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DEVELOPMENT OF TRAVELING WAVES IN AN INTERACTING TWO-SPECIES CHEMOTAXIS MODEL

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Abstract. By constructing sub and super solutions, we establish the existence of traveling wave solutions to a two-species chemotaxis model, which describes two interacting species chemotactically reacting to a chemical signal that is degraded by the two species. We identify the full parameter regime in which the traveling wave solutions exist, derive the asymptotical decay rates of traveling wave solutions at far field and show that the traveling wave solutions are convergent as the chemical diffusion coefficient goes to zero.

1. **Introduction.** The chemotaxis system, a well-known mathematical model to describe cell motion in response to a chemical cue, has many applications in biological systems. For instance, the propagation of traveling bands of chemotactic bacterial along a capillary tube toward the chemical energy sources consumed by the bacteria can be described by the following model proposed by Keller and Segel [13]

$$\begin{cases} u_t = [u_x - \chi u \phi(w)_x]_x, \\ w_t = \varepsilon w_{xx} - u \end{cases} \tag{1.1}$$

where $u(x,t)$ denotes the bacterial population densities, $w(x,t)$ the chemical concentration, $\varepsilon > 0$ is the chemical diffusion coefficient, and $\chi \in \mathbb{R}$ the chemotactic coefficients measuring the strength of chemotactic interactions. The function $\phi = \phi(w)$ is the chemotactic sensitivity function describing the signalling mechanism by which bacteria response to the chemical. The model (1.1) is usually referred to as a one-species chemotaxis model since only one species u is considered in the dynamics. It was shown in [13] that if $\phi(w) = \log w$, namely the sensitivity function is logarithmic, the model (1.1) can reproduce the propagating wave bands observed in the

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experiment of [1]. Subsequently the application of the logarithmic sensitivity function in chemotaxis models stimulated a large number of studies on the existence and (linear) stability of traveling wave solutions (see [3, 10, 22, 23, 24, 25, 26, 27, 28, 31] and references therein). Recently both experiments and model simulation were used in [12] to show that E. *coli* bacteria do sense the spatial gradient of the logarithmic ligand concentration, which in turn rationalizes the use of logarithmic sensitivity in previous studies. The nonlinear stability of traveling wave solutions to a one-species chemotaxis model with logarithmic sensitivity and exponential consumption for the chemical (i.e. replacing the term $-u$ in (1.1) by $-uw$) was established by the method of energy estimates in [11, 18, 20], and a generalization in [19] for the chemical consumption function.

Situations where the two populations react to one single chemical signal have been described experimentally in [14, 16, 21] and mathematically in [4, 7, 8, 17, 29, 32]. Recently a wide class of multi-species chemotaxis models and their mathematical properties have been discussed in [9] where, however, the traveling wave solution was not discussed. In the present paper, we shall study the traveling wave solutions of the following two interacting species chemotaxis model

$$\begin{cases} u_t = [u_x - \chi_1 u\phi(w)_x + \beta u\phi(v)_x]_x, \\ v_t = [v_x - \chi_2 v\phi(w)_x + \beta v\phi(u)_x]_x, \\ w_t = \varepsilon w_{xx} - (u+v) \end{cases} \tag{1.2}$$

where $u(x,t)$, $v(x,t)$ denote the two interacting cell population densities and $w(x,t)$ the chemical concentration. Besides, $\chi_1, \chi_2, \beta \in \mathbb{R}$ denote the chemotactic coefficients, $\varepsilon > 0$ is the chemical diffusion coefficient, and ϕ is the chemotactic sensitivity function. In particular, as $\chi_1 < 0$ and $\chi_2, \beta > 0$, (1.2) becomes a particularized case of [9, equation (4.1)] where both u and v consume w. The relevant biological background can be found in [9]. The interactions of u, v and w are governed by the coupling constants χ_1, χ_2 and β in (1.2). The sign of χ_1, χ_2 and β determines the chemotactic interaction (either attraction or repulsion) of pairs (u,w), (v,w) and (u,v), respectively. The details are listed as follows:

$\chi_1 > 0$ (or $\chi_1 < 0$): u is attracted (or repelled) by w;
$\chi_2 > 0$ (or $\chi_2 < 0$): v is attracted (or repelled) by w;
$\beta > 0$ (or $\beta < 0$): u and v are repelled (attracted) each other;
$\beta = 0$: u and v have no mutual chemotactic interaction.

