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Convergence to nonlinear diffusion waves for a hyperbolic-parabolic chemotaxis system modelling vasculogenesis

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Abstract

In this paper, we are concerned with a quasi-linear hyperbolic-parabolic system of persistence and endogenous chemotaxis modelling vasculogenesis. Under some suitable structural assumption on the pressure function, we first predict and derive the system admits a nonlinear diffusion wave in $\mathbb{R}$ driven by the damping effect. Then we show that the solution of the concerned system will locally and asymptotically converge to this nonlinear diffusion wave if the wave strength is small. By using the time-weighted energy estimates, we further prove that the convergence rate of the nonlinear diffusion wave is algebraic.

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1. Introduction

In order to depict the key characteristics of the *in vitro* experiment of blood vessels, showing that the cells randomly scattered on the gel matrix will automatically organize into a network of connected blood vessels, Ambrosi et al. in [1,14] proposed the following quasi-linear hyperbolic-parabolic chemotaxis system

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p(\rho) = \mu \rho \nabla \phi - \alpha \rho u, \\ \partial_t \phi = D \Delta \phi + a\rho - b\phi. \end{cases} \tag{1.1}$$

Here the unknowns $\rho = \rho(x,t) \geq 0$ and $u = u(x,t) \in \mathbb{R}^n (n \geq 1)$ denote the density and velocity of endothelial cells, respectively, and $\phi = \phi(x,t) \geq 0$ denotes the concentration of the chemoattractant secreted by the endothelial cells. The convection term $\nabla \cdot (\rho u \otimes u)$ models the cell movement persistence (inertial effect), $p(\rho)$ is the cell-density dependent pressure function accounting for the fact that closely packed cells resist to compression due to the impenetrability of cellular matter, the parameter $\mu > 0$ measures the intensity of cell response to the chemoattractant concentration gradient and $-\alpha\rho u$ corresponds to a damping (friction) force with coefficient $\alpha > 0$ as a result of the interaction between cells and the underlying substratum; $D > 0$ is the diffusivity of the chemoattractant, the positive constants a and b denote the secretion and death rates of the chemoattractant, respectively.

The hyperbolic or hyperbolic-parabolic chemotaxis models with different structures from (1.1) have been studied in the literature (cf. [20,34,40,50]). Nevertheless the mathematical structure of (1.1) is analogous to the well-known damped Euler-Poisson system where ϕ is called the flow potential satisfying a Poisson (elliptic) equation: $-\Delta\phi = \rho$, which has numerous essential applications such as propagation of electrons in semiconductor devices (cf. [39]) and the transport of ions in plasma physics (cf. [6]) when $\mu < 0$ and $\alpha \geq 0$, as well as the collapse of gaseous stars due to self-gravitation [4] when $\mu > 0$ and $\alpha = 0$. Here the sign of μ corresponds to attractive or repulsive forces similar to the attractive or repulsive chemotaxis (cf. [31]). Due to the essential difference of analysis between elliptic and parabolic equations, the analytical tools developed for the damped Euler-Poisson system is not directly applicable to the system (1.1). Up to date, there are not many analytical results available to the system (1.1). When the initial value $(\rho_0(x), u_0(x), \phi_0(x)) \in H^s(\mathbb{R}^n)(s > n/2+1)$ is a small perturbation of the constant ground state (i.e. equilibrium) $(\rho_c, 0, \phi_c)$ with $\rho_c > 0$ sufficiently small, it was shown in [9,10] that the system (1.1) admits global strong solutions without vacuum converging to $(\rho_c, 0, \phi_c)$ in L^2-norm with an algebraic decay rate. As $\alpha \to \infty$ (strong damping), it was formally derived in [5] by the asymptotic analysis and subsequently justified in [8] that the solution of (1.1) converges to that of a parabolic-elliptic Keller-Segel type chemotaxis system. The asymptotic behavior of solutions to (1.1) and its limiting Keller-Segel system was further compared numerically in [44]. By adding a viscous term Δu to the second equation of (1.1), the linear stability of the constant ground state $(\rho_c, 0, \phi_c)$ was obtained in [32] under the condition

$$bp'(\rho_c) - a\mu\rho_c > 0. \tag{1.2}$$

A typical form of p fulfilling (1.2) is $p(\rho) = \frac{K}{2}\rho^2$ with $K > \frac{a\mu}{b}$. The stationary solutions of one dimensional (1.1) with vacuum (bump solutions) in a bounded interval with zero-flux boundary condition were constructed in [2,3,43]. The model (1.1) with $p(\rho) = \rho$ and periodic boundary conditions in one dimension was numerically explored in [13]. Recently the stability of transition layer stationary solutions of (1.1) on $\mathbb{R}_+ = [0, \infty)$ was established in [17].

It is well known that damping is a factor triggering diffusion waves in many hyperbolic system such as the p-system or damped Euler equations (cf. [21,29,30,37,41,42,45,46,48,55] without vacuum and [7,22,38,49] with vacuum), as well as the Euler-Poisson system of semiconductors [15,23–26], bipolar Euler-Maxwell equation [11], Timoshenko system [28] and the radiating gas model [36]. Though the above-mentioned works are obtained under L^2 framework, the L^1 stability of diffusion waves can also be possibly obtained (cf. [47,53]). When the damping is time-dependent, we refer the relevant results to [16,18,19,33] and references therein. Motivated by the structural analogue between the Euler-Poisson system and (1.1) with dampings, the authors have shown recently in [35] that the hyperbolic-parabolic system (1.1) admits linear diffusion waves in $\mathbb{R}^3$ which are shown to be locally asymptotically stable by the Fourier and spectral analysis. Under the framework of [35], the time decay of solutions decreases with respect to the space dimension and the energy estimates will lose time integrability in $\mathbb{R}^n (n = 1, 2)$. Therefore whether the hyperbolic-parabolic chemotaxis system (1.1) admits stable diffusion waves in $\mathbb{R}$ or $\mathbb{R}^2$ remains unknown. The purpose of this paper is to show that the system (1.1) in $\mathbb{R}$ admits nonlinear diffusion waves which are stable against a small perturbation by using the technique of taking antiderivative, unlike the framework of [35].

To demonstrate our ideas, we set $m = \rho u$ for convenience, namely m denotes the momentum of cells, and recast the system (1.1) for $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ as

$$\begin{cases} \rho_t + m_x = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ m_t + \left(\frac{m^2}{\rho} + p(\rho)\right)_x = \mu\rho\phi_x - \alpha m, & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ \phi_t = D\phi_{xx} + a\rho - b\phi, & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \end{cases} \tag{1.3}$$

where we prescribe the initial data

$$U|_{t=0} = U_0 := (\rho_0, m_0, \phi_0) \longrightarrow (\rho_\pm, 0, \phi_\pm), \qquad \text{as} \quad x \to \pm\infty, \tag{1.4}$$

with $\rho_\pm > 0$, $\rho_+ \neq \rho_-$ and $\phi_\pm = \frac{a}{b}\rho_\pm$, where we assume $\rho_\pm > 0$ to avoid possible vacuum. Moreover we impose the following conditions on the pressure function

$$p(\rho) \in C^3(\mathbb{R}^+), \quad p'(\rho) - \frac{a\mu}{b}\rho > 0, \ \text{ for any } \rho > 0. \tag{1.5}$$

Due to the external damping (frictional) force, one may expect that the inertial term in the momentum equation of (1.3) decays to zero faster than other terms so that the pressure gradient force is balanced by the frictional force plus the potential force. Hence we may predict that the solution (ρ, m, ϕ) of (1.3)-(1.4) will behave time asymptotically as the solution $(\bar\rho, \bar m, \bar\phi)$ to the following equations

$$\begin{cases} \bar\rho_t + \bar m_x = 0, \\ (p(\bar\rho))_x = \mu\bar\rho\bar\phi_x - \alpha\bar m, \\ a\bar\rho = b\bar\phi, \end{cases} \tag{1.6}$$

or equivalently, by denoting $q(\bar\rho) = p(\bar\rho) - \frac{a\mu}{2b}\bar\rho^2$,

$$\begin{cases} \bar\rho_t - \left(\frac{1}{\alpha}q(\bar\rho)\right)_{xx} = 0, & \text{(a)} \\ \bar m = -\frac{1}{\alpha}\left[p(\bar\rho) - \frac{a\mu}{2b}\bar\rho^2\right]_x, & \text{(b)} \\ \bar\phi = \frac{a}{b}\bar\rho, & \text{(c)} \end{cases} \tag{1.7}$$

with the following asymptotic states at far fields

$$(\bar\rho(x,t), \bar m(x,t), \bar\phi(x,t)) \longrightarrow \left(\rho_\pm, 0, \frac{a}{b}\rho_\pm\right), \text{ as } x \to \pm\infty. \tag{1.8}$$

In our paper, without loss of generality, we assume $\rho_- < \rho_+$. As shown in [52], the equation (1.7a) admits unique nonlinear diffusion wave $\bar\rho(x,t) = \varphi(\xi)$ with $\xi = \frac{x}{\sqrt{1+t}}$ under the condition (1.5) such that $\varphi(\pm) = \rho_\pm$. Then substituting this into (1.7b) and (1.7c), we find a unique solution $(\bar\rho, \bar m, \bar\phi)(x,t)$ of diffusion wave for (1.7) with

$$\bar\rho(x,t) = \varphi\left(\frac{x}{\sqrt{1+t}}\right), \ \bar m(x,t) = -\frac{1}{\alpha}\left[p(\bar\rho) - \frac{a\mu}{2b}\bar\rho^2\right]_x, \ \bar\phi(x,t) = \frac{a}{b}\bar\rho. \tag{1.9}$$

The aim of this paper is to show if the initial value (ρ_0, m_0, ϕ_0) satisfying (1.4) is a small perturbation of $(\bar\rho(x,0), \bar m(x,0), \bar\phi(x,0))$, then the system (1.3) with (1.5) admits a unique solution whose asymptotic profile is the nonlinear diffusion wave $(\bar\rho, \bar m, \bar\phi)(x,t)$ given by (1.9). To state our main results, we first define the perturbation of $(\bar\rho(x,0), \bar m(x,0), \bar\phi(x,0)$ as follows

$$\begin{cases} V_0(x) = \int_{-\infty}^x (\rho_0(y) - \bar\rho(y + x_0, 0))dy, \\ M_0(x) = m_0(x) - \bar m(x + x_0, 0), \\ \Phi_0(x) = \phi_0(x) - \bar\phi(x + x_0, 0), \end{cases} \tag{1.10}$$

where x_0 is a constant uniquely determined (see section 2.2) such that the initial perturbation from the spatially shifted diffusion waves with shift x_0 is of integral zero, namely $V_0(\infty) = 0$. Above we define $V_0(x)$ in a form of anti-derivative of the perturbation because the first equation of (1.3) is a conservation law for which the technique of taking anti-derivative is usually invoked (cf. [51]). By the method of weighted energy estimates, we shall prove the following results in this paper.

Theorem 1.1. *Let (1.5) hold. Then there exists a constant $\epsilon > 0$, such that if $(V_0, M_0, \Phi_0) \in H^3(\mathbb{R}) \times H^2(\mathbb{R}) \times H^4(\mathbb{R})$ satisfies*

$$\|V_0\|^2_{H^3(\mathbb{R})} + \|M_0\|^2_{H^2(\mathbb{R})} + \|\Phi_0\|^2_{H^4(\mathbb{R})} + |\rho_+ - \rho_-| \le \epsilon^2,$$

where (V_0, M_0, Φ_0) is defined in (1.10), the system (1.3)-(1.4) possesses a unique global classical solution $(\rho, m, \phi)(x,t)$ which converges to the shifted diffusion wave $(\bar\rho, \bar m, \bar\phi)(x + x_0, t)$ solving (1.7) and (1.8) in $L^\infty(\mathbb{R})$ with algebraic decay rates:

$$\begin{aligned}
&\|\partial_x^k(\rho-\bar{\rho})(t)\|_{L^\infty(\mathbb{R})} \le C\epsilon(1+t)^{-3/4-k/2},\ k=0,1,\\
&\|\partial_x^k(m-\bar{m})(t)\|_{L^\infty(\mathbb{R})} \le C\epsilon(1+t)^{-5/4-k/2},\ k=0,1,\\
&\|\partial_x^k(\phi-\bar{\phi})(t)\|_{L^\infty(\mathbb{R})} \le C\epsilon(1+t)^{-3/4-k/2},\ k=0,1,
\end{aligned} \tag{1.11}$$

where $C > 0$ is a constant independent of t.

The rest of the paper is organized as follows. In Section 2, we present some known results on the diffusion wave solution $\bar{\rho}(x,t)$ of (1.7) with far field states (1.8), and reformulate the original equation (1.3) against a suitable perturbation. In Section 3, we derive the uniform *a-priori* estimates and hence establish the existence of global solutions of reformulated problem. In Section 4, we show the algebraic time asymptotic rate of solutions convergent to the nonlinear diffusion wave and prove the main Theorem 1.1.

2. Reformulation of the problem

In this section, we shall prove the global existence of solutions to (1.6). We first introduce some notations frequently throughout the paper.

Notations. In the sequel, C denotes a generic positive constant where C may vary in the context. For two quantities a and b, $a \sim b$ means $\lambda a \le b \le \frac{1}{\lambda} a$ for some constant $0 < \lambda < 1$. For any integer $m \ge 0$, we use H^m to denote the usual Sobolev space $H^m(\mathbb{R})$. For simplicity, the norm of H^m is denoted by $\|\cdot\|_m$ with $\|\cdot\| = \|\cdot\|_0$, and we set

$$\|[A,B]\|_X = \|A\|_X + \|B\|_X.$$

Without confusion, we shall abbreviate $\|\cdot\|_{L^p(\mathbb{R})}$ as $\|\cdot\|_{L^p}$ for $1 \le p \le \infty$ in the sequel.

2.1. Nonlinear diffusion waves

In this subsection, we give some properties of nonlinear diffusion wave profile (1.9) that will be used in the paper. Let

$$\bar{\rho}(x,t) = \varphi\left(\frac{x}{\sqrt{1+t}}\right) = \varphi(\xi), \qquad -\infty < \xi < \infty.$$

Substituting the above self-similar structure form into (1.7a), we find that $\varphi(\xi)$ satisfies

$$\begin{cases}
\varphi''(\xi) + \dfrac{\frac{\alpha}{2}\xi + q''(\varphi(\xi))\varphi'(\xi)}{q'(\varphi(\xi))}\varphi'(\xi) = 0,\\
\varphi(\pm\infty) = \rho_\pm,
\end{cases}$$

where $q'(\bar{\rho}) = p'(\bar{\rho}) - \frac{a\mu}{b}\bar{\rho} > 0$. The existence of unique solution of the above equations has been shown in [21]. For any ξ_0, it further follows that

$$\varphi'(\xi) = \frac{\varphi'(\xi_0)q'(\varphi(\xi_0))}{q'(\varphi(\xi))}\exp\left(-\int_{\xi_0}^{\xi}\frac{\alpha\eta}{2q'(\varphi(\eta))}d\eta\right).$$

The solution $\varphi(\xi)$ is increasing if $\rho_- < \rho_+$ and decreasing if $\rho_- > \rho_+$, and satisfies

$$\sum_{k=1}^{6}\left|\frac{d^k}{d\xi^k}\varphi(\xi)\right| + |\varphi(\xi)-\rho_+|_{\xi>0} + |\varphi(\xi)-\rho_-|_{\xi<0} \le C|\rho_+-\rho_-|\exp(-c\alpha\xi^2),$$

where c is a positive constant independent of x and t. Moreover, we can obtain following L^p-estimates of the derivatives of $\bar{\rho}$ (cf. [22,27,54,55]).

