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Stationary and non-stationary patterns of the density-suppressed motility model

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ABSTRACT

In this paper, we first explore the stationary problem of the density-suppressed motility (DSM) model proposed in Fu et al. (2012) and Liu et al. (2011) where the diffusion rate of the bacterial cells is a decreasing function (motility function) of the concentration of a chemical secreted by bacteria themselves. We show that the DSM model does not admit non-constant steady states if either the chemical diffusion rate or the intrinsic growth rate of bacteria is large. We also prove that when the decay of the motility function is sub-linear or linear, the DSM model does not admit non-constant steady states if either the chemical diffusion rate or the intrinsic growth rate of bacteria is small. Outside these non-existence parameter regimes, we show that the DSM model will have non-constant steady states under some constraints on the parameters. Furthermore we numerically find the stable stationary patterns only when the parameter values are close to the critical instability regime. Finally by performing a delicate multiple-scale analysis, we derive that the DSM model may generate propagating oscillatory waves whose amplitude is governed by an explicit Ginzburg–Landau equation, which is further verified by numerical simulations.

1. Introduction

Turing (i.e., diffusion-driven) and chemotaxis-driven instabilities have been widely accepted as two major mechanisms reproducing many exquisite biological spatio-temporal patterns observed in nature or experiments [1]. Recently a so-called "self-trapping" mechanism was introduced into the engineered *E. coli* strains in the experiment by a synthetic biology approach and spatio-temporal patterns were observed (see [2]), where *E. coli* cells excrete a signaling molecule acyl-homoserine lactone (AHL) such that at low AHL level, *E. coli* cells undergo run-and-tumble random motion, while at high AHL levels *E. coli* cells tumble incessantly and become immotile as a result of a vanishing macroscopic motility. Later on a system of reaction–diffusion equations with density-suppressed motility was proposed in the paper [3] to explain the underlying stripe pattern formation process observed in the experiment of [2]. This paper is concerned with the model proposed in [3], which reads as

$$\begin{cases} u_t = \Delta(r(v)u) + \sigma u(1-u), & x \in \Omega, t > 0, \\ v_t = d\Delta v + u - v, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $u(x,t)$ and $v(x,t)$ represent the densities of *E. coli* cells and AHL, respectively. The first equation states that *E. coli* cells undertake a non-random diffusion with a logistic birth–death kinetics with intrinsic rate $\sigma > 0$ saturated at the normalized density 1, where the diffusion rate of *E. coli* cells depends on a motility function $r(v)$ satisfying $r'(v) < 0$ (suppressed effect of AHL concentration on cell's motility). $d > 0$ is a constant representing the diffusive rate of v. The model (1.1) is considered in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) with zero-Neumann boundary conditions warranting that no individual crosses the boundary of the habitat, which is well consistent with the experimental setting of [2] where the experiment was performed in an isolated apparatus.

Mathematically, system (1.1) may be degenerate due to the property $r'(v) < 0$ and hence its analysis becomes delicate. To the best of our knowledge, no much results have been known to (1.1) as of today. When $\sigma > 0$, the existence of global classical solutions of (1.1) was obtained first in [4] for $\Omega \subset \mathbb{R}^2$ with $r(v)$ satisfying the following hypotheses:

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(H1) $r(v) \in C^3([0,\infty))$, $r(v) > 0$ and $r'(v) < 0$ on $[0,\infty)$, $\lim_{v\to\infty} r(v) = 0$;

(H2) $\lim_{v\to\infty} \frac{r'(v)}{r(v)}$ exists.

It was further shown that if $d\sigma$ is suitably large such that

$$\sigma d > \frac{k_0}{16}, \text{ where } k_0 = \max_{0\le v\le\infty} \frac{|r'(v)|^2}{r(v)}, \tag{1.2}$$

then the constant steady state (1, 1) is globally asymptotically stable. Numerical simulations in [4] illustrated that system (1.1) can produce oscillating traveling waves and stable/unstable aggregation patterns under some conditions on the parameter values. Recently the global existence of classical solutions of (1.1) was extended to higher dimensions ($N \ge 3$) in [5] for large $\sigma > 0$ with a mildly weaker condition than (H1)–(H2). When $r(v)$ is a piecewise constant function, the analysis of (1.1) with $\sigma > 0$ was performed in [6] to study the dynamics of interface of discontinuity of solutions. When $\sigma = 0$, the global existence of class solutions of (1.1) was established in [7] for the case $r(v) = c_0/v^k (k > 0)$ with small constant $c_0 > 0$ in any dimensions with an extension in [8]. By assuming that $r(v)$ has positive lower and upper bounds, global classical solutions in two dimensions and global weak solutions in three dimensions of (1.1) with $\sigma = 0$ were obtained in [9]. Except the aforementioned results, no other results appear to be available and many interesting analytical questions on the density-suppressed motility model (1.1) have not been addressed, such as traveling wave solutions, stationary (pattern) solutions and so on.

Among other things, this paper is to explore the stationary problem of system (1.1) which reads as

$$\begin{cases} -\Delta(r(v)u) = \sigma u(1-u), & x \in \Omega, \\ -d\Delta v = u - v, & x \in \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

In view of the realistic meaning, only nonnegative solutions of (1.3) are of interest. By the well-known maximum principle and Hopf boundary lemma for elliptic equations, for any nonnegative classical solution (u, v) of (1.3) with $(u, v) \not\equiv (0, 0)$, it is easily seen that $u, v > 0$ on $\overline{\Omega}$. Hence in this paper we shall consider the existence and non-existence of non-constant positive classical solutions of (1.3) on $\overline{\Omega}$. For convenience, we let $w = r(v)u$ and transform problem (1.3) into the following equivalent one:

$$\begin{cases} -\Delta w = \frac{\sigma w}{r(v)}[1 - \frac{w}{r(v)}], & x \in \Omega, \\ -d\Delta v = \frac{w}{r(v)} - v, & x \in \Omega, \\ \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{1.4}$$

Throughout the paper, whenever we say a solution of (1.3) or (1.4), we always mean a positive classical solution. Clearly, system (1.4) has a unique trivial solution (0, 0) and a unique positive constant solution $(r(1), 1)$. The main purpose of this paper is to find the conditions for the non-existence and existence of non-constant solutions of the stationary problem (1.4) with $r(v)$ satisfying the condition (H1) in a bounded smooth domain $\Omega \subset \mathbb{R}^N (1 \le N \le 3)$. From the global stability result of [4] under condition (1.2), one can conclude that (1.3) and hence (1.4) under hypotheses (H1)–(H2) will not have non-constant positive solutions in two dimensions if σd is large. In this paper, we shall remove the condition (H2) and prove the non-existence of non-constant solutions of (1.4) in three or lower dimensions for large σd, see Theorem 3.1(a). More interestingly we find that (1.4) will not have non-constant solution either if σd is sufficiently small for some $r(v)$ satisfying certain additional conditions besides (H1), see Theorem 3.1(b). This result is somewhat counterintuitive, since for mathematical models of biology the patterns (i.e., non-constant solutions) will usually tend to arise when the diffusion rate d or the intrinsic growth rate σ is small (cf. [10–12]). We believe this is a distinctive phenomenon caused by the density-suppressed motility. For moderate value of σd, we show that non-constant positive solutions of (1.4) may exist, as shown in Theorem 4.1. Furthermore in this paper we use the multiple-scale analysis to derive that system (1.1) may generate pulsating (oscillating) traveling waves whose amplitude is shown to be governed by a Ginzburg–Landau equation (see Section 5). We verify our results by numerical simulations showing that the model (1.1) can reproduce the expanding strip (ring) patterns qualitatively similar to those observed in the experiment of [2]. This in turn justifies the system (1.1) in modeling the cell movement with density-suppressed motility.

Since the solutions of (1.4) and (1.3) are equivalent under the transformation $w = r(v)u$, we shall focus on the transformed system (1.4) in a bounded domain $\Omega \subset R^N (1 \le N \le 3)$ in the sequel unless otherwise stated. The rest of this paper is organized as follows. In Section 2, we derive a key priori estimate on solutions. In Section 3, we find the conditions under which non-constant positive solutions of (1.4) do not exist. Then we prove the existence of non-constant positive solutions of (1.4) under some conditions with numerical illustrations in Section 4. Finally we show that system (1.1) can generate oscillating waves whose amplitude is determined by a Ginzburg–Landau type equation with numerical verifications.

2. A key priori estimates

In order to establish the existence and nonexistence theorem of non-constant steady states for small σ or d, we need to derive some *priori* estimates for positive solutions of the system (1.4). Our result reads as follows:

Proposition 2.1. *Let Ω be a bounded domain in $\mathbb{R}^N (1 \le N \le 3)$ with smooth boundary. Then for any given constant $d_0 > 0$, there exists a positive constant $c > 1$, which depends only on d_0 and Ω, such that any positive solution (w, v) of (1.4) satisfies*

$$\frac{1}{c} \le w(x), \quad v(x) \le c, \quad \forall x \in \bar{\Omega},$$

provided that $d \ge d_0$. Moreover, if $\liminf_{v\to\infty} r(v)v \in (r(0), \infty]$, such c is independent of d_0 and Ω.

Proof. Assume that $w(x_0) = \max_{\overline{\Omega}} w$. By the maximum principle and Hopf boundary lemma (see Proposition 2.2 of [13]), it follows from the first equation of (1.4) that

$$1 - \frac{w(x_0)}{r(v(x_0))} \ge 0, \text{ so } w(x_0) \le r(v(x_0)) \le r(0),$$

which yields

$$w(x) \le r(0), \quad \forall x \in \bar{\Omega}. \tag{2.1}$$

Similarly, let $w(y_0) = \min_{\bar{\Omega}} w$ and we have

$$1 - \frac{w(y_0)}{r(v(y_0))} \le 0, \quad \text{i.e., } w(y_0) \ge r(v(y_0)).$$

This gives

$$w(x) \ge \min r(v(x)), \quad \forall x \in \bar{\Omega}. \tag{2.2}$$

In order to derive the lower bound of w and the upper/lower bounds of v, we first consider the case that $\liminf_{s\to\infty} r(s)s \in (r(0), \infty]$. In this case, by assuming $v(x^0) = \max_{\overline{\Omega}} v$, we can conclude from the maximum principle with Hopf lemma (Proposition 2.2 of [13]) as applied to the second equation of (1.4) and

(2.1) that $r(v(x^0))v(x^0) \le w(x^0) \le r(0)$. Then there exists a large constant $c > 0$, independent of σ, d and Ω, such that

$$v(x) \le v(x^0) \le c, \quad \forall x \in \overline{\Omega}. \tag{2.3}$$

As $r(v)$ is decreasing on $[0,\infty)$ with respect to v, (2.2) and (2.3) yield $w(x) \ge r(c) > 0, \quad \forall x \in \overline{\Omega}$. Arguing similarly as above, letting $v(y^0) = \min_{\overline{\Omega}} v(y)$, we get from the second equation of (1.4) that $r(v(y^0))v(y^0) \ge w(y^0) \ge r(c)$, and so we find a positive constant $c_* = \frac{r(c)}{r(0)}$ independent of σ, d and Ω such that

$$v(x) \ge v(y^0) \ge c_* > 0, \forall x \in \overline{\Omega}. \tag{2.4}$$

For the general case, to obtain the lower bound of w and the upper/lower bounds of v, we rewrite the equation of v as

$$-\Delta v + \frac{1}{d}v = \frac{1}{d}\frac{w}{r(v)}, \; x \in \Omega; \; \frac{\partial v}{\partial \nu} = 0, \; x \in \partial\Omega. \tag{2.5}$$

On the other hand, integrating the equation of w over Ω yields $\int_\Omega \frac{w}{r(v)}[1 - \frac{w}{r(v)}]dx = 0$, which gives

$$\int_\Omega \frac{w^2}{r^2(v)}dx = \int_\Omega \frac{w}{r(v)}dx \le \left(\int_\Omega \frac{w^2}{r^2(v)}dx\right)^{\frac{1}{2}} |\Omega|^{\frac{1}{2}}.$$

Hence, we have

$$\int_\Omega \frac{w^2}{r^2(v)}dx \le C_0. \tag{2.6}$$

Hereafter, the positive constant C_0 depends only on d_0, Ω and may be different from line to line.

