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NONLINEAR STABILITY OF LARGE AMPLITUDE VISCOUS SHOCK WAVES OF A GENERALIZED HYPERBOLIC–PARABOLIC SYSTEM ARISING IN CHEMOTAXIS

TONG LI

Department of Mathematics,
University of Iowa, Iowa City, IA 52242, USA
tli@math.uiowa.edu

ZHI-AN WANG*

Department of Applied Mathematics,
Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong
mawza@polyu.edu.hk

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Traveling wave (band) behavior driven by chemotaxis was observed experimentally by Adler[1,2] and was modeled by Keller and Segel.[15] For a quasilinear hyperbolic−parabolic system that arises as a non-diffusive limit of the Keller−Segel model with nonlinear kinetics, we establish the existence and nonlinear stability of traveling wave solutions with large amplitudes. The numerical simulations are performed to show the stability of the traveling waves under various perturbations.

Keywords: Nonlinear conservation laws; nonlinear stability; chemotaxis; traveling waves; non-diffusive signals; large amplitude; nonlinear kinetics; energy estimates; Shizuta−Kawashima condition.

AMS Subject Classification: 35L45, 35L60, 92B05, 92B99, 92C15, 92C17

1. Introduction

Chemotaxis is a process in which cells change their states of movement reacting to the presence of a chemical substance, approaching chemically favorable environment and avoiding unfavorable ones. It is a fundamental cellular process which plays essential roles in embryonic development, immune response, progression of diseases, tissue homeostasis, wound healing, as well as in finding food, repellent action and forming

*Corresponding author

the multicellular body of protozoa.[37] The chemotaxis is called attractive (positive) if the chemotactic movement is toward higher chemical concentration, and repulsive (negative) if the movement direction is opposite.

The mathematical modeling of chemotaxis dates to Keller and Segel[13] from the macroscopic perspective, and to Patlak[28] based on the microscopic (or statistical) description. The original model in Ref. 13 comprised four strongly coupled partial differential equations and was proposed to describe the aggregation of cellular slime molds *Dictyostelium discoideum*. A simplified formulation, however, also most extensively studied in the literature,[8] is the following two strongly coupled parabolic equations

$$\begin{aligned} u_t &= \nabla \cdot (D\nabla u - \chi u \nabla\Phi(c)), \\ \tau c_t &= d\Delta c + g(u,c), \end{aligned} \tag{1.1}$$

where $u(t,x), c(t,x)$ denote the cell density and the chemical concentration, respectively. D is the diffusivity of cells and d is the diffusion rate of the chemical substance. $\tau \geq 0$ is a relaxation timescale such that $1/\tau$ is the rate towards equilibrium. The function $\Phi(c)$ is called the chemotactic potential function describing the mechanism of signal detection. The function $g(u,c)$ describes the chemical kinetics. The constant χ, often referred to as chemosensitivity, is a measure of the strength of chemical signals, and $\chi > 0$ (< 0) corresponds to attractive (repulsive) chemotaxis.

There are two major limiting cases of the Keller–Segel model (1.1). The first one is when the chemical substance v relaxes so fast that it reaches its equilibrium instantaneously. This means $\tau \to 0$ and gives rise to the following so-called parabolic–elliptic system

$$\begin{aligned} u_t &= \nabla \cdot (D\nabla u - \chi u \nabla\Phi(c)), \\ 0 &= d\Delta c + g(u,c). \end{aligned} \tag{1.2}$$

There were vast results to the model (1.2) on the global existence, blowup behavior and the stationary solutions. The most relevant results are summarized in a review article[8] and textbook.[29]

The other limiting case is when the diffusion of the chemical substance is so small that it is negligible. Then the model becomes a partial differential equation coupled with an ordinary differential equation (PDE–ODE)

$$\begin{aligned} u_t &= \nabla \cdot (D\nabla u - \chi u \nabla\Phi(c)), \\ c_t &= g(u,c), \end{aligned} \tag{1.3}$$

where we have assumed $\tau = 1$ without loss of generality. The model (1.3) is also called the parabolic-degenerate chemotaxis model.[5] This model had already been used to study the chemotactic traveling bands of bacteria in Ref. 15 whereby $g(u,c) = -kuc^m$ $(m \geq 1)$ and has received increasing attention recently (see Ref. 33). A direct application of the above PDE–ODE coupled system (1.3) is the modeling of haptotaxis where cells move towards immobilized, substratum-bound such as laminin and fibronectin. It has essential applications in the modeling of cancer invasion,[3] as well as the gliding movement of myxobacteria[27] towards slime trails which means a non-diffusive chemical substance.

The objective of this paper is to study the existence and nonlinear stability of traveling wave solutions of the chemotaxis model of type (1.3). The experimental study of traveling wave (band) behavior driven by chemotaxis dates to Adler's works.[1,2] The mathematical modeling and analysis of the traveling waves was first presented by Keller and Segel[15] and a flurry of works then followed (see Refs. 9, 23 and references therein). However, the results on stability of traveling wave solutions of chemotaxis models are much less and there are only few results on linear stability,[30,7] linear instability[25] and nonlinear stability.[20]

To proceed, we need to specify the potential function $\Phi(v)$ and kinetic function $g(u,c)$ in (1.3). For the potential function, we consider a logarithmic representation

$$\Phi(c) = \log c. \tag{1.4}$$

Although the form of the potential function $\Phi(c)$ can be generic in principle, it has been shown early in Refs. 15, 14 and 31 that the singularity at small chemical concentration c is necessary to reproduce the biologically relevant traveling bands. Hence a natural choice is (1.4) which has been used throughout most of the studies of traveling waves of chemotaxis in the literature.[9,35,23] The logarithmic potential function (1.4) was first used successfully by Keller and Segel[15] to interpret the traveling band behavior of bacteria. It was shown in Ref. 23 that the logarithmic function (1.4) is also appropriate from the mathematical viewpoint[23] although it is singular at $c = 0$. Depending on the specific modeling goals, the kinetic function $g(u,c)$ has a wide variability. When $g(u,c)$ is linear with respect to u and c, the existence of traveling waves of both microscopic and macroscopic Keller–Segel chemotaxis models has been discussed in Ref. 23. For other possible forms of the kinetic function, we refer the readers to Refs. 9 and 10 and references therein. Here we consider a class of nonlinear kinetic function $g(u,c)$

$$g(u,c) = \delta f(u)c, \tag{1.5}$$

where $\delta \in \mathbb{R}$ is a constant and f is a smooth function.

The motivation of considering the kinetic function (1.5) follows from specific applications. In a chemotactic–haptotactic model of cancer invasion,[3] and in models of formation of capillary networks of blood vessels,[17,5,4] the basic kinetics for the extracellular matrix is $g(u,c) = -uc$, which corresponds to $f(u) = u, \delta = -1$ in (1.5). In the modeling of self-organization of myxobacteria[27] and angiogenesis,[18] $g(u,c) = \lambda uc - \mu c$ where $\lambda > 0$ and $\mu \geq 0$ which is equivalent to $f(u) = \lambda u - \mu$ with $\delta = 1$ in (1.5). In these examples, $f(u)$ is linear. When $\chi < 0, \delta > 0$ and when $f(u)$ is linear, the existence and the nonlinear stability of traveling wave solutions of (1.3) have been recently established in Refs. 36 and 20, respectively.

In this paper, we will consider more general kinetic function g as defined in (1.5) where f satisfies

$$f(u) \geq 0, \quad f'(u) > 0, \quad f''(u) \geq -\beta \tag{1.6}$$

for all u under considerations, with β being a positive constant which will be given in Sec. 5, and will extend the parameter space to

$$\chi\delta < 0.$$

In summary, we consider the following one-dimensional chemotaxis model

$$\begin{aligned} u_t &= Du_{xx} - (\chi u\phi(c)c_x)_x, \\ c_t &= \delta f(u)c, \end{aligned} \tag{1.7}$$

where $\phi(c) = 1/c$, f satisfies (1.6) and $\chi\delta < 0$. The kinetic function considered here is more applicable than that in Refs. 36 and 20. We need to point out that (1.7) with $f(u) = \lambda u - \mu$ was the model considered in Refs. 18 and 27 which is now a special case of (1.7) with (1.6).

To establish the existence and nonlinear stability of traveling wave solutions to the system (1.7), we convert (1.7), through the transformation

$$v = -\frac{c_x}{c}, \tag{1.8}$$

into an equivalent quasilinear hyperbolic–parabolic system as follows:

$$\begin{aligned} u_t - \chi(uv)_x &= Du_{xx}, \\ v_t + \delta f(u)_x &= 0. \end{aligned} \tag{1.9}$$

Making the following scalings

$$\lambda = -\delta\chi, \quad \tilde{t} = \lambda t, \quad \tilde{x} = \sqrt{\lambda}x, \quad \tilde{v} = \frac{\sqrt{\lambda}}{\delta}v, \tag{1.10}$$

(1.9) becomes the following system of nonlinear conservation laws

$$\begin{aligned} u_t - (uv)_x &= Du_{xx}, \\ v_t - f(u)_x &= 0, \end{aligned} \tag{1.11}$$

where the tilde has been dropped for convenience. System (1.11) is usually called a hyperbolic–parabolic system whereby the viscosity matrix is nonnegative-definite.

We prescribe the initial condition for system (1.11) as follows:

$$(u,v)(x,0) = (u_0,v_0)(x) \to \begin{cases} (u^-,v^-) & \text{as } x \to -\infty \\ (u^+,v^+) & \text{as } x \to +\infty \end{cases} \tag{1.12}$$

with $u_0(x) > 0$ for all x and $u^\pm > 0$ in the region of biological interest.

The nonlinear stability theory of viscous shock profiles in systems of conservation laws is important and has been developed in the literature, see Refs. 6, 12, 19, 21, 22, 24 and 34. The small wave strength is generally an assumption for the nonlinear stability of the wave in the literature. We are able to prove the nonlinear stability of traveling waves for (1.11) without the smallness condition on wave strengths. More precisely, we show that, if the initial data (1.12) is a small perturbation of a traveling wave such that $\alpha = 0$ in (3.1), then the solution to the Cauchy problem (1.11), (1.12) exists globally and converges to a shifted traveling wave as $t \to +\infty$. We need to point out that when f is linear, the nonlinear stability of traveling waves (1.11), (1.12) has been established by Li and Wang in Ref. 20. In this paper we consider nonlinear function $f(u)$. As will be seen in the L^2-estimates (see Lemma 5.2 and its proof), the sign and magnitude of $f''(u)$ are critical. However, when $f(u)$ is linear, $f''(u) = 0$ and the estimates will be significantly simplified and there is not any issue with $f''(u)$. So some techniques and efforts are made here to determine the number β in condition (1.6).

The rest of the paper is organized as follows. In Sec. 2, we shall first present some preliminary results, and then state and prove the existence theorem of viscous shock waves of system (1.11)–(1.12). In Sec. 3, we state the stability theorem of the viscous shock waves. In Secs. 4 and 5, we derive the energy estimates and prove the stability theorem. In Sec. 6, we show numerical simulations to verify our theoretical results. Finally, we give a brief summary in Sec. 7.

2. Existence of Viscous Shock Profile

In this section, we devote ourselves to establishing the existence of viscous shock waves of (1.11)–(1.12). It can be verified that system (1.11) fulfills the Shizuta–Kawashima condition[32] which says that any vector in the kernel of viscosity matrix

$$B = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

is not an eigenvector of the Jacobian matrix

$$J = \begin{pmatrix} -v & -u \\ -f'(u) & 0 \end{pmatrix}.$$

Since waves of (1.11)–(1.12) propagate along the characteristics which are dissipative, the stability of viscous shock waves can then be expected.