Lemma 2.1. *Let $\bar{\rho}(x,t)$ be the self-similar solution of* (1.7a) *and* (1.8)*. Then for $p \in [2,+\infty]$ and $1 \le l+k \le 6$, we have*

$$\left\|\partial_t^l\partial_x^k\bar{\rho}(\cdot,t)\right\|_{L^p} \le C|\rho_+-\rho_-|(1+t)^{-\frac{k}{2}-l+\frac{1}{2p}}.$$

2.2. Reformulation of the problem

Inspired by the work [21], we set the perturbation function around the diffusion wave $(\bar{\rho},\bar{m},\bar{\phi})$ as

$$\begin{cases} V(x,t) = \int_{-\infty}^{x}(\rho(y,t)-\bar{\rho}(y+x_0,t))dy, \\ M(x,t) = m(x,t)-\bar{m}(x+x_0,t), \\ \Phi(x,t) = \phi(x,t)-\bar{\phi}(x+x_0,t), \end{cases} \tag{2.1}$$

where x_0 is a constant uniquely determined by

$$\int_{-\infty}^{+\infty}(\rho(x,0)-\bar{\rho}(x+x_0,0))dx = 0,$$

namely,

$$x_0 = \frac{1}{\rho_+-\rho_-}\int_{-\infty}^{+\infty}(\rho_0(y)-\bar{\rho}(y,0))\,dy.$$

Define the initial perturbation functions as

$$\begin{cases} V_0(x) = V(x,0) = \int_{-\infty}^{x}(\rho_0(y)-\bar{\rho}(y+x_0,0))dy, \\ M_0(x) = M(x,0) = m_0(x)-\bar{m}(x+x_0,0), \\ \Phi_0(x) = \Phi(x,0) = \phi_0(x)-\bar{\phi}(x+x_0,0). \end{cases}$$

Then upon the substitution of (2.1), we reformulate our problem (1.3)-(1.4) as

$$
\begin{cases}
V_t + M = 0, \\
M_t + \left(\dfrac{(M+\bar{m})^2}{V_x + \bar{\rho}} \right)_x + [p(V_x+\bar{\rho}) - p(\bar{\rho})]_x = \mu V_x \Phi_x + \mu V_x \bar{\phi}_x + \mu \bar{\rho} \Phi_x - \alpha M - \bar{m}_t, \\
\Phi_t = D\Phi_{xx} + aV_x - b\Phi - \bar{\phi}_t + D\bar{\phi}_{xx},
\end{cases}
\tag{2.2}
$$

with initial data (V_0, M_0, Φ_0) satisfying

$$
(V(x,0), M(x,0), \Phi(x,0)) = (V_0(x), M_0(x), \Phi_0(x)) \to 0, \quad \text{as } x \to \pm\infty. \tag{2.3}
$$

Rewrite (2.2)-(2.3) as

$$
\begin{cases}
V_{tt} - (p'(\bar{\rho})V_x)_x + \alpha V_t + \mu V_x \Phi_x + \mu V_x \bar{\phi}_x + \mu \bar{\rho} \Phi_x + h_x + f_x = 0, \\
\Phi_t = D\Phi_{xx} + aV_x - b\Phi + g,
\end{cases}
\tag{2.4}
$$

with initial data

$$
(V(x,0), V_t(x,0), \Phi(x,0)) = (V_0(x), -M_0(x), \Phi_0(x)) \to 0, \quad \text{as } x \to \pm\infty, \tag{2.5}
$$

where

$$
\begin{cases}
h = -\dfrac{(V_t + \frac{1}{\alpha} q(\bar{\rho})_x)^2}{V_x + \bar{\rho}}, \\
f = \frac{1}{\alpha}(q(\bar{\rho}))_t - \left[p(V_x+\bar{\rho}) - p(\bar{\rho}) - p'(\bar{\rho})V_x \right], \\
g = -\bar{\phi}_t + D\bar{\phi}_{xx}.
\end{cases}
\tag{2.6}
$$

For system (2.2)-(2.3), we shall establish the following results.

Proposition 2.2. *Let $\delta_0 = |\rho_+ - \rho_-|$ and (1.5) hold. If $(V_0, M_0, \Phi_0) \in H^3 \times H^2 \times H^3$ and*

$$
\|[V_0, \Phi_0]\|_3^2 + \|M_0\|_2^2 + \delta_0 \le \varepsilon_0^2,
$$

for some sufficiently small constant $\varepsilon_0 > 0$, then problem (2.2)-(2.3) admits a unique global classical solutions (V, M, Φ) satisfying

$$
(V(x,t), M(x,t), \Phi(x,t)) \in L^\infty\left([0,\infty); H^3 \times H^2 \times H^3\right),
$$

and

$$
\sup_{t\ge 0} \|V(t)\|_3^2 + \|V_t(t)\|_2^2 + \|\Phi(t)\|_3^2 \le C\left(\|V_0\|_3^2 + \|M_0\|_2^2 + \|\Phi_0\|_3^2 + \delta_0 \right) \le C\varepsilon_0^2.
$$

Moreover, if there exists a constant ϵ such that

$$
\delta_0 + \|V_0\|_3^2 + \|M_0\|_2^2 + \|\Phi_0\|_4^2 \le \epsilon^2,
$$

then the solution (V, M, Φ) has the following decay

$$\begin{aligned} &\|\partial_x^k V_x(t)\|_{L^2} \le C\epsilon(1+t)^{-(k+1)/2}, \ k=0,1,2, \\ &\|\partial_x^k M(t)\|_{L^2} \le C\epsilon(1+t)^{-(k+2)/2}, \ k=0,1,2, \\ &\|\partial_x^k \Phi(t)\|_{L^2} \le C\epsilon(1+t)^{-(k+1)/2}, \ k=0,1,2, \end{aligned} \tag{2.7}$$

where $C > 0$ is a positive constant independent of t.

In view of (2.1), Theorem 1.1 is a direct consequence of Proposition 2.2. Hence we will focus on the proof of Proposition 2.2. Before proceeding, we briefly outline the ideas of proving Proposition 2.2 where part of them are inspired from works [45,47]. First we establish the global existence of smooth solutions to (2.2)-(2.3), and then we derive time decay rates of the solution toward diffusion waves. Although such procedures are routine, the desired results are not easy to be achieved due to the coupling of Φ and V. Since we can not expect the exponential decay of Φ like the electronic field E as in [15] or do not want to impose smallness assumption on the constant equilibrium as in [10] either, some new ideas need to be developed in order to control the linear terms $\mu\bar{\rho}\Phi_x$ in the first equation of (2.4) and aV_x in the second equation of (2.4). To this end, we take up the assumption (1.5) which implies that the following matrix

$$\begin{pmatrix} p'(\bar{\rho}) & -\mu\bar{\rho} \\ -\mu\bar{\rho} & \frac{b\mu\bar{\rho}}{a} \end{pmatrix} =: A(\bar{\rho}),$$

is positive definite. Then for any (x_1, x_2), we have

$$\min\{\lambda_1(\bar{\rho}), \lambda_2(\bar{\rho})\}(x_1^2+x_2^2) \le p'(\bar{\rho})x_1^2 - 2\mu\bar{\rho}x_1x_2 + \frac{b\mu\bar{\rho}}{a}x_2^2 \le \max\{\lambda_1(\bar{\rho}), \lambda_2(\bar{\rho})\}(x_1^2+x_2^2),$$

where $\lambda_1(\bar{\rho}) > 0$ and $\lambda_2(\bar{\rho}) > 0$ are the eigenvalues of $A(\bar{\rho})$. Since $\rho_- < \bar{\rho} < \rho_+$, there exist two constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1(x_1^2+x_2^2) \le p'(\bar{\rho})x_1^2 - 2\mu\bar{\rho}x_1x_2 + \frac{b\mu\bar{\rho}}{a}x_2^2 \le C_2(x_1^2+x_2^2). \tag{2.8}$$

Under the condition (2.8), the two linear terms $\mu\bar{\rho}\Phi_x$ and aV_x can be absorbed or eliminated in the energy estimates. Due to the coupling effect, we find that the decay properties of Φ and V_x can be obtained synchronously. Moreover the term g in the second equation of (2.4) will complicate the weighted energy estimates in the sequel.

For later use, we recall a Sobolev inequality about the L^p estimate on products of any two or several terms with the sum of the order of their derivatives equal to a given integer (cf. [12]).

Lemma 2.3. *Let $\alpha^1 = (\alpha_1^1, \cdots, \alpha_n^1)$ and $\alpha^2 = (\alpha_1^2, \cdots, \alpha_n^2)$ be two multi-indices with $|\alpha^1| = k_1$, $|\alpha^2| = k_2$ and set $k = k_1 + k_2$. Then, for $1 \le p, q, r \le \infty$ with $1/p = 1/q + 1/r$, we have*

$$\left\|\partial^{\alpha^1}u_1\partial^{\alpha^2}u_2\right\|_{L^p(\mathbb{R}^n)}\le C\Big(\left\|u_1\right\|_{L^q(\mathbb{R}^n)}\left\|\nabla^k u_2\right\|_{L^r(\mathbb{R}^n)}+\left\|u_2\right\|_{L^q(\mathbb{R}^n)}\left\|\nabla^k u_1\right\|_{L^r(\mathbb{R}^n)}\Big), \tag{2.9}$$

where C is a positive constant.

3. Global existence. Proof of Proposition 2.2

The existence of local-in-time solutions to (2.4)-(2.5) can be readily established by the standard iteration argument and hence will be stated without proof details.

Proposition 3.1 *(Local existence). Let the conditions of Proposition 2.2 hold. Then there exists a positive constant T_0 depending on ε_0 such that the problem (2.2)-(2.3) admits a unique solution $(V(x,t), M(x,t), \Phi(x,t)) \in L^\infty\left([0,T_0); H^3\times H^2\times H^3\right)$ satisfying*

$$\sup_{t\in[0,T_0]}\left(\|V\|_3^2+\|M\|_2^2+\|\Phi\|_3^2\right)\le 2\varepsilon_0^2.$$

To extend local solutions to be global in time, it suffices to derive the uniform *a priori* estimates of solutions to (2.4)-(2.5) where $V_t=-M$. For any given $T>0$, we denote the solution space for the Cauchy problem (2.4)-(2.5) by

$$X(T)=\left\{(V,V_t,\Phi)\in H^3\times H^2\times H^3,\ 0\le t\le T\right\}.$$

Assume that the following *a priori* assumption holds:

$$N(T)=\sup_{0<t<T}\{\|[V,\Phi](\cdot,t)\|_3+\|V_t(\cdot,t)\|_2\}\ll 1. \tag{3.1}$$

By the Sobolev inequality and $\rho_-<\bar\rho<\rho_+$, it follows from (3.1) that

$$\frac{1}{2}\rho_-\le V_x+\bar\rho\le\frac{3}{2}\rho_+. \tag{3.2}$$

Then we prove the following uniform *a priori* estimate.

Proposition 3.2 *(A priori estimate). Let the conditions of Proposition 2.2 hold and $(V,V_t,\Phi)\in X(T)$ be a smooth solution of (2.4)-(2.5). Then there exists a constant $C>0$ independent of t such that*

$$\begin{aligned}&\|V\|_3^2+\|V_t\|_2^2+\|\Phi\|_3^2+\int_0^T\left(\|V_x\|_2^2+\|\Phi\|_3^2+\|\Phi_t\|_2^2+\|V_t\|_2^2\right)dt\\ \le& C\left(\|V_0\|_3^2+\|M_0\|_2^2+\|\Phi_0\|_3^2+\delta_0\right)\le C\varepsilon_0^2.\end{aligned} \tag{3.3}$$

Proof. We divide the proof into three steps.

Step 1. We claim the following inequality hold

$$\begin{aligned}&\frac{d}{dt}\sum_{0\le k\le 2}\left\{\frac{\alpha}{2}\|\partial_x^k V\|^2+\int_{\mathbb{R}}\partial_x^k V_t\partial_x^k V dx+\frac{\mu}{2a}\int_{\mathbb{R}}\bar{\rho}\left(\partial_x^k\Phi\right)^2dx\right\}+\frac{\mu D}{a}\sum_{0\le k\le 2}\int_{\mathbb{R}}\bar{\rho}\left(\partial_x^k\Phi_x\right)^2dx\\&+\sum_{0\le k\le 2}\left\{\int_{\mathbb{R}}p'(\bar{\rho})\left(\partial_x^kV_x\right)^2dx-2\mu\int_{\mathbb{R}}\bar{\rho}\partial_x^k\Phi\partial_x^kV_xdx+\frac{b\mu}{a}\int_{\mathbb{R}}\bar{\rho}\left(\partial_x^k\Phi\right)^2dx\right\}\\&\le-\frac{\mu}{b}\frac{d}{dt}\int_{\mathbb{R}}\bar{\rho}_x\Phi V dx+\|V_t\|_2^2+C(N(T)+\delta_0)(\|V_x\|_2^2+\|V_t\|_2^2+\|\Phi\|_3^2)+C\delta_0(1+t)^{-\frac{5}{4}}.\end{aligned}\tag{3.4}$$

To this end, we rewrite (2.4) as

$$\begin{cases}V_{tt}-(p'(\bar{\rho})V_x)_x+\alpha V_t+\mu(\bar{\rho}\Phi)_x=-\mu V_x\Phi_x-\mu V_x\bar{\phi}_x+\mu\bar{\rho}_x\Phi-h_x-f_x,\\ \Phi_t-D\Phi_{xx}+b\Phi-aV_x=g.\end{cases}\tag{3.5}$$

Applying ∂_x^k for $0\le k\le 2$ to the first equation of (3.5) and multiplying the result by $\partial_x^k V$, one has

$$\begin{aligned}&\partial_x^kV_{tt}\partial_x^kV-\partial_x^k(p'(\bar{\rho})V_x)_x\partial_x^kV+\alpha\partial_x^kV_t\partial_x^kV+\mu\partial_x^k(\bar{\rho}\Phi)_x\partial_x^kV\\=&-\mu\partial_x^k(V_x\Phi_x)\partial_x^kV-\mu\partial_x^k(V_x\bar{\phi}_x-\bar{\rho}_x\Phi)\partial_x^kV-\partial_x^kh_x\partial_x^kV-\partial_x^kf_x\partial_x^kV.\end{aligned}\tag{3.6}$$

