Research Article

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Logistic damping effect in chemotaxis models with density-suppressed motility

https://doi.org/10.1515/anona-2022-0263
received December 23, 2021; accepted July 14, 2022

Abstract: This paper is concerned with a parabolic-elliptic chemotaxis model with density-suppressed motility and general logistic source in an n-dimensional smooth bounded domain with Neumann boundary conditions. Under the minimal conditions for the density-suppressed motility function, we explore how strong the logistic damping can warrant the global boundedness of solutions and further establish the asymptotic behavior of solutions on top of the conditions.

Keywords: chemotaxis, density-suppressed motility, logistic source, global boundedness, asymptotic behavior

MSC 2020: 35A01, 35B35, 35B45, 35B51, 35Q92

1 Introduction and main results

To explain the aggregation phase of *Dictyostelium discoideum* cells in response to the secreted chemical signal cyclic adenosine monophosphate, Keller and Segel [18] proposed the following well-known system:

$$\begin{cases} u_t = \nabla\cdot(\gamma(v)\nabla u - u\phi(v)\nabla v), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \tag{1.1}$$

in 1971, where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with smooth boundary, u denotes the cell density, and v is the concentration of the chemical signal emitted by cells. The parameter $\tau \in \{0, 1\}$ represents the relaxation time, that is, the rate of the time scale of v relative to u. $\gamma(v)$ and $\phi(v)$ are motility functions representing diffusion and chemotactic coefficients, respectively, and both of them depend on the chemical signal concentration linked through the following relation:

$$\phi(v) = (\alpha - 1)\gamma'(v)$$

and α denotes the ratio of effective body length (i.e., distance between receptors) to step size. Of particular interest is the case $\alpha = 0$, namely, the distance between receptors is zero, where the chemotaxis occurs because of an undirected effect on activity due to the presence of a chemical sensed by a single receptor (cf. [18, p. 228]), the aforementioned system is reduced to

$$\begin{cases} u_t = \Delta(\gamma(v)u), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega,\ t > 0. \end{cases} \tag{1.2}$$

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When $\gamma(v)$ and $\phi(v)$ are constant, the model (1.1) is called the minimal chemotaxis model (cf. [26]), which has been extensively studied in the literature, where a prominent phenomenon is the existence of critical mass and critical dimension for the blowup of solutions. For a broad overview on various types of chemotactic processes and relevant mathematical results, we refer the readers to the survey papers [12,13,30,31] and the references therein. When a logistic damping source is included to the minimal Keller-Segel model, there are many studies on the question whether and how the logistic damping may preclude the blowup of solutions. Since there are many works in this direction and we only mention a few of early ones [27,33,39,40,42] exploring this question and refer the reader to a survey paper [5].

In contrast to abundant results obtained for the minimal chemotaxis system, the progresses made to the original Keller-Segel model (1.1) with nonconstant $\gamma(v)$ and $\phi(v)$ are very limited. To describe the stripe pattern formation of bacterial movement observed in the experiment of [20], a so-called density-suppressed motility model as below was proposed in [20] (see the Supplemental material of [20]) and formally analyzed in [8]

$$\begin{cases} u_t = \Delta(\gamma(v)u) + \mu u(1-u), & x \in \Omega, \ t > 0, \\ \tau v_t = \Delta v + u - v, & x \in \Omega, \ t > 0, \end{cases} \tag{1.3}$$

where the parameter $\mu > 0$ denotes the intrinsic cell growth rate, and $\gamma'(v) < 0$ accounting for the repressive effect of chemical signal concentration on the cell motility (cf. [20]). Of interest is that (1.3) with $\mu = 0$ coincides with the simplified Keller-Segel model (1.2). Indeed the density-suppressed motility mechanism has been previously used to model other biological processes, such as the predator-prey system describing the inhomogeneous co-existence distributions of ladybugs (predators) and aphids (prey) populations in the field (cf. [17,37]). Under the condition $\gamma'(v) < 0$ (i.e., density-suppressed motility), the main challenge of the analysis lies in the possible diffusion degeneracy since $\gamma(v)$ could have no positive lower bound. Therefore, many conventional methods for reaction–diffusion equations or chemotaxis models are inapplicable. The dynamics of the aforementioned models with density-suppressed motility have not been well understood until recently. Below we shall give a brief review of existing results on (1.1)–(1.3) and then raise our question to explore. In what follows, we shall always assume the homogenous Neumann boundary conditions unless otherwise stated.

Case of $\mu > 0$. It was first shown in [16] when the motility function $\gamma(v)$ satisfies $\gamma(v) \in C^3([0, +\infty))$, $\gamma(v) > 0, \quad \gamma'(v) < 0 \quad \text{on} \quad [0, \infty)$, $\lim_{v\to+\infty}\gamma(v) = 0$ and $\frac{|\gamma'(v)|^2}{\gamma(v)}$ is bounded for all $v > 0$, system (1.3) with $\tau = 0$ has a unique global classical solution in two dimensions ($n = 2$) which globally asymptotically converges to the equilibrium $(1, 1)$ if $\mu > \frac{K_0}{16}$ with $K_0 = \max_{0\le v\le+\infty}\frac{|\gamma'(v)|^2}{\gamma(v)}$. Later, similar results were extended to higher dimensions ($n \ge 3$) for large $\mu > 0$ in [19] and [35]. The condition that $\frac{|\gamma'(v)|^2}{\gamma(v)}$ is bounded for all $v > 0$ was later removed in [10] for the parabolic-elliptic case model (i.e., $\tau = 0$). On the other hand, for small $\mu > 0$, the existence/nonexistence of nonconstant steady states of (1.3) was rigorously established in [25] in appropriate parameter regimes, and the periodic pulsating wave is analytically obtained by the multiscale analysis.

Case of $\mu = 0$. The analysis for the case $\mu = 0$ was much more delicate due to the loss of logistic damping. It was first shown in [32] that globally bounded solutions exist in two dimensions if the motility function $\gamma(v)$ has both positive lower and upper bounds. However, $\gamma(v)$ may not have priori positive lower/upper bound, for example, $\gamma(v) = \frac{c_0}{v^k}$, for which it was proved in [44] that global bounded solutions exist in all dimensions for any $k > 0$ provided that $c_0 > 0$ is small. The smallness of c_0 is later removed in [3] for the parabolic-elliptic case model (i.e., $\tau = 0$ in (1.3)) for $0 < k < \frac{2}{(n-2)_+}$. The global existence of weak solutions of (1.3) with large initial data was established in [7] for $\gamma(v) = \frac{1}{c+v^k}$ with $c \ge 0, \quad k > 0$ and $n = 1$ or $k \in (0, 2)$ and $n = 2$ or $k \in \left(0, \frac{3}{4}\right)$ and $n = 3$. When $\gamma(v) = e^{-\chi v}$, a critical mass phenomenon was identified (cf. [36]): If $n = 2$, there is a critical number $m = 4\pi/\chi > 0$ such that the solution of (1.3) with $\tau = 1$ may blow up if the initial cell mass $\|u_0\|_{L^1(\Omega)} > m$, while global bounded solutions exist if $\|u_0\|_{L^1(\Omega)} < m$. This result was further refined in [10,11] showing that the blowup occurs at the infinity time. It was also proved in [9,36,38] that global bounded classical solutions exist in all dimensions with some additional conditions on $\gamma(v)$. Recently global weak solutions of (1.3) in all dimensions, and the blow-up of solutions of (1.3) in two dimensions was investigated in [6].

We mention that in addition to the two-component density suppressed motility model (1.3), a three-component density-suppressed motility model was also proposed in [20] and has been studied recently in [14,15,24,43]. Except the studies on the bounded domain with zero Neumann boundary conditions, there are some results obtained in the whole space $\mathbb{R}$. When $\gamma(v)$ is a piecewise constant function, the dynamics of discontinuity interface of (1.3) was studied in [29], and discontinuous traveling wave solutions were constructed in [21].

The present work is motivated in the following way. When $\mu = 0$, the solution of (1.3) (i.e., the model (1.2)) may blow up in two dimensions with the motility function $\gamma(v)$ decaying exponentially (cf. [10,11,36]), while the blowup is immediately prevented by the quadratic logistic damping $\mu u(1-u)$ with $\mu > 0$ (cf. [10,16]). Hence, we ask how strong the logistic damping is actually adequate to preclude the blowup of solutions for the density-suppressed motility function satisfying only the basic (minimal) assumptions in multidimensions. This question amounts to consider the system (1.3) by replacing $\mu u(1-u)$ with $\mu(u-u^\sigma)$ and explore how large the damping exponent $\sigma > 0$ can ensure the global boundedness of solutions. Therefore, we are motivated to consider the following system:

$$\begin{cases} u_t = \Delta(\gamma(v)u) + au - bu^\sigma, & x \in \Omega,\ t > 0, \\ -\Delta v + v = u, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

where constants $a > 0$, $b > 0$, $\sigma > 1$, and ν is the unit outer normal vector of $\partial\Omega$.