System (1.11) is hyperbolic when $D = 0$. Indeed, the eigenvalues of the Jacobian matrix of (1.11) are

$$\lambda_1(u,v) = -\frac{v}{2} - \frac{1}{2}\sqrt{v^2 + 4uf'(u)}, \quad \lambda_2(u,v) = -\frac{v}{2} + \frac{1}{2}\sqrt{v^2 + 4uf'(u)} \tag{2.1}$$

with the corresponding eigenvectors

$$\mathbf{r}_1(u,v) = (-\lambda_1(u,v), f'(u)), \quad \mathbf{r}_2(u,v) = (-\lambda_2(u,v), f'(u)). \tag{2.2}$$

Clearly $\lambda_1 < 0 < \lambda_2$ if $u > 0$. Hence the system (1.11) is strictly hyperbolic if $u > 0$. The fact that u remains positive if $u_0 > 0$ will be established in Sec. 4. Furthermore, it can be verified that the characteristic families $(\lambda_1, \mathbf{r}_1)$ and $(\lambda_2, \mathbf{r}_2)$ satisfy

$$\begin{aligned} \nabla\lambda_1 \cdot \mathbf{r}_1 &= \frac{-v - \sqrt{v^2 + 4uf'(u)}}{2\sqrt{v^2 + 4uf'(u)}}(2f'(u) + uf''(u)) < 0, \\ \nabla\lambda_2 \cdot \mathbf{r}_2 &= \frac{v - \sqrt{v^2 + 4uf'(u)}}{2\sqrt{v^2 + 4uf'(u)}}(2f'(u) + uf''(u)) < 0 \end{aligned} \tag{2.3}$$

if

$$f''(u) > -\max_{u \in I} \frac{2f'(u)}{u} := -\beta_1, \tag{2.4}$$

where $\beta_1 > 0$ and I is a bounded interval containing $[\min\{u^+, u^-\}, \max\{u^+, u^-\}]$ such that $u_0 \in I$ and $0 \notin I$ which will be determined in Sec. 4. Under condition (2.4), system (1.11) is a genuinely nonlinear hyperbolic system for $u \in I$. We hereafter assume $0 < \beta \leq \beta_1$ in (1.6), see (5.20).

To study the traveling wave solutions for (1.11), we define the traveling wave ansatz

$$(u, v)(x, t) = (U, V)(z), \quad z = x - st$$

where s denotes the traveling wave speed and z is the traveling wave variable. Substituting the above ansatz into (1.11), one obtains the following system of differential equations

$$\begin{cases} -sU_z - (UV)_z = DU_{zz}, \\ -sV_z - f(U)_z = 0 \end{cases} \tag{2.5}$$

with boundary conditions

$$(U, V)(z) \to (u^{\pm}, v^{\pm}) \quad \text{as } z \to \pm\infty, \tag{2.6}$$

where we require $u^{\pm} > 0$ due to the biological interest.

Integrating system (2.5) with respect to z over $(-\infty, \infty)$ and using the fact $U_z \to 0$ as $z \to \pm\infty$, we have

$$\begin{cases} s(u^+ - u^-) = -u^+v^+ + u^-v^-, \\ s(v^+ - v^-) = -f(u^+) + f(u^-) \end{cases} \tag{2.7}$$

which corresponds to the Rankine–Hugonoit jump condition of the following shock wave

$$(u, v) = \begin{cases} (u^-, v^-), & x - st < 0 \\ (u^+, v^+), & x - st > 0 \end{cases} \tag{2.8}$$

for hyperbolic system (1.11) with $D = 0$. Canceling v^+ in (2.7), we obtain a quadratic equation for shock speed s

$$s^2 + v^-s - u^+ \frac{f(u^+) - f(u^-)}{u^+ - u^-} = 0.$$

Note that the condition $f'(u) > 0$ in (1.6) ensures that $\frac{f(u^+) - f(u^-)}{u^+ - u^-} > 0$. Hence the above equation gives two solutions for s with opposite signs, where $s > 0$ corresponds to the second characteristic field of shocks whereas $s < 0$ corresponds to the first characteristic field. In this paper, we restrict our attention to the case of $s > 0$ and the analysis can be extended to the case $s < 0$ without changes. That is, we consider the second characteristic family of shock waves with speed

$$s = -\frac{v^-}{2} + \frac{1}{2}\sqrt{(v^-)^2 + 4u^+\frac{f(u^+) - f(u^-)}{u^+ - u^-}}. \tag{2.9}$$

It is easy to check that the shock speed s defined in (2.9) satisfies the *entropy condition*[16]

$$\lambda_2(u^+, v^+) < s < \lambda_2(u^-, v^-). \tag{2.10}$$

Thus shock wave (2.8) for hyperbolic system (1.11) with $D = 0$ is an admissible shock.

Our first result concerning the existence of traveling wave solutions of (1.11), namely, the existence of solutions to (2.5)–(2.6), is as follows.

Theorem 2.1. *Let* (1.6) *hold. Then there exists a monotone shock profile* $(U, V)(x - st)$ *to system* (2.5)–(2.6) *with* s *defined in* (2.9), *which is unique up to a translation and satisfies* $U_z < 0$ *and* $V_z > 0$. *Moreover, the following relations hold*

$$\begin{cases} sDU_z = Uf(U) - (s^2 + \varrho_1)U + \varrho_2, \\ sV = \varrho_1 - f(U), \end{cases} \tag{2.11}$$

where ϱ_1 *and* ϱ_2 *are constants defined by*

$$\varrho_1 = sv^+ + f(u^+) = sv^- + f(u^-), \quad \varrho_2 = (s^2 + sv^+)u^+ = (s^2 + sv^-)u^-. \tag{2.12}$$

Proof. From the second equation of (2.5) and (2.6), it follows that

$$sV + f(U) = \varrho_1 \tag{2.13}$$

with ϱ_1 being defined in (2.12), which immediately implies the second equation of (2.11). Then from the first equation of (2.5), one obtains

$$sDU_{zz} = (-s^2 - \varrho_1 + f(U) + Uf'(U))U_z. \tag{2.14}$$

Introducing $U_z = W$, (2.14) can be written as a first-order ODE system as follows:

$$\begin{cases} U_z = W, \\ sDW_z = W(-s^2 - \varrho_1 + f(U) + Uf'(U)), \end{cases} \tag{2.15}$$

from which, one derives that

$$sD\frac{dW}{dU} = f(U) + Uf'(U) - s^2 - \varrho_1. \tag{2.16}$$

Integrating (2.16) with respect to U, using (2.6) and $W(u^\pm) = U_z(u^\pm) = 0$ yield

$$sDW = sDU_z = Uf(U) - (s^2 + \varrho_1)U + \varrho_2, \tag{2.17}$$

where ϱ_2 is a constant as given in (2.12). Equation (2.17) immediately gives the first equation of (2.11).

We now show that there is a trajectory connecting u^- and u^+ with the claimed monotonicity property: $U_z < 0$. Indeed, we define the right-hand side of (2.17) as a new function

$$h(U) = Uf(U) - (s^2 + \varrho_1)U + \varrho_2.$$

Then

$$h(u^-) = h(u^+) = 0 \tag{2.18}$$

and

$$h''(U) = 2f'(U) + Uf''(U) > 0 \tag{2.19}$$

provided that assumption (2.4) holds. Thus for U between u^- and $u^+, h(U) < 0$. Therefore $U_z = \frac{1}{sD}h(U) < 0$ for all U between u^- and u^+. Hence $u^+ < u^-$. By (2.13), it is straightforward to verify that $V_z > 0$ and hence $v^- < v^+$. Since $h''(U) > 0$, Eq. (2.17) only has two equilibria u^+ and u^-. Due to the fact that Eq. (2.17) is a first-order scalar ordinary differential equation of U, the trajectory of (2.17) satisfying boundary condition (2.6) necessarily connects equilibria u^- and u^+ by the standard argument (see e.g. Refs. 12 and 16). Indeed the trajectory on the U-W phase plane has been given by (2.13). The proof is then complete. □

Remark 2.1. Due to $U_z < 0$ and $V_z > 0$, it follows that $0 < u^+ \le U \le u^-$ and $v^- \le V \le v^+$. Thus (2.11) implies that $|U_z|$ and $|V_z|$ are bounded for any given diffusion rate $D > 0$, and hence $\frac{1}{U}$ and all of its derivatives are bounded. Moreover, $\frac{1}{f'(U)}$ is bounded under assumption (1.6).

3. Asymptotic Stability of Viscous Shock Waves

The second main result of our paper is the nonlinear stability of traveling wave solutions obtained in Theorem 2.1. We first define the appropriate initial perturbations. As in Refs. 12, 21, 22 and 34, we write the initial perturbation of traveling waves of the system (1.11) as

$$\int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x) - U(x) \\ v_0(x) - V(x) \end{pmatrix} dx = x_0 \begin{pmatrix} u^+ - u^- \\ v^+ - v^- \end{pmatrix} + \alpha \mathbf{r}_1(u^-, v^-), \tag{3.1}$$

where $\mathbf{r}_1(u^-, v^-)$ is the first eigenvector in (2.2) evaluated at (u^-, v^-). The coefficients x_0 and α are uniquely determined by the initial data $(u_0(x), v_0(x))$. For stability results of small amplitude waves corresponding to the case $\alpha \neq 0$, the reader is referred to Refs. 22 and 34 and references therein. In this paper, we assume that $\alpha = 0$ as in Refs. 6 and 12. Now by the conservation laws in (1.11) and (2.5), one readily derives that

$$\int_{-\infty}^{+\infty} \begin{pmatrix} u(x,t) - U(x + x_0 - st) \\ v(x,t) - V(x + x_0 - st) \end{pmatrix} dx$$
$$= \int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x) - U(x + x_0) \\ v_0(x) - V(x + x_0) \end{pmatrix} dx$$

$$
- \int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x) - U(x) \\ v_0(x) - V(x) \end{pmatrix} dx + \int_{-\infty}^{+\infty} \begin{pmatrix} U(x) - U(x+x_0) \\ V(x) - V(x+x_0) \end{pmatrix} dx
$$

$$
= \int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x) - U(x) \\ v_0(x) - V(x) \end{pmatrix} dx - x_0 \begin{pmatrix} u^+ - u^- \\ v^+ - v^- \end{pmatrix} = \mathbf{0}.
$$

We decompose the solution (u, v) of (1.11) as

$$
(u, v)(x, t) = (U, V)(x - st + x_0) + (\phi_x, \psi_x)(x, t), \tag{3.2}
$$

where

$$
\phi(x, t) = \int_{-\infty}^{x} (u(y, t) - U(y + x_0 - st)) dy,
$$

$$
\psi(x, t) = \int_{-\infty}^{x} (v(y, t) - V(y + x_0 - st)) dy
$$

for all $x \in \mathbb{R}$ and $t \geq 0$.