In view of (2.6), by the standard L^p-estimates for elliptic equations (see, for instance, [14]) as applied to (2.5), it holds $\|v\|_{W^{2,2}(\Omega)} \le C_0, \forall d \ge d_0$. In the following, we always assume that $d \ge d_0$. Then, the well-known embedding theorem (see [14]) allows us to conclude that

$$\|v\|_{L^\infty(\Omega)} \le C_0, \quad \text{i.e., } v(x) \le C_0, \quad \forall x \in \overline{\Omega}, \quad \text{if } 1 \le N \le 3. \tag{2.7}$$

Now, (2.7), together with (2.2) and the monotonicity of r, indicates $w(x) \ge r(C_0) > 0, \quad \forall x \in \overline{\Omega}$. Thus, by a similar argument as obtaining (2.4), we get a positive constant $c^* = \frac{r(C_0)}{r(0)}$ depending only on d_0, Ω such that

$$v(x) \ge c^* > 0, \quad \forall x \in \overline{\Omega}.$$

The proof is now complete. □

3. Nonexistence of nonconstant steady states

In this section, we shall use two different approaches to establish the nonexistence result of non-constant solutions of (1.4) for large or small d or σ. Hereafter, whenever we say d is large or small, it should be understood that σ is fixed first; the same convention applies to the case when we say σ is large or small. The main results we shall prove in this section are stated in the following theorem.

Theorem 3.1. *Let $r(v)$ satisfy the condition* (H1)*. Then we have the following results on the stationary system* (1.4)*.*

(a) *Given $d > 0$, there is a constant σ^* such that system (1.4) has no non-constant positive solution if $\sigma > \sigma^*$; Conversely there is a constant d^* for given $\sigma > 0$ such that (1.4) has no non-constant positive solution if $d > d^*$.*

(b) *Assume further that $r(v)v$ is increasing in $v \in [0,\infty)$. Then the following assertions hold.*

  (i) *Given $d > 0$, there is a constant σ_* such that (1.4) has no non-constant positive solution if $0 < \sigma \le \sigma_*$;*

  (ii) *Given $\sigma > 0$, there is a constant d_* such that (1.4) has no non-constant positive solution if $0 < d \le d_*$ and $\lim_{v\to\infty} r(v)v \in (r(0), \infty]$.*

Remark 3.1. It has been shown in [4] that the constant solution $(r(1), 1)$ is globally asymptotically stable if $d\sigma > \frac{k_0}{16}$ where k_0 is defined in (1.2), which indicates that the stationary problem (1.4) has only positive constant solution $(r(1), 1)$ if $d\sigma > \frac{k_0}{16}$. This result was proved in two dimensions under the hypotheses (H1)–(H2), however, the results in Theorem 3.1(a) hold for three dimensions without condition (H2). The conditions on $r(v)$ imposed in Theorem 3.1(b) basically requires the decay rate of $r(v)$ must be linear or sublinear. For instance $r(v) = \frac{1}{(\alpha+\beta v)^\xi}$ with $\xi < 1$ or $\beta < \alpha$ and $\xi = 1$ is a candidate.

Theorem 3.1 is a consequence of the following series of lemmas.

Lemma 3.1. *Let $r(v)$ satisfy the condition* (H1)*. Then the following assertions hold.*

(i) *If $r(v)v$ is non-decreasing in $v \in [0,\infty)$, then there exists a positive constant σ_* such that (1.4) has no non-constant positive solution provided that $0 < \sigma \le \sigma_*$.*

(ii) *There exists a positive constant d^* such that (1.4) has no non-constant positive solution provided that $d \ge d^*$.*

Proof. We first verify (i). To this end, we first claim that any positive solution (w, v) of (1.4) satisfies

$$(w, v) \to (r(1), 1) \text{ in } C^2(\bar{\Omega}) \times C^2(\bar{\Omega}), \quad \text{as } \sigma \to 0. \tag{3.1}$$

Notice that c is independent of $\sigma > 0$ in Proposition 2.1. By a standard compactness argument, there exists a sequence σ_i with $\sigma_i \to 0$ as $i \to \infty$ such that the corresponding positive solution sequence $(w_{\sigma_i}, v_{\sigma_i})$ of (1.4) with $\sigma = \sigma_i$ satisfies

$$(w_{\sigma_i}, v_{\sigma_i}) \to (\hat{w}, \hat{v}) \text{ in } C^2(\bar{\Omega}) \times C^2(\bar{\Omega}), \quad \text{as } i \to \infty,$$

where $0 < \hat{w}, \hat{v}$ on $\bar{\Omega}$. Clearly, $(\hat{w}, \hat{v})$ solves

$$\begin{cases} -\Delta \hat{w} = 0, & x \in \Omega, \\ -d\Delta \hat{v} = \frac{\hat{w}}{r(\hat{v})} - \hat{v}, & x \in \Omega, \\ \frac{\partial \hat{w}}{\partial \nu} = \frac{\partial \hat{v}}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.2}$$

Hence, $\hat{w}$ must be a positive constant. Furthermore, by setting $\hat{v}(x_*) = \min_{\bar{\Omega}} \hat{v}$ and $\hat{v}(x^*) = \max_{\bar{\Omega}} \hat{v}$, we have from the maximum principle and Hopf boundary Lemma applied to the second equation of (3.2) (see also Proposition 2.2 of [13]) that

$$\frac{\hat{w}}{r(\hat{v}(x_*))} - \hat{v}(x_*) \le 0, \quad \frac{\hat{w}}{r(\hat{v}(x^*))} - \hat{v}(x^*) \ge 0.$$

These inequalities, together with the facts that $r(v)v$ is non-decreasing on $[0,\infty)$ and $\hat{w}$ is a positive constant, immediately imply

$$\hat{v}(x_*) = \hat{v}(x^*), \quad \text{i.e., } \hat{v} \equiv \text{a positive constant}.$$

On the other hand, for any fixed $i \ge 1$, integrating over $\bar{\Omega}$ the equation satisfied by w_{σ_i} and v_{σ_i}, respectively, we have

$$\int_\Omega \left\{ \frac{w_{\sigma_i}(x)}{r(v_{\sigma_i}(x))}\left(1 - \frac{w_{\sigma_i}(x)}{r(v_{\sigma_i}(x))}\right)\right\}dx = 0,$$
$$\int_\Omega \left\{ \frac{w_{\sigma_i}(x)}{r(v_{\sigma_i}(x))} - v_{\sigma_i}(x)\right\}dx = 0.$$

Since $\hat{w}$ and $\hat{v}$ are positive constants, we send $i \to \infty$ to obtain $(\hat{w}, \hat{v}) = (r(1), 1)$. Therefore, the above analysis implies the claim (3.1).

Let us now define

$$W_\nu^{2,2}(\Omega)=\left\{g\in W^{2,2}(\Omega):\ \frac{\partial g}{\partial\nu}=0\ \text{ on }\partial\Omega\right\},$$
$$L_0^2(\Omega)=\left\{g\in L^2(\Omega):\ \int_\Omega gdx=0\right\}$$

and

$$\mathcal{F}(\sigma,\ z,\ v,\ \xi)=(f_1,\ f_2,\ f_3)(\sigma,\ z,\ v,\ \xi)$$

with

$$f_1(\sigma,\ z,\ v,\ \xi)=\Delta z+\sigma\frac{(z+\xi)}{r(v)}\left(1-\frac{z+\xi}{r(v)}\right),$$
$$f_2(\sigma,\ z,\ v,\ \xi)=d\Delta v+\frac{z+\xi}{r(v)}-v,$$
$$f_3(\sigma,\ z,\ v,\ \xi)=\int_\Omega\frac{(z+\xi)}{r(v)}\left(1-\frac{z+\xi}{r(v)}\right)dx,$$

where $w=z+\xi$ with $\int_\Omega zdx=0$ and $\xi\in\mathbb{R}_+^1:=[0,\infty)$.

Then

$$\mathcal{F}:\ \mathbb{R}_+^1\times(L_0^2(\Omega)\cap W_\nu^{2,2}(\Omega))\times W_\nu^{2,2}\times\mathbb{R}_+^1\longmapsto L_0^2(\Omega)\times L^2(\Omega)\times\mathbb{R}^1.$$

It is observed that finding the positive solution of (1.4) is equivalent to solving $\mathcal{F}(\sigma,\ z,\ v,\ \xi)=0$. Moreover, $\mathcal{F}(\sigma,\ z,\ v,\ \xi)=0$ has a unique solution $(z,\ v,\ \xi)=(0,\ 1,\ r(1))$ when $\sigma=0$.

By elementary calculation, we have

$$\Phi\equiv D_{(z,\ v,\ \xi)}\mathcal{F}(0,\ 0,\ 1,\ r(1)):\ (L_0^2(\Omega)\cap W_\nu^{2,2}(\Omega))\times W_\nu^{2,2}\times\mathbb{R}_+^1\longmapsto L_0^2(\Omega)\times L^2(\Omega)\times\mathbb{R}^1,$$

where

$$\Phi(h,k,\tau)=\begin{pmatrix}\Delta h\\ d\Delta k+\frac{1}{r(1)}(h+\tau)-\frac{r(1)+r'(1)}{r(1)}k\\ -\int_\Omega\left\{\frac{1}{r(1)}(h+\tau)-\frac{r'(1)}{r(1)}k\right\}dx\end{pmatrix}.$$

In order to use the implicit function theorem, we need to verify that Φ is both invertible and surjective. Indeed, assume that $\Phi(h,\tau,k)=(0,0,0)$. Clearly, $h=0$. Integrating the second equation over Ω, and using the third equation and the fact of $h=0$, we further have

$$\int_\Omega\tau dx=r'(1)\int_\Omega kdx=[r'(1)+r(1)]\int_\Omega kdx,$$

which gives $\int_\Omega kdx=0$ due to $\tau\in\mathbb{R}^1$, and in turn $\tau=0$. In addition, k solves

$$-d\Delta k=-\frac{r(1)+r'(1)}{r(1)}k\ \text{ in }\Omega;\quad\frac{\partial k}{\partial\nu}=0\ \text{ on }\partial\Omega.$$

In light of $r(1)+r'(1)\geq 0$ (since $r(v)v$ is non-decreasing in $v\in[0,\infty)$) and $\int_\Omega kdx=0$, it is obvious that $k=0$. Hence, $h=\tau=k=0$ and Φ is invertible. On the other hand, one can easily check that Φ is also a surjection.

As a consequence, the implicit function theorem allows us to conclude that there exists a positive constant σ_* such that, for each $\sigma\in[0,\sigma_*]$, $(0,\ 1,\ r(1))$ is the unique solution of $\mathcal{F}(\sigma,\ z,\ v,\ \xi)=0$ in $B_{\sigma_*}(0,\ 1,\ r(1))$, where $B_{\sigma_*}(0,\ 1,\ r(1))$ is the ball in $(L_0^2(\Omega)\cap W_\nu^{2,2}(\Omega))\times W_\nu^{2,2}\times\mathbb{R}_+^1$ centered at $(0,\ 1,\ r(1))$ with radius σ_*. Taking smaller σ_* if necessary, we can see that the assertion (i) holds by using the claim (3.1).

The proof of (ii) is similar to that of (i). First of all, it can be shown that any positive solution $(w,\ v)$ of (1.4) satisfies

$$(w,\ v)\to(r(1),1)\ \text{ in }C^2(\bar\Omega)\times C^2(\bar\Omega),\quad\text{as }d\to\infty.\tag{3.3}$$

Then, we set $\rho=d^{-1}$ and define the analogous operator $\mathcal{F}$:

$$\mathcal{F}(\rho,\ w,\ z,\ \xi)=(f_1,\ f_2,\ f_3)(\rho,\ w,\ z,\ \xi):\ \mathbb{R}_+^1\times(L_0^2(\Omega)\cap W_\nu^{2,2}(\Omega))\times W_\nu^{2,2}(\Omega)\times\mathbb{R}_+^1\longmapsto L^2(\Omega)\times L_0^2(\Omega)\times\mathbb{R}^1,$$

where

$$f_1(\rho,\ w,\ z,\ \xi)=\Delta w+\frac{\sigma w}{r(z+\xi)}\left(1-\frac{w}{r(z+\xi)}\right),$$
$$f_2(\rho,\ w,\ z,\ \xi)=\Delta z+\rho\left(\frac{w}{r(z+\xi)}-(z+\xi)\right),$$
$$f_3(\rho,\ w,\ z,\ \xi)=\int_\Omega\left(\frac{w}{r(z+\xi)}-(z+\xi)\right)dx,$$

where $v=z+\xi$ with $\int_\Omega zdx=0$ and $\xi\in\mathbb{R}_+^1:=[0,\infty)$. Clearly, $(w,\ z,\ \xi)=(r(1),0,1)$ is the unique nonnegative nontrivial solution of $F(0,\ w,\ z,\ \xi)=0$, and moreover, it is easily verified that $D_{(w,\ z,\ \xi)}F(0,r(1),0,1)$ is a bijection. Thus, combined with (3.3), one can use the implicit function theorem to yield the desired assertion (ii). □

Below we shall derive some nonexistence result of nonconstant steady states when the parameter $d>0$ is small. To highlight the dependence of solution $(w,\ v)$ of (1.4) on the parameter d, we use $(w_d,\ v_d)$ instead of $(w,\ v)$ below. First we determine the asymptotic behavior of any positive solution $(w_d,\ v_d)$ as $d\to 0$.