The main goal of this paper is to find under what conditions on parameters χ_1, χ_2, β, the traveling wave solutions of (1.2) with logarithmic sensitivity $\phi(w) = \ln w$ exist for $(x,t) \in \mathbb{R} \times [0,\infty)$. It would be natural to believe that the existence of traveling wave solutions of (1.2) may be affected by the sign of χ_1, χ_2 and β. Indeed in this paper we shall show that the sign conditions of χ_1, χ_2 and β are not sufficient to guarantee the existence of traveling wave solutions, and the traveling wave solutions exist only for the parameters in certain parameter regimes (see Theorem 2.1) plotted in Fig. 1 which gives the regions of (χ_1, χ_2) for $-1 < \beta \neq 1$. When $\beta = 1$, traveling wave solutions exist if $\chi_1 = \chi_2 \geq 2$, see Theorem 2.2.

Due to the high dimensionality of the traveling wave system of ordinary differential equations (ODEs) corresponding to the chemotaxis model, the study of traveling wave solutions to the two-species chemotaxis model (1.2) faces much greater challenges than one-species models in general. In this paper, taking advantage of logarithmic sensitivity which may reduce the dimensionality of the system and employing the sub and super solution method, we are able to establish the existence

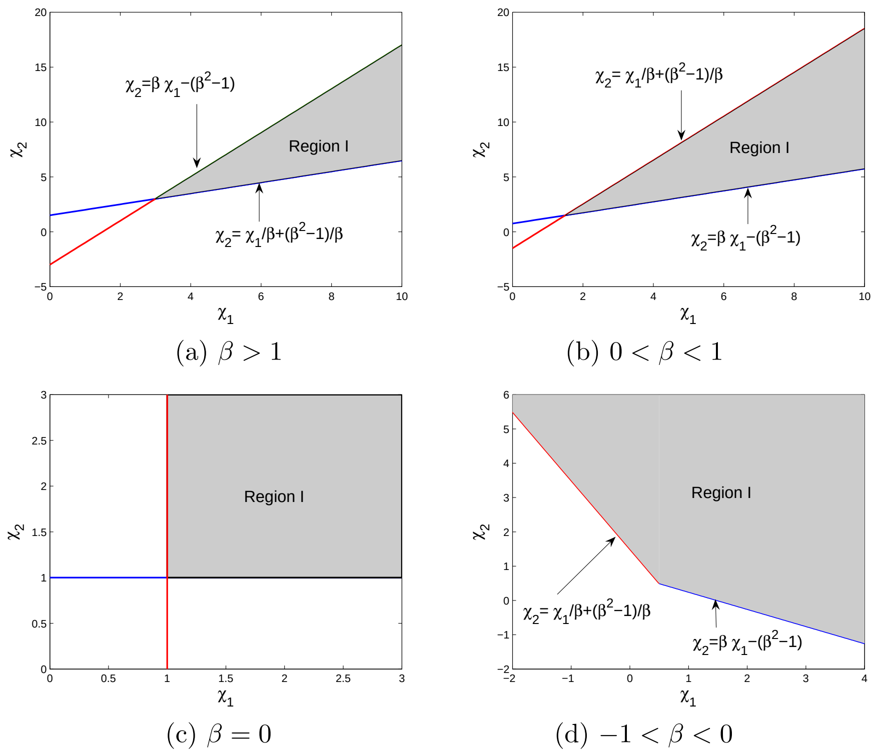


FIGURE 1. Numerical plots of parameter Region I in which the traveling wave solutions exist.

and asymptotic behavior of traveling wave solutions to the two-species chemotaxis model (1.2). Furthermore we may extend the idea of [23, 30] for one-species model to two species model (1.2) and obtain the ε-convergence of traveling wave solutions using the fact that the wave speed is independent of the chemical diffusion rate ε.

The rest of this paper is organized as follows. In section 2, main results of this paper are stated. In section 3, we establish several key Theorems which will be essentially used in section 4 where we prove our main results and identify the explicit parameter regimes in which traveling wave solutions exist. Finally we establish the ε-convergence of traveling wave solutions in section 5.