It then holds that

$$
\phi(\pm\infty, t) = 0, \quad \psi(\pm\infty, t) = 0 \quad \text{for all } t > 0.
$$

We further assume without loss of generality that the translation $x_0 = 0$, namely,

$$
\int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x) - U(x) \\ v_0(x) - V(x) \end{pmatrix} dx = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.3}
$$

The initial condition of the perturbation (ϕ, ψ) is then given by

$$
(\phi_0, \psi_0)(x) = \int_{-\infty}^{x} (u_0 - U, v_0 - V)(y) dy. \tag{3.4}
$$

The asymptotic stability of the traveling wave solutions means that $(\phi_x, \psi_x)(x, t) \to 0$ as $t \to \infty$. Before stating our result, we introduce a notation L_w^2 which denotes the weighted space of measurable function f so that for weight function $w \geq 0$, $\sqrt{w} f \in L^2$ with norm

$$
\|f\|_{L_w^2} = \left(\int w(x) |f(x)|^2 dx \right)^{1/2}.
$$

Then the main result of the asymptotic stability is the following.

Theorem 3.1. *Suppose that* (1.6) *holds. Let* $(U, V)(x - st)$ *be a viscous shock profile of* (1.11) *obtained in Theorem* 2.1. *Then there exists a constant* $\varepsilon_0 > 0$ *such that if* $\|u_0 - U\|_1 + \|v_0 - V\|_1 + \|(\phi_0, \psi_0)\| \leq \varepsilon_0$ *and that* $\alpha = 0$ *in* (3.1), *then the Cauchy problem* (1.11)–(1.12) *has a unique global solution* $(u, v)(x, t)$ *satisfying* $u(x, t) \geq \delta_0 > 0$ *for all* $x \in \mathbb{R}$, $t \geq 0$ *for some* $\delta_0 > 0$, *and*

$$
(u - U, v - V) \in C([0, \infty); H^1) \cap L^2([0, \infty); H^1).
$$

Furthermore, the solution has the following asymptotic nonlinear stability

$$\sup_{x\in\mathbb{R}} |(u,v)(x,t) - (U,V)(x-st)| \to 0 \quad \text{as } t \to +\infty. \tag{3.5}$$

Remark 3.1. The above nonlinear stability results hold true regardless of the strengths of the waves, i.e. the wave amplitude $|u^+ - u^-| + |v^+ - v^-|$ can be large, in contrast to the previous results related to the nonlinear stability of traveling waves to hyperbolic−parabolic systems, where various smallness conditions on wave strengths were imposed (see e.g. Refs. 6, 19, 24, 11, 21, 22 and 38).

Remark 3.2. When the system (1.11) is not genuinely nonlinear hyperbolic, i.e. when (2.4) does not hold, the nonlinear stability of the traveling waves can be established by weighted energy methods as in Ref. 19.

4. Reformulation of the Stability Problem

The proof of Theorem 3.1 is based on iterative L^2 energy estimates due to partial diffusion in the hyperbolic−parabolic system (1.11). The energy estimate approach for the nonlinear stability viscous shock profiles was first introduced independently by Matsumura and Nishihara in Ref. 24 and by Goodman in Ref. 6. It has been further developed over the years, see Refs. 12, 19, 21, 22 and 34.

In view of (3.2), we seek a solution of the following form

$$(u,v)(x,t) = (U,V)(x-st) + (\phi_x,\psi_x)(x,t) = (U,V)(z) + (\bar{\phi}_z,\bar{\psi}_z)(z,t) \tag{4.1}$$

with $(\bar{\phi},\bar{\psi})$ in some functional space which will be defined below. For simplicity of notation, we will omit the bars in $(\bar{\phi},\bar{\psi})$ in the rest of the paper.

Substituting (4.1) into (1.11), using (2.5), and integrating the resulting equations with respect to z, we derive that the perturbation (ϕ,ψ) satisfies

$$\begin{cases}\phi_t = D\phi_{zz} + s\phi_z + U\psi_z + V\phi_z + \phi_z\psi_z,\\ \psi_t = s\psi_z + f'(U)\phi_z + F(U,\phi_z)\end{cases} \tag{4.2}$$

with initial data given by

$$(\phi,\psi)(z,0) = (\phi_0,\psi_0)(z), \quad z \in \mathbb{R} \tag{4.3}$$

where (ϕ_0,ψ_0) is defined as in (3.4), and $F(U,\phi_z) = f(U+\phi_z) - f(U) - f'(U)\phi_z$.

We seek solutions of the reformulated problem (4.2)−(4.3) in the following solution space

$$\begin{aligned} X(0,T) = \{&(\phi(z,t),\psi(z,t)) : \phi \in C([0,T];H^2), \psi \in C([0,T];H^2) \cap C^1((0,T);H^1),\\ &\phi_z \in L^2((0,T);H^2), \psi_z \in L^2((0,T);H^1)\}\end{aligned}$$

with $0 \le T < +\infty$.

Let

$$N(t) = \sup_{0\le\tau\le t}\{\|\phi(\cdot,\tau)\|_2 + \|\psi(\cdot,\tau)\|_2\}, \tag{4.4}$$

where $\|\cdot\|_p$ denotes the H^p norm for $p > 0$ and for $t \ge 0$.

By the Sobolev embedding theorem, we have

$$\sup_{z\in\mathbb{R}}\{|\phi|, |\phi_z|, |\psi|, |\psi_z|\} \le N(t) \tag{4.5}$$

for $t \ge 0$.

Thus Theorem 3.1 is a consequence of the following theorem.

Theorem 4.1. *Let the assumptions of Theorem 3.1 hold. Then there exists a constant $\delta_1 > 0$ such that if*

$$N(0) \le \delta_1, \tag{4.6}$$

then the Cauchy problem (4.2)–(4.3) has a unique global solution $(\phi,\psi) \in X(0,+\infty)$ such that for any $t \in [0,+\infty)$, it holds that for some constant $C > 0$

$$\begin{aligned} &\|\phi(\cdot,t)\|_2^2 + \|\psi(\cdot,t)\|_2^2 + \int_0^t \|(\phi(\cdot,\tau),\psi(\cdot,\tau))\|_{L_w^2}^2 d\tau \\ &\quad + \int_0^t \|\phi_z(\cdot,\tau)\|_2^2 d\tau + \int_0^t \|\psi_z(\cdot,\tau)\|_1^2 d\tau \\ &\le CN^2(0), \end{aligned} \tag{4.7}$$

where $w = |U_z|$. Moreover, the following asymptotic stability holds

$$\sup_{z\in\mathbb{R}} |(\phi_z,\psi_z)(z,t)| \to 0 \quad \text{as } t \to +\infty. \tag{4.8}$$

With Theorem 4.1, we can immediately show the positivity of u claimed in Theorem 3.1. In fact, if the initial perturbation (4.3) satisfies (4.6), then by (4.7) there is a constant $C > 0$ such that

$$|\phi_z(z,t)| \le \sqrt{2}N(t) \le CN(0) \le C\delta_1.$$

Thus for all $x \in R$ and $t \ge 0$, it follows from (4.1) that

$$\begin{aligned} u(x,t) &= (u(x,t) - U(z)) + U(z) \\ &= \phi_z(z,t) + U(z) \ge -CN(0) + u^+ \\ &\ge -C\delta_1 + u^+ = \delta_0 > 0 \end{aligned}$$

provided that δ_1 is suitably small and u^+ is positive. Moreover, the bounded interval I in condition (2.4) can be chosen as $[\delta_0, u^- + C\delta_1]$ so that $u(x,t) \in I$ for all $x \in R$ and $t \ge 0$.

The global existence of (ϕ,ψ) announced in Theorem 4.1 follows from the local existence theorem and from the *a priori* estimates which are given below.

Proposition 4.2. (Local existence) *For any $\delta_2 > 0$, there exists a positive constant T depending on δ_2 such that if $(\phi_0,\psi_0) \in H^2$ with $N(0) \le \delta_2/2$, then the* *problem* (4.2)–(4.3) *has a unique solution* $(\phi, \psi) \in X(0, T)$ *satisfying*

$$N(t) < 2N(0) \tag{4.9}$$

for any $0 \leq t \leq T$.

Proposition 4.3. (*A priori* estimates) *Assume that* $(\phi, \psi) \in X(0, T)$ *is a solution obtained in Proposition* 4.2 *for a positive constant* T. *Then there is a positive constant* $\delta_3 > 0$, *independent of* T, *such that if*

$$N(t) < \delta_3$$

for any $0 \leq t \leq T$, *then the solution* (ϕ, ψ) *of* (4.2)–(4.3) *satisfies* (4.7) *for any* $0 \leq t \leq T$.

With the solution (ϕ, ψ) obtained in Theorem 4.1 and traveling wave solution (U, V) in Lemma 2.1, we have the desired solution of the problem (1.11)–(1.12) through relation (4.1).

The local existence in Proposition 4.2 can be shown in a standard way (cf. Ref. 26) and we omit the proof. Theorem 4.1 is a consequence of Propositions 4.2 and 4.3 by the continuation argument. So it remains to prove Proposition 4.3. The following section is devoted to the proof of Proposition 4.3 based on iterative L^2 energy estimates.

5. Energy Estimates

In this section, we derive the *a priori* estimates for the solutions of system (4.2)–(4.3) and prove Proposition 4.3. In what follows, we use C to denote a generic constant which changes from one line to another. An integral lacking limits of integration means an integral over the whole real line $\mathbb{R}$. For simplicity, we use $\|\cdot\|$ to denote the L^2 norm. We take advantage of the explicit formula (2.11) for U_z and V_z and avoid the smallness assumptions on the wave strengths as imposed for hyperbolic–parabolic systems in the literature (e.g. Refs. 11, 19, 24 and 38). The stability result is a consequence of the following *a priori* estimates.

Lemma 5.1. *Let f satisfy* (1.6) *and* $w = |U_z|$. *Assume* $(\phi_0, \psi_0) \in H^2$ *and let* (ϕ, ψ) *be a solution of* (4.2)–(4.3). *Then there exists a constant* $C > 0$ *such that*

$$\begin{aligned}
&\|\phi(\cdot,t)\|_2^2 + \|\psi(\cdot,t)\|_2^2 + \int_0^t \|(\phi(\cdot,\tau), \psi(\cdot,\tau))\|_{L_w^2}^2 d\tau \\
&\qquad + \int_0^t \|\phi_z(\cdot,\tau)\|_2^2 d\tau + \int_0^t \|\psi_z(\cdot,\tau)\|_1^2 d\tau \\
&\quad \leq C(\|\phi_0\|_2^2 + \|\psi_0\|_2^2) + C\int_0^t \int (|\phi| + |\phi_z| + |\psi_z| + |\phi_{zz}|)|\phi_z \psi_z| dz d\tau \\
&\qquad + C\int_0^t \int (|\phi_{zz}| + |\psi_{zz}| + |\phi_{zzz}|)|(\phi_z \psi_z)_z| dz d\tau
\end{aligned}$$

$$+ C\int_0^t \int |F|(|\psi| + |\psi_z| + |\psi_{zz}|)dzd\tau + C\int_0^t \int (|F| + |F_{zz}|)|\phi_z|dzd\tau$$
$$+ C\int_0^t \int (|F_z| + |F_{zz}|)|\psi_{zz}|dzd\tau. \tag{5.1}$$

The proof of Lemma 5.1 consists of a series of energy estimates which are given in the following subsections.

5.1. L^2-estimate

In this subsection, we shall derive the L^2 estimates for (ϕ, ψ).