We imposed the minimal structural assumptions on the density-suppressed motility function:

$$\gamma \in C^3([0,+\infty)), \quad \gamma(v) > 0, \quad \gamma'(v) < 0 \quad \text{on } [0,+\infty), \tag{1.5}$$

and assume that the initial value u_0 satisfies

$$u_0 \in W^{1,\infty}(\Omega), \quad u_0 \geq 0 \quad \text{and} \quad u_0 \not\equiv 0. \tag{1.6}$$

Then we shall investigate under what conditions on the damping exponent σ associated with the dimension n, the global boundedness of solutions to (1.4) is ensured, and further find the asymptotic behavior of solutions as time tends to infinity.

For the convenience of presentation, we let

$$Q = Q(b,\sigma) = \|(-\Delta + I)^{-1}u_0\|_{L^\infty(\Omega)} + \frac{\sigma-1}{\gamma(0)b^{\frac{1}{\sigma-1}}}\left(\frac{a+2\gamma(0)}{\sigma}\right)^{\frac{\sigma}{\sigma-1}}, \tag{1.7}$$

and $G := G(p) > 0$ be a constant satisfying the following Gagliardo-Nirenberg inequality

$$\|\nabla v\|_{L^{2(p+1)}(\Omega)} \leq G(p)\|v\|^{\frac{1}{2}}_{W^{2,p+1}(\Omega)}\|v\|^{\frac{1}{2}}_{L^\infty(\Omega)}.$$

By applying the Agmon-Douglis-Nirenberg regularity theorem (cf. [1,2]) to the elliptic equation $-\Delta v + v = u$, we find a constant $R := R(p) > 0$ such that

$$\|v\|_{W^{2,p}(\Omega)} \leq R(p)\|u\|_{L^p(\Omega)}.$$

We write

$$\kappa = \kappa(p,b,\sigma) = (G^{2(p+1)}(p)R^{p+1}(p)Q^{p+1}(b,\sigma) + 1)\sup_{0 \leq v \leq Q(b,\sigma)} \frac{|\gamma'(v)|^2}{\gamma(v)}, \tag{1.8}$$

and

$$b_1 := \frac{\left[\frac{n}{2}\right]\kappa\left(\left[\frac{n}{2}\right]+1,1,2\right)}{2}+1.$$

Then our main results are stated as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Assume $\gamma(v)$ satisfies the hypothesis* (1.5), *and the initial value satisfies the condition* (1.6). *Then if one of the following conditions holds*:
(1) $n \le 2, \sigma > 1$;
(2) $n \ge 3, \sigma > 2$;
(3) $n \ge 3, \sigma = 2$, *and* $b > b_1$,

then there exists a couple (u, v) *of functions*:

$$u \in C^0(\bar{\Omega} \times [0, +\infty)) \cap C^{2,1}(\bar{\Omega} \times (0, +\infty)), \quad v \in C^{2,1}(\bar{\Omega} \times (0, +\infty)),$$

which solves (1.4) *in the classical sense and* $u, v > 0$ *in* $\Omega \times (0, +\infty)$. *Moreover, the solution of* (1.4) *is uniformly bounded in* $\Omega \times (0, +\infty)$; *namely, there exists a constant* $C > 0$ *independent of t such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leqslant C \quad \text{for all } t > 0.$$

Remark 1.1. The results of Theorem 1.1 indicate that if $\sigma > 2$ (strong damping), then the minimal assumption (1.5) on the motility function is sufficient to warrant the global boundedness of solution to (1.4) in all dimensions. If $1 < \sigma < 2$ (weak damping), then the minimal assumption (1.5) can guarantee the global boundedness of solutions for $n \le 2$. If $n = 2$, $\gamma(v) = e^{-\lambda v}$, and $a = b = 0$, a critical mass $m_0 = \frac{4\pi}{\lambda}$ exists and solution may blow up at infinite time if the initial cell mass is greater than m_0 (cf. [10,11,36]). Therefore, Theorem 1.1 entails that the mere logistic damping ($\sigma > 1$) can preclude the blowup in two dimensions, in contrast to the results in [10, 16] with $\sigma = 2$. In [22,23], the global boundedness of solutions to the parabolic-parabolic version of (1.4) in all dimensions was obtained for $\sigma > \max\{2, \frac{n+2}{2}\}$. Clearly, these results are improved by our results in Theorem 1.1 in the parabolic-elliptic case model.

Remark 1.2. In our assumption (1.5) for the motility function $\gamma(v)$, we have ruled out the possibility that $\gamma(v)$ is singular at $v = 0$, for instance, $\gamma(v) = v^{-\alpha}$ with $\alpha > 0$, since we are unable to prove that v has a positive lower bound for any $t > 0$ in this case. Hence, our results cannot cover such singular $\gamma(v)$.

Next, we investigate the large time behavior of solutions to (1.4). For convenience, let

$$u_* := \left(\frac{a}{b}\right)^{\frac{1}{\sigma-1}}, \quad \bar{v} := \frac{1}{|\Omega|}\int_\Omega v, \tag{1.9}$$

$$K_1 := \begin{cases} \frac{(2|\Omega|)^{\frac{\sigma+1}{3-\sigma}}2^{2-\sigma}}{\sigma-1} + \frac{2^{\frac{4}{3-\sigma}}(2|\Omega|)^{2-\sigma}}{1-2^{-(\sigma-1)}}, & \text{if } \sigma < 2, \\ 1, & \text{if } \sigma \ge 2, \end{cases} \tag{1.10}$$

and

$$K_2 := \begin{cases} \xi^2, & \text{if } \sigma < 2, \\ 1, & \text{if } \sigma \ge 2, \end{cases} \tag{1.11}$$

where $\xi > 0$ is a Poincaré constant satisfying the following inequality:

$$\|v - \bar{v}\|_{L^d(\Omega)} \le \xi\|\nabla v\|_{L^2(\Omega)}, \quad d \in \left[1, \frac{2n}{(n-2)_+}\right).$$

Then we write

$$b_2 := \left(\frac{K_1K_2}{4}\right)^{\frac{\sigma-1}{2}} a^{-\frac{\sigma-3}{2}}\left(\sup_{v\geq 0}\frac{|\gamma'(v)|^2}{\gamma(v)}\right)^{\frac{\sigma-1}{2}}. \tag{1.12}$$

Theorem 1.2. *Let the assumptions in* Theorem 1.1 *hold. If one of the following conditions holds*:
(1) $n \leq 2$, $\sigma > 1$ *and* $b > b_2$
(2) $n \geq 3$, $\sigma > 2$ *and* $b > b_2$
(3) $n \geq 3$, $\sigma = 2$ *and* $b > \max\{b_1, b_2\}$
(4) $n \geq 3$, $2 - \frac{2}{n} < \sigma < 2$, $b > b_2$ *and* $\gamma(v)$ *satisfies* $\inf_{v\geq 0}\frac{\gamma(v)\gamma''(v)}{|\gamma'(v)|^2} > \frac{n}{2}$,

then the solution of (1.4) *obtained in* Theorem 1.1 *satisfies*

$$\|u(\cdot,t) - u_*\|_{L^\infty} \to 0 \quad \text{and} \quad \|v(\cdot,t) - u_*\|_{L^\infty} \to 0 \quad \text{as } t \to +\infty. \tag{1.13}$$

The key of proving Theorem 1.1 is to derive that v has an upper bound to rule out the possible diffusion degeneracy (see Section 2.3). Inspired by an idea from the work [10], we use the maximum principle for the inverse operator $(I - \Delta)^{-1}$ to construct a simple differential inequality on v, which yields an upper bound of v. In deriving the *a priori* estimates to obtain the global solutions, the case $\sigma > 2$ is easier to handle due to the strong damping effect, while in the case $1 < \sigma \leq 2$, we shall further exploit the structure of the equation governing v. To study the large time behavior of solutions, the Lyapunov functional method and Poincaré's inequality are essentially used.

The rest of this paper is organized as follows. In Section 2, we first state the local existence of solutions to (1.4) with an extensibility condition, and derive a criterion of global boundedness of solutions. Then we deduce some *a priori* estimates to meet the criterion and obtain the global bounded solutions of (1.4) in Section 3. Finally, we show the large time behavior of solutions to (1.4) in Section 4.

2 Preliminaries

This section is devoted to introducing some preliminary results for proving the global boundedness of solutions to (1.4), including the existence of local solutions and extensibility criterion for (1.4) as well as some basic estimates. In the following, we shall use C_i $(i = 1, 2, \ldots)$ to denote a generic positive constant, which may vary in the context. For simplicity, we shall abbreviate $\int_\Omega f\,\mathrm{d}x$ as $\int_\Omega f$ and $\int_0^t\int_\Omega f\,\mathrm{d}x\mathrm{d}s$ as $\int_0^t\int_\Omega f$ without confusion.

2.1 Local existence and extensibility criterion

First, we present the existence of local solutions and extensibility criterion for (1.4), which can be proved by the fixed-point theorem directly.

Lemma 2.1. (Local existence) *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with smooth boundary*, $a > 0$, $b > 0$, $\sigma > 1$ *and assume* $\gamma(v)$ *satisfies the hypothesis* (1.5). *If the initial data satisfy the condition* (1.6), *then there exists* $T_{\max} \in (0, +\infty]$ *such that* (1.4) *admits a unique classical solution* (u, v) *satisfying*

$$u \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})),$$
$$v \in C^{2,1}(\bar{\Omega} \times (0, T_{\max})),$$

and $u, v > 0$ *in* $\Omega \times (0, T_{\max})$. *Moreover, if* $T_{\max} < +\infty$, *then*

$$\limsup_{t\nearrow T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = +\infty.$$

Below is a uniform Grönwall inequality [34], which will help us to derive the uniform in-time estimates of solutions.