2. **Statement of main results.** Integrating the first two equations of (1.2) yields the conservation of mass of u and v, namely

$$\int_{\mathbb{R}} u(x,t)dx = \int_{\mathbb{R}} u_0(x)dx =: N_1, \ \int_{\mathbb{R}} v(x,t)dx = \int_{\mathbb{R}} v_0(x)dx =: N_2 \tag{2.1}$$

where N_1, N_2 are prescribed positive numbers denoting the cell mass of u and v, respectively. This requires $u(\pm\infty) = v(\pm\infty) = 0$. Hence the traveling wave solution of (1.2) in $(x,t) \in \mathbb{R}\times[0,\infty)$ is a non-constant special solution in $C^2(\mathbb{R})$ with ansatz

$$(u,v,w)(x,t) = (U,V,W)(z), \ z = x - ct$$

where c is the wave speed and z is called the wave (or moving) variable. Upon the substitution of above ansatz into (1.2), the traveling wave equations are obtained as

$$\begin{cases} -cU' = U'' - \chi_1(U\phi'(W))' + \beta(U\phi'(V))', \\ -cV' = V'' - \chi_2(V\phi'(W))' + \beta(V\phi'(U))', \\ -cW' = \varepsilon W'' - (U+V) \end{cases} \tag{2.2}$$

where $' = \frac{d}{dz}$. Next one needs to determine the appropriate boundary conditions for the traveling wave solutions. Since u, v and w denote the densities of biological species, we restrict our attention to the non-negative solution $U, V, W \geq 0$ only. Moreover the mass preservation (2.1) implies that

$$\int_{\mathbb{R}} U(z)dz = N_1, \ \int_{\mathbb{R}} V(z)dz = N_2 \tag{2.3}$$

which indicates $U(\pm\infty) = V(\pm\infty) = 0$. Moreover the integration of the third equation of (2.2) on (y, z) yields

$$\varepsilon[W'(z) - W'(y)] = \int_y^z (U+V)(\xi)d\xi - c[W(z) - W(y)],$$

which implies that $W'(\pm\infty)$ exists. Since $W(\pm\infty)$ exists as boundary conditions, we obtain $W'(\pm\infty)$=0. If we write the third equation of (4.1) as $(e^{\frac{c}{\varepsilon}z}W')' = \frac{1}{\varepsilon}e^{\frac{c}{\varepsilon}z}(U+V)$, then the integration of above equation leads to

$$W' = \frac{1}{\varepsilon}e^{-\frac{c}{\varepsilon}z}\int_{-\infty}^z e^{\frac{c}{\varepsilon}\xi}(U+V)(\xi)d\xi > 0$$

since $U, V \geq 0$. This implies that $W(z)$ is monotone increasing if it exists. Therefore the appropriate boundary conditions for $(U, V, W)(z)$ would be:

$$U(\pm\infty) = V(\pm\infty) = 0, W(-\infty) = 0, W(\infty) = m, \tag{2.4}$$

where $m > 0$ is a constant.

In the following, we adopt the convention $a(z) \sim b(z)$ as $z \to \pm\infty$ if and only if $\lim_{z\to\pm\infty} \frac{a(z)}{b(z)}$
$= c$ with c being a constant. In the sequel, we shall use C to denote a generic constant which may vary in the context. Then our first main result concerning the existence and asymptotical behavior of traveling wave solutions for $\beta \neq \pm 1$ is as follows.

Theorem 2.1. *Let $\varepsilon \geq 0$ and $\beta^2 \neq 1$. Denote $\lambda_1 = \frac{\beta\chi_2 - \chi_1}{\beta^2 - 1}$, $\lambda_2 = \frac{\beta\chi_1 - \chi_2}{\beta^2-1}$. Then for each $c > 0$, the traveling wave solution (U, V, W) to model (1.2) satisfying the boundary condition (2.4) exists if and only if $\beta > -1$ and $\min\{\lambda_1, \lambda_2\} \geq 1$. Moreover if we let C denote a generic positive constant and $s = \frac{c}{\beta+1} > 0$, then the traveling wave solution has the following asymptotic behavior:*