Lemma 3.2. *Assume that $r(v)v$ is increasing on $[0,\infty)$, and $\lim_{v\to\infty}r(v)v=\theta\in(r(0),\infty]$. Then given $\sigma>0$, any positive solution $(w_d,\ v_d)$ of (1.4) satisfies*

$$(w_d,\ v_d)\to(r(1),1)\ \textit{uniformly on }\bar\Omega,\ \textit{ as }d\to 0.$$

Proof. Under our assumption, Proposition 2.1 tells us that c is independent of σ, $d>0$ in Proposition 2.1. Thus, a standard compactness argument, as applied to the first equation in (1.4), allows one to conclude that there exists a sequence d_i with $d_i\to 0$ as $i\to\infty$ such that the corresponding sequence $(w_{d_i},\ v_{d_i})$ of positive solutions of (1.4) with $d=d_i$ satisfies

$$w_{d_i}\to\widetilde{w}\ \text{uniformly on }\bar\Omega,\quad\text{as }i\to\infty,\tag{3.4}$$

where $0<\widetilde{w}\leq r(0)$ on $\bar\Omega$ (due to (2.1)). As a result, given any small $\varepsilon>0$ with $\varepsilon<\widetilde{w}$ and $\tilde{w}+\varepsilon\leq\theta$ on $\bar\Omega$, we have $\widetilde{w}(x)-\varepsilon\leq w_{d_i}(x)\leq\widetilde{w}(x)+\varepsilon,\quad\forall x\in\bar\Omega$ for all large i.

Next, let us consider the following two auxiliary problems

$$-d_i\Delta\bar{v}=\frac{\widetilde{w}(x)+\varepsilon}{r(\bar{v})}-\bar{v},\ x\in\Omega;\ \frac{\partial\bar{v}}{\partial\nu}=0,\ x\in\partial\Omega,\tag{3.5}$$

and

$$-d_i\Delta\underline{v}=\frac{\widetilde{w}(x)-\varepsilon}{r(\underline{v})}-\underline{v},\ x\in\Omega;\ \frac{\partial\underline{v}}{\partial\nu}=0,\ x\in\partial\Omega.\tag{3.6}$$

Clearly, v_{d_i} is a lower solution to (3.5) and an upper solution to (3.6) for all large i.

Now, we fix any such large i and treat problem (3.5). Since $\lim_{v\to\infty}r(v)v>r(0)$ and $\widetilde{w}(x)\leq r(0)$ on $\bar\Omega$, it is easily checked that a sufficiently large positive constant M with $M>r(0)$ is an upper solution to (3.5). Therefore, the well-known theory of upper–lower solutions ensures that (3.5) admits at least one positive solution.

In the sequel, we will show that (3.5) has a unique positive solution. Notice that any small positive constant is also a lower solution of (3.5). Assume that v_1 and v_2 are any two positive solutions of (3.5). Then, by the iteration theory of upper–lower solutions. We know that (3.5) admits a maximal positive solution

v_{max} and a minimal positive solution v_{min} with $v_{min} \le v_{max}$ on $\bar{\Omega}$ satisfying

$$v_{min} \le v_1,\ v_2 \le v_{max} \quad \text{on } \bar{\Omega}.$$

Thus, it suffices to verify $v_{min} = v_{max}$ on $\bar{\Omega}$.

To this end, let us denote $\rho^* = \inf\{\rho > 0 : v_{max} \le \rho v_{min} \text{ on } \bar{\Omega}\}$. Obviously, $\rho^* \ge 1$ and $v_{max} \le \rho^* v_{min}$ on $\bar{\Omega}$. If we can show $\rho^* = 1$, it is clear that $v_{min} = v_{max}$ on $\bar{\Omega}$, as we wanted. We proceed indirectly and suppose that $\rho^* > 1$. First note that $\rho^* r(\rho^* v) > r(v)$, $\forall v > 0$ since $r(v)v$ is increasing in $v > 0$. To reach a contradiction, we set $h = \rho^* v_{min} - v_{max}$. So $h \ge 0$ on $\bar{\Omega}$, and satisfies

$$\begin{aligned}-d_i\Delta h + h = \rho^* \frac{\widetilde{w}+\varepsilon}{r(v_{min})} - \frac{\widetilde{w}+\varepsilon}{r(v_{max})} &= \frac{(\widetilde{w}+\varepsilon)[\rho^* r(v_{max}) - r(v_{min})]}{r(v_{min}) r(v_{max})} \\ &\ge \frac{(\widetilde{w}+\varepsilon)[\rho^* r(\rho^* v_{min}) - r(v_{min})]}{r(v_{min}) r(v_{max})} \\ &> 0 \quad \text{in } \Omega.\end{aligned}$$

In the above, we also used the facts that $v_{max} \le \rho^* v_{min}$ on $\bar{\Omega}$ and $r(v)$ is decreasing in $v > 0$.

Since $\frac{\partial h}{\partial \nu} = 0$ on $\partial\Omega$, it immediately follows from the strong maximum principle and Hopf boundary lemma for elliptic equation that $h > 0$ on $\bar{\Omega}$. This implies that $h \ge \varepsilon_0 v_{max}$ on $\bar{\Omega}$ for some constant $\varepsilon_0 > 0$, which in turn gives

$$\frac{\rho*}{1+\varepsilon_0} v_{min} \ge v_{max} \quad \text{on } \bar{\Omega},$$

which contradicts the definition of ρ^*.

So far, we have proved that for any given large i, (3.5) has a unique positive solution, denoted by $\bar{v}_i$. Similarly, it can be proved that (3.6) also has a unique positive solution denoted by $\underline{v}_i$, for any given large i. To proceed further, we need to introduce some notations. As $r(v)v$ is increasing in $v \in [0,\infty)$ and $\lim_{v\to\infty} r(v)v = \theta \in [r(0),\infty]$, given $\tau \in [0,\theta)$, there exists a unique $v \in [0,\infty)$ such that $r(v)v = \tau$. Hence, we can define the C^2-function $v = g(\tau)$, $\tau \in [0,\theta)$, which satisfies $r(g(\tau))g(\tau) = \tau$, and so $g(\tau)$ is also increasing in $\tau \in [0,\theta)$.

Given large i, we rewrite (3.5) as follows:

$$-d_i\Delta[g(\bar{\tau}_i)] = \frac{\widetilde{w}+\varepsilon-\bar{\tau}_i}{r(g(\bar{\tau}_i))},\ x\in\Omega;\quad \frac{\partial \bar{\tau}_i}{\partial \nu} = 0,\ x \in \partial\Omega,$$

with $r(\bar{v}_i)\bar{v}_i = \bar{\tau}_i$. On the other hand, an application of the maximum principle yields $0 < \min_{\bar{\Omega}} \widetilde{w} \le r(\bar{v}_i(x))\bar{v}_i(x) \le \max_{\bar{\Omega}} \widetilde{w} + \varepsilon \le \theta$, $\forall x \in \bar{\Omega}$. Hence for all large i and for some constant $C_0 > 1$, we have

$$\frac{1}{C_0} \le g(\bar{\tau}_i) \le C_0.$$

As a consequence, a similar analysis as in Lemma 2.4 of [15] concludes that

$$r(\bar{v}_i)\bar{v}_i = \bar{\tau}_i \to \widetilde{w} + \varepsilon \quad \text{uniformly on } \bar{\Omega}, \quad \text{as } i \to \infty. \tag{3.7}$$

Similarly, making use of (3.6), we have

$$r(\underline{v}_i)\underline{v}_i \to \widetilde{w} - \varepsilon \quad \text{uniformly on } \bar{\Omega}, \quad \text{as } i \to \infty. \tag{3.8}$$

Combining (3.7) and (3.8) and the arbitrariness of ε, we obtain that

$$r(v_{d_i})v_{d_i} \to \widetilde{w} \ \text{ uniformly on } \bar{\Omega}, \quad \text{as } i \to \infty. \tag{3.9}$$

Hence it follows that

$$v_{d_i} \to g(\widetilde{w}) := \tilde{v} \ \text{ uniformly on } \bar{\Omega}, \quad \text{as } i \to \infty, \tag{3.10}$$

where $\tilde{v} \in C(\bar{\Omega})$ and $\tilde{v} > 0$ on $\bar{\Omega}$.

Together with (3.4) and (3.9), we can send $i \to \infty$ in the first equation in (1.4) to find that $\widetilde{w}$ solves (in the weak and then in the classical sense)

$$-\Delta\widetilde{w} = \frac{\sigma\widetilde{w}}{r(g(\widetilde{w}))}\left[1 - \frac{\widetilde{w}}{r(g(\widetilde{w}))}\right],\ x\in\Omega;\ \frac{\partial\widetilde{w}}{\partial\nu} = 0,\ x\in\partial\Omega. \tag{3.11}$$

Observe that $r(g(\tau))$ is decreasing in $\tau \ge 0$, and so $\frac{\tau}{r(g(\tau))}$ is increasing in $\tau \ge 0$. Let $\widetilde{w}(\underline{x}) = \min_{\bar{\Omega}} \widetilde{w}$ and $\widetilde{w}(\bar{x}) = \max_{\bar{\Omega}} \widetilde{w}$. Then thanks to the maximum principle, we get from (3.11) that

$$1 \ge \frac{\widetilde{w}(\bar{x})}{r(g(\widetilde{w}(\bar{x})))} \quad \text{and} \quad 1 \le \frac{\widetilde{w}(\underline{x})}{r(g(\widetilde{w}(\underline{x})))}.$$

This clearly implies that $\tilde{w}(\bar{x}) = \tilde{w}(\underline{x})$, that is, $\tilde{w}$ is a positive constant, and then $\tilde{v}$ is also a positive constant due to (3.10). Furthermore, it is necessary that $(\tilde{w},\tilde{v}) = (r(1),1)$. Therefore, we have proved

$$(w_d, v_d) \to (r(1),1) \ \text{ uniformly on } \bar{\Omega}, \quad \text{as } d \to 0. \tag{3.12}$$

The proof is thus complete. □

Lemma 3.3. *Assume that $r(v)v$ is increasing in $v \in [0,\infty)$ and $\lim_{v\to\infty} r(v)v = \theta \in (r(0),\infty]$. Given $\sigma > 0$, there exists a small constant $d_* > 0$, depending only on σ and Ω, such that (1.4) has no nonconstant positive solution if $0 < d \le d_*$.*

Proof. We shall employ the topological degree technique to establish the desired result. Denote

$$\mathfrak{D} = \left\{(w,v) \in C(\bar{\Omega}) \times C(\bar{\Omega}) : \frac{1}{c+1} < w, v < c+1\right\},$$

where c is given in Proposition 2.1, which is now independent of $d > 0$. Let us define the operator

$$\begin{aligned}\mathcal{A}(d,(w,v)) = (-\Delta + I)^{-1}\Bigg(&w + \frac{\sigma w}{r(v)}\left[1 - \frac{w}{r(v)}\right],\ v \\ &+ \frac{1}{d}\left[\frac{w}{r(v)} - v\right]\Bigg),\end{aligned}$$

where $(-\Delta+I)^{-1}$ represents the inverse operator of $-\Delta+I$ with the zero Neumann boundary condition over $\partial\Omega$. Clearly, (w,v) is a positive solution if and only if $\mathcal{A}(d,(w,v)) = (w,v)$. In addition, $\mathcal{A}$ is compact from $[d_1,d_2] \times \mathfrak{D}$ to $C(\bar{\Omega}) \times C(\bar{\Omega})$ for any given $0 < d_1 < d_2 < \infty$, and $\mathcal{A}(d,(w,v)) \ne (w,v)$ for all $d \in (0,\infty)$ and $(w,v) \in \partial\mathfrak{D}$. This implies that the Leray–Schauder degree $\deg(I - \mathcal{A}(d\cdot),\mathfrak{D})$ is well defined and its value does not depend on $d \in (0,\infty)$.