Lemma 5.2. *Let the assumptions in Lemma 5.1 hold. Then there exist constants $\nu_0 > 0$ and $C > 0$ such that the solution (ϕ, ψ) of (4.2)–(4.3) satisfies*

$$\|\phi(\cdot, t)\|^2 + \|\psi(\cdot, t)\|^2 + \nu_0 \int_0^t \|(\phi(\cdot, \tau), \psi(\cdot, \tau))\|^2_{L^2_w} d\tau + \int_0^t \|\phi_z(\cdot, \tau)\|^2 d\tau$$
$$\leq C\left(\|\phi_0\|^2 + \|\psi_0\|^2 + \int_0^t \int |\phi\phi_z\psi_z|dzd\tau + C\int_0^t \int |F\psi|dzd\tau \right). \tag{5.2}$$

Proof. Multiplying the first equation of (4.2) by ϕ/U and the second by $\psi/f'(U)$, and adding them, we end up with the following equation after integrating the result w.r.t. z

$$\frac{1}{2}\frac{d}{dt}\int \left(\frac{\phi^2}{U} + \frac{\psi^2}{f'(U)}\right)dz + D\int \frac{\phi_z^2}{U}dz + \frac{1}{2}\int \delta(z)\phi^2 dz$$
$$= -\frac{s}{2}\int \left(\frac{1}{f'(U)}\right)_z \psi^2 dz + \int \frac{\phi\phi_z\psi_z}{U}dz + \int \frac{F\psi}{f'(U)}dz \tag{5.3}$$

where

$$\delta(z) = -\left(\frac{D}{U}\right)_{zz} + \left(\frac{s+V}{U}\right)_z.$$

From (2.5), (2.11), (2.12) and noticing that $U_z < 0$, we derive that

$$\delta(z) = -\nu(z)U_z = \nu(z)|U_z|,$$

where

$$\nu(z) = \frac{2}{U^3}\left[\left(s + \frac{\varrho_1 - f(U)}{s}\right)U + DU_z\right]. \tag{5.4}$$

Substituting s in (2.9), U_z in (2.11) and (2.12) into (5.4), one derives that

$$\nu(z) = \frac{2u^-}{U^3}(s + v^-) \geq \frac{1}{(u^-)^2}\left(v^- + \sqrt{(v^-)^2 + 4u^+\frac{f(u^+) - f(u^-)}{u^+ - u^-}}\right) = \nu_0 > 0,$$

where the assumption $f'(u) > 0$ in (1.6) has been used. Hence

$$\delta(z) \geq \nu_0 |U_z|. \tag{5.5}$$

Substituting (5.5) into (5.3), one has that

$$\frac{1}{2}\frac{d}{dt}\int\left(\frac{\phi^2}{U}+\frac{\psi^2}{f'(U)}\right)dz+D\int\frac{\phi_z^2}{U}dz+\frac{1}{2}\nu_0\int|U_z|\phi^2dz$$

$$\le -\frac{s}{2}\int\frac{f''(U)}{f'(U)^2}|U_z|\psi^2dz+\int\frac{\phi\phi_z\psi_z}{U}dz+\int\frac{F\psi}{f'(U)}dz. \tag{5.6}$$

Observe that under assumption (1.6), $\frac{f''(U)}{f'(U)^2}$ is bounded but may change sign. Thus the first term on the right-hand side of (5.6) needs to be estimated. To this end, one needs to estimate $\int|U_z|\psi^2dx$. We multiply the first equation of (4.2) by ϕ and the second equation by $\psi U/f'(U)$, add them, integrate the result w.r.t. z and notice that $U_z<0$, to obtain

$$\frac{1}{2}\frac{d}{dt}\int\left(\phi^2+\frac{U}{f'(U)}\psi^2\right)dz+D\int\phi_z^2dz+\frac{1}{2}\int V_z\phi^2dz$$

$$=\int|U_z|\phi\psi dz+\frac{s}{2}\int\frac{f'(U)-Uf''(U)}{f'(U)^2}|U_z|\psi^2dz+\int\phi\phi_z\psi_zdz$$

$$+\int\frac{U}{f'(U)}F\psi dz. \tag{5.7}$$

Multiplying (5.6) by u^- and noting $U\in[u^+,u^-]$, we derive that

$$\frac{1}{2}\frac{d}{dt}\int\left(\phi^2+\frac{U\psi^2}{f'(U)}\right)dz+D\int\phi_z^2dz+\frac{1}{2}\nu_0u^-\int|U_z|\phi^2dz$$

$$\le -\frac{su^-}{2}\int\frac{f''(U)}{f'(U)^2}|U_z|\psi^2dz+u^-\int\frac{\phi\phi_z\psi_z}{U}dz+u^-\int\frac{F\psi}{f'(U)}dz. \tag{5.8}$$

Since $f'(U)>0$ and f is smooth, $f'(U)$ is bounded away from 0 due to the fact that $0<u^+\le U\le u^-$. Then one has from (5.7) and (5.8) that

$$\frac{s}{2}\int\frac{f'(U)-Uf''(U)}{f'(U)^2}|U_z|\psi^2dz$$

$$=\frac{1}{2}\frac{d}{dt}\int\left(\phi^2+\frac{U}{f'(U)}\psi^2\right)dz$$

$$+D\int\phi_z^2dz+\frac{1}{2}\int V_z\phi^2dz-\int|U_z|\phi\psi dz-\int\phi\phi_z\psi_zdz$$

$$\le -\frac{su^-}{2}\int\frac{f''(U)}{f'(U)^2}|U_z|\psi^2dz+\frac{1}{2}\int V_z\phi^2dz$$

$$-\int|U_z|\phi\psi dz+\left(\frac{u^-}{u^+}+1\right)\int|\phi\phi_z\psi_z|dz+C\int|F\psi|dz \tag{5.9}$$

which leads to

$$\frac{s}{2}\int \frac{f'(U)+(u^- - U)f''(U)}{f'(U)^2}|U_z|\psi^2 dz \le \frac{1}{2s}\int f'(U)|U_z|\phi^2 dz + \int |U_z\phi\psi|dz + \left(\frac{u^-}{u^+}+1\right)\int |\phi\phi_z\psi_z|dz + C\int |F\psi|dz, \tag{5.10}$$

where we have used the second equation of (2.5)

$$V_z = \frac{-f'(U)U_z}{s} = |U_z|\frac{f'(U)}{s}$$

and the fact that $s > 0$, see (2.9).

Next we use the Cauchy–Schwarz inequality to estimate

$$\int |U_z\phi\psi|dz \le \frac{s}{4}\int \frac{f'(U)+(u^+ - U)f''(U)}{f'(U)^2}|U_z|\psi^2 dz + \frac{1}{s}\int \frac{f'(U)^2}{f'(U)+(u^+ - U)f''(U)}|U_z|\phi^2 dz. \tag{5.11}$$

Substituting (5.11) into (5.10), we have that

$$\begin{aligned}
&\frac{s}{4}\int \frac{f'(U)+(u^- - U)f''(U)}{f'(U)^2}|U_z|\psi^2 dz \\
&\le \frac{1}{s}\int \left(\frac{f'(U)^2}{f'(U)+(u^+ - U)f''(U)} + \frac{1}{2}f'(U)\right)|U_z|\phi^2 dz \\
&\quad + \left(\frac{u^-}{u^+}+1\right)\int |\phi\phi_z\psi_z|dz \\
&\le \frac{1}{s}\int \left(\frac{m_1^2}{m_2+(u^- - u^+)f''(U)} + \frac{1}{2}m_1\right)|U_z|\phi^2 dz \\
&\quad + \left(\frac{u^-}{u^+}+1\right)\int |\phi\phi_z\psi_z|dz \\
&\le \frac{1}{s}\int \left(\frac{m_1^2}{m_2-(u^- - u^+)\beta} + \frac{1}{2}m_1\right)|U_z|\phi^2 dz \\
&\quad + \left(\frac{u^-}{u^+}+1\right)\int |\phi\phi_z\psi_z|dz + C\int |F\psi|dz,
\end{aligned} \tag{5.12}$$

where

$$m_1 = \max_{u^+\le U\le u^-} f'(U), \quad m_2 = \min_{u^+\le U\le u^-} f'(U)$$

with $m_1 \ge m_2 > 0, 0 < u^+ \le U \le u^-$ and $\beta > 0$ is defined in (1.6).

We denote

$$M = \frac{m_1^2}{m_2 - (u^- - u^+)\beta} + \frac{1}{2} m_1.$$

Then

$$M \leq \frac{2m_1^2}{m_2} + \frac{m_1}{2} \tag{5.13}$$

provided that $\beta > 0$ in (1.6) satisfies

$$0 < \beta \leq \frac{m_2}{2(u^- - u^+)} := \beta_2. \tag{5.14}$$

On the other hand, we derive from (5.6) that

$$\begin{aligned} \frac{M}{s} \int_0^t \int |U_z| \phi^2 dx d\tau \leq \frac{M}{\nu_0 s} \int \left(\frac{\phi_0^2}{U} + \frac{\psi_0^2}{f'(U)} \right) dz \\ - \frac{M}{\nu_0} \int_0^t \int \frac{f''(U)}{f'(U)^2} |U_z| \psi^2 dz d\tau \\ + \frac{2M}{u^+ \nu_0 s} \int_0^t \int |\phi \phi_z \psi_z| dz d\tau + C \int_0^t \int |F\psi| dz d\tau. \end{aligned} \tag{5.15}$$

Substituting (5.15) into (5.12), one has

$$\begin{aligned} &\frac{1}{4} \int_0^t \int \frac{s(f'(U) + (u^- - U) f''(U)) + \frac{4M}{\nu_0} f''(U)}{f'(U)^2} |U_z| \psi^2 dz d\tau \\ &\leq C(\|\phi_0\|^2 + \|\psi_0\|^2) + C \int_0^t \int |\phi \phi_z \psi_z| dz d\tau + C \int_0^t \int |F\psi| dz d\tau. \end{aligned} \tag{5.16}$$

Note that (5.13), (5.14) and (1.6) imply that

$$\begin{aligned} &s(f'(U) + (u^- - U) f''(U)) + \frac{4M}{\nu_0} f''(U) \\ &> s m_2 - s(u^- - u^+)\beta - \frac{4}{\nu_0} \left(\frac{2m_1^2}{m_2} + \frac{m_1}{2} \right) \beta \\ &\geq \tilde{\delta}_0 > 0 \end{aligned} \tag{5.17}$$

provided that $\beta > 0$ in (1.6) satisfies

$$0 < \beta \leq \frac{s m_2 - \tilde{\delta}_0}{s(u^- - u^+) + \left(\frac{2m_1^2}{m_2} + \frac{m_1}{2} \right) \frac{4}{\nu_0}} := \beta_3 \tag{5.18}$$

for some $0 < \tilde{\delta}_0 < s m_2$.

Now assuming that (5.14) and (5.18) hold, then it follows from (5.16) and (5.17) that

$$\int_0^t\int |U_z|\psi^2 dzd\tau \le C\left(\|\phi_0\|^2+\|\psi_0\|^2+\int_0^t\int |\phi\phi_z\psi_z|dzd\tau\right)+C\int_0^t\int |F\psi|dzd\tau. \tag{5.19}$$

Substituting (5.19) back into (5.8), we obtain the desired inequality (5.2).

This completes the proof of Lemma 5.2. □

Regarding the value of parameter $\beta > 0$ in assumption (1.6), we can now choose $\beta > 0$ as follows

$$\beta = \min\{\beta_1,\beta_2,\beta_3\} > 0, \tag{5.20}$$

where $\beta_1 > 0, \beta_2 > 0$ and $\beta_3 > 0$ are given in (2.4), (5.14) and (5.18), respectively.