By simple calculations, we have

$$\begin{aligned}\partial_x^kV_{tt}\partial_x^kV=&\frac{d}{dt}(\partial_x^kV_t\partial_x^kV)-|\partial_x^kV_t|^2,\\-\partial_x^k(p'(\bar{\rho})V_x)_x\partial_x^kV=&-[\partial_x^k(p'(\bar{\rho})V_x)\partial_x^kV]_x+\partial_x^k(p'(\bar{\rho})V_x)\partial_x^kV_x\\=&-[\partial_x^k(p'(\bar{\rho})V_x)\partial_x^kV]_x+p'(\bar{\rho})\left(\partial_x^kV_x\right)^2\\&+\sum_{\ell<k}C_k^\ell\partial_x^{k-\ell}(p'(\bar{\rho}))\partial_x^\ell V_x\partial_x^kV_x,\\\mu\partial_x^k(\bar{\rho}\Phi)_x\partial_x^kV=&\mu[\partial_x^k(\bar{\rho}\Phi)\partial_x^kV]_x-\mu\partial_x^k(\bar{\rho}\Phi)\partial_x^kV_x\\=&\mu[\partial_x^k(\bar{\rho}\Phi)\partial_x^kV]_x-\mu\bar{\rho}\partial_x^k\Phi\partial_x^kV_x-\mu\sum_{\ell<k}C_k^\ell\partial_x^{k-\ell}\bar{\rho}\partial_x^\ell\Phi\partial_x^kV_x,\\\mu\partial_x^k(V_x\bar{\phi}_x-\bar{\rho}_x\Phi)\partial_x^kV=&\frac{\mu}{b}\partial_x^k\left[\bar{\rho}_x\left(aV_x-b\Phi\right)\right]\partial_x^kV,\end{aligned}$$

where we have used the relationship between $\bar{\phi}$ and $\bar{\rho}$, i.e., $\bar{\phi}=\frac{a}{b}\bar{\rho}$. Substituting the above equality into (3.6) and integrating the equation over $\mathbb{R}$, we get

$$
\begin{aligned}
&\frac{d}{dt}\left\{\frac{\alpha}{2}\|\partial_x^k V\|^2+\int_{\mathbb{R}}\partial_x^k V_t\partial_x^k V dx\right\}+\int_{\mathbb{R}} p'(\bar\rho)\left(\partial_x^k V_x\right)^2 dx-\mu\int_{\mathbb{R}}\bar\rho\partial_x^k\Phi\partial_x^k V_x dx\\
=&\|\partial_x^k V_t\|^2-\sum_{\ell<k}C_k^\ell\int_{\mathbb{R}}\partial_x^{k-\ell}(p'(\bar\rho))\partial_x^\ell V_x\partial_x^k V_x dx+\mu\sum_{\ell<k}C_k^\ell\int_{\mathbb{R}}\partial_x^{k-\ell}\bar\rho\partial_x^\ell\Phi\partial_x^k V_x dx\\
&-\int_{\mathbb{R}}\mu\partial_x^k(V_x\Phi_x)\partial_x^k V dx-\frac{\mu}{b}\int_{\mathbb{R}}\partial_x^k\left[\bar\rho_x\left(aV_x-b\Phi\right)\right]\partial_x^k V dx\\
&-\int_{\mathbb{R}}\partial_x^k h_x\partial_x^k V dx-\int_{\mathbb{R}}\partial_x^k f_x\partial_x^k V dx.
\end{aligned}
\tag{3.7}
$$

Applying ∂_x^k for $0\le k\le 2$ to the second equation of (3.5), multiplying the result by $\frac{\mu\bar\rho}{a}\partial_x^k\Phi$, and integrating the resulting equation with respect to x give

$$
\begin{aligned}
&\frac{\mu}{2a}\frac{d}{dt}\int_{\mathbb{R}}\bar\rho\left(\partial_x^k\Phi\right)^2dx+\frac{\mu D}{a}\int_{\mathbb{R}}\bar\rho\left(\partial_x^k\Phi_x\right)^2dx+\frac{b\mu}{a}\int_{\mathbb{R}}\bar\rho\left(\partial_x^k\Phi\right)^2dx-\mu\int_{\mathbb{R}}\partial_x^k V_x\bar\rho\partial_x^k\Phi dx\\
=&\frac{\mu}{2a}\int_{\mathbb{R}}\bar\rho_t\left(\partial_x^k\Phi\right)^2dx-\frac{\mu D}{a}\int_{\mathbb{R}}\bar\rho_x\partial_x^k\Phi_x\partial_x^k\Phi dx+\frac{\mu}{a}\int_{\mathbb{R}}\bar\rho\partial_x^k\Phi\partial_x^k g dx.
\end{aligned}
\tag{3.8}
$$

Taking summation of (3.7) and (3.8) and using integration by parts for the last two terms of (3.7), we have

$$
\begin{aligned}
&\frac{d}{dt}\left\{\frac{\alpha}{2}\|\partial_x^k V\|^2+\int_{\mathbb{R}}\partial_x^k V_t\partial_x^k V dx+\frac{\mu}{2a}\int_{\mathbb{R}}\bar\rho\left(\partial_x^k\Phi\right)^2dx\right\}+\frac{\mu D}{a}\int_{\mathbb{R}}\bar\rho\left(\partial_x^k\Phi_x\right)^2dx\\
&+\int_{\mathbb{R}}p'(\bar\rho)\left(\partial_x^k V_x\right)^2dx-2\mu\int_{\mathbb{R}}\bar\rho\partial_x^k\Phi\partial_x^k V_x dx+\frac{b\mu}{a}\int_{\mathbb{R}}\bar\rho\left(\partial_x^k\Phi\right)^2dx\\
=&\|\partial_x^k V_t\|^2-\int_{\mathbb{R}}\mu\partial_x^k(V_x\Phi_x)\partial_x^k V dx-\frac{\mu}{b}\int_{\mathbb{R}}\partial_x^k\left[\bar\rho_x\left(aV_x-b\Phi\right)\right]\partial_x^k V dx\\
&+\int_{\mathbb{R}}\partial_x^k h\partial_x^k V_x dx+\int_{\mathbb{R}}\partial_x^k f\partial_x^k V_x dx+\frac{\mu}{2a}\int_{\mathbb{R}}\bar\rho_t\left(\partial_x^k\Phi\right)^2dx-\frac{\mu D}{a}\int_{\mathbb{R}}\bar\rho_x\partial_x^k\Phi_x\partial_x^k\Phi dx\\
&+\frac{\mu}{a}\int_{\mathbb{R}}\bar\rho\partial_x^k\Phi\partial_x^k g dx+\sum_{\ell<k}C_k^\ell I_{k,\ell}\\
=:&\|\partial_x^k V_t\|^2+\sum_{j=1}^{7}I_j+\sum_{\ell<k}C_k^\ell I_{k,\ell},
\end{aligned}
\tag{3.9}
$$

with

$$I_{k,\ell} = -\int_{\mathbb{R}} \partial_x^{k-1-\ell}(p''(\bar\rho)\bar\rho_x)\partial_x^\ell V_x \partial_x^k V_x dx + \mu \int_{\mathbb{R}} \partial_x^{k-1-\ell}\bar\rho_x \partial_x^\ell \Phi \partial_x^k V_x dx,$$

where the fact $\ell < k$ has been used.

When $k = 0$, we estimate I_1-I_7 as follows. By using the Cauchy-Schwartz inequality, Sobolev inequality and (3.1), I_1 can be estimated as follows,

$$I_1 \le C\|V\|_{L^\infty}\|V_x\|\|\Phi_x\| \le CN(T)\left(\|V_x\|^2 + \|\Phi_x\|^2\right).$$

For I_2, we have from the second equation of (3.5) and Lemma 2.1 that

$$\begin{aligned}
I_2 =& -\frac{\mu}{b}\int_{\mathbb{R}} \bar\rho_x (aV_x - b\Phi) V dx \\
=& -\frac{\mu}{b}\int_{\mathbb{R}} \bar\rho_x(\Phi_t - D\Phi_{xx} - g)V dx \\
=& -\frac{\mu}{b}\frac{d}{dt}\int_{\mathbb{R}} \bar\rho_x \Phi V dx + \frac{\mu}{b}\int_{\mathbb{R}} \bar\rho_{xt}\Phi V dx + \frac{\mu}{b}\int_{\mathbb{R}} \bar\rho_x \Phi V_t dx \\
&- \frac{\mu D}{b}\int_{\mathbb{R}} \bar\rho_{xx}\Phi_x V dx - \frac{\mu D}{b}\int_{\mathbb{R}} \bar\rho_x \Phi_x V_x dx + \frac{\mu}{b}\int_{\mathbb{R}} \bar\rho_x(-\bar\phi_t + D\bar\phi_{xx})V dx \\
\le& -\frac{\mu}{b}\frac{d}{dt}\int_{\mathbb{R}} \bar\rho_x \Phi V dx + C\left(\|\bar\rho_{xt}\|^2 + \|\bar\rho_{xx}\|^2\right) + C\|V\|_{L^\infty}^2\left(\|\Phi\|^2 + \|\Phi_x\|^2\right) \\
&+ C\|\bar\rho_x\|_{L^\infty}\left(\|\Phi\|^2 + \|V_t\|^2 + \|\Phi_x\|^2 + \|V_x\|^2\right) + C\|V\|\|\bar\rho_x\|_{L^\infty}\left(\|\bar\phi_t\| + \|\bar\phi_{xx}\|\right) \\
\le& -\frac{\mu}{b}\frac{d}{dt}\int_{\mathbb{R}} \bar\rho_x \Phi V dx + C\delta_0(1+t)^{-\frac{3}{2}} + CN(T)\delta_0(1+t)^{-\frac{5}{4}} \\
&+ C(N(T)+\delta_0)\left(\|\Phi\|^2 + \|V_t\|^2 + \|\Phi_x\|^2 + \|V_x\|^2\right).
\end{aligned}$$

Recall the definition of h and f in (2.6) and (3.2), we have

$$h \sim V_t^2 + \bar\rho_x^2, \quad f \sim \bar\rho_t + V_x^2,$$

which implies that

$$\begin{aligned}
I_3 + I_4 \le& C\int_{\mathbb{R}} |V_t^2 + \bar\rho_x^2||V_x| dx + C\int_{\mathbb{R}} |\bar\rho_t + V_x^2||V_x| dx \\
\le& C(N(T)+\delta_0)\left(\|V_t\|^2 + \|V_x\|^2\right) + C\delta_0(1+t)^{-\frac{3}{2}}.
\end{aligned}$$

Moreover, by using Cauchy-Schwartz inequality and Lemma 2.1, the other three terms can be estimated as

$$I_5 + I_6 + I_7 \le C(N(T)+\delta_0)\left(\|\Phi\|^2 + \|\Phi_x\|^2\right) + C\delta_0(1+t)^{-\frac{3}{2}}.$$

When $1 \le k \le 2$, we estimate I_1-I_7 term by term. By Lemma 2.3, it is easy to have

$$\begin{aligned} I_1 \le& C\left(\|V_x\|_{L^\infty}\|\partial_x^k\Phi_x\| + \|\Phi_x\|_{L^\infty}\|\partial_x^k V_x\|\right)\|\partial_x^k V\| \\ \le& CN(T)\left(\|\partial_x^k\Phi_x\|^2 + \|\partial_x^k V_x\|^2 + \|\partial_x^k V\|^2\right) \end{aligned}$$

and

$$\begin{aligned} I_2 =& -\frac{\mu}{b}\int_{\mathbb{R}} \partial_x^k\left[\bar\rho_x\left(aV_x - b\Phi\right)\right]\partial_x^k V dx \\ \le& C\left(\|\bar\rho_x\|_{L^\infty}\|\partial_x^p(aV_x - b\Phi)\| + \|\left(aV_x - b\Phi\right)\|_{L^\infty}\|\partial_x^k\bar\rho_x\|\right)\|\partial_x^k V\| \\ \le& C\delta_0\left(\|\Phi\|_2^2 + \|V_x\|_2^2\right). \end{aligned}$$

For I_3, I_4, recalling the definition of h and f in (2.6) and using (3.1), Cauchy-Schwartz inequality and Lemma 2.1, we can derive that

$$I_3 + I_4 \le C(N(T)+\delta_0)\left(\|V_t\|_2^2 + \|V_x\|_2^2\right) + C\delta_0(1+t)^{-\frac{3}{2}-k}.$$

Due to the properties of $\bar\rho_t$ and $\bar\rho_x$ in Lemma 2.1, I_5, I_6 and I_7 can be estimated as follows.

$$I_5 + I_6 + I_7 \le C\delta_0\left(\|\partial_x^k\Phi\|^2 + \|\partial_x^k\Phi_x\|^2\right) + C\delta_0(1+t)^{-\frac{3}{2}-k}.$$

In order to complete step 1, we still need to estimate the last term $I_{k,\ell}$. It follows from (2.9), (3.1) and Lemma 2.1 that

$$\begin{aligned} |I_{k,\ell}| =& \left| -\int_{\mathbb{R}} \partial_x^{k-1-\ell}(p''(\bar\rho)\bar\rho_x)\partial_x^\ell V_x \partial_x^k V_x dx + \mu\int_{\mathbb{R}} \partial_x^{k-1-\ell}\bar\rho_x \partial_x^\ell\Phi\partial_x^k V_x dx \right| \\ \le& C(\|\bar\rho_x\|_{L^\infty}\|\partial_x^{k-1}V_x\| + \|V_x\|_{L^\infty}\|\partial_x^{k-1}\bar\rho_x\|)\|\partial^k V_x\| \\ &+ C(\|\bar\rho_x\|_{L^\infty}\|\partial_x^{k-1}\Phi\| + \|\Phi\|_{L^\infty}\|\partial_x^{k-1}\bar\rho_x\|)\|\partial^k V_x\| \\ \le& C\delta_0\left(\|V_x\|_2^2 + \|\Phi\|_2^2\right). \end{aligned}$$

Then (3.4) is obtained by substituting all the estimates I_1-I_7 and $I_{k,\ell}$ into (3.9) and then taking summation over $0 \le k \le 2$.

Step 2. It is easy to check that

$$
\begin{aligned}
&\frac{d}{dt}\sum_{0\le k\le 2}\left\{\frac{1}{2}\|\partial_x^k V_t\|^2+\frac{\mu D}{2a}\int_{\mathbb{R}}\bar\rho\left(\partial_x^k\Phi_x\right)^2dx\right\}+\alpha\|V_t\|_2^2+\frac{\mu}{a}\sum_{0\le k\le 2}\int_{\mathbb{R}}\bar\rho\left(\partial_x^k\Phi_t\right)^2dx\\
&+\frac{1}{2}\frac{d}{dt}\sum_{0\le k\le 2}\left\{\int_{\mathbb{R}}p'(\bar\rho)\left(\partial_x^k V_x\right)^2dx-2\mu\int_{\mathbb{R}}\bar\rho\partial_x^k\Phi\partial_x^k V_x dx+\frac{\mu b}{a}\int_{\mathbb{R}}\bar\rho\left(\partial_x^k\Phi\right)^2dx\right\}\\
\le&\frac{1}{2}\frac{d}{dt}\sum_{0\le k\le 2}\left\{\int_{\mathbb{R}}\frac{(V_t+\frac{1}{\alpha}(q(\bar\rho))_x)^2}{(V_x+\bar\rho)^2}\left(\partial_x^k V_x\right)^2dx-\int_{\mathbb{R}}[p'(V_x+\bar\rho)-p'(\bar\rho)]\left(\partial_x^k V_x\right)^2dx\right\}\\
&+C(\delta_0+N(T))\left(\|V_x\|_2^2+\|V_t\|_2^2+\|\Phi\|_3^2+\|\Phi_t\|_2^2\right)+C\delta_0(1+t)^{-\frac{3}{2}}.
\end{aligned}
\tag{3.10}
$$

Applying ∂_x^k for $0\le k\le 2$ to the first equation of (3.5) and multiplying it by $\partial_x^k V_t$, we get

$$
\begin{aligned}
&\partial_x^k V_{tt}\partial_x^k V_t-\partial_x^k(p'(\bar\rho)V_x)_x\partial_x^k V_t+\alpha\partial_x^k V_t\partial_x^k V_t+\mu\partial_x^k(\bar\rho\Phi)_x\partial_x^k V_t\\
=&-\mu\partial_x^k(V_x\Phi_x)\partial_x^k V_t-\mu\partial_x^k(V_x\bar\phi_x-\bar\rho_x\Phi)\partial_x^k V_t-\partial_x^k h_x\partial_x^k V_t-\partial_x^k f_x\partial_x^k V_t.
\end{aligned}
\tag{3.11}
$$