Let d^* be given as in Lemma 3.1(ii). Then $(r(1),1)$ is the unique positive solution of $\mathcal{A}(d^*,(w,v)) = (w,v)$, which in turn yields

$$\deg(I - \mathcal{A}(d^*,\cdot),\mathfrak{D}) = \operatorname{index}(I - \mathcal{A}(d^*,\cdot),(r(1),1)),$$

where $\operatorname{index}(I - \mathcal{A}(d,\sigma,\cdot),(r(1),1))$ is the index of the operator $I - \mathcal{A}(d,\cdot)$ at the point $(r(1),1)$. Furthermore, the routine computation as in [4] (see Lemma 4.1 (1)) shows that $(r(1),1)$ is linearly stable provided that $r'(1) + r(1) \ge 0$ for any $d > 0$ which is fulfilled due to our assumption that $r(v)v$ is increasing in $v \in [0,\infty)$. It then follows from the Leray–Schauder degree formula (see for instance Theorem 2.8.1 of [16]) that

$$\deg(I - \mathcal{A}(d^*,\cdot),\mathfrak{D}) = \operatorname{index}(I - \mathcal{A}(d^*,\cdot),(r(1),1)) = 1.$$

As a result, it holds

$$\deg(I - \mathcal{A}(d,\cdot),\mathfrak{D}) = \operatorname{index}(I - \mathcal{A}(d^*,\cdot),\mathfrak{D}) = 1, \tag{3.13}$$

for any $d > 0$.

In view of Lemma 3.2, it is also easily computed that only possible positive solution (w_d,v_d) of (1.4) is linearly stable provided that $0 < d \le d_*$ for some small $d_* > 0$. This indicates that

$$\operatorname{index}(I - \mathcal{A}(d,\cdot),(w_d,v_d)) = 1, \tag{3.14}$$

if $0 < d \le d_*$. Moreover, the linear stability of (w_d, v_d) implies that for any given $0 < d \le d_*$, (1.4) admits at most finitely many such positive solutions, denoted by $\{(w_i, v_i)\}_{i=1}^{\ell}$. Hence it follows from (3.14) that

$$\deg(I - \mathcal{A}(d, \cdot), \mathfrak{D}) = \sum_{i=1}^{\ell} \mathrm{index}(I - \mathcal{A}(d, \cdot), (w_i, v_i)) = \ell. \quad (3.15)$$

Therefore (3.13) and (3.15) yield $\ell = 1$. This implies that (1.4) admits a unique positive solution, which must be $(r(1), 1)$ if $0 < d \le d_*$. The proof is now complete. □

By a similar analysis to that of Lemma 3.3, we are able to show the following result.

Lemma 3.4. *Given $d > 0$, there exists a large constant $\sigma^* > 0$ depending only on d and Ω, such that (1.4) has no nonconstant positive solution if $\sigma \ge \sigma^*$.*

Proof. First of all, it follows from Proposition 2.1 and the v-equation that, up to a sequence of σ if necessary, any positive solution (w, v) of (1.4) satisfies

$$v \to \tilde{v} \ \text{ in } C^1(\bar{\Omega}), \quad \text{as } \sigma \to \infty, \quad (3.16)$$

where $\tilde{v} > 0$ on $\bar{\Omega}$. Now, given small $\epsilon > 0$, consider the following two auxiliary problems:

$$-\Delta \underline{w} = \sigma \underline{w}\Big(\frac{1}{r(\tilde{v}) + \epsilon} - \frac{\underline{w}}{(r(\tilde{v}) - \epsilon)^2}\Big), \ x \in \Omega; \ \frac{\partial \underline{w}}{\partial \nu} = 0, \ x \in \partial\Omega, \quad (3.17)$$

and

$$-\Delta \overline{w} = \sigma \overline{w}\Big(\frac{1}{r(\tilde{v}) - \epsilon} - \frac{\overline{w}}{(r(\tilde{v}) + \epsilon)^2}\Big), \ x \in \Omega; \ \frac{\partial \overline{w}}{\partial \nu} = 0, \ x \in \partial\Omega. \quad (3.18)$$

In view of (3.16), given large $\sigma > 0$, w is an upper-solution to problem (3.17) and a lower solution to problem (3.18). In addition, (3.17) and (3.18) have a unique positive solution, denoted by $\underline{w}_\sigma$ and $\overline{w}_\sigma$, respectively. A simple upper–lower solution argument shows that

$$\underline{w}_\sigma \le w \le \overline{w}_\sigma \quad \text{on } \bar{\Omega}, \quad \text{for all large } \sigma. \quad (3.19)$$

Furthermore, a similar analysis as in Lemma 2.4 of [15] concludes that

$$\underline{w}_\sigma \to \frac{(r(\tilde{v}) - \epsilon)^2}{r(\tilde{v}) + \epsilon}, \quad \overline{w}_\sigma \to \frac{(r(\tilde{v}) + \epsilon)^2}{r(\tilde{v}) - \epsilon} \ \text{ uniformly on } \bar{\Omega}, \ \text{as } \sigma \to \infty. \quad (3.20)$$

Sending $\epsilon \to 0$ in (3.20), it then follows from (3.19) that $w \to r(\tilde{v})$ uniformly on $\bar{\Omega}$, as $\sigma \to \infty$. Notice that for all $\sigma > 0$, it holds

$$\int_\Omega \Big(\frac{w}{r(v)} - v\Big) dx = 0. \quad (3.21)$$

By means of (3.16) and (3.21), we let $\sigma \to \infty$ and find that $\tilde{v} = 1$. That is, we have proved that

$$(w, v) \to (r(1), 1) \ \text{ uniformly on } \bar{\Omega}, \ \text{ as } \sigma \to \infty. \quad (3.22)$$

Recall again that c depends neither on $\sigma > 0$ nor on $d \ge 1$ in Proposition 2.1, and as in the proof of Lemma 3.3, it can be shown that $(r(1), 1)$ is linearly stable for all large σ (see Lemma 4.1(1)). Now, with the aid of (3.22), one can adapt the argument of Lemma 3.4 to yield the desired conclusion. The details are omitted here. □

4. Existence of non-constant stationary solutions

This section is devoted to establishing the existence of non-constant positive solutions to the stationary problem (1.4) by applying the Leray–Schauder degree theory.

4.1. Preliminaries

We first present the decomposition in the function space based on the elliptic operator $-\Delta$ subject to the zero Neumann boundary condition on Ω. Let

$$0 = \mu_0 < \mu_1 < \mu_2 < \mu_3 < \cdots < \mu_i < \cdots \quad (4.1)$$

be the sequence of eigenvalues for this elliptic operator $-\Delta$ and each μ_i has multiplicity $m_i \ge 1$. Let φ_{ij}, $i \ge 0$, $1 \le j \le m_i$, be the normalized eigenfunctions corresponding to μ_i. Let X_i be the eigenspace associated with μ_i in $H^1(\Omega; \mathbb{R}^2)$. Then the set $\{\varphi_{ij}, \ i \ge 0, \ j = 1, 2, \ldots, m_i\}$ forms a complete orthogonal basis in $L^2(\Omega)$. Let $X = [H^1(\Omega)]^2$ and $X_{ij} = \{c\varphi_{ij} : 1 \le j \le m_i, c \in \mathbb{R}^2\}$. Then

$$\mathbf{X} = \bigoplus_{i=1}^{\infty} X_i, \qquad \mathbf{X}_i = \bigoplus_{j=1}^{m_i} X_{ij}, \quad (4.2)$$

where $\bigoplus$ denotes the direct sum of subspaces.

It is obvious that system (1.4) has two constant solutions $(0, 0)$ and $(r(1), 1)$. In order to compute the degree index of $(r(1), 1)$, we linearize (1.4) at $(r(1), 1)$ and have the eigenvalue problem associated with the linearized system as follows:

$$\begin{cases} \Delta w - \frac{\sigma}{r(1)} w + \frac{\sigma r'(1)}{r(1)} v = \rho w, & x \in \Omega, \\ \Delta v + \frac{1}{dr(1)} w - \frac{1}{d}\left(1 + \frac{r'(1)}{r(1)}\right) v = \rho v, & x \in \Omega, \\ \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \quad (4.3)$$

It is easily verified that eigenvalues ρ_i of (4.3) satisfy

$$\rho_i^2 + D_i \rho_i + E_i = 0, \ \ i = 0, 1, 2, \ldots, \quad (4.4)$$

where

$$D_i = \frac{1}{dr(1)}[r'(1) + r(1) + 2dr(1)\mu_i + d\sigma],$$
$$E_i = \frac{1}{dr(1)}\left\{(1 + d\mu_i)\sigma + \left[r'(1) + (1 + d\mu_i) r(1)\right]\mu_i\right\}.$$

It is obvious that

$$E_i < 0 \Leftrightarrow \sigma < -\left[\frac{r'(1)}{1 + d\mu_i} + r(1)\right]\mu_i \overset{def}{=} \sigma_i. \quad (4.5)$$

Now we give some results on eigenvalues in (4.4), which is easy to be verified.

Lemma 4.1. *For $d > 0$ and $\sigma > 0$, the following statements are true:*

(1) If either

$$r'(1) + r(1) \ge 0$$

or

$$r'(1) + r(1) < 0 \ \text{ and } \ \sigma > -\frac{r'(1) + r(1)}{d}$$

Then $Re\rho_i < 0$ for all $i \in \{0, 1, 2, \ldots, \infty\}$, namely $(r(1), 1)$ (resp.$(1, 1)$) is linearly stable to (1.4)(resp.(1.3)).

(2) Assume that

$$0 < \sigma < \sigma_i \ \text{ for some } \ i \in \{1, 2, \ldots, \infty\}, \quad (4.6)$$

then ρ_i^+ is a positive eigenvalue and ρ_i^- is a negative one, and the algebraic multiplicity of each one is m_i, where $\rho_i^{\pm} = \frac{-D_i \pm \sqrt{D_i^2 - 4E_i}}{2}$. This implies that $(r(1), 1)$ (resp.$(1, 1)$) is linearly unstable to (1.4)(resp. (1.3)) if (4.6) holds.

(3) If (4.6) is satisfied, then there exists an integer $i^c \geq 1$ such that

$$\sigma_i > 0 \;\; for \;\; i \in [1, i^c], \quad \sigma_{i^c+j} \leq 0 \;\; for \;\; j = 1, 2, 3, \ldots; \tag{4.7}$$

Moreover, let

$$\sigma_a = \max_{1 \leq i \leq i^c} \sigma_i = -\left[\frac{r'(1)}{1 + d\mu_{i_a}} + r(1)\right]\mu_{i_a} \;\; and \;\; 0 < \sigma < \sigma_a. \tag{4.8}$$

Then (4.4) has at least one positive root $\rho_{i_a}^+$ with the algebraic multiplicity m_{i_a}. Usually, we call $k_a = \sqrt{\mu_{i_a}}$ the admissible wave number.

(4) If (4.7) holds and $\sigma > \sigma_a$, then $E_i > 0$ for all $i \in \{0, 1, 2, \ldots, \infty\}$, so that for each i Eq. (4.4) has two distinct roots with either positive real parts or negative real parts, which implies that the number of positive real eigenvalues (counting multiplicity) must be even.