5.2. *H^1-estimates*

In this subsection, we shall derive the L^2 estimates for the first-order derivatives of the solution (ϕ,ψ).

Lemma 5.3. *Let f satisfy (1.6) and the assumptions in Theorem 3.1 hold. Then there exists a constant $C > 0$ such that the solution (ϕ,ψ) of (4.2)–(4.3) satisfies*

$$\begin{aligned}&\|\phi_z(\cdot,t)\|^2+\|\psi_z(\cdot,t)\|^2+\int_0^t\|\phi_{zz}(\cdot,\tau)\|^2d\tau+\int_0^t\|\psi_z(\cdot,\tau)\|^2d\tau\\&\quad\le C\left(\|\phi_0\|_1^2+\|\psi_0\|_1^2+\int_0^t\int|\phi_z\psi_z|(|\phi|+|\phi_z|+|\phi_{zz}|+|\psi_z|)dzd\tau\right)\\&\qquad+C\int_0^t\int|F|(|\psi|+|\psi_z|+|\psi_{zz}|+|\phi_z|)dzd\tau.\end{aligned} \tag{5.21}$$

Proof. Multiplying the first equation of (4.2) by $-\phi_{zz}/U$ and the second equation by $-\psi_{zz}/f'(U)$, and adding them, using (2.11), we obtain the following equation after integrating the result w.r.t. z

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\left(\frac{\phi_z^2}{U}+\frac{\psi_z^2}{f'(U)}\right)dz+D\int\frac{\phi_{zz}^2}{U}dz\\&\quad=\frac{s}{2}\int\frac{f''(U)}{f'(U)^2}|U_z|\psi_z^2dz-\int\left(\frac{1}{f'(U)}\right)_z\psi_t\psi_zdz-\int\left(\frac{1}{U}\right)_z\phi_t\phi_zdz\\&\qquad-\frac{1}{2}\int\left(s+\frac{\varrho_1+Uf'(U)-f(U)}{s}\right)\frac{U_z}{U^2}\phi_z^2dz\\&\qquad+\int\frac{\phi_z\phi_{zz}\psi_z}{U}dz-\int\frac{F\psi_{zz}}{f'(U)}dz.\end{aligned} \tag{5.22}$$

Next we estimate the first three terms on the right-hand side of (5.22). For convenience we denote these terms by

$$I_1 = \frac{s}{2}\int \frac{f''(U)}{f'(U)^2}|U_z|\psi_z^2 dz, \quad I_2 = -\int \left(\frac{1}{f'(U)}\right)_z \psi_t\psi_z dz, \quad I_3 = -\int \left(\frac{1}{U}\right)_z \phi_t\phi_z dz.$$

We first look at I_2. In fact, by using the second equation of (4.2) and the Young's inequality, one has

$$\begin{aligned} -\left(\frac{1}{f'(U)}\right)_z \psi_t\psi_z &= -s\frac{f''(U)}{f'(U)^2}|U_z|\psi_z^2 + \frac{f''(U)}{f'(U)}U_z\phi_z\psi_z + \frac{f''(U)}{f'(U)^2}FU_z\psi_z \\ &\le -\frac{3s}{4}\frac{f''(U)}{f'(U)^2}|U_z|\psi_z^2 + \frac{4}{s}f''(U)|U_z|\phi_z^2 + \frac{f''(U)}{f'(U)^2}FU_z\psi_z. \end{aligned} \tag{5.23}$$

Then by (5.23), we have that

$$I_1 + I_2 \le -\frac{s}{4}\int \frac{f''(U)}{f'(U)^2}|U_z|\psi_z^2 dz + \frac{4}{s}\int f''(U)|U_z|\phi_z^2 dz + \frac{f''(U)}{f'(U)^2}FU_z\psi_z. \tag{5.24}$$

Next we estimate the term I_3. To this end we use the first equation of (4.2) to have that

$$\begin{aligned} -\left(\frac{1}{U}\right)_z \phi_t\phi_z = &-\frac{D}{2}\left(\left(\frac{1}{U}\right)_z \phi_z^2\right)_z + \left(\frac{D}{2}\left(\frac{1}{U}\right)_{zz} - (s+V)\left(\frac{1}{U}\right)_z\right)\phi_z^2 \\ &- U\left(\frac{1}{U}\right)_z \phi_z\psi_z - \left(\frac{1}{U}\right)_z \phi_z^2\psi_z. \end{aligned}$$

From Remark 2.1, we know that $U \in [u^+, u^-]$ and $|U_z|$ is bounded. Thus condition (1.6) implies that $f'(U)$ and $f''(U)$ are bounded. Then using (2.14) and the Young's inequality, we derive that

$$\begin{aligned} -U\left(\frac{1}{U}\right)_z \phi_z\psi_z &= -\left(U\left(\frac{1}{U}\right)_z \phi_z\psi\right)_z - \frac{U_z}{U}\phi_{zz}\psi \\ &\quad + \left(\frac{1}{D}\left(s + \frac{\varrho_1 - f(U) - Uf'(U)}{s}\right) + \frac{U_z}{U}\right)\frac{U_z}{U}\phi_z\psi \\ &\le -\left(U\left(\frac{1}{U}\right)_z \phi_z\psi\right)_z + \frac{D}{2}\frac{\phi_{zz}^2}{U} + C(|U_z|\psi^2 + \phi_z^2) \end{aligned}$$

for some $C > 0$.

Using the boundedness of V, $(\frac{1}{U})_z$, $(\frac{1}{U})_{zz}$ and Young's inequality, we infer that

$$I_3 \le \frac{D}{2}\int \frac{\phi_{zz}^2}{U}dz + C\int (|U_z|\psi^2 + \phi_z^2)dz + C\int \phi_z^2|\psi_z|dz. \tag{5.25}$$

Now we substitute (5.24) and (5.25) into (5.22) to have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\left(\frac{\phi_z^2}{U}+\frac{\psi_z^2}{f'(U)}\right)dz+\frac{D}{2}\int\frac{\phi_{zz}^2}{U}dz \\
&\quad\leq -\frac{s}{4}\int\frac{f''(U)}{f'(U)^2}|U_z|\psi_z^2dz+C\int(|U_z|\psi^2+\phi_z^2)dz \\
&\qquad+C\int\phi_z^2|\psi_z|dz+C\int|\phi_z\phi_{zz}\psi_z|dz+C\int|F|(|\psi_z|+|\psi_{zz}|)dz, \qquad (5.26)
\end{aligned}$$

where we have used the boundedness of $U, f'(U), f''(U), |U_z|$, see Remark 2.1.

Next integrating (5.26) and using (5.2), we have

$$\begin{aligned}
&\|\phi_z(\cdot,t)\|^2+\alpha\|\psi_z(\cdot,t)\|^2+D\int_0^t\|\phi_{zz}(\cdot,\tau)\|^2d\tau \\
&\quad\leq \beta M\int_0^t\|\psi_z(\cdot,\tau)\|^2d\tau \\
&\qquad+C\left(\|\phi_0\|_1^2+\|\psi_0\|_1^2+\int_0^t\int(|\phi|+|\phi_z|+|\phi_{zz}|)|\phi_z\psi_z|dzd\tau\right) \\
&\qquad+C\int_0^t\int|F|(|\psi_z|+|\psi_{zz}|)dzd\tau, \qquad (5.27)
\end{aligned}$$

where

$$\alpha=\frac{1}{\max\limits_{u^+\leq u\leq u^-}f'(u)},\quad M=su^-\frac{\max\limits_{z\in\mathbb{R}}|U_z|}{\left|\min\limits_{u^+\leq u\leq u^-}f'(u)\right|^2}.$$

Now we proceed to estimate $\int_0^t\|\psi_z(\cdot,\tau)\|^2d\tau$. To this end, we multiply the first equation of (4.2) by ψ_z to obtain

$$\phi_t\psi_z=D\phi_{zz}\psi_z+s\phi_z\psi_z+U\psi_z^2+V\phi_z\psi_z+\phi_z\psi_z^2. \qquad (5.28)$$

On the other hand, from the second equation of (4.2), we can derive that

$$\phi_t\psi_z=(\phi\psi_z)_t-s(\phi\psi_z)_z-(\phi\phi_zf'(U))_z+s\phi_z\psi_z+f'(U)\phi_z^2. \qquad (5.29)$$

Equating (5.29) with (5.28) yields

$$U\psi_z^2=(\phi\psi_z)_t+f'(U)\phi_z^2-D\phi_{zz}\psi_z-V\phi_z\psi_z-\phi_z\psi_z^2-s(\phi\psi_z)_z-(\phi\phi_zf'(U))_z. \qquad (5.30)$$

Since $0<u^+\leq U\leq u^-$, $v^-\leq V\leq v^+$, we apply the Young's inequality to deduce that

$$\begin{aligned}
|-D\phi_{zz}\psi_z|&\leq\frac{u^+}{4}\psi_z^2+\frac{4D^2}{u^+}\phi_{zz}^2, \\
|V\phi_z\psi_z|&\leq\frac{u^+}{4}\psi_z^2+\frac{4(|v^-|+|v^+|)^2}{u^+}\phi_z^2.
\end{aligned} \qquad (5.31)$$

Inserting (5.31) into (5.30) and integrating the result w.r.t. z and t, one has

$$\begin{aligned}\frac{u^+}{2}\int_0^t\int \psi_z^2 dzd\tau &\le -\int \phi_z\psi dz - \int \phi_0\psi_{z,0}dz \\ &\quad + C\int_0^t\int \phi_z^2 dzd\tau + \frac{4D^2}{u^+}\int_0^t\int \phi_{zz}^2 dzd\tau \\ &\quad + C\int_0^t\int |\phi_z|\psi_z^2 dzd\tau + C\int_0^t\int |F\phi_z|dzd\tau. \qquad (5.32)\end{aligned}$$

Applying the Cauchy–Schwarz inequality to $-\int \phi_z\psi dz$ and employing (5.2), we conclude that

$$\begin{aligned}\int_0^t\int \psi_z^2 dzd\tau &\le C\left(\|\phi_0\|_1^2 + \|\psi_0\|_1^2 + \int_0^t\int |\phi_z\psi_z|(|\phi| + |\psi_z|)dzd\tau\right) \\ &\quad + \frac{8D^2}{|u^+|^2}\int_0^t\int \phi_{zz}^2 dzd\tau + C\int_0^t\int |F\psi|dzd\tau \\ &\quad + C\int_0^t\int |F\phi_z|dzd\tau. \qquad (5.33)\end{aligned}$$

Inserting (5.33) into (5.27), and choosing β such that $\beta \le \beta_4 = \frac{|u^+|^2}{8MD}$, we have

$$\begin{aligned}&\|\phi_z(\cdot,t)\|^2 + \|\psi_z(\cdot,t)\|^2 + \int_0^t\int \phi_{zz}^2 dzd\tau \\ &\le C\left(\|\phi_0\|_1^2 + \|\psi_0\|_1^2 + \int_0^t\int |\phi_z\psi_z|(|\phi| + |\phi_z| + |\phi_{zz}| + |\psi_z|)dzd\tau\right) \\ &\quad + C\int_0^t\int |F|(|\psi| + |\psi_z| + |\psi_{zz}| + |\phi_z|)dzd\tau. \qquad (5.34)\end{aligned}$$

The combination of (5.33) and (5.34) concludes the proof. □

5.3. H^2-estimates

In this subsection, we shall use the similar arguments as used in the preceding subsections to estimate the second-order derivatives of (ϕ, ψ).