Noting that

$$
\begin{aligned}
\partial_x^k V_{tt}\partial_x^k V_t=&\frac{1}{2}\frac{d}{dt}|\partial_x^k V_t|^2,\\
-\partial_x^k(p'(\bar\rho)V_x)_x\partial_x^k V_t=&-\partial_x^k(p''(\bar\rho)\bar\rho_x V_x)\partial_x^k V_t-\partial_x^k(p'(\bar\rho)V_{xx})\partial_x^k V_t\\
=&-\partial_x^k(p''(\bar\rho)\bar\rho_x V_x)\partial_x^k V_t-p'(\bar\rho)\partial_x^k V_{xx}\partial_x^k V_t\\
&-\sum_{\ell<k}C_k^\ell\partial_x^{k-\ell}(p'(\bar\rho))\partial_x^\ell V_{xx}\partial_x^k V_t\\
=&-\left(p'(\bar\rho)\partial_x^k V_x\partial_x^k V_t\right)_x+p''(\bar\rho)\bar\rho_x\partial_x^k V_x\partial_x^k V_t+p'(\bar\rho)\partial_x^k V_x\partial_x^k V_{xt}\\
&-\partial_x^k(p''(\bar\rho)\bar\rho_x V_x)\partial_x^k V_t-\sum_{\ell<k}C_k^\ell\partial_x^{k-\ell}(p'(\bar\rho))\partial_x^\ell V_{xx}\partial_x^k V_t\\
=&-\left(p'(\bar\rho)\partial_x^k V_x\partial_x^k V_t\right)_x+p''(\bar\rho)\bar\rho_x\partial_x^k V_x\partial_x^k V_t+\frac{1}{2}\frac{d}{dt}\left(p'(\bar\rho)(\partial_x^k V_x)^2\right)\\
&-\frac{1}{2}p''(\bar\rho)\bar\rho_t\left(\partial_x^k V_x\right)^2-\partial_x^k(p''(\bar\rho)\bar\rho_x V_x)\partial_x^k V_t\\
&-\sum_{\ell<k}C_k^\ell\partial_x^{k-\ell}(p'(\bar\rho))\partial_x^\ell V_{xx}\partial_x^k V_t,\\
\mu\partial_x^k(\bar\rho\Phi)_x\partial_x^k V_t=&\mu[\partial_x^k(\bar\rho\Phi)\partial_x^k V_t]_x-\mu\partial_x^k(\bar\rho\Phi)\partial_x^k V_{xt}\\
=&\mu[\partial_x^k(\bar\rho\Phi)\partial_x^k V_t]_x-\mu\bar\rho\partial_x^k\Phi\partial_x^k V_{xt}-\mu\sum_{\ell<k}C_k^\ell\partial_x^{k-\ell}\bar\rho\partial_x^\ell\Phi\partial_x^k V_{xt}
\end{aligned}
$$

$$
\begin{aligned}
&=\mu[\partial_x^k(\bar\rho\Phi)\partial_x^k V_t]_x-\mu\frac{d}{dt}\left(\bar\rho\partial_x^k\Phi\partial_x^k V_x\right)+\mu\bar\rho_t\partial_x^k\Phi\partial_x^k V_x\\
&\quad+\mu\bar\rho\partial_x^k\Phi_t\partial_x^k V_x-\mu\sum_{\ell<k}C_k^\ell\left(\partial_x^{k-\ell}\bar\rho\partial_x^\ell\Phi\partial_x^k V_t\right)_x\\
&\quad+\mu\sum_{\ell<k}C_k^\ell\left(\partial_x^{k-\ell}\bar\rho_x\partial_x^\ell\Phi\partial_x^k V_t+\partial_x^{k-\ell}\bar\rho\partial_x^\ell\Phi_x\partial_x^k V_t\right),
\end{aligned}
$$

$$
\mu\partial_x^k(V_x\bar\phi_x-\bar\rho_x\Phi)\partial_x^k V_t=\frac{\mu}{b}\partial_x^k\left[\bar\rho_x\left(aV_x-b\Phi\right)\right]\partial_x^k V_t.
$$

Substituting the above equalities into (3.11) and integrating the equation over $\mathbb{R}$, we have

$$
\begin{aligned}
&\frac{d}{dt}\left\{\frac12\|\partial_x^k V_t\|^2+\frac12\int_{\mathbb{R}}p'(\bar\rho)\left(\partial_x^k V_x\right)^2dx-\mu\int_{\mathbb{R}}\bar\rho\partial_x^k\Phi\partial_x^k V_x dx\right\}+\alpha\|\partial_x^k V_t\|^2\\
&+\mu\int_{\mathbb{R}}\bar\rho\partial_x^k\Phi_t\partial_x^k V_x dx\\
=&\frac12\int_{\mathbb{R}}p''(\bar\rho)\bar\rho_t\left(\partial_x^k V_x\right)^2dx-\int_{\mathbb{R}}p''(\bar\rho)\bar\rho_x\partial_x^k V_x\partial_x^k V_t dx+\int_{\mathbb{R}}\partial_x^k\left(p''(\bar\rho)\bar\rho_x V_x\right)\partial_x^k V_t dx\\
&+\sum_{\ell<k}C_k^\ell\int_{\mathbb{R}}\partial_x^{k-\ell}(p'(\bar\rho))\partial_x^\ell V_{xx}\partial_x^k V_t dx \qquad (3.12)\\
&-\mu\sum_{\ell<k}C_k^\ell\int_{\mathbb{R}}\left(\partial_x^{k-\ell}\bar\rho_x\partial_x^\ell\Phi\partial_x^k V_t+\partial_x^{k-\ell}\bar\rho\partial_x^\ell\Phi_x\partial_x^k V_t\right)dx-\mu\int_{\mathbb{R}}\bar\rho_t\partial_x^k\Phi\partial_x^k V_x dx\\
&-\int_{\mathbb{R}}\mu\partial_x^k(V_x\Phi_x)\partial_x^k V_t dx-\frac{\mu}{b}\int_{\mathbb{R}}\partial_x^k\left[\bar\rho_x\left(aV_x-b\Phi\right)\right]\partial_x^k V_t dx-\int_{\mathbb{R}}\partial_x^k h_x\partial_x^k V_t dx\\
&-\int_{\mathbb{R}}\partial_x^k f_x\partial_x^k V_t dx.
\end{aligned}
$$

Applying ∂_x^k for $0\le k\le 2$ to the second equation of (3.5), multiplying the resultant equation by $\frac{\mu}{a}\bar\rho\partial_x^k\Phi_t$ and taking integration in x give

$$
\begin{aligned}
&\frac{d}{dt}\left\{\frac{\mu b}{2a}\int_{\mathbb{R}}\bar\rho\left(\partial_x^k\Phi\right)^2dx+\frac{\mu D}{2a}\int_{\mathbb{R}}\bar\rho\left(\partial_x^k\Phi_x\right)^2dx\right\}+\frac{\mu}{a}\int_{\mathbb{R}}\bar\rho\left(\partial_x^k\Phi_t\right)^2dx\\
&-\mu\int_{\mathbb{R}}\partial_x^k V_x\bar\rho\partial_x^k\Phi_t dx \qquad (3.13)\\
=&\frac{\mu b}{2a}\int_{\mathbb{R}}\bar\rho_t\left(\partial_x^k\Phi\right)^2dx+\frac{\mu D}{2a}\int_{\mathbb{R}}\bar\rho_t\left(\partial_x^k\Phi_x\right)^2dx-\frac{\mu D}{a}\int_{\mathbb{R}}\bar\rho_x\partial_x^k\Phi_x\partial_x^k\Phi_t dx
\end{aligned}
$$

$$+\frac{\mu}{a}\int_{\mathbb{R}}\bar\rho\partial_x^k\Phi_t\partial_x^k g dx.$$

Combining (3.12) with (3.13) yields

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{2}\|\partial_x^k V_t\|^2+\frac{\mu D}{2a}\int_{\mathbb{R}}\bar\rho\left(\partial_x^k\Phi_x\right)^2dx\right\}+\alpha\|\partial_x^k V_t\|^2+\frac{\mu}{a}\int_{\mathbb{R}}\bar\rho\left(\partial_x^k\Phi_t\right)^2dx\\&+\frac{1}{2}\frac{d}{dt}\left\{\int_{\mathbb{R}}p'(\bar\rho)\left(\partial_x^k V_x\right)^2dx-2\mu\int_{\mathbb{R}}\bar\rho\partial_x^k\Phi\partial_x^k V_x dx+\frac{\mu b}{a}\int_{\mathbb{R}}\bar\rho\left(\partial_x^k\Phi\right)^2dx\right\}\\&=:J_1^{(k)}+J_2^{(k)}+J_3^{(k)}+J_4^{(k)}+\sum_{\ell<k}C_k^\ell J_{k,\ell},\end{aligned}\tag{3.14}$$

where

$$\begin{aligned}J_1^{(k)}=&\frac{1}{2}\int_{\mathbb{R}}p''(\bar\rho)\bar\rho_t\left(\partial_x^k V_x\right)^2dx-\int_{\mathbb{R}}p''(\bar\rho)\bar\rho_x\partial_x^k V_x\partial_x^k V_t dx-\mu\int_{\mathbb{R}}\bar\rho_t\partial_x^k\Phi\partial_x^k V_x dx\\&+\frac{\mu b}{2a}\int_{\mathbb{R}}\bar\rho_t\left(\partial_x^k\Phi\right)^2dx+\frac{\mu D}{2a}\int_{\mathbb{R}}\bar\rho_t\left(\partial_x^k\Phi_x\right)^2dx-\frac{\mu D}{a}\int_{\mathbb{R}}\bar\rho_x\partial_x^k\Phi_x\partial_x^k\Phi_t dx,\end{aligned}$$

$$J_2^{(k)}=\int_{\mathbb{R}}\partial_x^k\left(p''(\bar\rho)\bar\rho_x V_x\right)\partial_x^k V_t dx-\int_{\mathbb{R}}\mu\partial_x^k(V_x\Phi_x)\partial_x^k V_t dx-\frac{\mu}{b}\int_{\mathbb{R}}\partial_x^k\left[\bar\rho_x\left(aV_x-b\Phi\right)\right]\partial_x^k V_t dx,$$

$$J_3^{(k)}=-\int_{\mathbb{R}}\partial_x^k h_x\partial_x^k V_t dx-\int_{\mathbb{R}}\partial_x^k f_x\partial_x^k V_t dx=:J_{3h}^{(k)}+J_{3f}^{(k)},$$

$$J_4^{(k)}=\frac{\mu}{a}\int_{\mathbb{R}}\bar\rho\partial_x^k\Phi_t\partial_x^k g dx$$

and

$$J_{k,\ell}=\int_{\mathbb{R}}\partial_x^{k-\ell}(p'(\bar\rho))\partial_x^\ell V_{xx}\partial_x^k V_t dx-\mu\int_{\mathbb{R}}\partial_x^{k-\ell}\bar\rho_x\partial_x^\ell\Phi\partial_x^k V_t dx-\mu\int_{\mathbb{R}}\partial_x^{k-\ell}\bar\rho\partial_x^\ell\Phi_x\partial_x^k V_t dx.$$

It follows from the Cauchy-Schwartz inequality and Lemma 2.1 that

$$J_1^{(k)}\le C\delta_0(\|\partial_x^k V_x\|^2+\|\partial_x^k V_t\|^2+\|\partial_x^k\Phi\|^2+\|\partial_x^k\Phi_x\|^2+\|\partial_x^k\Phi_t\|^2).$$

Due to Lemma 2.3, we can estimate $J_2^{(k)}$ and $J_{k,\ell}$ as follows,

$$
\begin{aligned}
J_2^{(k)} \leq & C\left(\|\bar{\rho}_x\|_{L^\infty}\|\partial_x^k[V_x,\Phi]\| + \|[V_x,\Phi]\|_{L^\infty}\|\partial_x^k\bar{\rho}_x\|\right)\|\partial_x^k V_t\| \\
& + \left(\|V_x\|_{L^\infty}\|\partial_x^k\Phi_x\| + \|\Phi_x\|_{L^\infty}\|\partial_x^k V_x\|\right)\|\partial_x^k V_t\| \\
\leq & C(N(T)+\delta_0)(\|V_x\|_2^2 + \|\partial_x^k V_t\|^2 + \|\Phi\|_3^2).
\end{aligned}
$$

Thanks to $\ell < k$, $J_{k,\ell}$ can be rewritten in the following form

$$
\begin{aligned}
J_{k,\ell} = & -\int_{\mathbb{R}} \partial_x^{k-1-\ell}(p''(\bar{\rho})\bar{\rho}_x)\partial_x^\ell V_{xx}\partial_x^k V_t dx - \mu\int_{\mathbb{R}} \partial_x^{k-\ell}\bar{\rho}_x\partial_x^\ell\Phi\partial_x^k V_t dx \\
& - \mu\int_{\mathbb{R}} \partial_x^{k-1-\ell}\bar{\rho}_x\partial_x^\ell\Phi_x\partial_x^k V_t dx \\
\leq & C\left(\|\bar{\rho}_x\|_{L^\infty}\|\partial_x^{k-1}[V_{xx}\Phi_x]\| + \|[V_{xx},\Phi_x]\|_{L^\infty}\|\partial_x^{k-1}\bar{\rho}_x\|\right)\|\partial_x^k V_t\| \\
& + \left(\|\bar{\rho}_x\|_{L^\infty}\|\partial_x^k\Phi\| + \|\Phi\|_{L^\infty}\|\partial_x^k\bar{\rho}_x\|\right)\|\partial_x^k V_t\| \\
\leq & C\delta_0(\|V_x\|_2^2 + \|\partial_x^k V_t\|^2 + \|\Phi\|_3^2).
\end{aligned}
$$

For $J_4^{(k)}$, with the definition of g in (2.6), using the Cauchy-Schwartz inequality and Lemma 2.1, we have

$$
J_4^{(k)} \leq C\delta_0\|\partial_x^k\Phi_t\|^2 + \frac{C}{\delta_0}\|\partial_x^k g\|^2 \leq C\delta_0\|\partial_x^k\Phi_t\|^2 + C\delta_0(1+t)^{-\frac{3}{2}-k}.
$$

In order to estimate $J_3^{(k)}$, we calculate h_x and f_x first. Note that

$$
\begin{aligned}
h_x = -\left(\frac{(V_t + \frac{1}{\alpha}q(\bar{\rho})_x)^2}{V_x+\bar{\rho}}\right)_x = & \frac{(V_t+\frac{1}{\alpha}(q(\bar{\rho}))_x)^2}{(V_x+\bar{\rho})^2}V_{xx} - \frac{2(V_t+\frac{1}{\alpha}q(\bar{\rho})_x)}{V_x+\bar{\rho}}V_{xt} \\
& + \frac{(V_t+\frac{1}{\alpha}(q(\bar{\rho}))_x)^2}{(V_x+\bar{\rho})^2}\bar{\rho}_x - \frac{2(V_t+\frac{1}{\alpha}q(\bar{\rho})_x)}{V_x+\bar{\rho}}\frac{1}{\alpha}q(\bar{\rho})_{xx},
\end{aligned}
\tag{3.15}
$$

and

$$
\begin{aligned}
f_x = & \frac{1}{\alpha}(q(\bar{\rho}))_{xt} - \left[p(V_x+\bar{\rho}) - p(\bar{\rho}) - p'(\bar{\rho})V_x\right]_x \\
= & \frac{1}{\alpha}(q(\bar{\rho}))_{xt} - [p'(V_x+\bar{\rho}) - p'(\bar{\rho})]V_{xx} - (p'(V_x+\bar{\rho}) - p'(\bar{\rho}) - p''(\bar{\rho})V_x)\bar{\rho}_x.
\end{aligned}
\tag{3.16}
$$