We now treat μ_i as a real number, then at

$$\mu_i = \frac{1}{d}\left(\sqrt{\frac{-r'(1)}{r(1)}} - 1\right), \tag{4.9}$$

the maximum of σ_i, $i \in [1, i^c]$ is attained as

$$\sigma_c = \max_{\mu_i} \sigma_i = \frac{1}{d}\left(\sqrt{-r'(1)} - \sqrt{r(1)}\right)^2. \tag{4.10}$$

It is obvious that $\sigma_c \geq \sigma_a$, and $\sigma_c = \sigma_a$ only when there exists $i = i_a$ such that (4.9) holds. Since the odd number of the positive eigenvalues plays a critical role on the computation of the degree index of $(r(1), 1)$, by (3) and (4) in Lemma 4.1, in what follows we shall always assume the condition (4.7) holds and let σ satisfy

$$0 < \sigma < \sigma_c. \tag{4.11}$$

4.2. Main results

We now define a set as

$$\mathbf{X}^+ = \{(w, v) \in \mathbf{X} : \; w, \; v > 0, \quad x \in \Omega\},$$

and rewrite the system (1.4) in $\mathbf{X}^+$ in the matrix form

$$\begin{cases} W = (I - \Delta)^{-1}[W + H(\sigma, d, W)], & x \in \Omega, \\ \nabla W \cdot \nu = 0, & x \in \partial\Omega, \end{cases} \tag{4.12}$$

where $W = \begin{pmatrix} w \\ v \end{pmatrix}$, $(I - \Delta)^{-1}$ represents the inverse of $I - \Delta$ with homogeneous Neumann boundary conditions, and

$$H(\sigma, d, W) = \begin{pmatrix} \frac{\sigma w}{r(v)}\left(1 - \frac{w}{r(v)}\right) \\ \frac{1}{d}\left(\frac{w}{r(v)} - v\right) \end{pmatrix}.$$

Then we define an operator P by

$$P(\sigma, d, W) = W - (I - \Delta)^{-1}[W + H(\sigma, d, W)], \quad W \in \mathbf{X}^+. \tag{4.13}$$

Now we look for solutions of (1.4), which is equivalent to finding zero points of the operator P. We will apply the topological degree theory to prove the existence of non-constant zero points of P. By a computation, the linearized system of $P(\sigma, d, W) = 0$ at the constant steady state $W^* = (r(1), 1)$ is in the form of

$$W - (I - \Delta)^{-1}\left(I + \nabla_W H(\sigma, d, W^*)\right)W = 0,$$

where $\nabla_W = \frac{\partial}{\partial W}$ and

$$\nabla_W H(\sigma, d, W^*) = \frac{1}{dr(1)}\begin{pmatrix} -d\sigma & d\sigma r'(1) \\ 1 & -(r'(1) + r(1)) \end{pmatrix}. \tag{4.14}$$

It is well known that if the linear operator

$$\nabla_W P(\sigma, d, W^*) = I - (I - \Delta)^{-1}\left(I + \nabla_W H(\sigma, d, W^*)\right)$$

is invertible, then the index of P at W^* is computed by

$$\text{index}(P, W^*) = (-1)^{\gamma}, \tag{4.15}$$

where γ stands for the number of the negative eigenvalues of $\nabla_W P(\sigma, d, W^*)$. The decomposition of function space (4.2) implies that, for each integer $i \geq 0$ and $1 \leq j \leq m_i$, the subspace $\mathbf{X}_{ij}$ is an invariant space under $\nabla_W P(\sigma, d, W^*)$ and $\lambda \in \mathbb{R}$ is an eigenvalue of $\nabla_W P(\sigma, d, W^*)$ in $\mathbf{X}_i$ if and only if λ is an eigenvalue of $(1 + \mu_i)^{-1}(\mu_i I - \nabla_W H(\sigma, d, W^*))$. Hence, the invertibility of $\nabla_W P(\sigma, d, W^*)$ is equivalent to that of the matrix $(\mu_i I - \nabla_W H(\sigma, d, W^*)) \overset{def}{=} M_i$ for all $i \geq 0$. We now define a function by

$$\Gamma(\sigma, d, \chi) = \det(\chi I - \nabla_W H(\sigma, d, W^*)).$$

If $\Gamma(\sigma, d, \mu_i) \neq 0$, then the number of negative eigenvalues of Mi is 1 if and only if $\Gamma(\sigma, d, \mu_i) < 0$; When $\Gamma(\sigma, d, \mu_i) > 0$, the number of the negative eigenvalues of M_i is 0 or 2. The algebraic multiplicity of each of them is m_i. As such, if $\Gamma(\sigma, d, \mu_i) \neq 0$ for any $i \geq 0$, then by (4.15), we have (see [10,17])

$$\gamma = \sum_{i \geq 0, \Gamma(\sigma, d, \mu_i) < 0} m_i. \tag{4.16}$$

It is easy to see that the eigenvalue ρ_i of (4.3) and the eigenvalue λ_i of $\nabla_W P(\sigma, d, W^*)$ satisfy $\rho_i = -(1 + \mu_i)\lambda_i$. Thus, by (3) and (4) in Lemma 4.1, under the condition (4.11) the formula (4.16) is well posed.

Using (4.14), we have

$$\Gamma(\sigma, d, \chi) = \chi^2 + \frac{r'(1) + r(1) + d\sigma}{dr(1)}\chi + \frac{\sigma}{dr(1)}. \tag{4.17}$$

Note that (4.6) implies that $r'(1) + r(1) < 0$. Therefore if (4.11) is true, then $d\sigma < -(r'(1) + r(1))$ and $[r'(1) + r(1) + d\sigma]^2 - 4d\sigma r(1) > 0$. Thus, the equation $\Gamma(\sigma, d, \chi) = 0$ has two positive distinct roots $\chi_{\pm}(\sigma, d)$ given by

$$\chi_{\pm}(\sigma, d) = \frac{-[r'(1) + r(1) + d\sigma] \pm \sqrt{[r'(1) + r(1) + d\sigma]^2 - 4d\sigma r(1)}}{2dr(1)}. \tag{4.18}$$

We are now in a position to present the main result of this section.

Theorem 4.1. *Let (4.11) hold and further assume that there exist $j > i \geq 0$ such that*

(i) *$\mu_i < \chi_-(\sigma, d) < \mu_{i+1}$ and $\mu_j < \chi_+(\sigma, d) < \mu_{j+1}$;*

(ii) *$\sum_{k=i+1}^{j} m_k$ is odd.*

Then (1.4) has at least one non-constant solution for $d > d_0$ and $\sigma > 0$, where d_0 is a given positive constant.

Proof. Proposition 2.1 indicates that there exists a constant $c > 1$ independent of $d > d_0$ and $\sigma > 0$ such that all positive solutions of (1.4) are located in the following set

$$\mathfrak{D} = \left\{(w, v) \in C(\bar{\Omega}) \times C(\bar{\Omega}) : \; \frac{1}{c+1} < w, v < c + 1\right\}. \tag{4.19}$$

By Lemmas 3.1(ii) and 3.4, one can choose sufficiently large positive constants d^* and σ^* such that

(A_1) $\Gamma(\sigma^*, d^*, \chi) > 0$ for all $\chi \geq 0$, i.e., $d^*\sigma^* > \left(\sqrt{-r'(1)} - \sqrt{r(1)}\right)^2$;

(A_2) The system (1.4) has no non-constant solutions if $d\sigma \geq d^*\sigma^*$.

Let us define an operator $\Phi : [0,1] \times \mathfrak{D} \to C(\overline{\Omega}) \times C(\overline{\Omega})$ by

$$\Phi(t, W) = (I - \Delta)^{-1} \begin{pmatrix} w + \frac{((1-t)\sigma^* + t\sigma)w}{r(v)} \left(1 - \frac{w}{r(v)}\right) \\ v + \left(\frac{1-t}{d^*} + \frac{t}{d}\right)\left(\frac{w}{r(v)} - v\right) \end{pmatrix}.$$

It is obvious that $\Phi(t, W)$ is compact for each $t \in [0, 1]$, a solution of (1.4) is just a fixed point of $\Phi(1, \cdot)$, and $\Phi(t, \cdot)$ has no fixed points in $\partial\mathfrak{D}$ for all $t \in [0, 1]$. Thus the Leray–Schauder degree $\deg(I - \Phi(t, \cdot), \mathfrak{D}, \mathbf{0})$ is well-defined and is a constant for all $t \in [0, 1]$ due to the homotopy invariance of the topological degree, that is,

$$\deg(I - \Phi(1, \cdot), \mathfrak{D}, \mathbf{0}) = \deg(I - \Phi(0, \cdot), \mathfrak{D}, \mathbf{0}). \tag{4.20}$$

By (4.13), we know that $I - \Phi(1, \cdot) = P(\sigma, d, \cdot)$. Thus, if we assume that (1.4) has no non-constant solutions, then (4.15), (4.16) and the given condition (ii) lead to

$$\begin{aligned} \deg(I - \Phi(1, \cdot), \mathfrak{D}, \mathbf{0}) &= \mathrm{index}(P(\sigma, d, \cdot), W^*) = (-1)^{\sum_{k=i+1}^{j} m_k} \\ &= -1. \end{aligned} \tag{4.21}$$

On the other hand, by (A_2), we have that W^* is the unique fixed point of $\Phi(0, \cdot)$. Moreover, (A_1) and (4.15) yield γ is even, then

$$\deg(I - \Phi(0, \cdot), \mathfrak{D}, \mathbf{0}) = \mathrm{index}(I - \Phi(0, \cdot), W^*) = 1. \tag{4.22}$$

Obviously, (4.20) contradicts (4.21)–(4.22). Hence, our assumption is false, and as a consequence (1.4) has at least one non-constant solution. The proof is complete. □

Theorem 4.1 gives general sufficient conditions on parameters σ and d such that system (1.4) admits non-constant solutions. If we fix one of them, then the conditions can be more specifically expressed. As an illustration, below we shall fix the diffusion rate $d > 0$ and identify the more specific conditions on σ to guarantee the existence of non-constant solutions to (1.4).

Hereafter, we shall use the notation $\chi_\pm(\sigma) =: \chi_\pm(\sigma, d)$ to emphasize the dependence on σ only by fixing the value of $d > 0$, where $\chi_\pm(\sigma, d)$ is defined in (4.18). Furthermore, for the sake of presentation, we re-denote by

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \cdots \leq \mu_i \leq \cdots \tag{4.23}$$

all eigenvalues (counting multiplicity) of the elliptic operator $-\Delta$ with zero Neumann boundary condition. In order to calculate the degree index in (4.15), we give some properties of $\chi_\pm(\sigma)$ below, which can be easily verified by simple calculations.

Lemma 4.2. *Let the positive constant d be fixed and (4.11) hold. The following statements are true.*

(i) $\chi_\pm(\sigma) > 0$, $\chi_-(\sigma_c) = \chi_+(\sigma_c) = \sqrt{\frac{\sigma_c}{dr(1)}}$.

(ii) *$\chi_-(\sigma)$ is monotone decreasing and $\chi_+(\sigma)$ is monotone increasing. Furthermore, it holds that*

$$\lim_{\sigma\to 0^+} \chi_-(\sigma) = -\frac{r'(1) + r(1)}{dr(1)}, \quad \lim_{\sigma\to\sigma_c^-} \chi_-(\sigma) = \sqrt{\frac{\sigma_c}{dr(1)}},$$

and

$$\lim_{\sigma\to 0^+} \chi_+(\sigma) = 0, \quad \lim_{\sigma\to\sigma_c^-} \chi_+(\sigma) = \sqrt{\frac{\sigma_c}{dr(1)}}.$$

By the discussion above, we know that $\chi = \mu_i - (1 + \mu_i)\lambda$, where λ is an eigenvalue of $\nabla_W P(\sigma, d, W^*)$ and μ_i is defined in (4.23). Hence, $\nabla_W P(\sigma, d, W^*)$ has negative eigenvalues $\lambda_i(\sigma)$ if and only if

$$\lambda_i(\sigma) = \frac{\mu_i - \chi(\sigma)}{1 + \mu_i} < 0, \; i = 0, 1, 2, \ldots.$$

To proceed, we define two indicators as

$$\gamma_-(\sigma) = \text{the number of } \{i \in \mathbb{N} \cup \{0\} : \chi_-(\sigma) > \mu_i\}$$

and

$$\gamma_+(\sigma) = \text{the number of } \{i \in \mathbb{N} \cup \{0\} : \chi_+(\sigma) > \mu_i\},$$

where $\mathbb{N}$ represents the set of positive integers. Obviously, $\gamma_-(\sigma) + \gamma_+(\sigma)$ is the number of negative eigenvalues (counting multiplicity) of $\nabla_W P(\sigma, d, W^*)$. Then the power exponent γ in (4.15), now denoted by $\gamma(\sigma)$ to emphasize the dependence on σ, is

$$\gamma(\sigma) = \gamma_+(\sigma) + \gamma_-(\sigma).$$

Since $\chi_\pm(\sigma) > 0 = \mu_0$, we have $\gamma(\sigma) \geq 2$ for $\sigma \in (0, \sigma_c)$. To compute $\gamma(\sigma)$, we set

$$i_c = \max\{i : \chi_\pm(\sigma_c) > \mu_i\}, \quad j_0 = \max\{j : \chi_-(0) > \mu_{i_c + j}\}, \tag{4.24}$$

and

$$\begin{aligned} \overline{\sigma}^i &= \inf\{0 < \sigma < \sigma_c : \chi_+(\sigma) > \mu_i\}, \\ \underline{\sigma}_j &= \sup\{0 < \sigma < \sigma_c : \chi_-(\sigma) > \mu_{i_c+j}\}. \end{aligned} \tag{4.25}$$

Then Lemma 4.2 yields

$$\overline{\sigma}^1 \leq \overline{\sigma}^2 \leq \cdots \leq \overline{\sigma}^i \leq \cdots \to \sigma_c \text{ as } i \to i_c \text{ and}$$
$$\underline{\sigma}_1 \geq \underline{\sigma}_2 \geq \cdots \geq \underline{\sigma}_j \geq \cdots \to 0 \text{ as } j \to j_0.$$

Let $\overline{\sigma}^{i_c} \overset{def}{=} \sigma_c$ and $\overline{\sigma}^0 = 0$, then

$$\gamma_+(\sigma) = i + 1 \;\text{ for }\; \sigma \in (\overline{\sigma}^i, \overline{\sigma}^{i+1}), \; i = 0, 1, \ldots, i_c - 1, \tag{4.26}$$

and let $\underline{\sigma}_0 \overset{def}{=} \sigma_c$ and $\underline{\sigma}_{j_0} = 0$, then

$$\gamma_-(\sigma) = j + i_c + 1 \;\text{ for }\; \sigma \in (\underline{\sigma}_{j+1}, \underline{\sigma}_j), \; j = 0, 1, \ldots, j_0 - 1, \tag{4.27}$$

where $\overline{\sigma}^i \neq \overline{\sigma}^{i+1}$ and $\underline{\sigma}_{j+1} \neq \underline{\sigma}_j$. For sufficiently large σ the degree index of the operator $P(\sigma, d, \cdot)$ is given in the following lemma.