Lemma 2.2. *Let $T_{\max} > 0$, $\tau \in (0, T_{\max})$. Suppose that c_1, c_2, y are three positive locally integrable functions on $(0, T_{\max})$ such that y' is locally integrable on $(0, T_{\max})$ and satisfies*

$$y'(t) \le c_1(t)y(t) + c_2(t) \quad \text{for all } t \in (0, T_{\max}).$$

If

$$\int_t^{t+\tau} c_1 \le C_1, \quad \int_t^{t+\tau} c_2 \le C_2, \quad \int_t^{t+\tau} y \le C_3 \quad \text{for all } t \in [0, T_{\max} - \tau),$$

where C_i $(i = 1, 2, 3)$ are positive constants, then

$$y(t) \le \left(\frac{C_3}{\tau} + C_2\right)e^{C_1} \quad \text{for all } t \in [\tau, T_{\max}).$$

2.2 The uniform in-time L^1 boundedness of u

The uniform in-time L^1 boundedness of u below is a basic property of solutions and will play a role in our analysis.

Lemma 2.3. *Let (u, v) be a solution of (1.4). If $a > 0$, $b > 0$, and $\sigma > 1$ are constants, then there exists a constant $C > 0$ such that*

$$\int_\Omega u \le C \quad \text{for all } t \in (0, T_{\max}). \tag{2.1}$$

Proof. Integrating the first equation of (1.4) over Ω with the boundary condition, one finds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u \le a\int_\Omega u - b\int_\Omega u^\sigma \quad \text{for all } t \in (0, T_{\max}). \tag{2.2}$$

Due to $\sigma > 1$ and Hölder's inequality, we conclude the fact

$$\int_\Omega u \le |\Omega|^{\frac{\sigma-1}{\sigma}}\left(\int_\Omega u^\sigma\right)^{\frac{1}{\sigma}},$$

which, along with (2.2), implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u \le a\int_\Omega u - \frac{b}{|\Omega|^{\sigma-1}}\left(\int_\Omega u\right)^\sigma.$$

Solving this ordinary differential inequality and noticing the positivity of u, we have

$$\int_\Omega u \le \max\left\{\int_\Omega u_0, \left(\frac{a}{b}\right)^{\frac{1}{\sigma-1}}|\Omega|\right\}.$$

This finishes the proof of (2.1). □

2.3 The upper bound of *v*

To rule out the possible degeneracy of $\gamma(v)$, we shall derive the upper bound of v in this section. Our method is essentially inspired from the paper [10]. The following lemma asserts that the upper bound of v is independent of $T_{\max}$.

Lemma 2.4. *Let* (u, v) *be a solution of* (1.4). *If* $a > 0$, $b > 0$, *and* $\sigma > 1$ *are constants, then there is a constant* $C > 0$ *such that* v *satisfies*

$$v_t + \gamma(v)u + \gamma(0)v \le C \quad \text{for any } (x, t) \in \Omega \times (0, T_{\max}).$$

Moreover, we have

$$v \le Q \quad \text{for any } (x, t) \in \Omega \times (0, T_{\max}),$$

where the constant $Q > 0$ *is defined in* (1.7).

Proof. Step 1: We claim v satisfies the following equation:

$$v_t + \gamma(v)u = (I - \Delta)^{-1}\{\gamma(v)u + au - bu^{\sigma}\} \tag{2.3}$$

for all $(x, t) \in \Omega \times (0, T_{\max})$. Indeed, the first equation of (1.4) can be rewritten as follows:

$$u_t = -(I - \Delta)\gamma(v)u + \gamma(v)u + au - bu^{\sigma}.$$

Taking the operator $(I - \Delta)^{-1}$ on both sides of the aforementioned equation, and using the second equation of (1.4), we obtain (2.3) directly.

Step 2: By the nonincreasing property of $\gamma(v)$ and Young's inequality, we can find a constant

$$C_1 := \frac{\sigma - 1}{b^{\frac{1}{\sigma-1}}}\left(\frac{a + 2\gamma(0)}{\sigma}\right)^{\frac{\sigma}{\sigma-1}} > 0$$

such that

$$\gamma(v)u + au - bu^{\sigma} \le \gamma(0)u + au - bu^{\sigma} = -\gamma(0)u + (a + 2\gamma(0))u - bu^{\sigma} \le -\gamma(0)u + C_1,$$

which, combined with the comparison principle for elliptic equations, yields

$$(I - \Delta)^{-1}\{\gamma(v)u + au - bu^{\sigma}\} \le -\gamma(0)v + C_1. \tag{2.4}$$

Inserting (2.4) into (2.3), we obtain

$$v_t + \gamma(v)u + \gamma(0)v \le C_1.$$

Step 3: Owing to the nonnegativity of u and $\gamma(v)$, it holds that

$$\gamma(v)u \ge 0,$$

which, in conjunction with the result obtained in Step 2, leads to

$$v_t + \gamma(0)v \le C_1.$$

Therefore, by the Grönwall inequality, this completes the proof of the lemma. □

2.4 The time-space L^2 boundedness of *u*

Thanks to the vital inequality of v obtained in Lemma 2.4, we have the following improved time-space L^2 boundedness of u for $\sigma > 1$.

Lemma 2.5. *Let* (u, v) *be a solution of* (1.4). *If* $a > 0$, $b > 0$ *and* $\sigma > 1$ *are constants, then there exists a constant* $C > 0$ *such that*

$$\int_\Omega |\nabla v|^2 \le C \quad \textit{for all } t \in (0, T_{\max})$$

and

$$\int_t^{t+\tau} \int_\Omega u^2 \le C \quad \textit{for all } t \in (0, T_{\max} - \tau),$$

where $\tau = \min\left\{1, \frac{T_{\max}}{2}\right\}$.

Proof. Thanks to Lemma 2.4, we can find a constant $C_1 > 0$ such that

$$v_t + \gamma(v)u + \gamma(0)v \le C_1.$$

Then by the second equation of (1.4), integration by parts and Lemma 2.3, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (|\nabla v|^2 + v^2) = \int_\Omega uv_t \le -\int_\Omega \gamma(v)u^2 - \gamma(0)\int_\Omega uv + C_1\int_\Omega u \le -\int_\Omega \gamma(v)u^2 - \gamma(0)\int_\Omega (|\nabla v|^2 + v^2) + C_2 \tag{2.5}$$

for some constant $C_2 > 0$. Therefore, the Grönwall inequality gives a constant $C_3 > 0$ such that

$$\int_\Omega (|\nabla v|^2 + v^2) \le C_3.$$

Hence, integrating (2.5) over $[t, t+\tau]$, we find some constant $C_4 > 0$ so that

$$\int_t^{t+\tau} \int_\Omega \gamma(v)u^2 \le C_4. \tag{2.6}$$

In view of the hypothesis (1.5) and Lemma 2.4, we can find some constant $C_5 > 0$ such that

$$\gamma(v) \ge \gamma(Q) = C_5,$$

which along with (2.6) gives the desired result. □

2.5 A criterion of global existence

To prove our main result, we shall deduce a criterion of global boundedness of solutions for the system (1.4). To this end, we first derive an important inequality which will be used in the sequel frequently.

Lemma 2.6. *Let* (u, v) *be a solution of* (1.4). *If* $a > 0$, $b > 0$, *and* $\sigma > 1$ *are constants, then there exist constants* $C > 0$ *and* $\kappa > 0$ *defined on* (1.8) *such that for any* $p \ge 2$, *we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + \frac{2(p-1)C}{p}\int_\Omega \left|\nabla u^{\frac{p}{2}}\right|^2 + bp\int_\Omega u^{p+\sigma-1} \le \frac{p(p-1)\kappa}{2}\int_\Omega u^p|\nabla v|^2 + ap\int_\Omega u^p \quad \textit{for all } t \in (0, T_{\max}). \tag{2.7}$$

Furthermore, for any $p \ge \max\left\{2, \frac{n}{2} - 1\right\}$, *one can find constants* $C > 0$ *and* $\kappa > 0$, *which is defined in* (1.8) *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + \frac{2(p-1)C}{p}\int_\Omega \left|\nabla u^{\frac{p}{2}}\right|^2 + bp\int_\Omega u^{p+\sigma-1} \le \frac{p(p-1)\kappa}{2}\int_\Omega u^{p+1} + ap\int_\Omega u^p \quad \textit{for all } t \in (0, T_{\max}).$$

Proof. Using u^{p-1} with $p \ge 2$ as a test function to test the first equation in (1.4), integrating the result by parts and using Young's inequality, we obtain

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + (p-1)\int_\Omega \gamma(v)u^{p-2}|\nabla u|^2 + b\int_\Omega u^{p+\sigma-1} &= -(p-1)\int_\Omega \gamma'(v)u^{p-1}\nabla u\cdot\nabla v + a\int_\Omega u^p\\ &\le \frac{p-1}{2}\int_\Omega \gamma(v)u^{p-2}|\nabla u|^2 + \frac{p-1}{2}\int_\Omega \frac{|\gamma'(v)|^2}{\gamma(v)}u^p|\nabla v|^2 + a\int_\Omega u^p,\end{aligned}$$

which, along with the basic fact

$$\left|\nabla u^{\frac{p}{2}}\right|^2 = \frac{p^2}{4}u^{p-2}|\nabla u|^2,$$

yields that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + \frac{2(p-1)}{p^2}\int_\Omega \gamma(v)\left|\nabla u^{\frac{p}{2}}\right|^2 + b\int_\Omega u^{p+\sigma-1} \le \frac{p-1}{2}\int_\Omega \frac{|\gamma'(v)|^2}{\gamma(v)}u^p|\nabla v|^2 + a\int_\Omega u^p. \tag{2.8}$$