Then, for $J_{3h}^{(k)}$, we focus on the term which contains the highest-order derivative V_{xx} and V_{xt},

$$
J_{3h}^{(k)} = -\int_{\mathbb{R}} \partial_x^k h_x \partial_x^k V_t dx
$$

$$
\begin{aligned}
&= - \int_{\mathbb{R}} \frac{(V_t + \frac{1}{\alpha}(q(\bar\rho))_x)^2}{(V_x+\bar\rho)^2} \partial_x^k V_{xx} \partial_x^k V_t dx + \int_{\mathbb{R}} \frac{2(V_t + \frac{1}{\alpha} q(\bar\rho)_x)}{V_x+\bar\rho} \partial_x^k V_{xt} \partial_x^k V_t dx \\
&\quad - \int_{\mathbb{R}} (\text{O.T.H.}) \partial_x^k V_t dx \qquad (3.17) \\
&= \int_{\mathbb{R}} \left(\frac{(V_t + \frac{1}{\alpha}(q(\bar\rho))_x)^2}{(V_x+\bar\rho)^2} \right)_x \partial_x^k V_x \partial_x^k V_t dx + \frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}} \frac{(V_t + \frac{1}{\alpha}(q(\bar\rho))_x)^2}{(V_x+\bar\rho)^2} \left(\partial_x^k V_x\right)^2 dx \\
&\quad - \frac{1}{2} \int_{\mathbb{R}} \left(\frac{(V_t + \frac{1}{\alpha}(q(\bar\rho))_x)^2}{(V_x+\bar\rho)^2} \right)_t \left(\partial_x^k V_x\right)^2 dx - \int_{\mathbb{R}} \left(\frac{(V_t + \frac{1}{\alpha} q(\bar\rho)_x)}{V_x+\bar\rho} \right)_x \left(\partial_x^k V_t\right)^2 dx \\
&\quad - \int_{\mathbb{R}} (\text{O.T.H.}) \partial_x^k V_t dx,
\end{aligned}
$$

where (O.T.H.) as an abbreviation for "other terms of $\partial_x^k h_x$" reading as

$$
\begin{aligned}
(\text{O.T.H.}) = &\sum_{\ell<k} C_k^\ell \partial_x^{k-\ell} \left(\frac{(V_t + \frac{1}{\alpha}(q(\bar\rho))_x)^2}{(V_x+\bar\rho)^2} \right) \partial_x^\ell V_{xx} - \sum_{\ell<k} C_k^\ell \partial_x^{k-\ell} \left(\frac{2(V_t + \frac{1}{\alpha} q(\bar\rho)_x)}{V_x+\bar\rho} \right) \partial_x^\ell V_{xt} \\
&+ \partial_x^k \left(\frac{(V_t + \frac{1}{\alpha}(q(\bar\rho))_x)^2}{(V_x+\bar\rho)^2} \bar\rho_x - \frac{2(V_t + \frac{1}{\alpha} q(\bar\rho)_x)}{V_x+\bar\rho} \frac{1}{\alpha} q(\bar\rho)_{xx} \right).
\end{aligned}
$$

Similarly, for $J_{3f}^{(k)}$, we focus on the term which contains the highest-order derivative V_{xx},

$$
\begin{aligned}
J_{3f}^{(k)} &= - \int_{\mathbb{R}} \partial_x^k f_x \partial_x^k V_t dx = \int_{\mathbb{R}} [p'(V_x+\bar\rho) - p'(\bar\rho)] \partial_x^k V_{xx} \partial_x^k V_t dx - \int_{\mathbb{R}} (\text{O.T.F.}) \partial_x^k V_t dx, \\
&= - \int_{\mathbb{R}} [p'(V_x+\bar\rho) - p'(\bar\rho)] \partial_x^k V_x \partial_x^k V_{xt} dx - \int_{\mathbb{R}} [p'(V_x+\bar\rho) - p'(\bar\rho)]_x \partial_x^k V_x \partial_x^k V_t dx \\
&\quad - \int_{\mathbb{R}} (\text{O.T.F.}) \partial_x^k V_t dx \qquad (3.18) \\
&= - \frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}} [p'(V_x+\bar\rho) - p'(\bar\rho)] \left(\partial_x^k V_x\right)^2 dx + \frac{1}{2} \int_{\mathbb{R}} [p'(V_x+\bar\rho) - p'(\bar\rho)]_t \left(\partial_x^k V_x\right)^2 dx \\
&\quad - \int_{\mathbb{R}} [p'(V_x+\bar\rho) - p'(\bar\rho)]_x \partial_x^k V_x \partial_x^k V_t dx - \int_{\mathbb{R}} (\text{O.T.F.}) \partial_x^k V_t dx,
\end{aligned}
$$

with

$$(\text{O.T.F.}) = -\sum_{\ell<k} C_k^\ell \partial_x^{k-\ell}[p'(V_x+\bar\rho)-p'(\bar\rho)]\partial_x^\ell V_{xx} + \partial_x^k\left(\frac{1}{\alpha}(q(\bar\rho))_{xt} - (p'(V_x+\bar\rho)-p'(\bar\rho)-p''(\bar\rho)V_x)\bar\rho_x\right).$$

Here we use (O.T.F.) as an abbreviation for "other terms of $\partial_x^k f_x$". Clearly the total order of spatial derivatives for terms which are the product between V_x and V_t is not greater than 3 in (O.T.H.) and (O.T.F.), and the highest order of derivatives for V_x and V_t is not greater than 2. Then we can use Lemma 2.3 and Cauchy-Schwartz inequality to get

$$\int_{\mathbb{R}} (\text{O.T.H.})\partial_x^k V_t dx + \int_{\mathbb{R}} (\text{O.T.F.})\partial_x^k V_t dx \le C(\delta_0+N(T))\left(\|V_x\|_2^2+\|V_t\|_2^2\right) + C\delta_0(1+t)^{-\frac{5}{2}-k},$$

where we have used the following fact

$$|V_{tt}| \le C\left(|V_{xx}|+|V_t|+|V_x||\Phi_x|+(|\bar\rho_x|+|\bar\phi_x|)|V_x|+|V_{xt}|+|\Phi_x|+|\bar\rho_x|^3+|\bar\rho_t||\bar\rho_x|\right) \le C(N(T)+\delta_0).$$

Based on the above calculations for $J_{3h}^{(k)}$ and $J_{3f}^{(k)}$, we can estimate $J_3^{(k)}$ as follows,

$$J_3^{(k)} \le \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \frac{(V_t+\frac{1}{\alpha}(q(\bar\rho))_x)^2}{(V_x+\bar\rho)^2}\left(\partial_x^k V_x\right)^2 dx - \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} [p'(V_x+\bar\rho)-p'(\bar\rho)]\left(\partial_x^k V_x\right)^2 dx + C(\delta_0+N(T))\left(\|V_x\|_2^2+\|V_t\|_2^2\right) + C\delta_0(1+t)^{-\frac{5}{2}-k}.$$

Feeding (3.14) on all the estimations of $J_1^{(k)}$-$J_4^{(k)}$ and $J_{k,\ell}$, and taking summation of (3.14) over $0\le k\le 2$, one has (3.10).

Step 3. First, under the assumptions $p'(\bar\rho) - \frac{a\mu}{b}\bar\rho > 0$ and $\rho_- < \bar\rho < \rho_+$, we have

$$\int_{\mathbb{R}} p'(\bar\rho)\left(\partial_x^k V_x\right)^2 dx - 2\mu\int_{\mathbb{R}} \bar\rho\partial_x^k\Phi\partial_x^k V_x dx + \frac{\mu b}{a}\int_{\mathbb{R}} \bar\rho\left(\partial_x^k\Phi\right)^2 dx \sim \|[\partial_x^k\Phi, \partial_x^k V_x]\|^2. \quad (3.19)$$

Second, adding (3.4) with (3.10) multiplied by a constant K which will be determined later, it follows that

$$\begin{aligned}&\frac{d}{dt}\mathcal{E}(t) + C\left(\|V_x\|_2^2+\|\Phi\|_3^2+\|\Phi_t\|_2^2\right) + (\alpha K-1)\|V_t\|_2^2 \\ \le& C\delta_0(1+t)^{-\frac{5}{4}} + C(N(T)+\delta_0)(\|V_x\|_2^2+\|V_t\|_2^2+\|\Phi\|_3^2+\|\Phi_t\|_2^2),\end{aligned} \quad (3.20)$$

where $\alpha K > 1$, $\mathcal{E}(t)$ is given by

$$
\begin{aligned}
\mathcal{E}(t) = & \sum_{0\le k\le 2}\left\{\frac{\alpha}{2}\|\partial_x^k V\|^2 + \int_{\mathbb{R}} \partial_x^k V_t \partial_x^k V dx + \frac{K}{2}\|\partial_x^k V_t\|^2\right\} \\
& + \frac{K}{2}\sum_{0\le k\le 2}\left\{\int_{\mathbb{R}} p'(\bar\rho)\left(\partial_x^k V_x\right)^2 dx - 2\mu\int_{\mathbb{R}} \bar\rho \partial_x^k \Phi \partial_x^k V_x dx + \frac{\mu b}{a}\int_{\mathbb{R}} \bar\rho\left(\partial_x^k\Phi\right)^2 dx\right\} \\
& + \sum_{0\le k\le 2}\left\{\frac{\mu}{2a}\int_{\mathbb{R}} \bar\rho\left(\partial_x^k\Phi\right)^2 dx + \frac{\mu DK}{2a}\int_{\mathbb{R}} \bar\rho\left(\partial_x^k\Phi_x\right)^2 dx\right\} \\
& + \frac{\mu}{b}\int_{\mathbb{R}} \bar\rho_x \Phi V dx - \frac{K}{2}\sum_{0\le k\le 2}\int_{\mathbb{R}} \frac{(V_t + \frac{1}{\alpha}(q(\bar\rho))_x)^2}{(V_x+\bar\rho)^2}\left(\partial_x^k V_x\right)^2 dx \\
& + \frac{K}{2}\sum_{0\le k\le 2}\int_{\mathbb{R}} [p'(V_x+\bar\rho) - p'(\bar\rho)]\left(\partial_x^k V_x\right)^2 dx.
\end{aligned}
\tag{3.21}
$$

On the one hand, we can choose K large enough such that

$$
\frac{\alpha}{2}\|\partial_x^k V\|^2 + \int_{\mathbb{R}} \partial_x^k V_t \partial_x^k V dx + \frac{K}{2}\|\partial_x^k V_t\|^2 \ge C\left(\|\partial_x^k V_t\|^2 + \|\partial_x^k V\|^2\right)
$$

for some constant $C > 0$ independent of t. On the other hand, with the Cauchy-Schwartz inequality, Lemma 2.1 and *a priori* assumption, we have

$$
\int_{\mathbb{R}} \bar\rho_x \Phi V dx \le C\delta_0\left(\|\Phi\|^2 + \|V\|^2\right),
$$

$$
\int_{\mathbb{R}} \frac{(V_t + \frac{1}{\alpha}(q(\bar\rho))_x)^2}{(V_x+\bar\rho)^2}\left(\partial_x^k V_x\right)^2 dx \le C(N(T)+\delta_0)\|\partial_x^k V_x\|^2
$$

and

$$
\int_{\mathbb{R}} [p'(V_x+\bar\rho) - p'(\bar\rho)]\left(\partial_x^k V_x\right)^2 dx \le CN(T)\|\partial_x^k V_x\|^2.
$$

Choosing $N(T)$ and δ_0 small enough, one gets from (3.19) and (3.21) that

$$
\mathcal{E}(t) \sim \|V\|_3^2 + \|V_t\|_2^2 + \|\Phi\|_3^2.
$$

With the above equivalence for $\mathcal{E}(t)$, after integrating (3.20) over $(0, t)$, and taking $N(T)$ and δ_0 sufficiently small, we get (3.3). Thus, the proof of Proposition 3.2 is completed. □

Finally, we need to verify that the *a priori* assumption (3.1) is achievable. Since under the *a priori* assumption (3.1), we have proved that (3.3) holds true when $N(T)$ is appropriately small. So as long as ε_0 is small enough, (3.1) is ensured by (3.3). As such under the conditions of Proposition 2.2, we close the *a priori* assumption (3.1).

4. The time-decay rate of solutions

In this section, we are devoted to establishing the decay rate of the solution (V, M, Φ) or (V, V_t, Φ) to (2.4). First, we make the following *a priori* assumption on (V, V_t, Φ)

$$\sum_{k=0}^{2}(1+t)^{k+1}\left\|\partial_x^k[V_x,\Phi]\right\|^2+\sum_{k=0}^{2}(1+t)^{k+2}\left\|\partial_x^k V_t\right\|^2+\sum_{k=0}^{1}(1+t)^{k+3}\left\|\partial_x^k\Phi_t\right\|^2\ll 1. \quad (4.1)$$

Then we turn to prove the time-decay rate of (V, V_t, Φ) which is indicated by the following energy estimates.

Proposition 4.1. *Under the assumption of Proposition 2.2, we have*

$$\begin{aligned}&\sum_{k=0}^{2}(1+t)^{k+1}\left\|\partial_x^k[V_x,\Phi,\Phi_x]\right\|^2+\sum_{k=0}^{2}\int_0^t(1+s)^k\left\|\partial_x^k[V_x,\Phi,\Phi_x](\cdot,s)\right\|^2ds\\&\le C\left(\|V_0\|_3^2+\|M_0\|_2^2+\|\Phi_0\|_4^2+\delta_0\right)\le C\epsilon^2\end{aligned} \quad (4.2)$$

and

$$\begin{aligned}&\sum_{k=0}^{2}(1+t)^{k+2}\left\|\partial_x^k V_t\right\|^2+\sum_{k=0}^{2}\int_0^t(1+s)^{k+1}\left\|\partial_x^k V_t(\cdot,s)\right\|^2ds\\&+\sum_{k=0}^{1}(1+t)^{k+3}\left\|\partial_x^k[\Phi_t,\Phi_{xt}]\right\|^2+\sum_{k=0}^{1}\int_0^t(1+s)^{k+2}\left\|\partial_x^k[\Phi_t,\Phi_{xt}](\cdot,s)\right\|^2ds\\&\le C\left(\|V_0\|_3^2+\|M_0\|_2^2+\|\Phi_0\|_4^2+\delta_0\right)\le C\epsilon^2.\end{aligned}$$

Similar to the proof for the global existence, with the help of the assumption $bp'(\bar\rho)-a\mu\bar\rho>0$, the decay rates of V_x and Φ can be obtained. The main ideas of the proof for Proposition 4.1 come from [45] and [47], but additional efforts are needed to deal with the term g in $(2.4)_2$. The time-weighted energy estimates (4.2) for V_x and Φ follows from Lemmas 4.2-4.4 and a coarse decay rate for V_t can also be obtained simultaneously. The refined decay rate of V_t follows from Lemma 4.5. Here, due to the similarity, we only give the detailed proof of Lemmas 4.2-4.4 to see how we deal with the coupling of V and Φ.