Lemma 4.3. *Let $d > 0$ be fixed and let σ^* be a suitably large constant such that $\sigma \geq \sigma^* > \sigma_c$. Then it follows that*

$$\deg(P(\sigma, d, \cdot), \mathfrak{D}, \mathbf{0}) = 1,$$

where the set $\mathfrak{D}$ is defined as before, but here it is enough for the constant c to be independent of σ, σ_c is the same as in (4.10).

Proof. By Lemma 3.4, we can choose σ^* sufficiently large so that (1.4) has a unique solution W^* in $\mathfrak{D}$ for $\sigma \geq \sigma^* > \sigma_c$. Then we have

$$\deg(P(\sigma, \cdot), \mathfrak{D}, \mathbf{0}) = \mathrm{index}(P(\sigma, \cdot), W^*) = (-1)^\gamma.$$

Since $\sigma > \sigma_c$, Lemma 4.1(4) indicates that γ is even. Thus, the desired result follows. □

We now present a more specific version of Theorem 4.1 by fixing the value of $d > 0$.

Corollary 4.1. *Let d be fixed. Then the system (1.4) has at least one non-constant solution provided that*

(i) $0 < \sigma < \sigma_c$, *where* σ_c *is defined in (4.11).*

(ii) *if* $\sigma \in (\overline{\sigma}^i, \overline{\sigma}^{i+1}) \cap (\underline{\sigma}_{j+1}, \underline{\sigma}_j)$ *with* $\overline{\sigma}^i \neq \overline{\sigma}^{i+1}$ *for some* $i \in \{0, 1, \ldots, i_c - 1\}$ *and* $\underline{\sigma}_{j+1} \neq \underline{\sigma}_j$ *for some* $j \in \{0, 1, \ldots, j_0 - 1\}$, *then* $\gamma(\sigma) = i_c + j + i$ *is odd.*

Proof. Define $\overline{\Psi}(t,\cdot):[0,1]\times\mathfrak{D}\to C(\overline{\Omega})\times C(\overline{\Omega})$ by

$$\overline{\Psi}(t,W)=(I-\Delta)^{-1}[W+H((1-t)\sigma^*+t\sigma,d,W)],$$

where the set $\mathfrak{D}$ and σ^* are taken from Lemma 4.3.

It is clear that $\overline{\Psi}(t,\cdot)$ is a compact operator and its all fixed points are located in the interior of $\mathfrak{D}$ for $t\in[0,1]$. Thus, by the homotopy invariance of the topological degree, we have

$$\deg(I-\overline{\Psi}(1,\cdot),\mathfrak{D},\mathbf{0})=\deg(I-\overline{\Psi}(0,\cdot),\mathfrak{D},\mathbf{0}). \tag{4.28}$$

Since $I-\overline{\Psi}(0,\cdot)=P(\sigma^*,d,\cdot)$, from Lemma 4.3 it follows that

$$\deg(I-\overline{\Psi}(0,\cdot),\mathfrak{D},\mathbf{0})=\mathrm{index}(P(\sigma^*,\cdot),W^*)=1. \tag{4.29}$$

Obviously, $I-\overline{\Psi}(1,\cdot)=P(\sigma,d,\cdot)$. Thus, if we suppose, to the contrary, that there is no other solution for (1.4) except the constant one W^* in $\mathfrak{D}$, then, from the condition (*ii*) it follows that

$$\begin{aligned}&\deg(I-\overline{\Psi}(1,.),\mathfrak{D},\mathbf{0})=\mathrm{index}(I-\overline{\Psi}(1,.),W^*)\\ &=\mathrm{index}(P(\sigma,d,.),W^*)=(-1)^{\gamma_+(\sigma)+\gamma_-(\sigma)}=(-1)^{i_c+j+i}=-1,\end{aligned}$$

which contradicts (4.28) and (4.29). Thus, our assumption is false, which in turn asserts that there is at least one non-constant solution to (1.4). The proof is complete. □

Similarly if σ is fixed, more specified conditions on d can be found for the existence of non-constant solutions. But we shall not explore this cumbersome procedure again here. For illustration, below we shall present an example to verify Corollary 4.1 in one dimensional interval $[0,l]$. It is worthwhile to remind that the Neumann operator $-\Delta$ has eigenvalues $\mu_i=\left(\frac{i\pi}{l}\right)^2, i=0,1,2,\ldots$ in $[0,l]$.

4.3. Example

Taking into account condition (H1), we consider the following motility function

$$r(v)=\frac{1}{1+e^{8(v-1)}}. \tag{4.30}$$

Let us consider problem (1.1) in $(0,l)$ with $l=2\pi$ and choosing $d=0.4$. Then we have

$$\begin{cases}\sigma_c=\dfrac{1}{d}\left(\sqrt{-r'(1)}-\sqrt{r(1)}\right)^2=1.25,\\ \chi_\pm(\sigma_c)=\sqrt{\dfrac{\sigma_c}{dr(1)}}=2.5,\quad \chi_-(0)=\dfrac{r'(1)+r(1)}{dr(1)}=7.5.\end{cases}$$

Thus (4.24) leads to $i_c=3$, $i_a=3$ and the admissible wave number $k_a=1.5$. If we choose $\sigma=0.2$, then $\chi_+(\sigma)\approx0.1438$ and $\chi_-(\sigma)\approx6.9562$. By (4.25)–(4.27), we have $i=0,j=2$, and the degree index $\gamma=i_c+j+i=5$. If we choose $\sigma=0.6$, similarly we will have $i=1,j=1$ and hence $\gamma=i_c+j+i=5$. Thus, by Corollary 4.1, the non-constant positive steady state of (1.1) will arise, as numerically shown in Fig. 1(a) and (b). We find that when the value of σ (like $\sigma=0.2$) is far away from the critical value, the pattern is unstable and second bifurcation will occur and evolve into periodic-like patterns as observed in Fig. 1(a). However if the value of σ is close to the critical value, the pattern becomes stable as shown in Fig. 1(b). On the other hand, when the condition (ii) of Corollary 4.1 is not satisfied, the pattern may still arise as shown in Fig. 1(c) for $\sigma=1$ which gives degree index $\gamma=6$. This indicates that the condition (ii) in Corollary 4.1 may not be necessary for pattern formation. In Fig. 2, we increase the domain size and find similar patterning processes. In both figures, we observe that when the value of σ is away from the critical value $\sigma_c=1.25$, the pattern may keep changing in time and periodic (or chaotic) patterns arise. However if the value of σ is close to the critical value, the pattern appears to be stable. This implies within the instability region, the pattern dynamics will be more delicate as the parameter value is getting away from the critical regime. However how to qualitatively characterize this patterning phenomenon seems to be very challenging.

5. Propagation of pulsating waves

In this section, based on the results obtained in the previous sections, we will investigate how the pattern invades the whole domain when the size of the spatial domain is large for the system (1.1). To this end, we have to take into account the slow modulation of the amplitude of wave-pattern solutions in space. Hence below we shall distinguish the slow and fast spatial variables, and employ a weakly nonlinear multiple scale analysis to derive a Ginzburg–Landau type equation which captures the evolution of the amplitude of the propagating waves. This method has been widely used in the literature (e.g. see [18,19]). For simplicity, we shall consider the problem in one dimension only below.

5.1. Multiple scale analysis

Let $\Omega=(0,l)$ with some positive constant l. Then (1.1) can be rewritten in the form of

$$\begin{cases}u_t=\partial_{xx}(r(v)u)+\sigma u(1-u), & x\in(0,1),\ t>0,\\ v_t=d\partial_{xx}v-v+u, & x\in(0,l),\ t>0,\\ \partial_x u=\partial_x v=0, & x=0,l,\ t>0,\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x), & x\in(0,l).\end{cases} \tag{5.1}$$

Since the pattern bifurcates from the constant steady state $(1,1)$, we set

$$U=u-1,\ V=v-1.$$

Then (5.1) is changed to

$$\begin{cases}U_t=\partial_{xx}(r(V+1)(U+1))-\sigma U(U+1),\\ V_t=d\partial_{xx}V+U-V.\end{cases} \tag{5.2}$$

Introduce the time and space scales as

$$\begin{cases}t=t(T_1,T_2,T_3,\ldots),\quad T_i=\varepsilon^i t, i=1,2,\ldots,\\ x=x(x,X),\ X=\varepsilon x,\end{cases} \tag{5.3}$$

where X and x are the slow and the fast spatial variables, respectively, and $0<\varepsilon\ll1$. Then the corresponding derivatives decouple as

$$\begin{cases}\partial_t\to\varepsilon\partial_{T_1}+\varepsilon^2\partial_{T_2}+\varepsilon^3\partial_{T_3}+\cdots,\\ \partial_x\to\partial_x+\varepsilon\partial_X,\\ \partial_{xx}\to\partial_{xx}+2\varepsilon\partial_{xX}+\varepsilon^2\partial_{XX}.\end{cases} \tag{5.4}$$

The asymptotic expansion of $W=\begin{pmatrix}U\\V\end{pmatrix}$ and σ in the parameter ε are taken as

$$\begin{cases}\sigma=\sigma_a+\varepsilon\sigma_1+\varepsilon^2\sigma_2+\varepsilon^2\sigma_3+\cdots,\\ W=\varepsilon W_1+\varepsilon^2W_2+\varepsilon^3W_3+\cdots,\end{cases} \tag{5.5}$$

where σ_a is defined in (4.8), $W_i=\begin{pmatrix}W_{1i}\\W_{2i}\end{pmatrix}, i=1,2,3,\ldots$. Substituting of (5.4) and (5.5) into (5.2) and equating of the coefficients with the same powers of ε lead to the following systems