In view of hypothesis (1.5) and Lemma 2.4, we can find constants $C_1, C_2 > 0$ such that

$$\gamma(v) \ge \gamma(Q) = C_1 \quad \text{and} \quad \frac{|\gamma'(v)|^2}{\gamma(v)} \le C_2.$$

Therefore, (2.8) yields (2.7). Now, we estimate the terms on the right hand side of (2.7). First, the Young inequality implies

$$\int_\Omega u^p|\nabla v|^2 \le \int_\Omega u^{p+1} + \int_\Omega |\nabla v|^{2(p+1)}. \tag{2.9}$$

In addition, since $p \ge \max\left\{2, \frac{n}{2}-1\right\}$, we may invoke the Gagliardo-Nirenberg inequality, the regularity theory of elliptic equations and Lemma 2.4 to find some constants $C_{GN}, C_R > 0$ such that

$$\|\nabla v\|^{2(p+1)}_{L^{2(p+1)}(\Omega)} \le G^{2(p+1)}\|v\|^{p+1}_{W^{2,p+1}(\Omega)}\|v\|^{p+1}_{L^\infty(\Omega)} \le G^{2(p+1)}Q^{p+1}\|v\|^{p+1}_{W^{2,p+1}(\Omega)} \le G^{2(p+1)}R^{p+1}Q^{p+1}\|u\|^{p+1}_{L^{p+1}(\Omega)}.$$

This along with (2.9) updates (2.7) as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + \frac{2(p-1)C_1}{p}\int_\Omega \left|\nabla u^{\frac{p}{2}}\right|^2 + bp\int_\Omega u^{p+\sigma-1} \le \frac{p(p-1)C_2(G^{2(p+1)}R^{p+1}Q^{p+1}+1)}{2}\int_\Omega u^{p+1} + ap\int_\Omega u^p,$$

which gives the desired result of the lemma. □

The following lemma asserts that the global solution can be obtained as long as one can find a constant $p_0 > \frac{n}{2}$ such that the L^{p_0}-norm of u is bounded uniformly in time, which is very similar to a corresponding result well-known for quite general (semilinear) Keller-Segel systems (see [5]). The idea of the proof is directly motivated from [36].

Lemma 2.7. *Let* (u, v) *be a solution of* (1.4). *Assume* $a > 0, b > 0$ *and* $\sigma > 1$ *are constants. If there exist constants* $M > 0$ *and* $p_0 > \frac{n}{2}$ *such that*

$$\int_\Omega u^{p_0} \le M \quad \textit{for all } t \in (0, T_{\max}), \tag{2.10}$$

then the solution of the problem (1.4) *is global, i.e., we can find a constant* $C > 0$ *such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \quad \textit{for all } t \in (0, T_{\max}).$$

Proof. Step 1: We claim that there exists a constant $C > 0$ such that

$$\int_\Omega u^{2p_0} \leq C \quad \text{for all } t \in (0, T_{\max}).$$

Indeed, taking $p = 2p_0$ in Lemma 2.6, we obtain the following estimate

$$\frac{d}{dt}\int_\Omega u^{2p_0} + \frac{(2p_0-1)C_1}{p_0}\int_\Omega |\nabla u^{p_0}|^2 + 2bp_0\int_\Omega u^{2p_0+\sigma-1} \leq p_0(2p_0-1)C_2\int_\Omega u^{2p_0+1} + 2ap_0\int_\Omega u^{2p_0} \tag{2.11}$$

for some constants $C_1, C_2 > 0$. With Young's inequality, we can find a constant $C_3 > 0$ such that

$$(1+2ap_0)\int_\Omega u^{2p_0} \leq 2bp_0\int_\Omega u^{2p_0+\sigma-1} + C_3.$$

Then we have from (2.11) that

$$\frac{d}{dt}\int_\Omega u^{2p_0} + \int_\Omega u^{2p_0} + \frac{(2p_0-1)C_1}{p_0}\int_\Omega |\nabla u^{p_0}|^2 \leq p_0(2p_0-1)C_2\int_\Omega u^{2p_0+1} + C_3. \tag{2.12}$$

By the Gagliardo-Nirenberg inequality and (2.10), we can find some constants $C_4, C_5 > 0$ such that

$$\begin{aligned}\int_\Omega u^{2p_0+1} = \|u^{p_0}\|_{L^{\frac{2p_0+1}{p_0}}(\Omega)}^{\frac{2p_0+1}{p_0}} &\leq C_4\left(\|u^{p_0}\|_{L^1(\Omega)}^{\frac{2p_0+1}{p_0}(1-\theta)}\|\nabla u^{p_0}\|_{L^2(\Omega)}^{\frac{2p_0+1}{p_0}\theta} + \|u^{p_0}\|_{L^1(\Omega)}^{\frac{2p_0+1}{p_0}}\right)\\ &\leq C_4\left(M^{\frac{2p_0+1}{p_0}(1-\theta)}\|\nabla u^{p_0}\|_{L^2(\Omega)}^{\frac{2p_0+1}{p_0}\theta} + M^{\frac{2p_0+1}{p_0}}\right)\\ &\leq \frac{C_1}{p_0^2C_2}\int_\Omega |\nabla u^{p_0}|^2 + C_5,\end{aligned} \tag{2.13}$$

where $\theta = \frac{n}{n+2}\frac{2p_0+2}{2p_0+1} \in (0,1)$ and $\frac{2p_0+1}{2p_0}\theta < 1$ due to $p_0 > \frac{n}{2}$. Substituting (2.13) into (2.12), we obtain

$$\frac{d}{dt}\int_\Omega u^{2p_0} + \int_\Omega u^{2p_0} \leq p_0(2p_0-1)C_2C_5 + C_3,$$

which, together with the Grönwall's inequality, proves the claim.

Step 2: Now noticing the estimate in Step 1 and the regularity theory of elliptic equations, we obtain from the second equation of (1.4) that $v \in W^{2,2p_0}(\Omega) \hookrightarrow C^{1,1-\frac{n}{2p_0}}(\Omega)$ by the Sobolev embedding theorem. In other words, there exists a constant $C_6 > 0$ such that

$$\|\nabla v\|_{L^\infty(\Omega)} \leq C_6 \quad \text{for all } t \in (0, T_{\max}), \tag{2.14}$$

which, together with Lemma 2.6, implies that for any $p \geq 2$,

$$\frac{d}{dt}\int_\Omega u^p + \frac{(p-1)C_7}{p}\int_\Omega \left|\nabla u^{\frac{p}{2}}\right|^2 + bp\int_\Omega u^{p+\sigma-1} \leq C_8p(p-1)\int_\Omega u^p + ap\int_\Omega u^p$$

with some constants $C_7, C_8 > 0$. Then, we can employ the standard Moser iteration (cf. [4]) to prove that there exists a constant $C_9 > 0$ such that

$$\|u\|_{L^\infty(\Omega)} \leq C_9 \quad \text{for all } t \in (0, T_{\max}), \tag{2.15}$$

where the details are omitted here for brevity. Then we complete the proof by using (2.14), (2.15), and the extensibility criterion in Lemma 2.1. □

3 Global existence: Proof of Theorem 1.1

Now, we shall use the boundedness criterion in Lemma 2.7 to show (1.4) admits a global classical solution. If $\sigma > 2$, then we use the Young inequality to obtain the following estimate.

Lemma 3.1. *Let* (u, v) *be a solution of* (1.4). *If* $a > 0$, $b > 0, \sigma > 2$, *and* $\gamma(v)$ *satisfies hypothesis* (1.5), *then for any* $p \geq \max\left\{2, \frac{n}{2} - 1\right\}$, *there exists a constant* $C > 0$ *such that*

$$\int_\Omega u^p \leq C \quad \text{for all } t \in (0, T_{\max}).$$

Proof. By Lemma 2.6, we obtain the following estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + \frac{2(p-1)C_1}{p}\int_\Omega \left|\nabla u^{\frac{p}{2}}\right|^2 + bp\int_\Omega u^{p+\sigma-1} \leq \frac{p(p-1)C_2}{2}\int_\Omega u^{p+1} + ap\int_\Omega u^p \tag{3.1}$$

for some constants $C_1, C_2 > 0$. An application of the Young inequality gives some constant $C_3 > 0$ such that

$$\frac{p(p-1)C_2}{2}\int_\Omega u^{p+1} + (1+ap)\int_\Omega u^p \leq bp\int_\Omega u^{p+\sigma-1} + C_3,$$

which, combined with (3.1), yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + \int_\Omega u^p \leq C_3.$$

Hence, we can use the Grönwall inequality to prove the claim. □

Similarly, if $\sigma = 2$ and $b > 0$ is large enough, then the uniform in-time $L^{\left[\frac{n}{2}\right]+1}$–norm of u can be obtained.