Lemma 4.2. *Under the assumption of Proposition 2.2, we have*

$$(1+t)\|[V_x,V_t,\Phi,\Phi_x]\|^2+\int_0^t(1+s)(\|V_t\|^2+\|\Phi_t\|^2)ds\le C\left(\|V_0\|_3^2+\|M_0\|_2^2+\|\Phi_0\|_4^2+\delta_0\right).$$

Proof. Let $k=0$ in (3.14), we have

$$
\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\|V_t\|^2+\frac{\mu D}{2a}\int_{\mathbb{R}}\bar{\rho}|\Phi_x|^2dx\right\}+\alpha\|V_t\|^2+\frac{\mu}{a}\int_{\mathbb{R}}\bar{\rho}\Phi_t^2dx\\
&+\frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbb{R}}p'(\bar{\rho})V_x^2dx-\mu\int_{\mathbb{R}}\bar{\rho}\Phi V_xdx+\frac{\mu b}{2a}\int_{\mathbb{R}}\bar{\rho}\Phi^2dx\right\}\\
&=:J_1^{(0)}+J_2^{(0)}+J_3^{(0)}+J_4^{(0)},
\end{aligned}
\tag{4.3}
$$

with

$$
\begin{aligned}
J_1^{(0)}=&\frac{1}{2}\int_{\mathbb{R}}p''(\bar{\rho})\bar{\rho}_tV_x^2dx-\int_{\mathbb{R}}p''(\bar{\rho})\bar{\rho}_xV_xV_tdx-\mu\int_{\mathbb{R}}\bar{\rho}_t\Phi V_xdx\\
&+\frac{\mu b}{2a}\int_{\mathbb{R}}\bar{\rho}_t\Phi^2dx+\frac{\mu D}{2a}\int_{\mathbb{R}}\bar{\rho}_t\Phi_x^2dx-\frac{\mu D}{a}\int_{\mathbb{R}}\bar{\rho}_x\Phi_x\Phi_tdx,\\
J_2^{(0)}=&\int_{\mathbb{R}}p''(\bar{\rho})\bar{\rho}_xV_xV_tdx-\int_{\mathbb{R}}\mu V_x\Phi_xV_tdx-\frac{\mu}{b}\int_{\mathbb{R}}\bar{\rho}_x\left(aV_x-b\Phi\right)V_tdx,\\
J_3^{(0)}=&-\int_{\mathbb{R}}h_xV_tdx-\int_{\mathbb{R}}f_xV_tdx,\qquad J_4^{(0)}=\frac{\mu}{a}\int_{\mathbb{R}}\bar{\rho}\Phi_tgdx.
\end{aligned}
$$

Now we deal with the terms of the right hand side of (4.3) one by one. It follows from Lemma 2.1 that

$$
J_1^{(0)}\le C\delta_0(1+t)^{-1}\int_{\mathbb{R}}[V_x^2+\Phi^2+\Phi_x^2]dx+C\delta_0\int_{\mathbb{R}}[V_t^2+\Phi_t^2]dx,
$$

where we have used that

$$
\begin{aligned}
&-\int_{\mathbb{R}}p''(\bar{\rho})\bar{\rho}_xV_xV_tdx-\frac{\mu D}{a}\int_{\mathbb{R}}\bar{\rho}_x\Phi_x\Phi_tdx\\
\le&\delta_0\int_{\mathbb{R}}[V_t^2+\Phi_t]^2dx+\frac{C}{\delta_0}\|\bar{\rho}_x\|_{L^\infty}^2\int_{\mathbb{R}}[V_x^2+\Phi_x^2]dx\\
\le&\delta_0\int_{\mathbb{R}}[V_t^2+\Phi_t]^2dx+C\delta_0(1+t)^{-1}\int_{\mathbb{R}}[V_x^2+\Phi_x^2]dx.
\end{aligned}
$$

Similarly, we can estimate the first and the third term of $J_2^{(0)}$, and for the second term, by using the *a priori* assumption (4.1), the Sobolev and Cauchy-Schwartz inequalities, we have

$$
\begin{aligned}
-\mu\int_{\mathbb{R}} V_x\Phi_x V_t dx \leq& \frac{\alpha}{4}\int_{\mathbb{R}} V_t^2 dx + C\|\Phi_x\|_{L^\infty}^2\int_{\mathbb{R}} V_x^2 dx\\
\leq& \frac{\alpha}{4}\int_{\mathbb{R}} V_t^2 dx + C\|\Phi_x\|\|\Phi_{xx}\|\int_{\mathbb{R}} V_x^2 dx\\
\leq& \frac{\alpha}{4}\int_{\mathbb{R}} V_t^2 dx + C(1+t)^{-\frac{5}{2}}\int_{\mathbb{R}} V_x^2 dx,
\end{aligned}
$$

which implies that

$$
J_2^{(0)} \leq C(1+t)^{-1}\int_{\mathbb{R}}[V_x^2+\Phi^2]dx + C\left(\frac{\alpha}{4}+\delta_0\right)\int_{\mathbb{R}} V_t^2 dx.
$$

Recalling (3.18) and (3.17) when $k=0$, we can estimate $J_3^{(0)}$ as

$$
\begin{aligned}
J_3^{(0)} \leq& -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}[p'(V_x+\bar{\rho})-p'(\bar{\rho})]V_x^2 dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\frac{(V_t+\frac{1}{\alpha}(q(\bar{\rho}))_x)^2}{(V_x+\bar{\rho})^2}V_x^2 dx\\
&+ C\varepsilon_0\int_{\mathbb{R}} V_t^2 dx + C(1+t)^{-1}\int_{\mathbb{R}} V_x^2 dx + C\delta_0(1+t)^{-\frac{5}{2}},
\end{aligned}
$$

where we have used (4.1), Lemma 2.1, (2.4) and the following estimates:

$$
\left\|\left(\frac{(V_t+\frac{1}{\alpha}(q(\bar{\rho}))_x)^2}{(V_x+\bar{\rho})^2}\right)_x\right\|_{L^\infty} + \left\|\left(\frac{(V_t+\frac{1}{\alpha}(q(\bar{\rho}))_x)^2}{(V_x+\bar{\rho})^2}\right)_t\right\|_{L^\infty} \leq C(1+t)^{-1}.
$$

Noticing that

$$
\|g\|^2 \leq C\left(\|\bar{\phi}_t\|^2 + \|\bar{\phi}_{xx}\|^2\right) \leq C\delta_0^2(1+t)^{-\frac{3}{2}}
$$

is insufficient to warrant the decay rate $(1+t)^{-1}$ of $\|\Phi\|^2$, we estimate $J_4^{(0)}$ as follows

$$
\begin{aligned}
J_4^{(0)} =& \frac{\mu}{a}\int_{\mathbb{R}}\bar{\rho}\Phi_t g dx = \frac{d}{dt}\left(\frac{\mu}{a}\int_{\mathbb{R}}\bar{\rho}\Phi g dx\right) - \frac{\mu}{a}\int_{\mathbb{R}}\bar{\rho}_t\Phi g dx - \frac{\mu}{a}\int_{\mathbb{R}}\bar{\rho}\Phi g_t dx\\
\leq& \frac{d}{dt}\left(\frac{\mu}{a}\int_{\mathbb{R}}\bar{\rho}\Phi g dx\right) + C(1+t)^{-1}\int_{\mathbb{R}}\Phi^2 dx + C(1+t)\int_{\mathbb{R}}\left[g_t^2+\bar{\rho}_t^2 g^2\right]dx\\
\leq& \frac{d}{dt}\left(\frac{\mu}{a}\int_{\mathbb{R}}\bar{\rho}\Phi g dx\right) + C(1+t)^{-1}\int_{\mathbb{R}}\Phi^2 dx + C\delta_0(1+t)^{-\frac{5}{2}}.
\end{aligned}
$$

The estimation from $J_1^{(0)}$ to $J_4^{(0)}$ updates (4.3) as

$$
\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbb{R}} p'(\bar\rho)V_x^2dx-\mu\int_{\mathbb{R}}\bar\rho\Phi V_xdx+\frac{\mu b}{2a}\int_{\mathbb{R}}\bar\rho\Phi^2dx\right\}\\
&+\frac{d}{dt}\left\{\frac{1}{2}\|V_t\|^2+\frac{\mu D}{2a}\int_{\mathbb{R}}\bar\rho|\Phi_x|^2dx\right\}+\frac{\alpha}{2}\|V_t\|^2+\frac{\mu}{2a}\int_{\mathbb{R}}\bar\rho\Phi_t^2dx\\
\le& C(1+t)^{-1}\int_{\mathbb{R}}[V_x^2+\Phi^2+\Phi_x^2]dx+\frac{d}{dt}\left(\frac{\mu}{a}\int_{\mathbb{R}}\bar\rho\Phi gdx\right)+C\delta_0(1+t)^{-\frac{5}{2}}\\
&-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}[p'(V_x+\bar\rho)-p'(\bar\rho)]V_x^2dx+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\frac{(V_t+\frac{1}{\alpha}(q(\bar\rho))_x)^2}{(V_x+\bar\rho)^2}V_x^2dx.
\end{aligned}
\tag{4.4}
$$

Multiplying (4.4) with $(1+t)$, we have

$$
\begin{aligned}
&\frac{d}{dt}\left((1+t)\int_{\mathbb{R}}\left[\frac{1}{2}p'(\bar\rho)V_x^2-\mu\bar\rho\Phi V_x+\frac{\mu b}{2a}\bar\rho\Phi^2\right]dx\right)\\
&+\frac{d}{dt}\left((1+t)\int_{\mathbb{R}}\left[\frac{1}{2}V_t^2+\frac{D\mu}{2a}\bar\rho\Phi_x^2\right]dx\right)+\frac{1}{2}(1+t)\left(\int_{\mathbb{R}}\alpha V_t^2dx+\frac{\mu}{a}\int_{\mathbb{R}}\bar\rho\Phi_t^2dx\right)\\
=&\int_{\mathbb{R}}\left[\frac{1}{2}p'(\bar\rho)V_x^2-\mu\bar\rho\Phi V_x+\frac{\mu b}{2a}\bar\rho\Phi^2\right]dx+\int_{\mathbb{R}}\left[\frac{1}{2}V_t^2+\frac{D\mu}{2a}\bar\rho\Phi_x^2\right]dx\\
&+C\int_{\mathbb{R}}\left[V_x^2+\Phi^2+\Phi_x^2\right]dx+\frac{d}{dt}\left((1+t)\frac{\mu}{a}\int_{\mathbb{R}}\bar\rho\Phi gdx\right)-\frac{\mu}{a}\int_{\mathbb{R}}\bar\rho\Phi gdx+C\delta_0(1+t)^{-\frac{3}{2}}\\
&+\frac{1}{2}\frac{d}{dt}\left((1+t)\int_{\mathbb{R}}\frac{(V_t+\frac{1}{\alpha}(q(\bar\rho))_x)^2}{(V_x+\bar\rho)^2}V_x^2dx\right)-\frac{1}{2}\int_{\mathbb{R}}\frac{(V_t+\frac{1}{\alpha}(q(\bar\rho))_x)^2}{(V_x+\bar\rho)^2}V_x^2dx\\
&-\frac{1}{2}\frac{d}{dt}\left((1+t)\int_{\mathbb{R}}[p'(V_x+\bar\rho)-p'(\bar\rho)]V_x^2dx\right)+\frac{1}{2}\int_{\mathbb{R}}[p'(V_x+\bar\rho)-p'(\bar\rho)]V_x^2dx.
\end{aligned}
\tag{4.5}
$$

Integrating (4.5) over $[0,t]$ and using Proposition 3.2 give us that

$$
(1+t)\|[V_x,V_t,\Phi,\Phi_x]\|^2+\int_0^t(1+s)(\|V_t\|^2+\|\Phi_t\|^2)dt\le C\left(\|V_0\|_3^2+\|M_0\|_2^2+\|\Phi_0\|_3^2+\delta_0\right),
$$

where we have used the following estimates

$$\begin{aligned}(1+t)\frac{\mu}{a}\int_{\mathbb{R}}\bar{\rho}\Phi g dx \le& C\delta_0(1+t)\int_{\mathbb{R}}\Phi^2 dx+\frac{C}{\delta_0}(1+t)\int_{\mathbb{R}} g^2 dx\\ \le&\delta_0(1+t)\int_{\mathbb{R}}\Phi^2 dx + C\delta_0\end{aligned}$$

and

$$\frac{\mu}{a}\int_0^t\int_{\mathbb{R}}\bar{\rho}\Phi g dxdt \le C\int_0^t\int_{\mathbb{R}}\Phi^2 dxdt+\int_0^t\int_{\mathbb{R}} g^2 dxdt \le C\left(\|V_0\|_3^2+\|M_0\|_2^2+\|\Phi_0\|_3^2+\delta_0\right).$$

Thus, the proof of Lemma 4.2 is completed. □

Lemma 4.3. *Under the assumption of Proposition 2.2, we have*

$$\begin{aligned}&(1+t)^2\|[V_{xt},V_{xx},\Phi_x,\Phi_{xx}]\|^2+\int_0^t[(1+s)^2\|[V_{xt},\Phi_{xt}]\|^2+(1+s)\|[V_{xx},\Phi_x,\Phi_{xx}]\|^2]ds\\ \le& C\left(\|V_0\|_3^2+\|M_0\|_2^2+\|\Phi_0\|_3^2+\delta_0\right).\end{aligned} \tag{4.6}$$

Proof. The proof consists of the following three steps.

Step 1. Letting $k=1$ in (3.14), we have

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{2}\|V_{xt}\|^2+\frac{\mu D}{2a}\int_{\mathbb{R}}\bar{\rho}\Phi_{xx}^2 dx\right\}+\alpha\|V_{xt}\|^2+\frac{\mu}{a}\int_{\mathbb{R}}\bar{\rho}\Phi_{xt}^2 dx\\ &+\frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbb{R}}p'(\bar{\rho})V_{xx}^2 dx-\mu\int_{\mathbb{R}}\bar{\rho}\Phi_x V_{xx}dx+\frac{\mu b}{2a}\int_{\mathbb{R}}\bar{\rho}\Phi_x^2 dx\right\}\\ &=: J_1^{(1)}+J_2^{(1)}+J_3^{(1)}+J_4^{(1)}+J_{1,0},\end{aligned} \tag{4.7}$$

with

$$\begin{aligned}J_1^{(1)}=&\frac{1}{2}\int_{\mathbb{R}}p''(\bar{\rho})\bar{\rho}_t V_{xx}^2 dx-\int_{\mathbb{R}}p''(\bar{\rho})\bar{\rho}_x V_{xx}V_{xt}dx-\mu\int_{\mathbb{R}}\bar{\rho}_t\Phi_x V_{xx}dx\\ &+\frac{\mu b}{2a}\int_{\mathbb{R}}\bar{\rho}_t\Phi_x^2 dx+\frac{\mu D}{2a}\int_{\mathbb{R}}\bar{\rho}_t\Phi_{xx}^2 dx-\frac{\mu D}{a}\int_{\mathbb{R}}\bar{\rho}_x\Phi_{xx}\Phi_{xt}dx,\end{aligned}$$

$$J_2^{(1)} = \int_{\mathbb{R}} \left(p''(\bar\rho)\bar\rho_x V_x\right)_x V_{xt} dx - \int_{\mathbb{R}} \mu (V_x \Phi_x)_x V_{xt} dx - \frac{\mu}{b} \int_{\mathbb{R}} [\bar\rho_x (a V_x - b\Phi)]_x V_{xt} dx,$$

$$J_3^{(1)} = -\int_{\mathbb{R}} h_{xx} V_{xt} dx - \int_{\mathbb{R}} f_{xx} V_{xt} dx, \qquad J_4^{(1)} = \frac{\mu}{a} \int_{\mathbb{R}} \bar\rho \Phi_{xt} g_x dx$$

and

$$J_{1,0} = \int_{\mathbb{R}} p''(\bar\rho)\bar\rho_x V_{xx} V_{xt} dx - \mu \int_{\mathbb{R}} \bar\rho_{xx} \Phi V_{xt} dx - \mu \int_{\mathbb{R}} \bar\rho_x \Phi_x V_{xt} dx.$$

