2} \cdot b_1^{-\frac{n(q_k-1)}{2n+2-nq_k}} \left(b_3 p_k^2 A_{k-1}^{\frac{2n-(n-2)q_k}{n+2}}\right)^{\frac{n+2}{2n+2-nq_k}} + b_3 p_k^2 A_{k-1}^{q_k}. \end{aligned} \tag{A.11}$$

Repeatedly using (A.5) yields the following estimates for all $k \geq 1$:

$$\frac{n(q_k-1)}{n+2} \leq \frac{n}{n+2}, \quad \frac{2n+2-nq_k}{n+2} \leq \frac{2n+2}{n+2}, \quad \frac{n(q_k-1)}{2n+2-nq_k} \leq \frac{n}{2}$$

and

$$\frac{n+2}{2n+2-nq_k} \leq \frac{n+2}{2}, \quad \frac{2n+(2-n)q_k}{n+2} \cdot \frac{n+2}{2n+2-nq_k} = 1 + \frac{2(q_k-1)}{2n+2-nq_k} \leq 2.$$

With these estimates and facts that $b_1 < 1$, $p_k > 1$, $A_{k-1} \geq 1$ and (A.4), we can find a constant $b_4 > 0$ such that

$$b_2 p_k^2 \int_\Omega w^{-p_k+\alpha+2\beta-2} dx \leq \frac{n}{n+2} b_1 \int_\Omega |\nabla w^{-\frac{p_k+\alpha}{2}}|^2 dx + b_4 2^{(n+2)k} A_{k-1}^2. \quad \text{(A.12)}$$

Substituting (A.12) into (A.8) leads to

$$\frac{2b_1}{n+2} \int_\Omega |\nabla w^{-\frac{p_k+\alpha}{2}}|^2 dx \leq b_4 2^{(n+2)k} A_{k-1}^2 + \mu \int_\Omega u(1 - u/u_c) w^{-p_k-1} dx. \quad \text{(A.13)}$$

Using Lemma 2.2(ii) to $q = \widetilde{q}_k = \frac{2p_k}{p_k+\alpha}$, we can find a constant $b_5 > 1$ such that

$$\begin{aligned} \int_\Omega w^{-p_k} dx &= \left\| w^{-\frac{p_k+\alpha}{2}} \right\|_{L^{\widetilde{q}_k}(\Omega)}^{\widetilde{q}_k} \\ &\leq b_5 \left\| \nabla w^{-\frac{p_k+\alpha}{2}} \right\|_{L^2(\Omega)}^{\widetilde{q}_k \cdot a} \cdot \left\| w^{-\frac{p_k+\alpha}{2}} \right\|_{L^1(\Omega)}^{\widetilde{q}_k \cdot (1-a)} + b_5 \left\| w^{-\frac{p_k+\alpha}{2}} \right\|_{L^1(\Omega)}^{\widetilde{q}_k}, \end{aligned} \quad \text{(A.14)}$$

where $a = \frac{2n(\widetilde{q}_k - 1)}{(n+2)\widetilde{q}_k} < 1$. Then employing Young's inequality $fg \leq a f^{1/a} + (1-a) g^{1/(1-a)}$ for all $f, g \geq 0$, it follows from (A.10) and (A.14) that

$$\begin{aligned} \int_\Omega w^{-p_k} dx &\leq b_5 (\|\nabla w^{-\frac{p_k+\alpha}{2}}\|_{L^2(\Omega)}^{\widetilde{q}_k} + \|w^{-\frac{p_k+\alpha}{2}}\|_{L^1(\Omega)}^{\widetilde{q}_k}) + b_5 \|w^{-\frac{p_k+\alpha}{2}}\|_{L^1(\Omega)}^{\widetilde{q}_k} \\ &\leq b_5 \|\nabla w^{-\frac{p_k+\alpha}{2}}\|_{L^2(\Omega)}^{\widetilde{q}_k} + 2 b_5 A_{k-1}^{\widetilde{q}_k}, \end{aligned}$$

which gives rise to

$$\|\nabla w^{-\frac{p_k+\alpha}{2}}\|_{L^2(\Omega)}^2 \geq \left(\frac{1}{b_5} \int_\Omega w^{-p_k} dx - 2 A_{k-1}^{\widetilde{q}_k} \right)^{\frac{2}{\widetilde{q}_k}}.$$