Lemma 5.4. *Let f satisfy (1.6) and the assumptions in Theorem 3.1 hold. Then the solution (ϕ, ψ) of (4.2)–(4.3) satisfies*

$$\begin{aligned}&\|\phi_{zz}(\cdot,t)\|^2 + \|\psi_{zz}(\cdot,t)\|^2 + \int_0^t \|\phi_{zzz}(\cdot,\tau)\|^2 d\tau + \int_0^t \|\psi_{zz}(\cdot,\tau)\|^2 d\tau \\ &\le C(\|\phi_0\|_2^2 + \|\psi_0\|_2^2) + C\int_0^t\int (|\phi| + |\phi_z| + |\psi_z| + |\phi_{zz}|)|\phi_z\psi_z|dzd\tau \\ &\quad + C\int_0^t\int (|\phi_{zz}| + |\psi_{zz}| + |\phi_{zzz}|)|(\phi_z\psi_z)_z|dzd\tau\end{aligned}$$

$$
\begin{aligned}
&+C\int_0^t\int|F|(|\psi|+|\psi_z|+|\psi_{zz}|)dzd\tau \\
&+C\int_0^t\int(|F|+|F_{zz}|)|\phi_z|dzd\tau+C\int_0^t\int(|F_z|+|F_{zz}|)|\psi_{zz}|dzd\tau. \qquad (5.35)
\end{aligned}
$$

Proof. We multiply the first equation of (4.2) by $1/U$, differentiate the resulting equation with respect to z twice. Then we multiply the second equation of (4.2) by $1/f'(U)$ and then differentiate the result with respect to z twice. We end up with the following

$$
\begin{cases}
\left(\dfrac{1}{U}\phi_t\right)_{zz}=\left(\dfrac{D}{U}\phi_{zz}\right)_{zz}+\left(\dfrac{s}{U}\phi_z\right)_{zz}+\psi_{zzz}+\left(\dfrac{V}{U}\phi_z\right)_{zz}+\left(\dfrac{1}{U}\phi_z\psi_z\right)_{zz}, \\
\left(\dfrac{1}{f'(U)}\psi_t\right)_{zz}=s\left(\dfrac{\psi_z}{f'(U)}\right)_{zz}+\phi_{zzz}+\left(\dfrac{F}{f'(U)}\right)_{zz}.
\end{cases} \qquad (5.36)
$$

Then multiplying the first equation of (5.36) by ϕ_{zz}, the second equation by ψ_{zz}, adding them, and integrating the results, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\left(\frac{\phi_{zz}^2}{U}+\psi_{zz}^2\right)dz+\int\left\{\left(\frac{1}{U}\right)_{zz}\phi_t+2\left(\frac{1}{U}\right)_z\phi_{tz}\right\}\phi_{zz}dz \\
&\quad+\int\left\{\left(\frac{1}{f'(U)}\right)_{zz}\psi_t+2\left(\frac{1}{f'(U)}\right)_z\psi_{tz}\right\}\psi_{zz}dz \\
&=D\int\left(\frac{\phi_{zz}}{U}\right)_{zz}\phi_{zz}dz+s\int\left(\frac{\phi_z}{U}\right)_{zz}\phi_{zz}dz \\
&\quad+\int\left(\frac{V}{U}\phi_z\right)_{zz}\phi_{zz}dz+\int\left(\frac{1}{U}\phi_z\psi_z\right)_{zz}\phi_{zz}dz \\
&\quad+\int\left(\frac{s}{f'(U)}\psi\right)_{zz}\psi_{zz}dz+\int\left(\frac{F}{f'(U)}\right)_{zz}\psi_{zz}dz \\
&=: I_1+I_2+I_3+I_4+I_5+I_6. \qquad (5.37)
\end{aligned}
$$

By integration by parts, it can be readily verified that

$$
\begin{cases}
I_1=\dfrac{D}{2}\displaystyle\int\left(\frac{1}{U}\right)_{zz}\phi_{zz}^2dz-D\int\frac{1}{U}\phi_{zzz}^2dz, \\
I_2=-\dfrac{s}{2}\displaystyle\int\left(\frac{1}{U}\right)_{zzz}\phi_z^2dz+\frac{3s}{2}\int\left(\frac{1}{U}\right)_z\phi_{zz}^2dz, \\
I_3=-\dfrac{1}{2}\displaystyle\int\left(\frac{V}{U}\right)_{zzz}\phi_z^2dz+\frac{3}{2}\int\left(\frac{V}{U}\right)_z\phi_{zz}^2dz, \\
I_5=s\displaystyle\int\left(\frac{1}{f'(U)}\right)_{zz}\psi_z\psi_{zz}dz+\frac{3s}{2}\int\left(\frac{1}{f'(U)}\right)_z\psi_{zz}^2dz.
\end{cases} \qquad (5.38)
$$

Substituting (5.38) into (5.37) yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\left(\frac{\phi_{zz}^2}{U}+\psi_{zz}^2\right)dz+D\int\frac{1}{U}\phi_{zzz}^2dz\\
&\quad+\int\left\{\left(\frac{1}{U}\right)_{zz}\phi_t+2\left(\frac{1}{U}\right)_z\phi_{tz}\right\}\phi_{zz}dz\\
&\quad+\int\left\{\left(\frac{1}{f'(U)}\right)_{zz}\psi_t+2\left(\frac{1}{f'(U)}\right)_z\psi_{tz}\right\}\psi_{zz}dz\\
&\quad=-\frac{1}{2}\int\left(\frac{s+V}{U}\right)_{zzz}\phi_z^2dz+\frac{1}{2}\int\left\{3\left(\frac{s+V}{U}\right)_z+\left(\frac{D}{U}\right)_{zz}\right\}\phi_{zz}^2dz\\
&\quad+\int\left(\frac{1}{U}\phi_z\psi_z\right)_{zz}\phi_{zz}dz+s\int\left(\frac{1}{f'(U)}\right)_{zz}\psi_z\psi_{zz}dz\\
&\quad+\frac{3s}{2}\left(\frac{1}{f'(U)}\right)_z\psi_{zz}^2dz+\int\left(\frac{F}{f'(U)}\right)_{zz}\psi_{zz}dz.
\end{aligned}\tag{5.39}
$$

Now we estimate the last two terms on the left-hand side of (5.39). By substituting the first equation of (4.2), one can derive that

$$
\begin{aligned}
&\left\{\left(\frac{1}{U}\right)_{zz}\phi_t+2\left(\frac{1}{U}\right)_z\phi_{tz}\right\}\phi_{zz}\\
&\quad=\left(D\left(\frac{1}{U}\right)_z\phi_{zz}^2\right)_z+2(s+V)\left(\frac{1}{U}\right)_z\phi_{zz}^2\\
&\qquad+\left[(s+V)\left(\frac{1}{U}\right)_{zz}+2V_z\left(\frac{1}{U}\right)_z\right]\phi_z\phi_{zz}\\
&\qquad-U\left(\frac{1}{U}\right)_{zz}\psi_z\phi_{zz}+\left(\frac{1}{U}\right)_{zz}\phi_z\psi_z\phi_{zz}+2\left(U\left(\frac{1}{U}\right)_z\psi_z\phi_{zz}\right)_z\\
&\qquad-2U_z\left(\frac{1}{U}\right)_z\psi_z\phi_{zz}+2\left(\frac{1}{U}\right)_z(\phi_z\psi_z)_z\phi_{zz}.
\end{aligned}\tag{5.40}
$$

Using the second equation of (4.2), we infer that

$$
\begin{aligned}
&\left\{\left(\frac{1}{f'(U)}\right)_{zz}\psi_t+2\left(\frac{1}{f'(U)}\right)_z\psi_{tz}\right\}\psi_{zz}\\
&\quad=s\left(\frac{1}{f'(U)}\right)_{zz}\psi_z\psi_{zz}+f'(U)\left(\frac{1}{f'(U)}\right)_{zz}\phi_z\psi_{zz}
\end{aligned}
$$

$$+ 2s\left(\frac{1}{f'(U)}\right)_z \psi_{zz}^2 + 2\left(\frac{1}{f'(U)}\right)_z (f'(U)\phi_z)_z\psi_{zz}$$
$$+ \left(\frac{1}{f'(U)}\right)_{zz} F\cdot\psi_{zz} + 2\cdot\left(\frac{1}{f'(U)}\right)_z \cdot F_z\cdot\psi_{zz}. \tag{5.41}$$

Inserting (5.40) and (5.41) into (5.39), we obtain

$$\frac{1}{2}\frac{d}{dt}\int\left(\frac{\phi_{zz}^2}{U}+\psi_{zz}^2\right)dz + D\int\frac{1}{U}\phi_{zzz}^2dz$$
$$= -\frac{1}{2}\int\left(\frac{s+V}{U}\right)_{zzz}\phi_z^2dz$$
$$+ \frac{1}{2}\int\left\{3\left(\frac{s+V}{U}\right)_z + \left(\frac{D}{U}\right)_{zz} - 4(s+V)\left(\frac{1}{U}\right)_z\right\}\phi_{zz}^2dz$$
$$- \int\left[(s+V)\left(\frac{1}{U}\right)_{zz} + 2V_z\left(\frac{1}{U}\right)_z\right]\phi_z\phi_{zz}dz$$
$$- \int\left(\frac{1}{U}\right)_{zz}\phi_z\psi_z\phi_{zz}dz + \int\left\{U\left(\frac{1}{U}\right)_{zz} + 2U_z\left(\frac{1}{U}\right)_z\right\}\psi_z\phi_{zz}$$
$$- 2\int\left(\frac{1}{U}\right)_z(\phi_z\psi_z)_z\phi_{zz}dz + \int\left(\frac{1}{U}\phi_z\psi_z\right)_{zz}\phi_{zz}dz$$
$$- \frac{s}{2}\int\left(\frac{1}{f'(U)}\right)_z\psi_{zz}^2dz - \int f'(U)\left(\frac{1}{f'(U)}\right)_{zz}\phi_z\psi_{zz}dz$$
$$- 2\int\left(\frac{1}{f'(U)}\right)_z(f'(U)\phi_z)_z\psi_{zz}dz$$
$$+ \int\left(\frac{F}{f'(U)}\right)_{zz}\psi_{zz}dz - \int\left(\frac{1}{f'(U)}\right)_{zz}F\cdot\psi_{zz}dz$$
$$- 2\cdot\int\left(\frac{1}{f'(U)}\right)_z\cdot F_z\cdot\psi_{zz}dz. \tag{5.42}$$

It is worthwhile to note again that U and V as well as their derivatives are bounded. So we can apply Cauchy−Schwarz inequality to (5.42) to obtain

$$\frac{1}{2}\frac{d}{dt}\int\left(\frac{\phi_{zz}^2}{U}+\psi_{zz}^2\right)dz + D\int\frac{1}{U}\phi_{zzz}^2dz$$
$$\leq C\int(\phi_z^2+\phi_{zz}^2+\psi_z^2)dz - \frac{s}{4}\int\left(\frac{1}{f'(U)}\right)_z\psi_{zz}^2dz$$

$$- \int \left(\frac{1}{U}\right)_{zz} \phi_z \psi_z \phi_{zz} dz - 2 \int \left(\frac{1}{U}\right)_z (\phi_z \psi_z)_z \phi_{zz} dz$$
$$+ \int \left(\frac{1}{U} \phi_z \psi_z\right)_{zz} \phi_{zz} dz + C \int (|F| + |F_z| + |F_{zz}|)|\psi_{zz} dz. \qquad (5.43)$$