Similar to Lemma 4.2, the term $J_4^{(1)}$ can be estimated as

$$\begin{aligned} J_4^{(1)} =& \frac{\mu}{a} \int_{\mathbb{R}} \bar\rho \Phi_{xt} g_x dx = \frac{d}{dt}\left(\frac{\mu}{a}\int_{\mathbb{R}} \bar\rho \Phi_x g_x dx\right) - \frac{\mu}{a}\int_{\mathbb{R}} \bar\rho_t \Phi_x g_x dx - \frac{\mu}{a}\int_{\mathbb{R}} \bar\rho \Phi_x g_{xt} dx \\ \le& \frac{d}{dt}\left(\frac{\mu}{a}\int_{\mathbb{R}} \bar\rho \Phi_x g_x dx\right) + C(1+t)^{-1}\int_{\mathbb{R}} \Phi_x^2 dx + C(1+t)\int_{\mathbb{R}} \left[g_{xt}^2 + \bar\rho_t^2 g_x^2\right] dx \\ \le& \frac{d}{dt}\left(\frac{\mu}{a}\int_{\mathbb{R}} \bar\rho \Phi_x g_x dx\right) + C(1+t)^{-1}\int_{\mathbb{R}} \Phi_x^2 dx + C\delta_0 (1+t)^{-\frac{7}{2}}. \end{aligned}$$

Recalling (3.17) and (3.18), we have

$$\begin{aligned} J_3^{(1)} =& -\int_{\mathbb{R}} h_{xx} V_{xt} dx - \int_{\mathbb{R}} f_{xx} V_{xt} dx \\ \le& -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} [p'(V_x+\bar\rho) - p'(\bar\rho)] V_{xx}^2 dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \frac{(V_t + \frac{1}{\alpha}(q(\bar\rho))_x)^2}{(V_x+\bar\rho)^2} V_{xx}^2 dx \\ &+ C\varepsilon_0 \int_{\mathbb{R}} V_{xt}^2 dx + C(1+t)^{-1}\int_{\mathbb{R}} V_{xx}^2 dx + C(1+t)^{-2}\int_{\mathbb{R}} [V_x^2 + V_t^2] dx + C\delta_0 (1+t)^{-\frac{7}{2}}. \end{aligned}$$

Substituting the above inequalities into (4.7) leads to

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} V_{xt}^2 dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \left(p'(\bar\rho) V_{xx}^2 dx - 2\mu\bar\rho\Phi_x V_{xx} + \frac{\mu b \bar\rho}{a}\Phi_x^2\right) dx \\ &+ \frac{D\mu}{2a}\frac{d}{dt}\int_{\mathbb{R}} \bar\rho \Phi_{xx}^2 dx + \frac{\mu}{a}\int_{\mathbb{R}} \bar\rho \Phi_{xt}^2 dx + \alpha \int_{\mathbb{R}} V_{xt}^2 dx \end{aligned}$$

$$\leq \frac{1}{2}\int_{\mathbb{R}} \left(\frac{\mu\bar{\rho}}{a}\Phi_{xt}^2 + \alpha V_{xt}^2\right) dx + C(1+t)^{-1}\int_{\mathbb{R}} \left[V_{xx}^2 + \Phi_x^2 + \Phi_{xx}^2\right] dx \tag{4.8}$$

$$+ C\delta_0(1+t)^{-\frac{7}{2}} + C(1+t)^{-2}\int_{\mathbb{R}} [V_x^2 + V_t^2 + \Phi^2] dx + \frac{d}{dt}\left(\frac{\mu}{a}\int_{\mathbb{R}} \bar{\rho}\Phi_x g_x dx\right)$$

$$+ \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \frac{(V_t + \frac{1}{\alpha}(q(\bar{\rho}))_x)^2}{(V_x+\bar{\rho})^2} V_{xx}^2 dx - \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} [p'(V_x+\bar{\rho}) - p'(\bar{\rho})] V_{xx}^2 dx.$$

Step 2. Multiplying $(2.4)_1$ with $-V_{xx}$ and $(2.4)_2$ with $-\frac{\mu\bar{\rho}}{a}\Phi_{xx}$, integrating the result over $\mathbb{R}$, and using the integration by parts, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}} \left(\frac{\alpha}{2}V_x^2 + V_{xt}V_x\right) dx + \frac{\mu}{2a}\frac{d}{dt}\int_{\mathbb{R}} \bar{\rho}\Phi_x^2 dx$$

$$+ \int_{\mathbb{R}} \left(p'(\bar{\rho})V_{xx}^2 - 2\mu\bar{\rho}\Phi_x V_{xx} + \frac{b\mu\bar{\rho}}{a}\Phi_x^2\right) dx + \frac{\mu D}{a}\int_{\mathbb{R}} \bar{\rho}\Phi_{xx}^2 dx$$

$$= \int_{\mathbb{R}} V_{xt}^2 dx - \int_{\mathbb{R}} p''(\bar{\rho})\bar{\rho}_x V_x V_{xx} dx + \mu\int_{\mathbb{R}} V_x\Phi_x V_{xx} dx$$

$$+ \mu\int_{\mathbb{R}} V_x\bar{\phi}_x V_{xx} dx + \int_{\mathbb{R}} (h_x + f_x)V_{xx} dx + \mu\int_{\mathbb{R}} \bar{\rho}_x\Phi_x V_x dx$$

$$+ \frac{\mu}{2a}\int_{\mathbb{R}} \bar{\rho}_t\Phi_x^2 dx - \frac{\mu}{a}\int_{\mathbb{R}} \bar{\rho}_x\Phi_t\Phi_x dx - \frac{b\mu}{a}\int_{\mathbb{R}} \bar{\rho}_x\Phi_x\Phi dx - \int_{\mathbb{R}} \frac{\mu\bar{\rho}}{a} g\Phi_{xx} dx.$$

Integrating the last term by parts twice leads to

$$-\int_{\mathbb{R}} \frac{\mu\bar{\rho}}{a} g\Phi_{xx} dx = -\frac{\mu}{a}\int_{\mathbb{R}} (\bar{\rho}g)_{xx}\,\Phi dx$$

$$\leq (1+t)^{-1}\int_{\mathbb{R}} \Phi^2 dx + C(1+t)\int_{\mathbb{R}} \left(g_{xx}^2 + \bar{\rho}_{xx}^2 g^2 + \bar{\rho}_x^2 g_x^2\right) dx$$

$$\leq (1+t)^{-1}\int_{\mathbb{R}} \Phi^2 dx + C\delta_0(1+t)^{-\frac{5}{2}}.$$

Thus we arrive at

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left(\frac{\alpha}{2}V_x^2+V_{xt}V_x\right)dx+\frac{\mu}{2a}\frac{d}{dt}\int_{\mathbb{R}}\bar{\rho}\Phi_x^2dx\\
&+\int_{\mathbb{R}}\left[p'(\bar{\rho})V_{xx}^2-2\mu\bar{\rho}\Phi_xV_{xx}+\frac{b\mu\bar{\rho}}{a}\Phi_x^2\right]dx+\frac{\mu D}{a}\int_{\mathbb{R}}\bar{\rho}\Phi_{xx}^2dx\\
\le&2\int_{\mathbb{R}}V_{xt}^2dx+C(1+t)^{-1}\int_{\mathbb{R}}[V_x^2+\Phi^2+\Phi_t^2+V_t^2]dx\\
&+C\delta_0(1+t)^{-\frac{5}{2}}+C\varepsilon_0\int_{\mathbb{R}}[V_{xx}^2+\Phi_x^2]dx.
\end{aligned}
\tag{4.9}
$$

Here, we have used the expressions of h_x and f_x in (3.15) and (3.16).

Step 3. Multiplying $(1+t)$ to $[2\times(4.8)+\frac{\alpha}{4}\times(4.9)]$ and integrating the result over $[0,t]$, we get

$$
\begin{aligned}
&(1+t)\|[V_x,V_{xt},V_{xx},\Phi_x,\Phi_{xx}]\|^2+\int_0^t(1+s)\|[V_{xt},\Phi_{xt},V_{xx},\Phi_x,\Phi_{xx}]\|^2ds\\
\le&C\left(\|V_0\|_3^2+\|M_0\|_2^2+\|\Phi_0\|_3^2+\delta_0\right).
\end{aligned}
\tag{4.10}
$$

Similarly, multiplying (4.8) by $(1+t)^2$ and integrating the resulting equation over $[0,t]$, we have from (4.10) that

$$
\begin{aligned}
&(1+t)^2\|[V_{xt},V_{xx},\Phi_x,\Phi_{xx}]\|^2+\int_0^t(1+s)^2\|[V_{xt},\Phi_{xt}]\|^2ds\\
\le&C\left(\|V_0\|_3^2+\|M_0\|_2^2+\|\Phi_0\|_3^2+\delta_0\right),
\end{aligned}
\tag{4.11}
$$

where we have used the following estimates

$$
\begin{aligned}
(1+t)\frac{\mu}{a}\int_{\mathbb{R}}\bar{\rho}\Phi_xg_xdx\le&C\delta_0(1+t)\int_{\mathbb{R}}\Phi_x^2dx+\frac{C}{\delta_0}(1+t)\int_{\mathbb{R}}g_x^2dx\\
\le&\delta_0(1+t)\int_{\mathbb{R}}\Phi_x^2dx+C\delta_0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mu}{a}\int_0^t\int_{\mathbb{R}}\bar{\rho}\Phi_xg_xdxds\le&C\int_0^t\int_{\mathbb{R}}\Phi_x^2dxds+C\int_0^t\int_{\mathbb{R}}g_x^2dxds\\
\le&C\left(\|V_0\|_3^2+\|M_0\|_2^2+\|\Phi_0\|_3^2+\delta_0\right),
\end{aligned}
$$

$$
\begin{aligned}
(1+t)^2\frac{\mu}{a}\int_{\mathbb{R}}\bar{\rho}\Phi_x g_x dx \le& C\delta_0(1+t)^2\int_{\mathbb{R}}\Phi_x^2 dx+\frac{C}{\delta_0}(1+t)^2\int_{\mathbb{R}} g_x^2 dx\\
\le&\delta_0(1+t)^2\int_{\mathbb{R}}\Phi_x^2 dx + C\delta_0
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\mu}{a}\int_0^t(1+s)\int_{\mathbb{R}}\bar{\rho}\Phi_x g_x dxds \le& C\int_0^t(1+s)\int_{\mathbb{R}}\Phi_x^2 dxds + C\int_0^t(1+s)\int_{\mathbb{R}} g_x^2 dxds\\
\le& C\left(\|V_0\|_3^2+\|M_0\|_2^2+\|\Phi_0\|_3^2+\delta_0\right).
\end{aligned}
$$

Then the combination of (4.10) and (4.11) gives (4.6). The proof of Lemma 4.3 is completed. □

Lemma 4.4. *Under the assumption of Proposition 2.2, we have*

$$
\begin{aligned}
&(1+t)^3\|[V_{xxx},V_{xxt},\Phi_{xx},\Phi_{xxx}]\|^2+\int_0^t(1+s)^3\|[V_{xxt},\Phi_{xxt}]\|^2\\
&+(1+t)^2\|[V_{xxx},\Phi_{xx},\Phi_{xxx}]\|^2 ds\\
\le& C\left(\|V_0\|_3^2+\|M_0\|_2^2+\|\Phi_0\|_3^2+\delta_0\right).
\end{aligned}
$$

Proof. The proof consists of three steps.

Step 1. Let $k=2$ in (3.14), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} V_{xxt}^2 dx+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left[p'(\bar{\rho})V_{xxx}^2-2\mu\bar{\rho}\Phi_{xx}V_{xxx}+\frac{\mu b\bar{\rho}}{a}\Phi_{xx}^2\right]dx\\
&+\frac{D\mu}{2a}\frac{d}{dt}\int_{\mathbb{R}}\bar{\rho}\Phi_{xxx}^2 dx+\alpha\int_{\mathbb{R}} V_{xxt}^2 dx+\frac{\mu}{a}\int_{\mathbb{R}}\bar{\rho}\Phi_{xxt}^2 dx\\
&=: J_1^{(2)}+J_2^{(2)}+J_3^{(2)}+J_4^{(2)}+\sum_{\ell<2}J_{2,\ell},
\end{aligned}
$$

with

$$
\begin{aligned}
J_1^{(2)}=&\frac{1}{2}\int_{\mathbb{R}} p''(\bar{\rho})\bar{\rho}_t V_{xxx}^2 dx-\int_{\mathbb{R}} p''(\bar{\rho})\bar{\rho}_x V_{xxx}V_{txx}dx-\mu\int_{\mathbb{R}}\bar{\rho}_t\Phi_{xx}V_{xxx}dx\\
&+\frac{\mu b}{2a}\int_{\mathbb{R}}\bar{\rho}_t\Phi_{xx}^2 dx+\frac{\mu D}{2a}\int_{\mathbb{R}}\bar{\rho}_t\Phi_{xxx}^2 dx-\frac{\mu D}{a}\int_{\mathbb{R}}\bar{\rho}_x\Phi_{xxx}\Phi_{txx}dx,
\end{aligned}
$$

$$J_2^{(2)} = \int_{\mathbb{R}} \left(p''(\bar\rho)\bar\rho_x V_x\right)_{xx} V_{txx}dx - \int_{\mathbb{R}} (V_x\Phi_x)_{xx}V_{txx}dx - \frac{\mu}{b}\int_{\mathbb{R}} [\bar\rho_x\,(aV_x - b\Phi)]_{xx}V_{txx}dx,$$

$$J_3^{(2)} = -\int_{\mathbb{R}} h_{xxx}V_{txx}dx - \int_{\mathbb{R}} f_{xxx}V_{txx}dx, \qquad J_4^{(2)} = \frac{\mu}{a}\int_{\mathbb{R}} \bar\rho\Phi_{txx}g_{xx}dx$$

and

$$J_{2,\ell} = \int_{\mathbb{R}} \partial_x^{2-\ell}(p'(\bar\rho))\partial_x^\ell V_{xx}V_{txx}dx - \mu\int_{\mathbb{R}} \partial_x^{2-\ell}\bar\rho_x\partial_x^\ell\Phi V_{txx}dx - \mu\int_{\mathbb{R}} \partial_x^{2-\ell}\bar\rho\partial_x^\ell\Phi_x V_{txx}dx.$$

It is similar to estimate $J_4^{(2)}$ as in Lemma 4.2 that

$$\begin{aligned}
\frac{\mu}{a}\int_{\mathbb{R}} \bar\rho\Phi_{xxt}g_{xx}dx =& \frac{d}{dt}\left(\frac{\mu}{a}\int_{\mathbb{R}} \bar\rho\Phi_{xx}g_{xx}dx\right) - \frac{\mu}{a}\int_{\mathbb{R}} \bar\rho_t\Phi_{xx}g_{xx}dx - \frac{\mu}{a}\int_{\mathbb{R}} \bar\rho\Phi_{xx}g_{xxt}dx \\
\le& \frac{d}{dt}\left(\frac{\mu}{a}\int_{\mathbb{R}} \bar\rho\Phi_{xx}g_{xx}dx\right) + C(1+t)^{-1}\int_{\mathbb{R}} \Phi_{xx}^2dx \\
&+ C(1+t)\int_{\mathbb{R}} \left[g_{xxt}^2 + \bar\rho_t^2 g_{xx}^2\right]dx \\
\le& \frac{d}{dt}\left(\frac{\mu}{a}\int_{\mathbb{R}} \bar\rho\Phi_{xx}g_{xx}dx\right) + C(1+t)^{-1}\int_{\mathbb{R}} \Phi_{xx}^2dx + C\delta_0(1+t)^{-\frac{9}{2}}.
\end{aligned}$$