In view of (A.13), we obtain

$$\begin{aligned} &\frac{2b_1}{n+2} \left(\frac{1}{b_5} \int_\Omega w^{-p_k} dx - 2 A_{k-1}^{\widetilde{q}_k} \right)^{\frac{2}{\widetilde{q}_k}} \\ &\qquad \leq b_4 2^{(n+2)k} A_{k-1}^2 + \mu \int_\Omega u(1 - u/u_c) w^{-p_k-1} dx. \end{aligned} \quad \text{(A.15)}$$

Note that the last term in (A.15) is bounded. Since $A_{k-1} \geq 1$, we may find a constant $b_6 > 0$ such that $\mu \int_\Omega u(1 - u/u_c) w^{-p_k-1} dx \leq b_6 A_{k-1}^2$. Then

$$\frac{2b_1}{n+2} \left(\frac{1}{b_5} \int_\Omega w^{-p_k} dx - 2 A_{k-1}^{\widetilde{q}_k} \right)^{\frac{2}{\widetilde{q}_k}} \leq [b_4 2^{(n+2)k} + b_6] A_{k-1}^2,$$

which yields

$$\int_\Omega w^{-p_k} dx \leq b_5 \left\{ \left[\frac{(n+2)}{2b_1} (b_4 2^{(n+2)k} + b_6) \right]^{\frac{p_k}{p_k+\alpha}} + 2 \right\} A_{k-1}^{\frac{2p_k}{p_k+\alpha}}.$$

So we have some constant $b > 1$ independent of k such that

$$\int_\Omega w^{-p_k}\,dx \le b^k A_{k-1}^{2(1+\delta_k)} \quad \text{for all } k \ge 1$$

with $\delta_k = \frac{-\alpha}{p_k+\alpha}$. Therefore, we have by the definition of A_k

$$A_k \le \max\{1, b^k A_{k-1}^{2(1+\delta_k)}\}, \quad k \ge 1.$$

If $A_k \le 1$ for infinitely many $k \in \mathbb{N}$, we immediately conclude that (2.17) holds. Otherwise, we have $A_k \le b^k A_{k-1}^{2(1+\delta_k)}$ for all $k \ge 1$. By induction and (A.4), we have for all $k \ge 1$

$$\begin{aligned} A_k^{\frac{1}{p_k}} &\le [b^{\sum_{j=1}^k j\cdot\prod_{i=j+1}^k 2(1+\delta_i)} \cdot A_0^{\prod_{i=1}^k 2(1+\delta_i)}]^{\frac{1}{\sigma_1\cdot 2^k}} \\ &= b^{\frac{1}{\sigma_1}\cdot\sum_{j=1}^k j\cdot 2^{-j}\cdot\prod_{i=j+1}^k (1+\delta_i)} \cdot A_0^{\prod_{i=1}^k (1+\delta_i)}. \end{aligned}$$

Noticing $\delta_k \le \frac{1}{p_k} \cdot \frac{|\alpha|}{1-\frac{|\alpha|}{p_0}} \le c2^{-k}$ with $c = \frac{p_0|\alpha|}{\sigma_1(p_0-|\alpha|)}$, then $\Pi_{i=1}^\infty(1+\delta_i)$ is finite since $\sum_{i=1}^\infty \delta_i$ is convergent. Considering that Lemma 2.3 gives $A_0 \le C(K)$, (2.17) is concluded. The proof is completed.

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