Lemma 3.2. *Let* (u, v) *be a solution of* (1.4). *If* $a > 0, b > b_1, \sigma = 2$, *and* $\gamma(v)$ *satisfies hypothesis* (1.5), *then we can find a constant* $C > 0$ *such that*

$$\int_\Omega u^{\left[\frac{n}{2}\right]+1} \leq C \quad \text{for all } t \in (0, T_{\max}).$$

Proof. By Lemma 2.6, for some $\widetilde{p} = \left[\frac{n}{2}\right] + 1$, there exist some constants $C_1 > 0$ and $\kappa(\widetilde{p}, b, 2) > 0$ defined in (1.8) such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{\widetilde{p}} + \frac{2(\widetilde{p}-1)C_1}{\widetilde{p}}\int_\Omega \left|\nabla u^{\frac{\widetilde{p}}{2}}\right|^2 + b\widetilde{p}\int_\Omega u^{\widetilde{p}+1} \leq \frac{\widetilde{p}(\widetilde{p}-1)\kappa(\widetilde{p}, b, 2)}{2}\int_\Omega u^{\widetilde{p}+1} + a\widetilde{p}\int_\Omega u^{\widetilde{p}}. \tag{3.2}$$

The Young inequality gives a constant $C_2 > 0$ satisfying

$$(1 + a\widetilde{p})\int_\Omega u^{\widetilde{p}} \leq \widetilde{p}\int_\Omega u^{\widetilde{p}+1} + C_2. \tag{3.3}$$

Taking $b \geq b_1$, we have from the definition of κ that

$$b \geq \frac{(\widetilde{p}-1)\kappa(\widetilde{p}, b, 2)}{2} + 1,$$

which, along with (3.2) and (3.3), implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{\tilde{p}} + \int_\Omega u^{\tilde{p}} \le C_2.$$

Hence, we obtain the desired result by the Grönwall inequality. □

If $n = 2$, we make full use of the second equation of (1.4) to obtain the uniform-in-time L^2-norm of u.

Lemma 3.3. *Let* (u, v) *be a solution of* (1.4). *If* $n = 2$, $a > 0$, $b > 0$ *and* $\sigma > 1$ *are constants, then there exists a constant* $C > 0$ *such that*

$$\int_\Omega u^2 \le C \quad \text{for all } t \in (0, T_{\max}).$$

Proof. We multiply the second equation of (1.4) by $-\Delta v$ and integrate the result by parts to have

$$\int_\Omega |\Delta v|^2 + \int_\Omega |\nabla v|^2 = -\int_\Omega u\Delta v \le \frac{1}{2}\int_\Omega |\Delta v|^2 + \frac{1}{2}\int_\Omega u^2,$$

which yields

$$\frac{1}{2}\int_\Omega |\Delta v|^2 + \int_\Omega |\nabla v|^2 \le \frac{1}{2}\int_\Omega u^2. \tag{3.4}$$

Taking $p = 2$ in (2.7), we obtain the following estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2 + C_1\int_\Omega |\nabla u|^2 + 2b\int_\Omega u^{\sigma+1} \le C_2\int_\Omega u^2|\nabla v|^2 + 2a\int_\Omega u^2 \tag{3.5}$$

for some constants $C_1, C_2 > 0$. By (3.4), Lemmas 2.3, 2.4, and the Gagliardo-Nirenberg inequality subject to the Neumann boundary conditions, there exist some constants $C_3, C_4, C_5 > 0$ such that

$$\|u\|_{L^4(\Omega)} \le C_3\left(\|\nabla u\|_{L^2(\Omega)}^{\frac{1}{2}}\|u\|_{L^2(\Omega)}^{\frac{1}{2}} + \|u\|_{L^1(\Omega)}\right) \le C_4\left(\|\nabla u\|_{L^2(\Omega)}^{\frac{1}{2}}\|u\|_{L^2(\Omega)}^{\frac{1}{2}} + 1\right) \tag{3.6}$$

and

$$\|\nabla v\|_{L^4(\Omega)} \le C_3\left(\|\Delta v\|_{L^2(\Omega)}^{\frac{1}{2}}\|v\|_{L^\infty(\Omega)}^{\frac{1}{2}} + \|v\|_{L^\infty(\Omega)}\right) \le C_5\left(\|u\|_{L^2(\Omega)}^{\frac{1}{2}} + 1\right). \tag{3.7}$$

Using (3.6), (3.7), Lemma 2.5, Hölder's inequality, and Young's inequality, we can find some constant $C_6 > 0$ such that

$$\begin{aligned}
C_2\int_\Omega u^2|\nabla v|^2 &\le C_2\|u\|_{L^4(\Omega)}^2\|\nabla v\|_{L^4(\Omega)}^2 \\
&\le C_2C_4^2C_5^2\left(\|\nabla u\|_{L^2(\Omega)}^{\frac{1}{2}}\|u\|_{L^2(\Omega)}^{\frac{1}{2}} + 1\right)^2\left(\|u\|_{L^2(\Omega)}^{\frac{1}{2}} + 1\right)^2 \\
&\le 4C_2C_4^2C_5^2(\|\nabla u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} + 1) \\
&\le C_1\|\nabla u\|_{L^2(\Omega)}^2 + C_6(1 + \|u\|_{L^2(\Omega)}^2)\|u\|_{L^2(\Omega)}^2 + C_6,
\end{aligned}$$

which, combined with (3.5), implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2 \le C_6(1 + \|u\|_{L^2(\Omega)}^2)\int_\Omega u^2 + C_7$$

for some constant $C_7 > 0$. An application of Lemmas 2.2 and 2.5 gives the desired result. □

Proof of Theorem 1.1. Consolidating the results in Lemmas 2.3, 2.7, 3.1, 3.2, and 3.3, we obtain Theorem 1.1 directly. □

4 Large time behavior

In this section, we aim to study the large time behavior of (1.4). We first improve the regularity of the solution component u.

Lemma 4.1. *There exist constants $\theta \in (0,1)$ and $C > 0$ such that*

$$\|u\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} \leqslant C \quad \text{for all } t > 1.$$

Proof. Let $\psi_0 = C\gamma'^2(v)|\nabla v|^2$, $\psi_1 = \gamma'(v)\nabla v$, and $\psi_2 = au + bu^\sigma$ in conditions (A_1), (A_2), and (A_3) of [28]. With an application of the results in [28] to the solution of the first equation of (1.4) and the boundedness of u, v, and ∇v, we obtain the results directly. □

Based on the Lyapunov functionals method, the large time behavior of the solution can be studied. We first show a basic property on the solution component u.

Lemma 4.2. *For any $\alpha > 1$, there exists a constant $T_* > 0$ such that*

$$\int_\Omega u \leq \alpha u_* |\Omega| \quad \text{for all } t > T_*, \tag{4.1}$$

where u_* is defined in (1.9).

Proof. Integrating the first equation in (1.4) over Ω and using the boundary condition as well as Hölder's inequality, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u \leq a\int_\Omega u - \frac{b}{|\Omega|^{\sigma-1}}\left(\int_\Omega u\right)^\sigma \quad \text{for all } t \geqslant 0.$$

To solve this ordinary differential inequality, we first consider the following ordinary differential equation:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}y(t) = ay(t) - \dfrac{b}{|\Omega|^{\sigma-1}}y^\sigma(t), & t \geqslant 0,\\ y(0) = \displaystyle\int_\Omega u_0, \end{cases}$$

which is a Bernoulli's equation having the solution:

$$y(t) = \left\{\frac{b}{a|\Omega|^{\sigma-1}} + \left[\left(\int_\Omega u_0\right)^{-(\sigma-1)} - \frac{b}{a|\Omega|^{\sigma-1}}\right]\mathrm{e}^{-a(\sigma-1)t}\right\}^{-\frac{1}{\sigma-1}}.$$

This, along with a comparison argument, shows that

$$\int_\Omega u \leq \left\{\frac{b}{a|\Omega|^{\sigma-1}} + \left[\left(\int_\Omega u_0\right)^{-(\sigma-1)} - \frac{b}{a|\Omega|^{\sigma-1}}\right]\mathrm{e}^{-a(\sigma-1)t}\right\}^{-\frac{1}{\sigma-1}}. \tag{4.2}$$

Observing

$$\left\{\frac{b}{a|\Omega|^{\sigma-1}}+\left[\left(\int_\Omega u_0\right)^{-(\sigma-1)}-\frac{b}{a|\Omega|^{\sigma-1}}\right]\mathrm{e}^{-a(\sigma-1)t}\right\}^{-\frac{1}{\sigma-1}}\to u_*|\Omega|$$

as $t\to+\infty$, we can find a constant $T_*>0$ such that

$$\left\{\frac{b}{a|\Omega|^{\sigma-1}}+\left[\left(\int_\Omega u_0\right)^{-(\sigma-1)}-\frac{b}{a|\Omega|^{\sigma-1}}\right]e^{-a(\sigma-1)t}\right\}^{-\frac{1}{\sigma-1}}\le\alpha u_*|\Omega|\quad\text{for all}\quad t>T_*.$$

Together with (4.2), we obtain (4.1). □

We note that the result in Lemma 4.2 slightly improves the result of [41, Lemma 2.2], where $\alpha=2$. To prove the large time behavior of solutions, we shall prove an following inequality, which looks similar to [41, Lemma 3.4] derived for the classical logistic Keller-Segel system without density-suppressed motility. Here, we extend this inequality to the density-suppressed motility model with the logistic source. Since the proof has some differences from [41], we provide the details for clarity.