Then, by a direct calculation, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} V_{xxt}^2dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \left[p'(\bar\rho)V_{xxx}^2 - 2\mu\bar\rho\Phi_{xx}V_{xxx} + \frac{\mu b\bar\rho}{a}\Phi_{xx}^2\right]dx \\
&+ \frac{D\mu}{2a}\frac{d}{dt}\int_{\mathbb{R}} \bar\rho\Phi_{xxx}^2dx + \frac{\mu}{a}\int_{\mathbb{R}} \bar\rho\Phi_{xxt}^2dx + \alpha\int_{\mathbb{R}} V_{xxt}^2dx \\
\le& \frac{1}{2}\int_{\mathbb{R}} \left[\alpha V_{xxt}^2 + \frac{\mu\bar\rho}{a}\Phi_{xxt}^2\right]dx + C(1+t)^{-1}\int_{\mathbb{R}} [V_{xxx}^2 + \Phi_{xx}^2 + \Phi_{xxx}^2 + V_{xt}^2]dx \qquad (4.12)\\
&+ C(1+t)^{-2}\int_{\mathbb{R}} [V_{xx}^2 + \Phi_x^2]dx + C(1+t)^{-3}\int_{\mathbb{R}} [V_x^2 + V_t^2 + \Phi^2]dx \\
&+ C\delta_0(1+t)^{-\frac{9}{2}} + \frac{d}{dt}\left(\frac{\mu}{a}\int_{\mathbb{R}} \bar\rho\Phi_{xx}g_{xx}dx\right)
\end{aligned}$$

$$+\frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{R}}\frac{(V_t+\frac{1}{\alpha}(q(\bar\rho))_x)^2}{(V_x+\bar\rho)^2}V_{xxx}^2dx-\frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{R}}[p'(V_x+\bar\rho)-p'(\bar\rho)]V_{xxx}^2dx.$$

Step 2. Differentiating (2.4) with respect to x, we obtain

$$\begin{cases} V_{xtt}-(p'(\bar\rho)V_x)_{xx}+\alpha V_{xt}+\mu(\bar\rho\Phi_x)_x=-(\mu V_x\Phi_x+\mu V_x\bar\phi_x)_x-h_{xx}-f_{xx},\\ \Phi_{xt}-D\Phi_{xxx}-aV_{xx}+b\Phi_x=g_x. \end{cases} \tag{4.13}$$

Multiplying $(4.13)_1$ with $-V_{xxx}$ and multiplying $(4.13)_2$ with $-\frac{\mu\bar\rho}{a}\Phi_{xxx}$, we have

$$\begin{aligned}
&\frac{d}{dt}\int\limits_{\mathbb{R}}\left(\frac{\alpha}{2}V_{xx}^2+V_{xxt}V_{xx}\right)dx+\frac{\mu}{2a}\frac{d}{dt}\int\limits_{\mathbb{R}}\bar\rho\Phi_{xx}^2dx\\
&+\int\limits_{\mathbb{R}}\left[p'(\bar\rho)V_{xxx}^2-2\mu\bar\rho\Phi_{xx}V_{xxx}+\frac{b\mu\bar\rho}{a}\Phi_{xx}^2\right]dx+\frac{\mu D}{a}\int\limits_{\mathbb{R}}\bar\rho\Phi_{xxx}^2dx\\
=&\int\limits_{\mathbb{R}}V_{xxt}^2dx-2\int\limits_{\mathbb{R}}p''(\bar\rho)\bar\rho_xV_{xx}V_{xxx}dx-\int\limits_{\mathbb{R}}(p'(\bar\rho))_{xx}V_xV_{xxx}dx\\
&+\mu\int\limits_{\mathbb{R}}(V_x\Phi_x)_xV_{xxx}dx+\mu\int\limits_{\mathbb{R}}(V_x\bar\phi_x)_xV_{xxx}dx+\mu\int\limits_{\mathbb{R}}\bar\rho_x\Phi_xV_{xxx}dx\\
&+\int\limits_{\mathbb{R}}(h+f)_{xx}V_{xxx}dx+\frac{\mu}{2a}\int\limits_{\mathbb{R}}\bar\rho_t\Phi_{xx}^2dx-\frac{\mu}{a}\int\limits_{\mathbb{R}}\bar\rho_x\Phi_{xt}\Phi_{xx}dx\\
&+\mu\int\limits_{\mathbb{R}}\bar\rho_x\Phi_{xx}V_{xx}dx-\frac{b\mu}{a}\int\limits_{\mathbb{R}}\bar\rho_x\Phi_x\Phi_{xx}dx-\frac{\mu}{a}\int\limits_{\mathbb{R}}\bar\rho g_x\Phi_{xxx}dx.
\end{aligned} \tag{4.14}$$

Integrating the last term by parts twice, and then we have

$$\begin{aligned}
-\frac{\mu}{a}\int\limits_{\mathbb{R}}\bar\rho g_x\Phi_{xxx}dx=&-\frac{\mu}{a}\int\limits_{\mathbb{R}}\bar\rho g_{xxx}\Phi_xdx-\frac{\mu}{a}\int\limits_{\mathbb{R}}\bar\rho_{xx}g_x\Phi_xdx-\frac{2\mu}{a}\int\limits_{\mathbb{R}}\bar\rho_xg_{xx}\Phi_xdx\\
\le&C\delta_0(1+t)^{-1}\int\limits_{\mathbb{R}}\Phi_x^2dx+\frac{C}{\delta_0}(1+t)\int\limits_{\mathbb{R}}\left(g_{xxx}^2+\bar\rho_{xx}^2g_x^2+\bar\rho_x^2g_{xx}^2\right)dx\\
\le&C\delta_0(1+t)^{-1}\int\limits_{\mathbb{R}}\Phi_x^2dx+C\delta_0(1+t)^{-\frac{7}{2}},
\end{aligned}$$

which along with (4.14) yields

$$\begin{aligned}&\frac{d}{dt}\int\limits_{\mathbb{R}}\left(\frac{\alpha}{2}V_{xx}^2+V_{xxt}V_{xx}\right)dx+\frac{\mu}{2a}\frac{d}{dt}\int\limits_{\mathbb{R}}\bar{\rho}\Phi_{xx}^2dx\\&+\int\limits_{\mathbb{R}}\left[p'(\bar{\rho})V_{xxx}^2-2\mu\bar{\rho}\Phi_{xx}V_{xxx}+\frac{b\mu\bar{\rho}}{a}\Phi_{xx}^2\right]dx+\frac{\mu D}{a}\int\limits_{\mathbb{R}}\bar{\rho}\Phi_{xxx}^2dx\\\le&2\int\limits_{\mathbb{R}}V_{xxt}^2dx+C\varepsilon_0\int\limits_{\mathbb{R}}[V_{xxx}^2+\Phi_{xx}^2]dx+C\delta_0(1+t)^{-\frac{7}{2}}\\&+C(1+t)^{-1}\int\limits_{\mathbb{R}}[V_{xx}^2+\Phi_x^2+\Phi_{xt}^2+V_{xt}^2]dx+C(1+t)^{-2}\int\limits_{\mathbb{R}}[V_x^2+V_t^2]dx.\end{aligned}$$

Step 3. Multiplying $(1+t)^j$ $(j=1,2)$ to $[2\times(4.14)+\frac{\alpha}{4}\times(4.12)]$ and integrate the result over $[0,t]$, we get by Lemma 4.2 and Lemma 4.3 that

$$\begin{aligned}&(1+t)^2\|[V_{xx},V_{xxt},V_{xxx},\Phi_{xx},\Phi_{xxx}]\|^2+\int\limits_0^t(1+s)^2\|[V_{xxx},V_{xxt},\Phi_{xx},\Phi_{xxx},\Phi_{xxt}]\|^2ds\\\le&C\left(\|V_0\|_3^2+\|M_0\|_2^2+\|\Phi_0\|_3^2+\delta_0\right).\end{aligned}\tag{4.15}$$

Furthermore integrating (4.12) multiplied by $(1+t)^{j+1}$ $(j=1,2)$ over $[0,t]$, we obtain from (4.15) that

$$\begin{aligned}&(1+t)^3\|[V_{xxt},V_{xxx},\Phi_{xx},\Phi_{xxx}]\|^2+\int\limits_0^t(1+s)^3\|[V_{xxt},\Phi_{xxt}]\|^2ds\\\le&C\left(\|V_0\|_3^2+\|M_0\|_2^2+\|\Phi_0\|_3^2+\delta_0\right).\end{aligned}$$

Here, for $j=1,2$, we have used the following estimates

$$\begin{aligned}(1+t)^j\frac{\mu}{a}\int\limits_{\mathbb{R}}\bar{\rho}\Phi_{xx}g_{xx}dx\le&C\delta_0(1+t)^j\int\limits_{\mathbb{R}}\Phi_{xx}^2dx+\frac{C}{\delta_0}(1+t)^j\int\limits_{\mathbb{R}}g_{xx}^2dx\\\le&\delta_0(1+t)^j\int\limits_{\mathbb{R}}\Phi_x^2dx+C\delta_0(1+t)^{-\frac{5}{2}+j}\\\le&\delta_0(1+t)^j\int\limits_{\mathbb{R}}\Phi_x^2dx+C\delta_0,\end{aligned}$$

$$\frac{\mu}{a}\int\limits_0^t(1+s)^{j-1}\int\limits_{\mathbb{R}}\bar{\rho}\Phi_{xx}g_{xx}dxds\le C\int\limits_0^t(1+s)^{j-1}\int\limits_{\mathbb{R}}\Phi_{xx}^2dx+C\int\limits_0^t(1+s)^{j-1}\int\limits_{\mathbb{R}}g_{xx}^2dxds$$

$$
\begin{aligned}
&\le C\int_0^t (1+s)^{j-1}\int_{\mathbb{R}} \Phi_{xx}^2 dx + C\int_0^t (1+s)^{-\frac{9}{2}+j} ds\\
&\le C\left(\|V_0\|_3^2 + \|M_0\|_2^2 + \|\Phi_0\|_3^2 + \delta_0\right),
\end{aligned}
$$

$$
\begin{aligned}
(1+t)^{j+1}\frac{\mu}{a}\int_{\mathbb{R}} \bar\rho \Phi_{xx} g_{xx} dx \le& C\delta_0 (1+t)^{j+1}\int_{\mathbb{R}} \Phi_{xx}^2 dx + \frac{C}{\delta_0}(1+t)^{j+1}\int_{\mathbb{R}} g_{xx}^2 dx\\
\le&\delta_0 (1+t)^{j+1}\int_{\mathbb{R}} \Phi_x^2 dx + C\delta_0 (1+t)^{-\frac{5}{2}+j}\\
\le&\delta_0 (1+t)^{j+1}\int_{\mathbb{R}} \Phi_x^2 dx + C\delta_0
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\mu}{a}\int_0^t (1+s)^j \int_{\mathbb{R}} \bar\rho \Phi_{xx} g_{xx} dx ds \le& C\int_0^t (1+s)^j\int_{\mathbb{R}} \Phi_{xx}^2 dx + C\int_0^t (1+s)^j\int_{\mathbb{R}} g_{xx}^2 dx ds\\
\le& C\int_0^t (1+s)^j\int_{\mathbb{R}} \Phi_{xx}^2 dx + C\int_0^t (1+s)^{-\frac{7}{2}+j} ds\\
\le& C\left(\|V_0\|_3^2 + \|M_0\|_2^2 + \|\Phi_0\|_3^2 + \delta_0\right).
\end{aligned}
$$

Thus, we complete the proof of Lemma 4.4. □

Similar to Lemma 4.4, we can get the following higher-order estimates for which we only outline the procedures without details for brevity.

Lemma 4.5. *Under the assumption of Proposition 2.2, we have*

$$
\begin{aligned}
&(1+t)^3\|[V_{tt}, V_{xt}, \Phi_t, \Phi_{xt}]\|^2 + (1+t)^2\|V_t\|^2\\
&\quad + \int_0^t (1+s)^3\|[V_{tt}, \Phi_{tt}]\|^2 + (1+t)^2\|[V_{xt}, \Phi_{xt}, \Phi_t]\|^2 ds \qquad (4.16)\\
&\le C\left(\|V_0\|_3^2 + \|M_0\|_2^2 + \|\Phi_0\|_3^2 + \delta_0\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&(1+t)^4\|[V_{xtt}, V_{xxt}, \Phi_{xt}, \Phi_{xxt}]\|^2 + (1+t)^3\|V_{xt}\|^2\\
&\quad + \int_0^t (1+s)^4\|[V_{xtt}, \Phi_{xtt}]\|^2 ds + (1+t)^3\|[V_{xxt}, \Phi_{xt}, \Phi_{xxt}]\|^2 ds \qquad (4.17)
\end{aligned}
$$

$$\leq C\left(\|V_0\|_3^2+\|M_0\|_2^2+\|\Phi_0\|_4^2+\delta_0\right).$$

Proof. In fact, taking integration over $\mathbb{R}\times(0,t)$ of equations $(1+t)^i\times\{2[\partial_t(3.5)_1\times V_{tt}+\partial_t(3.5)_2\times\frac{\mu\bar\rho}{a}\Phi_{tt}]+\frac{\alpha}{4}[\partial_t(3.5)_1\times V_t+\partial_t(3.5)_2\times\frac{\mu\bar\rho}{a}\Phi_t]\}$, and $(1+t)^{i+1}\times[\partial_t(3.5)_1\times V_{tt}+\partial_t(3.5)_2\times\frac{\mu\bar\rho}{a}\Phi_{tt}]$ for $i=0,1,2$, we get (4.16). Taking integration over $\mathbb{R}\times(0,t)$ of equations $(1+t)^j\times\{2[\partial_{xt}(3.5)_1\times V_{xtt}+\partial_{xt}(3.5)_2\times\frac{\mu\bar\rho}{a}\Phi_{xtt}]+\frac{\alpha}{4}[\partial_{xt}(3.5)_1\times V_{xt}+\partial_t(3.5)_2\times\frac{\mu\bar\rho}{a}\Phi_{xt}]\}$, and $(1+t)^{j+1}\times[\partial_{xt}(3.5)_1\times V_{xtt}+\partial_{xt}(3.5)_2\times\frac{\mu\bar\rho}{a}\Phi_{xtt}]$ for $j=0,1,2,3$, we get (4.17). □

Note Proposition 4.1 is a direct consequence of Lemmas 4.2-4.5 shown above. Thus, we can close the *a priori* assumptions (4.1) by taking ϵ to be sufficiently small in Proposition 2.2.

Finally we are in a position to prove Proposition 2.2 and Theorem 1.1.

Proof of Proposition 2.2. The first part of Proposition 2.2 (global existence) is a consequence of Proposition 3.1 and Proposition 3.2. For the decay rate, we have from Proposition 4.1 directly.

Proof of Theorem 1.1. Noticing that $M(x,t)=-V_t(x,t)$ and the transformation (2.1), we get (1.11) from (2.7) by the Sobolev inequality $\|f\|^2_{L^\infty(\mathbb{R})}\leq 2\|f\|_{L^2(\mathbb{R})}\|f_x\|_{L^2(\mathbb{R})}$ and hence complete the proof.

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