$$O(\varepsilon):\mathcal{M}(\sigma_a)W_1=\mathbf{0}, \tag{5.6}$$

$$O(\varepsilon^2):\mathcal{M}(\sigma_a)W_2=K(W_1), \tag{5.7}$$

$$O(\varepsilon^3):\mathcal{M}(\sigma_a)W_3=L(W_1,W_2), \tag{5.8}$$

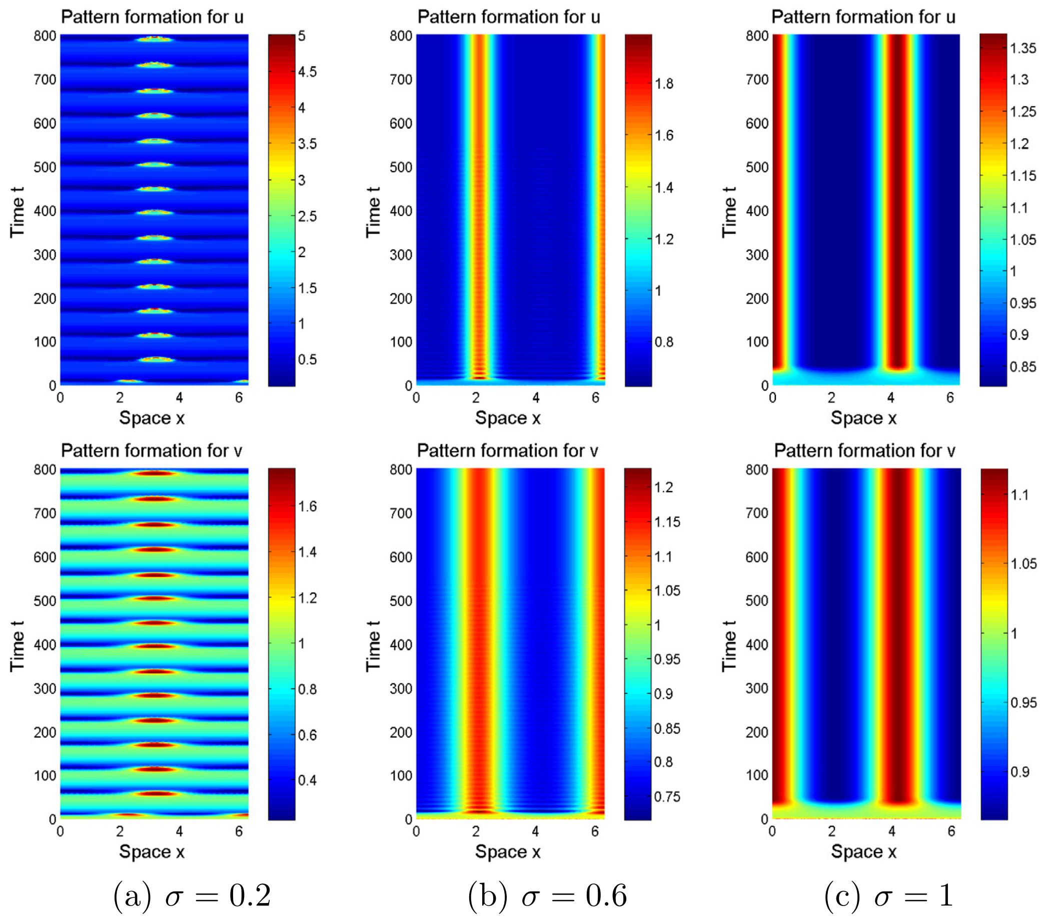

Fig. 1. Numerical simulations of pattern formation of system (1.1) in $[0, 2\pi]$, where $r(v)$ is given by (4.30), $d = 0.4$ and the initial value (u_0, v_0) is set as a small random perturbation of the constant steady state $(1, 1)$.

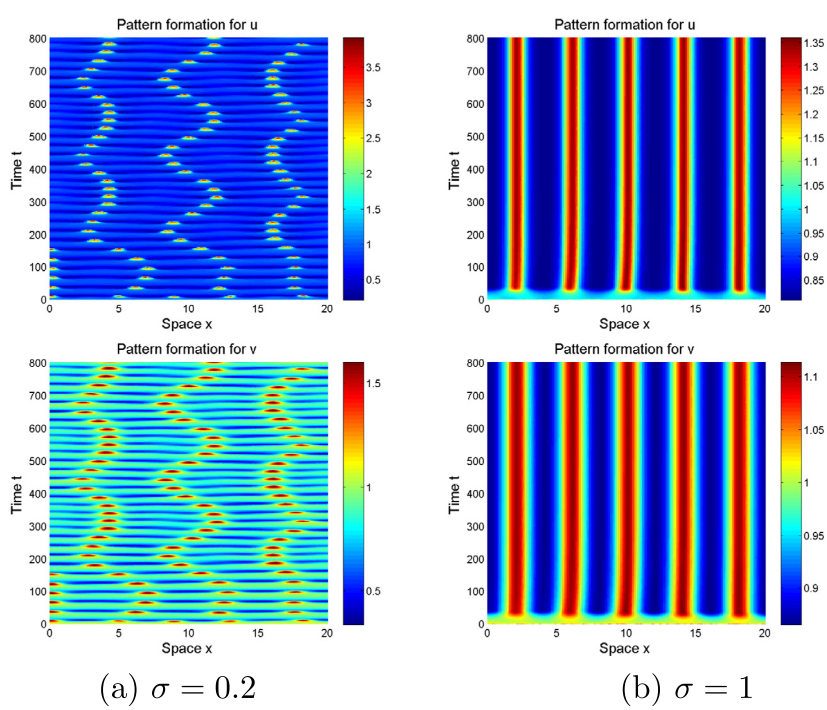

Fig. 2. Numerical simulations of pattern formation of system (1.1) in $[0, 20]$, where $r(v)$ is given by (4.30), $d = 0.4$ and the initial value (u_0, v_0) is set as a small random perturbation of the constant steady state $(1, 1)$.

where $K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$, $L = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$, and $\mathcal{M}(\sigma_a) = \begin{pmatrix} r(1)\partial_{xx} - \sigma_a & r'(1)\partial_{xx} \\ 1 & d\partial_{xx} - 1 \end{pmatrix}$ with

$$\begin{cases} K_1 &= \frac{\partial W_{11}}{\partial T_1} - r''(1)[(\partial_x W_{21})^2 + W_{21}\partial_{xx} W_{21}] \\ &\quad - r'(1)[\partial_x(W_{11}\partial_x W_{21}) + \partial_x(W_{21}\partial_x W_{11}) + 2\partial_{xx} W_{21}] \\ &\quad -2r(1)\partial_{xx} W_{11} + \sigma_a W_{11}^2 + \sigma_1 W_{11}, \\ K_2 &= \frac{\partial W_{21}}{\partial T_1} - 2d\partial_{xx} W_{21} \end{cases}$$

and

$$\begin{cases} L_1 &= \frac{\partial W_{11}}{\partial T_2} + \frac{\partial W_{12}}{\partial T_1} - r'''(1)[W_{21}(\partial_x W_{21})^2 + \frac{1}{2} W_{21}^2 \partial_{xx} W_{21}] \\ &\quad - r''(1)[W_{11}(\partial_x W_{21})^2 + W_{11}W_{21}\partial_{xx} W_{21} + W_{21}\partial_{xx} W_{22} \\ &\quad + W_{22}\partial_{xx} W_{21}] \\ &\quad - r''(1)[2W_{21}\partial_x W_{21}\partial_x W_{11} + 2\partial_x W_{21}\partial_x W_{22} + \frac{1}{2} W_{21}^2 \partial_{xx} W_{11}] \\ &\quad - r'(1)[\partial_x(W_{21}\partial_x W_{12}) + \partial_x(W_{12}\partial_x W_{21}) + \partial_x(W_{11}\partial_x W_{22}) \\ &\quad + \partial_x(W_{22}\partial_x W_{11})] \\ &\quad - 2[r''(1)\partial_X(W_{21}\partial_x W_{21}) + r'(1)\partial_X(W_{11}\partial_x W_{21}) \\ &\quad + r'(1)\partial_X(W_{21}\partial_x W_{11})] \\ &\quad - [2r'(1)\partial_{xX} W_{22} + 2r(1)\partial_{xX} W_{12} + r'(1)\partial_{XX} W_{21} \\ &\quad + r(1)\partial_{XX} W_{11}] \\ &\quad + 2\sigma_a W_{11}W_{12} + \sigma_1(W_{11}^2 + W_{12}) + \sigma_2 W_{11}, \\ L_2 &= \frac{\partial W_{21}}{\partial T_2} + \frac{\partial W_{22}}{\partial T_1} - 2d\partial_{xX} W_{22} - d\partial_{XX} W_{21}. \end{cases}$$

Here ∂_x and ∂_{xx} represent the first and the second partial derivatives with respect to x, respectively. Substituting (5.4) and (5.5) into the boundary condition $U' = V' = 0, x = 0, l$, we have

$$\frac{\partial W_1}{\partial x} = 0, \quad \frac{\partial W_2}{\partial x} = -\frac{\partial W_1}{\partial X}, \quad \frac{\partial W_3}{\partial x} = -\frac{\partial W_2}{\partial X}, \quad \ldots. \tag{5.9}$$

Solving (5.6) with (5.9) yields

$$W_1 = \rho A(X, T_1, T_2)\cos(k_a x), \quad \rho = \begin{pmatrix} 1 + dk_a^2 \\ 1 \end{pmatrix}, \tag{5.10}$$

where $k_a = \frac{i_a \pi}{l}$, the letter A represents the amplitude of the pattern and the vector $\rho \in \ker \mathcal{M}(\sigma_{\mathbf{a}})|_{\mathbf{W_1}}$ is defined up to a constant. Using (5.10), we have

$$\begin{cases} K_1 &= \left[(1 + dk_a^2)\frac{\partial A}{\partial T_1} + \sigma_1(1 + dk_a^2)A\right]\cos(k_a x) \\ &\quad + \left[r''(1)k_a^2 + 2r'(1)(1 + dk_a^2)k_a^2 + \frac{1}{2}\sigma_a(1 + dk_a^2)^2\right] \\ &\quad \times A^2\cos(2k_a x) \\ &\quad + \left[2r(1)(1 + dk_a^2)k_a\frac{\partial A}{\partial X} + 2r'(1)k_a\frac{\partial A}{\partial X}\right]\sin(k_a x) \\ &\quad + \frac{1}{2}\sigma_a(1 + dk_a^2)^2 A^2, \\ K_2 &= \frac{\partial A}{\partial T_1}\cos(k_a x) + 2dk_a\frac{\partial A}{\partial X}\sin(k_a x). \end{cases}$$

For the use of solving system (5.7), we find the solution of the adjoint system of (5.6) is

$$\overline{W}_1 = \begin{pmatrix} \overline{W}_{11} \\ \overline{W}_{12} \end{pmatrix} = \overline{\rho} A(X, T_1, T_2)\cos(k_a x), \quad \overline{\rho} = \begin{pmatrix} \frac{1 + dk_a^2}{-r'(1)k_a^2} \\ 1 \end{pmatrix}, \tag{5.11}$$

where $\overline{\rho}$ is the kernel of the adjoint of the operator $\mathcal{M}(\sigma_a)|_{W_1}$. We will apply the solvability condition for (5.7), i.e.,

$$\langle K, \overline{W}_1 \rangle = \int_0^{\frac{2\pi}{k_a}} K_1\overline{W}_1 + K_2\overline{W}_2 dx = 0. \tag{5.12}$$

Here we should mention that due to the homogeneous Neumann boundary condition, the Fredholm alternative has to be used on the interval $[0, \frac{2\pi}{k_a}]$. Then we can obtain a solution of (5.1) on the whole domain by employing the method of reflection symmetry and periodic extension.

By (5.12), we have $\sigma_1 = 0, T_1 = 0$. Thus, the expression of solution of (5.7) is in the form of

$$\begin{cases} W_{12} = A^2(c_{11} + c_{12}\cos(2k_a x)) + \frac{\partial A}{\partial X} c_{13}\sin(k_a x), \\ W_{22} = A^2(c_{21} + c_{22}\cos(2k_a x)) + \frac{\partial A}{\partial X} c_{23}\sin(k_a x), \end{cases} \tag{5.13}$$

where c_{ij} $(i = 1, 2, j = 1, 2, 3)$ solve, respectively, the following systems

$$\Phi_0(\sigma_a)\begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix} = -\begin{pmatrix} \frac{\sigma_a}{2}\left(1 + dk_a^2\right)^2 \\ 0 \end{pmatrix},$$

$$\Phi_2(\sigma_a)\begin{pmatrix} c_{12} \\ c_{22} \end{pmatrix} = -\begin{pmatrix} \frac{\sigma_a}{2}\left(1 + dk_a^2\right)^2 + \left[r''(1) + 2r'(1)(1 + dk_a^2)\right]k_a^2 \\ 0 \end{pmatrix},$$

$$\Phi_1(\sigma_a)\begin{pmatrix} c_{13} \\ c_{23} \end{pmatrix} = 2k_a\begin{pmatrix} -r'(1) - r(1)\left(1 + dk_a^2\right) \\ d \end{pmatrix},$$

with

$$\Phi_p(\sigma_a) = \begin{pmatrix} p^2 r(1)k_a^2 + \sigma_a & p^2 r'(1)k_a^2 \\ 1 & -1 - p^2 dk_a^2 \end{pmatrix}, \quad p = 0, 1, 2.$$