Lemma 4.3. *Let*

$$q:=\begin{cases}\dfrac{2}{3-\sigma}, & \text{if }\ \sigma\in(1,2),\\ 2, & \text{if }\ \sigma\in[2,+\infty).\end{cases}$$

Then there exist constants $K_1>0$, which is defined in (1.10) *and $T_*\ge 0$ such that*

$$\|u-u_*\|^2_{L^q(\Omega)}\le K_1u_*^{2-\sigma}\int_\Omega(u^{\sigma-1}-u_*^{\sigma-1})(u-u_*)\quad\text{for all }\ t>T_*,\tag{4.3}$$

where $u_*>0$ is defined in (1.9).

Proof. Step 1: If $\sigma\in[2,+\infty)$, then $u^{\sigma-2}\le u_*^{\sigma-2}$ for $u\in[0,u_*]$, and thus,

$$\begin{aligned}(u-u_*)(u^{\sigma-1}-u_*^{\sigma-1})&=(u_*-u)(u_*^{\sigma-1}-u^{\sigma-1})\ge(u_*-u)(u_*^{\sigma-1}-uu_*^{\sigma-2})\\&=u_*^{\sigma-2}(u-u_*)^2\quad\text{for all }\ u\in[0,u_*],\end{aligned}\tag{4.4}$$

whereas $u_*^{\sigma-2}\le u^{\sigma-2}$ for $u\in(u_*,+\infty)$, and hence,

$$(u-u_*)(u^{\sigma-1}-u_*^{\sigma-1})\ge(u-u_*)(uu_*^{\sigma-2}-u_*^{\sigma-1})=u_*^{\sigma-2}(u-u_*)^2\quad\text{for all }\ u\in(u_*,+\infty).$$

This, along with (4.4), establishes

$$(u-u_*)(u^{\sigma-1}-u_*^{\sigma-1})\ge u_*^{\sigma-2}(u-u_*)^2\quad\text{for any }\ \sigma\in[2,+\infty),$$

which implies (4.3).

Step 2: If $\sigma\in(1,2)$, then $\varphi(\xi):=\xi^{\sigma-1}$ is a continuous function on $[0,2u_*]$ and $\varphi'(\xi)=(\sigma-1)\xi^{\sigma-2}$ on $(0,2u_*)$. For $u\in[0,u_*]$, the mean value theorem yields that there exists $\theta\in(u,u_*)$ such that

$$u^{\sigma-1}-u_*^{\sigma-1}=(\sigma-1)\theta^{\sigma-2}(u-u_*)\le(\sigma-1)u_*^{\sigma-2}(u-u_*),$$

which implies

$$(u-u_*)(u^{\sigma-1}-u_*^{\sigma-1})\ge(\sigma-1)u_*^{\sigma-2}(u-u_*)^2.\tag{4.5}$$

Meanwhile, for $u\in(u_*,2u_*]$, an application of the mean value theorem gives $\vartheta\in(u_*,u)$ satisfying

$$u^{\sigma-1} - u_*^{\sigma-1} = (\sigma-1)\vartheta^{\sigma-2}(u-u_*) \ge (\sigma-1)(2u_*)^{\sigma-2}(u-u_*),$$

which yields

$$(u-u_*)(u^{\sigma-1} - u_*^{\sigma-1}) \ge (\sigma-1)(2u_*)^{\sigma-2}(u-u_*)^2.$$

This together with (4.5) proves

$$(u-u_*)(u^{\sigma-1} - u_*^{\sigma-1}) \ge (\sigma-1)(2u_*)^{\sigma-2}(u-u_*)^2 \quad \text{for any } u \in [0, 2u_*] \tag{4.6}$$

due to $2^{\sigma-2} \le 1$.

Since $q \le 2$, we have $(s+r)^{\frac{2}{q}} \le 2^{\frac{2}{q}-1}\big(s^{\frac{2}{q}} + r^{\frac{2}{q}}\big)$ for $s \ge 0$ and $r \ge 0$, which yields

$$\|u-u_*\|_{L^q(\Omega)}^2 = \Bigg(\int\limits_{\{u<2u_*\}} |u-u_*|^q + \int\limits_{\{u\ge 2u_*\}} (u-u_*)^q\Bigg)^{\frac{2}{q}} \le 2^{\frac{2}{q}-1}\Bigg(\int\limits_{\{u<2u_*\}} |u-u_*|^q\Bigg)^{\frac{2}{q}} + 2^{\frac{2}{q}-1}\Bigg(\int\limits_{\{u\ge 2u_*\}} u^q\Bigg)^{\frac{2}{q}} \tag{4.7}$$

for all $t \ge 0$. Now we estimate the inequalities on the right-hand side of (4.7).

For the first integral, noticing again $q \le 2$, we invoke the Hölder inequality and (4.6) to find that for all $t \ge 0$,

$$2^{\frac{2}{q}-1}\Bigg(\int\limits_{\{u<2u_*\}} |u-u_*|^q\Bigg)^{\frac{2}{q}} \le 2^{\frac{2}{q}-1}|\Omega|^{\frac{2-q}{q}} \int\limits_{\{u<2u_*\}} (u-u_*)^2 \le \frac{(2|\Omega|)^{\frac{\sigma+1}{3-\sigma}}(2u_*)^{2-\sigma}}{\sigma-1} \int\limits_{\{u<2u_*\}} (u^{\sigma-1} - u_*^{\sigma-1})(u-u_*). \tag{4.8}$$

Thanks to Lemma 4.2, we can pick a constant $T_* > 0$ satisfying

$$\|u\|_{L^1(\Omega)} \le 2u_*|\Omega| \quad \text{for all } t > T_*. \tag{4.9}$$

To estimate the second integral, we may utilize (4.9) to see that if $\sigma < 2$, it follows from the Hölder inequality and our special choice of q that

$$\Bigg(\int\limits_{\{u\ge 2u_*\}} u^q\Bigg)^{\frac{2}{q}} = \|u\|^2_{L^{\frac{2}{3-\sigma}}(\{u\ge 2u_*\})} \le \|u\|^{2-\sigma}_{L^1(\{u\ge 2u_*\})}\|u\|^{\sigma}_{L^{\sigma}(\{u\ge 2u_*\})} \le (2u_*|\Omega|)^{2-\sigma} \int\limits_{\{u\ge 2u_*\}} u^\sigma \quad \text{for all} \quad t > T_*. \tag{4.10}$$

Simple calculation shows

$$\int\limits_{\{u\ge 2u_*\}} (u^{\sigma-1} - u_*^{\sigma-1})(u-u_*) \ge \int\limits_{\{u\ge 2u_*\}} \bigg(u^{\sigma-1} - \Big(\frac{u}{2}\Big)^{\sigma-1}\bigg)\Big(u - \frac{u}{2}\Big) = \frac{1-2^{-(\sigma-1)}}{2} \int\limits_{\{u\ge 2u_*\}} u^\sigma \quad \text{for all } t \ge 0,$$

which, along with (4.10), indicates that

$$2^{\frac{2}{q}-1}\Bigg(\int\limits_{\{u\ge 2u_*\}} u^q\Bigg)^{\frac{2}{q}} \le \frac{2^{\frac{4}{3-\sigma}}(2u_*|\Omega|)^{2-\sigma}}{1-2^{-(\sigma-1)}} \int\limits_{\{u\ge 2u_*\}} (u^{\sigma-1} - u_*^{\sigma-1})(u-u_*) \tag{4.11}$$

for all $t > T_*$.

Collecting (4.7), (4.8), and (4.11), we obtain (4.3). □

Next we derive an estimate on v.

Lemma 4.4. *We have*

$$\int\limits_\Omega |\nabla v|^2 + \int\limits_\Omega (v-\bar{v})^2 \le K_2\|u-u_*\|^2_{L^q(\Omega)},$$

where q is defined in Lemma 4.3 *and $K_2 > 0$ is defined in* (1.11).