The last term on the right-hand side of (5.43) can be calculated as

$$\left(\frac{1}{U} \phi_z \psi_z\right)_{zz} \phi_{zz} = \left(\frac{1}{U}\right)_{zz} \phi_z \psi_z \phi_{zz} + \left(\frac{1}{U}\right)_z (\phi_z \psi_z)_z \phi_{zz}$$
$$+ \left(\frac{1}{U} (\phi_z \psi_z)_z \phi_{zz}\right)_z - \frac{1}{U} (\phi_z \psi_z)_z \phi_{zzz}. \qquad (5.44)$$

Note that $\left(\frac{1}{f'(U)}\right)_z = \frac{f''(U)}{f'(U)^2}|U_z|$ and $|U_z|, f'(U), f''(U)$ are bounded. Then substituting (5.44) into (5.43), integrating the result w.r.t. t and using (1.6), (5.2) and (5.21), we obtain

$$\begin{aligned} &\int (\phi_{zz}^2 + \psi_{zz}^2) dz + D \int_0^t \int \phi_{zzz}^2 dz d\tau \\ &\quad \leq C \int (\phi_{0,zz}^2 + \psi_{0,zz}^2) dz + \beta M \int_0^t \int \psi_{zz}^2 dz d\tau \\ &\qquad + C \int_0^t \int (|\phi| + |\phi_z| + |\psi_z| + |\phi_{zz}| + |\psi_{zz}|)|\phi_z \psi_z| dz d\tau \\ &\qquad + C \int_0^t \int |(\phi_z \psi_z)_z|(|\phi_{zz}| + |\phi_{zzz}|) dz d\tau \\ &\qquad + C \int (|F| + |F_z| + |F_{zz}|)|\psi_{zz}| dz. \end{aligned} \qquad (5.45)$$

To complete the proof, we need to estimate $\int_0^t \int \psi_{zz}^2 dz d\tau$. Toward this end, we multiply the first equation of (4.2) by ψ_{zzz} to obtain after some computations that

$$\begin{aligned} \phi_t \psi_{zzz} = &D(\phi_{zz} \psi_{zz})_z + (s \phi_z \psi_{zz})_z + (U \psi_z \psi_{zz})_z + (V \phi_z \psi_{zz})_z \\ &+ (\phi_z \psi_z \psi_{zz})_z - D \phi_{zzz} \psi_{zz} - (V_z \phi_z + (s + V) \phi_{zz}) \psi_{zz} \\ &- U \psi_{zz}^2 - U_z \psi_z \psi_{zz} - (\phi_z \psi_z)_z \psi_{zz}. \end{aligned} \qquad (5.46)$$

From the second equation of (4.2) one has that

$$\begin{aligned} \phi_t \psi_{zzz} = &(\phi \psi_{zzz})_t - s(\phi \psi_{zzz})_z + s(\phi_z \psi_{zz})_z - s(\phi_{zz} \psi_z)_z \\ &+ s \phi_{zzz} \psi_z - (\phi (f'(U) \phi_z)_{zz})_z + (\phi_z (f'(U) \phi_z)_z)_z \\ &- f'(U) \phi_{zz}^2 - (f'(U))_z \phi_z \phi_{zz} - (\phi F_{zz})_z + \phi_z F_{zz}. \end{aligned} \qquad (5.47)$$

Equating (5.46) with (5.47) and integrating the result w.r.t. z, we end up with

$$\begin{aligned}\int U\psi_{zz}^2dz &= -\frac{d}{dt}\int \phi\psi_{zzz}dz + s\int \phi_{zzz}\psi_z dz - \int f'(U)\phi_{zz}^2 dz\\&\quad - \int (f'(U))_z\phi_z\phi_{zz}dz - D\int \phi_{zzz}\psi_{zz}dz\\&\quad - \int (V_z\phi_z + (s+V)\phi_{zz})\psi_{zz}dz\\&\quad - \int U_z\psi_z\psi_{zz}dz - \int (\phi_z\psi_z)_z\psi_{zz}dz + \int \phi_z F_{zz}dz\\&\le \frac{d}{dt}\int \phi_z\psi_{zz}dz + C\int (\phi_z^2+\phi_{zz}^2+\psi_z^2)dz + \frac{4D^2}{u^+}\int \phi_{zzz}^2dz\\&\quad + \int \frac{u^+}{2}\psi_{zz}^2 - \int (\phi_z\psi_z)_z\psi_{zz}dz + \int |\phi_z F_{zz}|dz, \end{aligned}\tag{5.48}$$

where we have used Cauchy−Schwarz inequality.

Integrating (5.48) with respect to t and using the fact $0 < u^+ \le U \le u^-$, and choosing β such that $\beta \le \beta_4$, where β_4 is defined in Sec. 5.2 we have

$$\begin{aligned}\int_0^t\int \psi_{zz}^2dz &\le \frac{4}{u^+}\int \phi_z\psi_{zz}dz - \frac{4}{u^+}\int \phi_{0,z}\psi_{0,zz}dz\\&\quad + C\int_0^t\int (\phi_z^2+\phi_{zz}^2+\psi_z^2)dz - \int_0^t\int (\phi_z\psi_z)_z\psi_{zz}dz\\&\le C(\|\phi_0\|_2^2+\|\psi_0\|_2^2)\\&\quad + C\int_0^t\int (|\phi|+|\phi_z|+|\psi_z|+|\phi_{zz}|)|\phi_z\psi_z|dzd\tau\\&\quad + C\int_0^t\int (|\phi_{zz}|+|\psi_{zz}|+|\phi_{zzz}|)|(\phi_z\psi_z)_z|dzd\tau\\&\quad + C\int_0^t\int |F|(|\psi|+|\psi_z|+|\psi_{zz}|)dzd\tau\\&\quad + C\int_0^t\int (|F|+|F_{zz}|)|\phi_z|dzd\tau\\&\quad + C\int_0^t\int (|F_z|+|F_{zz}|)|\psi_{zz}|dzd\tau, \end{aligned}\tag{5.49}$$

where Cauchy–Schwarz inequality (5.2), (5.20) and (5.45) have been used. Finally the combination of (5.49) with (5.45) concludes the proof. □

Proof of Proposition 4.3. In fact it only remains to show that the *a priori estimate* (4.7) holds. To this end, we need to estimate the cubic nonlinear terms in (5.1). Indeed, by applying (4.4), (4.5), the Cauchy–Schwarz inequality, Taylor theorem and the Sobolev embedding theorems, all of these cubic terms can be bounded by $CN(t)(\int_0^t \|\phi_z(\cdot,\tau)\|_2^2 d\tau + \int_0^t \|\psi_z(\cdot,\tau)\|_1^2 d\tau)$ for some constant $C > 0$ and for $t \in [0,T]$. Then from Lemma 5.1 we have

$$
\begin{aligned}
N^2(t) &+ \int_0^t \|(\phi(\cdot,\tau),\psi(\cdot,\tau))\|_{L_w^2}^2 d\tau + \int_0^t \|\phi_z(\cdot,\tau)\|_2^2 d\tau + \int_0^t \|\psi_z(\cdot,\tau)\|_1^2 d\tau \\
&\le CN^2(0) + CN(t)\left(\int_0^t \|\phi_z(\cdot,\tau)\|_2^2 d\tau + \int_0^t \|\psi_z(\cdot,\tau)\|_1^2 d\tau\right)
\end{aligned}
$$

for $t \in [0,T]$ and for some constant $C > 0$.

Therefore, by using (4.9) and letting $N(t) \le 2N(0) < \frac{1}{2C} =: \delta_3$, we obtain the following estimate for any $t \in [0,T]$

$$
\begin{aligned}
N^2(t) &+ \int_0^t \|(\phi(\cdot,\tau),\psi(\cdot,\tau))\|_{L_w^2}^2 d\tau + \frac{1}{2}\int_0^t \|\phi_z(\cdot,\tau)\|_2^2 d\tau \\
&+ \frac{1}{2}\int_0^t \|\psi_z(\cdot,\tau)\|_1^2 d\tau \le CN^2(0)
\end{aligned}
$$

which gives the desired estimate (4.7). Thus the proof of Proposition 4.3 is complete. □

Proof of Theorem 4.1. Due to Proposition 4.3, it only remains to show (4.8). Indeed from global estimate (4.7), we derive

$$
\|(\phi_z(\cdot,t),\psi_z(\cdot,t))\|_1 \to 0 \quad \text{as } t \to +\infty. \tag{5.50}
$$

Consequently, for all $z \in \mathbb{R}$,

$$
\begin{aligned}
\phi_z^2(z,t) &= 2\int_{-\infty}^{z} \phi_z \phi_{zz}(y,t) dy \\
&\le 2\left(\int_{-\infty}^{+\infty} \phi_z^2 dy\right)^{1/2} \left(\int_{-\infty}^{+\infty} \phi_{zz}^2 dy\right)^{1/2} \to 0 \quad \text{as } t \to +\infty.
\end{aligned} \tag{5.51}
$$

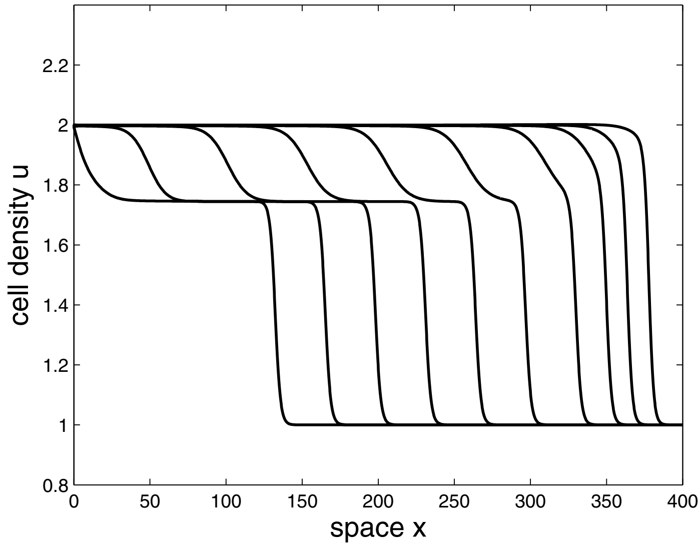
Applying the same argument to ψ_z leads to, for all $z \in \mathbb{R}$,

$$
\psi_z(z,t) \to 0 \quad \text{as } t \to +\infty.
$$

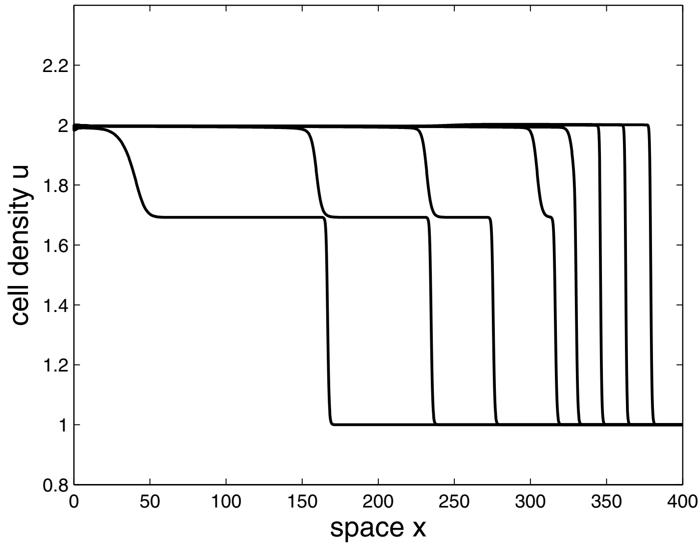
Hence (4.8) is proved.