Substituting (5.10) and (5.13) into L, we have

$$\begin{cases} L_1 &= \left[(1 + dk_a^2)\frac{\partial A}{\partial T_2} - L_1^{(1)}\frac{\partial^2 A}{\partial X^2} + H_1 A^3 + \sigma_2(1 + dk_a^2)A\right]\cos(k_a x) \\ &\quad + H_2 A^3 \sin^2(k_a x)\cos(k_a x) + H_3 A^3 \cos(k_a x)\cos(2k_a x) \\ &\quad + H_4 A^3 \cos^3(k_a x) + L_1^*, \\ L_2 &= \left(\frac{\partial A}{\partial T_2} - d\frac{\partial^2 A}{\partial X^2} - 2dk_a c_{23}\frac{\partial^2 A}{\partial X^2}\right)\cos(k_a x) + L_2^*, \end{cases}$$

where L_1^* and L_2^* satisfy $\langle L_1^*, \overline{W}_{11} \rangle = 0$ and $\langle L_2^*, \overline{W}_{21} \rangle = 0$, and their expressions depend on the parameters of the system (1.1) and are too cumbersome to give them here. Furthermore,

$$\begin{aligned} L_1^{(1)} &= r'(1)(1 + 2k_a c_{23}) + r(1)(1 + dk_a^2 + 2k_a c_{13}), \\ H_1 &= \left[r''(1)c_{21} + r'(1)c_{11}\right]k_a^2 + \left[r'(1)k_a^2 c_{21} + 2\sigma_a c_{11}\right](1 + dk_a^2), \\ H_2 &= -r'''(1)k_a^2 - 3r''(1)k_a^2(1 + dk_a^2), \\ H_3 &= \left[r''(1)c_{22} + r'(1)c_{12}\right]k_a^2 + \left[r'(1)k_a^2 c_{22} + 2\sigma_a c_{12}\right](1 + Dk_a^2), \\ H_4 &= \left[\frac{1}{2}r'''(1) + \frac{3}{2}r''(1)(1 + dk_a^2)\right]k_a^2. \end{aligned}$$

Again from the solvability condition of (5.8), i.e., $\langle L, \overline{W}_1 \rangle = 0$, it follows that

$$\frac{\partial A}{\partial T_2} = \kappa\frac{\partial^2 A}{\partial X^2} + \varsigma A - \varrho A^3, \tag{5.14}$$

where

$$\begin{aligned} \kappa &= \frac{(1 + dk_a^2)L_1^{(1)} - dr'(1)k_a^2(1 + 2k_a c_{23})}{(1 + dk_a^2)^2 - r'(1)k_a^2}, \\ \varsigma &= \frac{-\sigma_2(1 + dk_a^2)^2}{(1 + dk_a^2)^2 - r'(1)k_a^2} > 0, \\ \varrho &= \frac{(1 + k_a^2)\left(L_1^{(2)} + L_1^{(3)}\right)}{(1 + dk_a^2)^2 - r'(1)k_a^2}, \end{aligned}$$

$$L_1^{(2)} + L_1^{(3)} = H_1 + \frac{1}{4}H_2 + \frac{1}{2}H_3 + \frac{3}{4}H_4.$$

Eq. (5.14) gives the third order approximation of the amplitude of pattern solution for (1.1). It is the typical Ginzburg–Landau equation. The coefficient ς is always positive. However, ϱ can be positive or negative depending on the values of the system parameters. Usually, it is called that *if ϱ is positive (negative), one has a supercritical (subcritical) bifurcation.* Since the expression for ϱ is quite involved, it is hard to perform a general analytical study

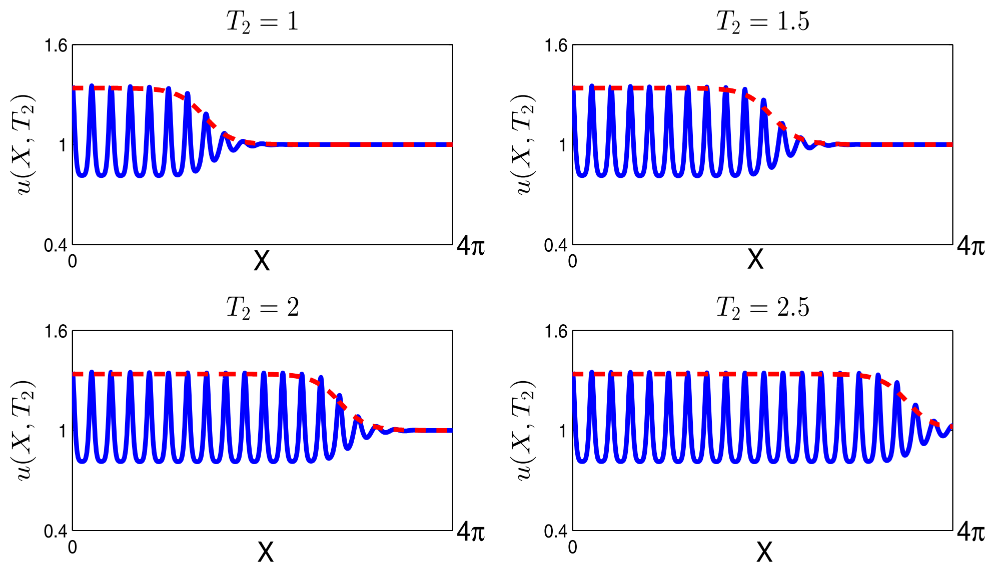

Fig. 3. An illustration of modulated progressing wave produced by system (1.1) where the pattern is formed sequentially and the traveling wavefront is the precursor to patterning. The red dash line is the third order approximate solution given by (5.17). The blue solid line is a numerical solution of system (1.1) with the initial value $(u_0, v_0) = (1, 1) + \varepsilon\rho A(X, 0)\cos(k_a x)$.

of its sign. In what follows, we will present an example with supercritical bifurcation to demonstrate that Eq. (5.14) can govern the evolution of the amplitude of pattern solution of (1.1). But for the subcritical case, it is naturally required to push the weakly nonlinear expansion at a higher order so that one can obtain a higher order Ginzburg–Landau equation to capture the evolution of the amplitude. This is extraordinarily difficult and will not be pursued here.

For the supercritical case, we can use "tanh" method to obtain the exact solution of (5.14) in $\mathbb{R}$ of the form

$$A(X, T_2) = \frac{1}{2}\sqrt{\frac{\varsigma}{\varrho}}\left(1 - \tanh\left(\sqrt{\frac{\varsigma}{\kappa}}\frac{Y - Y_0}{2\sqrt{2}}\right)\right), \tag{5.15}$$

where $Y = X - cT_2$, $c = 3\sqrt{\varsigma\kappa/2}$, which is a traveling wave solution of (5.14) connecting 0 and $\sqrt{\varsigma/\varrho}$.

5.2. Numerical verifications

Below shall numerically demonstrate that the Ginzburg–Landau equation (5.14) gives a very good approximation of the amplitude of propagating waves generated by the motility model (1.1). For definiteness, we consider

$$r(v) = \frac{1}{1 + e^{8(v-1)}}$$

and assume the system parameters $d = 1$, $\sigma = 0.4$, $l = 4\pi$. Then system (5.1) reads

$$\begin{cases} u_t = \partial_{xx}\left(\frac{1}{1+e^{8(v-1)}}u\right) + 0.4u(1-u), & x \in (0, 4\pi),\ t > 0, \\ v_t = \partial_{xx}v - v + u, & x \in (0, 4\pi),\ t > 0, \\ \partial_x u = \partial_x v = 0, & x = 0, 4\pi, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), & x \in (0, 4\pi). \end{cases} \tag{5.16}$$

By a direct computation, one can find that the admissible mode $i_a = 4$ and the corresponding admissible wave number $k_a = 1$, $\rho = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and the critical value of σ is $\sigma_c = \sigma_a = 0.5$. Based on the discussion in Section 4, we find that the degree index $\gamma = 7$. Then Corollary 4.1 guarantees that (5.16) admits at least one stationary non-constant solution.

Through a tedious computation, we have $c_{11} = -2$, $c_{12} \approx 7.7778$ and $c_{13} = 0.4$; $c_{21} = -2$, $c_{22} \approx 1.5556$ and $c_{23} = -0.8$; $\kappa \approx 0.6667$, $\varsigma \approx 6.6667$, and $\varrho \approx 4.2963$. Therefore, by (5.10) and (5.13), the third approximation of the solution component $u(X, T_2)$ now is given by

$$u = 1 + 2\varepsilon A(X, T_2) + 5.7778\varepsilon^2 A^2(X, T_2), \tag{5.17}$$

where $A(X, T_2)$ is a solution of (5.14) given by (5.15).

We plot the graph of u given by (5.17) in Fig. 3 (see the red dashed line), which quantitatively well captures the evolution of the amplitude of the propagating pattern (see the blue solid line in Fig. 3) generated by system (1.1). There is a subtle quantitative discrepancy at the first bent which results from the neglect of the higher terms. We observe that the pattern propagates into the whole domain in the form of oscillatory waves.

6. Summary and discussion

The concept "density-suppressed motility" has been advocated recently in [2,3] and the mathematical analysis of the relevant model (1.1) has been touched only on the existence of global solutions and stability of constant steady states in two dimensions or in higher dimensions for large growth rate σ in the work [4,5] for $\sigma > 0$. When $\sigma = 0$, the global existence of solutions in two or higher dimensions was established in [7–9] under various assumptions. The results on the stationary problem of (1.1) have been unavailable yet. Motivated by a previous work [4] where the numerical simulations have demonstrated that the model (1.1) is capable of producing various interesting patterns for appropriate values of d, σ and motility function $r(v)$, we investigate the existence/nonexistence of non-constant stationary solutions for the density-suppressed motility model (1.1) supplemented with numerical illustrations. The primary analytical results of this paper are given in Theorems 3.1 and 4.1. Moreover the multiple-scale analysis is performed in Section 5 to show that the model (1.1) can generate periodic traveling waves which qualitatively interpret the expanding strip patterns observed in the experiment of [2] and the model is hence justified. Given a class of decreasing motility function $r(v)$ satisfying the condition (H1), Theorem 3.1(a) asserts that for large chemical diffusion coefficient d or bacterial intrinsic growth rate σ, the model (1.1) is incapable of producing pattern formation. Furthermore if the decay of $r(v)$ is very slow (i.e., sublinear or linear) in v such as $r(v) = \frac{1}{(\alpha+\beta v)^\xi}$ with $\xi < 1$ or $\beta < \alpha$ and $\xi = 1$, Theorem 3.1(b) shows that no pattern formation can develop from the model (1.1) when $d > 0$ or $\sigma > 0$ is small. These results together reveal that the model (1.1) can generate pattern formation only in a moderate (narrow) regime of parameters of d and σ where the decay rate of $r(v)$ with respect to v will play a role. Therefore the rigorous proof of the existence of non-constant stationary solutions of (1.1) will be very intricate. In this paper, we explore this question and present some sufficient conditions on d, σ and $\gamma(v)$ warranting the existence of non-constant stationary solutions of (1.1) (see Theorem 4.1 and Corollary 4.1). By expanding the Laplacian term in the first question and set $\chi(v) = -r'(v) > 0$, (1.1) becomes a Keller–Segel type chemotaxis system as follows

$$\begin{cases} u_t = \nabla\cdot(r(v)u - u\chi(v)\nabla v) + \sigma u(1-u), & x\in\Omega, t>0,\\ v_t = d\Delta v + u - v, & x\in\Omega, t>0,\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega, t>0,\\ u(x,0)=u_0(x), v(x,0)=v_0(x), & x\in\Omega. \end{cases} \tag{6.1}$$

Even if we regard (6.1) as a chemotaxis system, it is also a new type of chemotaxis systems since the diffusion rate depends on the chemical (signal) concentration. As we know, the available analytical results of (6.1) are limited to the case of constant $r(v)$ (e.g., see [20–22] and references therein). When both $r(v)$ and $\chi(v)$ are constant, the complex patterns/dynamics of (6.1) have been numerically illustrated in [23,24] and only part of them have been analytically understood (cf. [10,25]). Hence we may anticipate that the system (6.1) with non-constant $r(v)$ and $\chi(v)$ will have rich dynamics and complex patterns as illustrated in the numerical simulations of [4]. Accordingly the mathematical analysis of these dynamics/patterns will be intriguing yet difficult. This paper takes a step forward to understand the complex dynamics underlying the model (6.1) for the case $\chi(v) = -r'(v)$ (namely the model (1.1)). However our results are far from the complete understanding of the dynamics of (1.1) in the whole parameter space. For instance, the critical regime of d and σ for the existence/non-existence of non-constant steady states is unknown, and how the decay rate of $r(v)$ in v will affect the pattern formation still remains poorly understood. There are many interesting analytical questions for future studies and our current work has provided an illuminating insight into the complex dynamics underlying the model (6.1) with general $r(v)$ and $\chi(v)$.

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