Proof. Due to

$$\bar{v} = \frac{1}{|\Omega|}\int_\Omega v,$$

it holds that

$$\int_\Omega (v - \bar{v}) = 0. \tag{4.12}$$

Multiplying the second equation of (1.4) by $v - \bar{v}$ and integrating the result over Ω, we obtain

$$\int_\Omega |\nabla(v - \bar{v})|^2 + \int_\Omega (v - \bar{v})v = \int_\Omega (v - \bar{v})u, \tag{4.13}$$

which, along with (4.12), updates (4.13) as follows:

$$\int_\Omega |\nabla v|^2 + \int_\Omega (v - \bar{v})^2 = \int_\Omega (v - \bar{v})(u - u_*). \tag{4.14}$$

We proceed with two cases. If $\sigma \in [2, +\infty)$, the Young inequality implies

$$\int_\Omega (v - \bar{v})(u - u_*) \le \frac{1}{2}\int_\Omega (v - \bar{v})^2 + \frac{1}{2}\|u - u_*\|^2_{L^q(\Omega)},$$

which, along with (4.14), yields the desired result. If $1 < \sigma < 2$ with

$$\sigma \in \begin{cases} (1, 2), & \text{if} \quad n \le 2, \\ \left(2 - \frac{2}{n}, 2\right), & \text{if} \quad n > 2, \end{cases}$$

thanks to the definition of q, we obtain from Poincaré's inequality and the Sobolev embedding theorem that there exists a constant $\xi > 0$ such that

$$\|v - \bar{v}\|_{L^{\frac{q}{q-1}}(\Omega)} \le \xi\|\nabla v\|_{L^2(\Omega)},$$

which, along with Hölder's inequality and the Young inequality, yields

$$\int_\Omega (v - \bar{v})(u - u_*) \le \|v - \bar{v}\|_{L^{\frac{q}{q-1}}(\Omega)}\|u - u_*\|_{L^q(\Omega)} \le C_p\|\nabla v\|_{L^2(\Omega)}\|u - u_*\|_{L^q(\Omega)} \le \frac{1}{2}\|\nabla v\|^2_{L^2(\Omega)} + \frac{\xi^2}{2}\|u - u_*\|^2_{L^q(\Omega)}.$$

Therefore, we use (4.14) to finish the proof of the lemma. □

To study the large time behavior of solutions, we shall construct a Lyapunov functional for (1.4). Given a positive number u_*, let $\varphi_{u_*} : (0, \infty) \to \mathbb{R}$ be defined by

$$\varphi_{u_*}(x) := x - u_* - u_* \ln\frac{x}{u_*}, \quad x > 0.$$

Then φ_{u_*} is convex with $\varphi_{u_*}(u_*) = \varphi'_{u_*}(u_*) = 0$, which implies $\varphi_{u_*}(x) \ge 0$ for all $x > 0$. For any nonnegative continuous function $u : \overline{\Omega} \to (0, \infty)$, we define an energy functional for (1.4) as follows:

$$F(u) = \int_\Omega \left(u - u_* - u_* \ln\frac{u}{u_*}\right),$$

which is nonnegative and nonincreasing. We further have the following results.

Lemma 4.5. *If $b > b_2$ where b_2 is defined in* (1.12), *then there exists a constant $C > 0$ such that*

$$\frac{d}{dt}F(u) + C\left\{\int_\Omega (v-\bar{v})^2 + \|u-u_*\|^2_{L^q(\Omega)}\right\} \le 0 \quad \text{for all } t > T^*,$$

where u_* is defined as in (1.9). *Moreover, there exists a constant $C > 0$ such that*

$$\int_{T^*}^{+\infty}\int_\Omega (v-\bar{v})^2 + \int_{T^*}^{+\infty} \|u-u_*\|^2_{L^q(\Omega)} \le C.$$

Proof. Noticing Lemmas 4.3 and 4.4, one can find constants $K_1, K_2 > 0$ such that

$$\|u-u_*\|^2_{L^q(\Omega)} \le K_1 u_*^{2-\sigma}\int_\Omega (u^{\sigma-1}-u_*^{\sigma-1})(u-u_*) \quad \text{for all } t > T_* \tag{4.15}$$

and

$$\int_\Omega |\nabla v|^2 + \int_\Omega (v-\bar{v})^2 \le K_2\|u-u_*\|^2_{L^q(\Omega)}. \tag{4.16}$$

Applying the first equation of the system (1.4) and integrating by parts, we find

$$\begin{aligned}\frac{d}{dt}\int_\Omega\left(u-u_*-u_*\ln\frac{u}{u_*}\right) &= \int_\Omega u_t - u_*\int_\Omega\frac{u_t}{u}\\ &= a\int_\Omega u - b\int_\Omega u^\sigma - u_*\int_\Omega \frac{1}{u}(\Delta(\gamma(v)u) + au - bu^\sigma)\\ &= a\int_\Omega u - b\int_\Omega u^\sigma - au_*|\Omega| + bu_*\int_\Omega u^{\sigma-1} - u_*\int_\Omega\gamma(v)\frac{|\nabla u|^2}{u^2} - u_*\int_\Omega \gamma'(v)\frac{\nabla u\cdot\nabla v}{u}\\ &= -b\int_\Omega (u^{\sigma-1}-u_*^{\sigma-1})(u-u_*) - u_*\int_\Omega\gamma(v)\frac{|\nabla u|^2}{u^2} - u_*\int_\Omega\gamma'(v)\frac{\nabla u\cdot\nabla v}{u}.\end{aligned} \tag{4.17}$$

Due to $b > b_2$, we have

$$\frac{|\gamma'(v)|^2 u_*}{4\gamma(v)} < \frac{bu_*^{\sigma-2}}{K_1K_2},$$

which implies that there exists a constant $\varepsilon > 0$ such that

$$\frac{|\gamma'(v)|^2 u_*}{4\gamma(v)} \le \varepsilon < \frac{bu_*^{\sigma-2}}{K_1K_2}. \tag{4.18}$$

Combining (4.15), (4.16), and (4.17), one obtains

$$\begin{aligned}\frac{d}{dt}\int_\Omega\left(u-u_*-u_*\ln\frac{u}{u_*}\right) &\le -b\int_\Omega (u^{\sigma-1}-u_*^{\sigma-1})(u-u_*) - \varepsilon\int_\Omega (v-\bar{v})^2 + \varepsilon K_2\|u-u_*\|^2_{L^q(\Omega)}\\ &\quad - u_*\int_\Omega \gamma(v)\frac{|\nabla u|^2}{u^2} - u_*\int_\Omega\gamma'(v)\frac{\nabla u\cdot\nabla v}{u} - \varepsilon\int_\Omega|\nabla v|^2\\ &\le -\left(\frac{bu_*^{\sigma-2}}{K_1} - \varepsilon K_2\right)\|u-u_*\|^2_{L^q(\Omega)} - \varepsilon\int_\Omega (v-\bar{v})^2 - \int_\Omega \Sigma^T B\Sigma,\end{aligned}$$

where

$$\Sigma = \begin{bmatrix} \frac{\nabla u}{u} \\ \nabla v \end{bmatrix}, \quad B = \begin{bmatrix} \gamma(v)u_* & \frac{1}{2}\gamma'(v)u_* \\ \frac{1}{2}\gamma'(v)u_* & \varepsilon \end{bmatrix}.$$

As (4.18) ensures that $\frac{bu_*^{\sigma-2}}{K_1} - \varepsilon K_2$ is positive, B and $F(u)$ are nonnegative, the results of Lemma 4.5 follow directly. □

Now, we use the argument of contradiction, Lemmas 4.1, and 4.5 to obtain the asymptotic behavior of (1.4) and prove Theorem 1.2.

Proof of Theorem 1.2. We use the argument of contradiction to prove the theorem. Supposing that $\|u(\cdot,t) - u_*\|_{L^\infty} \not\to 0$, we can find a constant $C_1 > 0$, some sequences $\{t_j\}_{j\in\mathbb{N}} \subset (1, +\infty)$ such that $t_j \to +\infty$ as $j \to +\infty$ and

$$\|u(\cdot,t_j) - u_*\|_{L^\infty} \geq C_1$$

for all $j \in \mathbb{N}$. Without loss of generality, we may assume that $t_{j+1} > t_j + 1$ for all $j \in \mathbb{N}$. Since $\{u(\cdot,t_j) - u_*\}_{j\in\mathbb{N}}$ is relatively compact in $C^0(\overline{\Omega})$ according to Lemma 4.1 and the Arzelà-Ascoli theorem, we may assume that

$$u(\cdot,t_j) - u_* \to \tilde{u} \quad \text{in } L^\infty(\Omega)$$

as $j \to +\infty$ with some nonnegative $\tilde{u} \in C^0(\overline{\Omega})$, where the subsequence of $\{u(\cdot,t_j) - u_*\}_{j\in\mathbb{N}}$ is denoted by $\{u(\cdot,t_j) - u_*\}_{j\in\mathbb{N}}$ for notational convenience. Hence, we have

$$u(\cdot,t_j) - u_* \to \tilde{u} \quad \text{in } L^q(\Omega)$$

as $j \to +\infty$, which implies

$$\|u(\cdot,t_j) - u_*\|_{L^q(\Omega)} \geq \frac{1}{2}\|\tilde{u}\|_{L^q(\Omega)}$$

for all sufficient large $j \in \mathbb{N}$. Now by using Lemma 4.1, we easily obtain the existence of $\tau \in (0, 1)$ and $j_0 \in \mathbb{N}$ such that

$$\|u(\cdot,t) - u_*\|_{L^q(\Omega)} \geq \frac{1}{4}\|\tilde{u}\|_{L^q(\Omega)}$$

for all $t \in [t_j, t_j + \tau]$ and each $j \geqslant j_0$, which in particular implies that for any $j \geqslant j_0$ we have

$$\int_{t_j}^{t_j+\tau} \|u(x, t) - u_*\|^2_{L^q(\Omega)} \geq \frac{\tau}{16}\|\tilde{u}\|^2_{L^q(\Omega)}. \tag{4.19}$$

From Lemma 4.5, we derive

$$\int_{t_j}^{t_j+\tau} \|u(x, t) - u_*\|^2_{L^q(\Omega)} \leq \int_{t_j}^{+\infty} \|u(x, t) - u_*\|^2_{L^q(\Omega)} \to 0$$

as $j \to +\infty$, which contradicts (4.19). Hence, (1.13) is verified.