6. Numerical Verifications

In this section, we will numerically verify the theoretical results obtained in foregoing sections. Since the numerical scheme is restricted to a finite domain, we consider the stability of traveling wave solutions under the perturbations that vanish outside the finite domain to mimic the real situation. The boundary conditions are set as Dirichlet conditions such that they are compatible with the initial data. We select



(a)



(b)

Fig. 1. The numerical illustration of the evolution of traveling wave solutions for system (1.11). Only profiles of u are displayed. The wave propagates from the left to the right. (a) A numerical solution u of (1.11), (6.1) with $D = 2, \chi = 1, \delta = 1$ and $f(u) = u$. The curves are plotted at time t equals to 50, 100, 150, 200, 250, 300, 350, 380, 400, 420. (b) A numerical solution of (1.11), (6.1) where $D = 1, \chi = 1, \delta = 1$ and $f(u) = u^2$. The curves are plotted at time t equals to 50, 100, 130, 160, 170, 180, 190, 200. The left state u^- and right state u^+ of cell density are set as 2 and 1, respectively. A subtle difference observed in these two figures is that the transition part of traveling wave from the left and right state in (a) is steeper than that in (b), which shows the fact that shock wave structure is more pronounced as the diffusion constant D becomes smaller.

various examples for function $f(u)$ such that condition (1.6) is satisfied. The MATLAB PDE package is used to perform numerical computations. The domain is set as $[0, 400]$ and mesh size is 0.2. We only plot the quantity of interest: cell density u.

The first initial datum is chosen as

$$\begin{aligned} u_0(x) &= \tilde{u}(x) = 1 + \frac{1}{1+\exp(2(x-100))}, \\ v_0(x) &= \tilde{v}(x) = 1 + \frac{1}{1+\exp(-2(x-100))}. \end{aligned} \tag{6.1}$$

The left and right states of the traveling wave solution (U, V) are

$$(u^-, v^-) = (2, 1), \quad (u^+, v^+) = (1, 2). \tag{6.2}$$

The amplitude of the wave is not small.

The first example of $f(u)$ satisfying condition (1.6) is chosen to be $f(u) = u$. For initial data given in (6.1), the evolution of a traveling wave solution of (1.11) is plotted in Fig. 1(a). We now perturb the initial data (6.1) in the form of

$$(u_0 - \tilde{u})(x) = (v_0 - \tilde{v})(x) = \frac{\sin x}{((x-100)/10)^2 + 1} \tag{6.3}$$

so that the perturbation is in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ as required by Theorem 3.1. With the numerical simulation shown in Fig. 2, we observe that the solution subject to such a perturbation (6.3) stabilizes to a traveling wave solution.

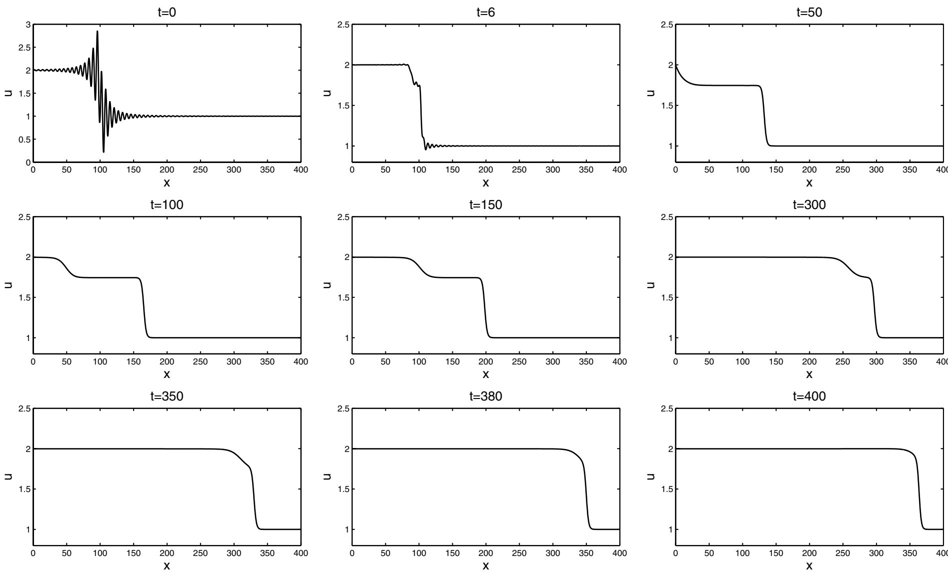


Fig. 2. The numerical illustration of the stability of the traveling wave solution u for system (1.11), (6.3) with $f(u) = u$ satisfying the condition (1.6), where $D = 2, \chi = 1, \delta = 1$.

The next example of $f(u)$ satisfying condition (1.6) is chosen to be $f(u) = u^2$. For the initial data given in (6.1), Fig. 1(b) shows the evolution of a numerical traveling wave. Then we perturb (6.1) such that

$$(u_0 - \tilde{u})(x) = (v_0 - \tilde{v})(x) = \sin x \exp(-0.0008(x-100)^2). \tag{6.4}$$

It is evident that such a perturbation is in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$. Figure 3 shows the stability of the traveling wave solution of system (1.11) under such a perturbation (6.4). It is clearly observed that the solution converges to a traveling wave as time increases. Lastly, we provide another example $f(u) = u + 0.1\sin(u)$ which satisfies condition (1.6) and $f''(u)$ changes the sign. The initial perturbation is given as

$$(u_0 - \tilde{u})(x) = (v_0 - \tilde{v})(x) = \cos x \exp(-0.0008(x-100)^2) \tag{6.5}$$

which is also in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$. Figure 4 illustrates the stability of the traveling wave solution of (1.11) under the perturbation (6.5).

The numerical simulations illustrated above show that traveling wave solutions of system (1.11) are stable for various functions $f(u)$ satisfying (1.6) under different perturbations, which confirms our theoretical results. Moreover, the numerical simulations show that the stability result still holds true without the restriction, $\alpha = 0$ in (3.1) and that the traveling waves are also stable under large perturbations, which are however not proved analytically in the paper.

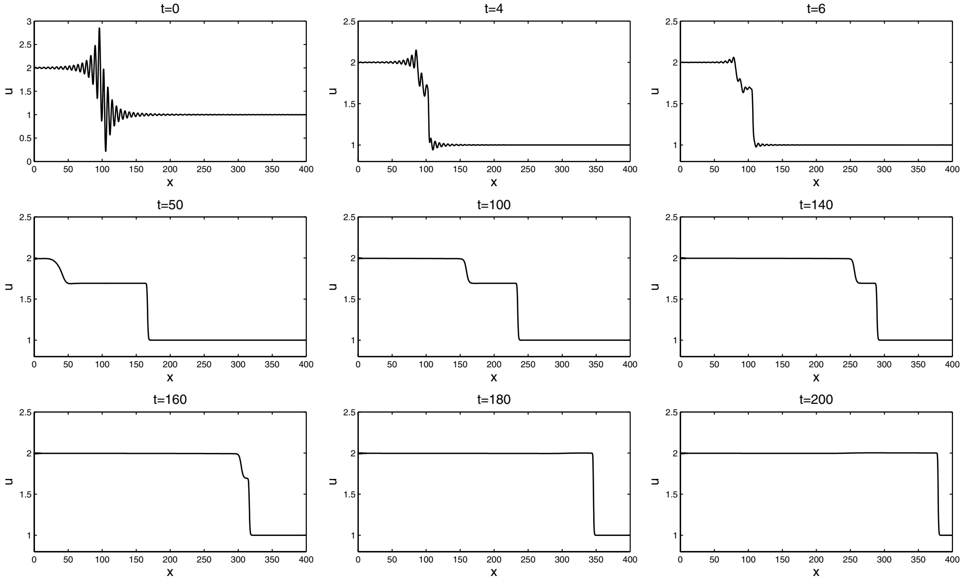


Fig. 3. The numerical illustration of the stability of the traveling wave solution u for system (1.11), (6.4), where $D = 1, \chi = 1, \delta = 1$ and $f(u) = u^2$. The left state u^- and right state u^+ of cell density are set as 2 and 1, respectively.

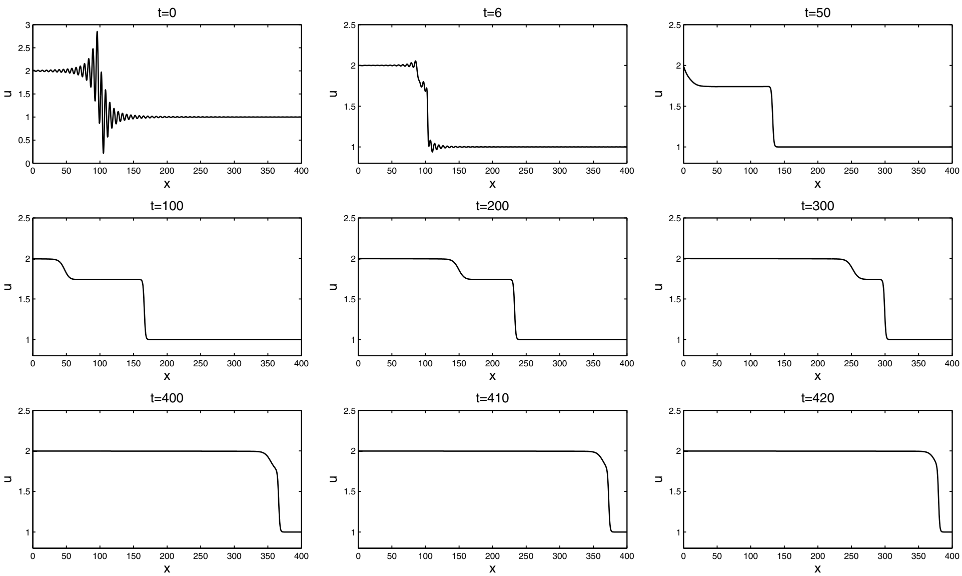


Fig. 4. The numerical simulations of the stability of the traveling wave solution u for system (1.11), (6.5) with $f(u) = u + 0.1\sin(u)$, where $D = 4, \chi = 1, \delta = 1$.

7. Conclusion

In this paper, we established the nonlinear stability of traveling wave (viscous shock) solutions with arbitrary wave amplitudes for a Keller–Segel type model (1.7) which describes the chemotactic movement of cells/organisms along the concentration gradient of the chemical which is non-diffusible. The L^2 energy estimates and theory of nonlinear conservation laws were applied to prove the results. Numerical simulations are performed which agree with the analytical results. Our results are applicable to both attractive ($\chi > 0$ and $\delta < 0$) and repulsive ($\chi < 0$ and $\delta > 0$) cases. We considered a more general chemical kinetic function $g(u, c)$ than that in the literature[18;27,36,38] and generalized the results in Ref. 20 where the function $f(u)$ is linear. The model considered in this paper has applications in cancer modeling[3–5,17] and self-organization of myxobacteria[27] as well as angiogenesis.[18] However, the case $\delta\chi > 0$ was not discussed in this paper. Our preliminary numerical simulations (not shown here) show that the traveling wave solutions are unstable under small perturbations. Hence the instability of traveling wave solutions when $\delta\chi > 0$ is expected. We shall study this in the future.

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