The convergence $\|v(\cdot,t) - u_*\|_{L^\infty} \to 0$ as $t \to \infty$ can be directly obtained by the maximum principle applied to the second equation of (1.4). □

Acknowledgments: The author thanks the reviewers for various nice comments and suggestions that improve the exposition of this paper.

Funding information: The research of the first author was supported by the National Nature Science Foundation of China (No. 12101377) and the Nature Science Foundation of Shanxi Province (No. 20210302124080). The research of the second author was supported by the Hong Kong RGC GRF grant No. 15303019 (Project ID P0030816) and an internal grant No. UAH0 (Project ID P0031504) from the Hong Kong Polytechnic University.

Conflict of interest: The authors state no conflict of interest.

References

[1] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions ofelliptic partial differential equations satisfying general boundary conditions I*, Commun. Pure Appl. Math. **12** (1959), 623–727.
[2] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions ofelliptic partial differential equations satisfying general boundary conditions II*, II. Commun. Pure Appl. Math. **17** (1964), 35–92.
[3] J. Ahn and C. W. Yoon, *Global well-posedness and stability of constant equilibria in parabolic-elliptic chemotaxis systems without gradient sensing*, Nonlinearity **32** (2019), no. 4, 1327–1351.
[4] N. D. Alikakos, *L^p bounds of solutions of reaction-diffusion equations*, Comm. Partial Differ. Equ. **4** (1979), 8, 827–868.
[5] N. Bellomo, A. Bellouquid, Y. S. Tao, and M. Winkler, *Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues*, Math. Models Methods Appl. Sci. **25** (2015), no. 9, 1663–1763.
[6] M. Burger, P. Laurençot, and A. Trescases, *Delayed blow-up for chemotaxis models with local sensing*, J. Lond. Math. Soc. (2) **103** (2021), no. 4, 1596–1617.
[7] L. Desvillettes, Y. J. Kim, A. Trescases, and C. W. Yoon, *A logarithmic chemotaxis model featuring global existence and aggregation*, Nonlinear Anal. Real World Appl. **50** (2019), 562–582.
[8] X. F. Fu, L. H. Tang, C. L. Liu, J. D. Huang, T. Hwa, and P. Lenz, *Stripe formation in bacterial systems with density-suppressed motility*, Phy.s Rev. Lett. **108** (2012), no. 19, 1981–1988.
[9] K. Fujie and J. Jiang, *Boundedness of classical solutions to a degenerate Keller-Segel type model with signal-dependent motilities*, Acta Appl. Math. **176** (2021), 3.
[10] K. Fujie and J. Jiang, *Global existence for a kinetic model of pattern formation with density-suppressed motilities*, J. Differ. Equ. **269** (2020), no. 6, 5338–5378.
[11] K. Fujie and J. Jiang, *Comparison methods for a Keller-Segel-type model of pattern formations with density-suppressed motilities*. Calc. Var. Partial Differ. Equ. **60** (2021), no. 3, Paper No. 92, 37.
[12] T. Hillen and K. J. Painter, *A user's guide to PDE models for chemotaxis*, J. Math. Biol. **58** (2009), no. 1–2, 183–217.
[13] D. Horstmann, *From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I*, Jahresber. Deutsch. Math.-Verein. **105** (2003), no. 3, 103–165.
[14] J. Jiang, P. Laurençot, and Y. Y. Zhang, *Global existence, uniform boundedness, and stabilization in a chemotaxis system with density-suppressed motility and nutrient consumption*, Comm. Partial Differ. Equ. **47** (2022), no. 5, 1024–1069.
[15] H. Y. Jin, S. J. Shi, and Z. A. Wang, *Boundedness and asymptotics of a reaction-diffusion system with density-dependent motility*, J. Differ. Equ. **269** (2020), no. 9, 6758–6793.
[16] H. Y. Jin and Z. A. Wang, *The Keller-Segel system with logistic growth and signal-dependent motility*, Discrete Contin. Dyn. Syst. Ser. B **26** (2021), no. 6, 3023–3041.
[17] P. Kareiva and G. Odell, *Swarms of predators exhibit p¨reytaxisïf individual predators use area-restricted search*, Am. Naturalist **130** (1987), no. 2, 233–270.
[18] E. F. Keller and L. A. Segel, *Model for chemotaxis*, J. Theoret. Biol. **30** (1971), no. 2, 225–234.
[19] Z. R. Liu and J. Xu, *Large time behavior of solutions for density-suppressed motility system in higher dimensions*, J. Math. Anal. Appl. **475** (2019), no. 2, 1596–1613.
[20] C. Liu, X. Fu, L. Liu, X. Ren, C. K. Chau, S. Li, et al. *Sequential establishment of stripe patterns in an expanding cell population*, Science, **334** (2011), 238–241.
[21] R. Lui and H. Ninomiya, *Traveling wave solutions for a bacteria system with density-suppressed motility*, Discrete Contin. Dyn. Syst. Ser. B **24** (2019), no. 2, 931–940.
[22] W. B. Lv and Q. Y. Wang, *Global existence for a class of Keller-Segel models with signal-dependent motility and general logistic term*, Evol. Equ. Control Theory **10** (2021), no. 1, 25–36.
[23] W. B. Lv and Q. Y. Wang, *An n-dimensional chemotaxis system with signal-dependent motility and generalized logistic source: global existence and asymptotic stabilization*, Proc. Roy. Soc. Edinburgh Sect. A **151** (2021), no. 2, 821–841.
[24] W. Lyu and Z. A. Wang, *Global classical solutions for a class of reaction-diffusion system with density-suppressed motility*, Electron. Res. Arch. **30** (2022), no. 3, 995–1015.
[25] M. J. Ma, R. Peng, and Z. A. Wang, *Stationary and non-stationary patterns of the density-suppressed motility model*, Phys. D **402** (2020), no. 132259, 13pp.
[26] N. Nanjundiah, *Signal relaying and aggregation morphology*, J. Theor. Biol. **42** (1970), 63–105.

[27] K. Osaki, T. Tsujikawa, A. Yagi, and M. Mimura, *Exponential attractor for a chemotaxis-growth system of equations*, Nonlinear Anal. **51** (2002), 119–144.
[28] M. M. Porzio and V. Vespri, *Hölder estimates for local solutions of some doubly nonlinear degenerate parabolic equations*, J. Differ. Equ. **103** (1993), no. 1, 146–178.
[29] J. Smith-Roberge, D. Iron, and T. Kolokolnikov, *Pattern formation in bacterial colonies with density-dependent diffusion*, European J. Appl. Math. **30** (2019), no. 1, 196–218.
[30] T. Suzuki, *Free energy and self-interacting particles*, in: Progress in Nonlinear Differential Equations and their Applications, vol. 62, Birkhäuser Boston, Inc., Boston, MA, 2005.
[31] T. Suzuki, *Chemotaxis, Reaction, Network*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018.
[32] Y. S. Tao and M. Winkler, *Effects of signal-dependent motilities in a Keller-Segel-type reaction-diffusion system*, Math. Models Methods Appl. Sci. **27** (2017), no. 9, 1645–1683.
[33] J. I. Tello and M. Winkler, *A chemotaxis system with logistic source*, Comm. Partial Differ. Equ. **32** (2007), no. 4–6, 849–877.
[34] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Applied Mathematical Sciences, vol. 68, Springer-Verlag, New York, 1988.
[35] J. P. Wang and M. X. Wang, *Boundedness in the higher-dimensional Keller-Segel model with signal-dependent motility and logistic growth*, J. Math. Phys. **60** (2019), no. 1, 011507, 14.
[36] Z. A. Wang, *On the parabolic-elliptic Keller-Segel system with signal-dependent motilities: a paradigm for global boundedness and steady states*, Math. Methods Appl. Sci. **44** (2021), no. 13, 10881–10898.
[37] Z. A. Wang and J. J.Xu, *On the Lotka-Volterra competition system with dynamical resources and density-dependent diffusion*, J. Math. Biol. **82** (2021), no. 1–2, Paper No. 7, 37.
[38] Z. A. Wang and J. S. Zheng, *Global boundedness of the fully parabolic Keller-Segel system with signal-dependent motilities*, Acta Appl. Math. **171** (2021), 25.
[39] M. Winkler, *Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source*, Comm. Partial Differ. Equ. **35** (2010), no. 8, 1516–1537.
[40] M. Winkler, *Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction*, J. Math. Anal. Appl. **384** (2011), no. 2, 261–272.
[41] M. Winkler, *Attractiveness of constant states in logistic-type Keller-Segel systems involving subquadratic growth restrictions*, Adv. Nonlinear Stud. **20** (2020), no. 4, 795–817.
[42] T. Xiang, *Boundedness and global existence in the higher-dimensional parabolic-parabolicchemotaxis system with/without growth source*, J. Differ. Equ. **258** (2015), 4275–4323.
[43] C. Xu and Y. F. Wang, *Asymptotic behavior of a quasilinear Keller-Segel system with signal-suppressed motility*, Calc. Var. Partial Differ. Equ. **60** (2021), no. 5, Paper No. 183.
[44] C. Yoon and Y. J. Kim, *Global existence and aggregation in a Keller-Segel model with Fokker-Planck diffusion*, Acta Appl. Math. **149** (2017), 101–123.