(i) If $\varepsilon = 0$ and λ_1=$\lambda_2 \geq 1$, then

$$W(z) = \begin{cases} \left(m^{1-\lambda_2} + \frac{C(\lambda_2-1)}{s}e^{-sz}\right)^{\frac{1}{1-\lambda_2}}, & \lambda_1 = \lambda_2 > 1, \\ me^{-\frac{C}{s}e^{-sz}}, & \lambda_1 = \lambda_2 = 1 \end{cases} \tag{2.5}$$

and if $\lambda_1 > \lambda_2$

$$W(z) \sim \left(m^{1-\lambda_2} + \frac{C(\lambda_2 - 1)}{s}e^{-sz}\right)^{\frac{1}{1-\lambda_2}}, \text{ as } z \to \pm\infty; \tag{2.6}$$

(ii) If $\varepsilon > 0$ and $\lambda_1 \geq \lambda_2 \geq 1$, then

(a) If $\lambda_2 > 1$*, then*

$$\begin{aligned} &W(z) \sim e^{-\mu_2 z}, &&\text{as } z \to -\infty,\\ &W(z) - m \sim e^{-\frac{c}{\varepsilon}z}, &&\text{as } z \to \infty \end{aligned} \tag{2.7}$$

where $\mu_2 = \frac{s}{1-\lambda_2} < 0$.

(b) If $\lambda_2 = 1$*, then*

$$\begin{aligned} &W(z) \sim e^{(\frac{s}{4} - \frac{c}{2\varepsilon})z - \frac{2C}{c\sqrt{\varepsilon}}e^{-\frac{s}{2}z}}, \text{ as } z \to -\infty,\\ &W(z) - m \sim e^{-\frac{c}{\varepsilon}z}, \text{ as } z \to \infty. \end{aligned} \tag{2.8}$$

The asymptotic behaviors of U *and* V *as* $z \to \pm\infty$ *in case (i) and (ii) are determined by*

$$U(z) \sim e^{-sz}W(z)^{\lambda_1}; \quad V(z) \sim e^{-sz}W(z)^{\lambda_2}.$$

Remark 1. In Theorem 2.1, we assume that $\lambda_1 \geq \lambda_2$ without loss of generality. When $\lambda_1 \leq \lambda_2$, the same results hold by interchanging λ_1 and λ_2 in the theorem.

Our next main result is the existence theorem of traveling wave solutions when $\beta = \pm 1$.

Theorem 2.2. *Let* $\phi(w) = \ln w$ *and let* $\varepsilon \geq 0, \beta^2 = 1$*. Then the following assertions hold.*

(i) If $\beta = -1$*, there is no traveling wave solution to (1.2);*

(ii) If $\beta = 1$ *and* $\chi_1 \neq \chi_2$*, then (1.2) does not have a traveling wave solution;*

(iii) If $\beta = 1$ *and* $\chi_1 = \chi_2$*, then*

(a) the system (1.2) has no traveling wave solution if $\chi_1 = \chi_2 < 2$*;*

(b) the system (1.2) has a traveling wave solution satisfying (2.4) if $\chi_1 = \chi_2 = 2$*, such that the asymptotic behavior of the solution component* (U, V) *as* $z \to \pm\infty$ *is given by*

$$U(z) = V(z) \sim e^{-\frac{c}{2}z}W \tag{2.9}$$

where the asymptotic behavior of W *is*

$$\begin{aligned} &W(z) \sim e^{(\frac{c}{8} - \frac{c}{2\varepsilon})z - \frac{2C}{c\sqrt{\varepsilon}}e^{-\frac{c}{4}z}}, \text{ as } z \to -\infty,\\ &W(z) - m \sim e^{-\frac{c}{\varepsilon}z}, \text{ as } z \to \infty; \end{aligned} \tag{2.10}$$

(c) the system (1.2) has a solution if $\chi_1 = \chi_2 > 2$*, such that the asymptotic behavior of the solution component* (U, V) *as* $z \to \pm\infty$ *is given by*

$$U(z) \sim e^{-\frac{c}{2}z}W^{\tilde{\chi}_1}, \; V(z) \sim e^{-\frac{c}{2}z}W^{\tilde{\chi}_2},$$

where the asymptotic behavior of W *is the same as those in Theorem 2.1 by replacing* s *with* $\frac{c}{2}$ *and* (λ_1, λ_2) *with* $(\tilde{\chi}_1, \tilde{\chi}_2)$*, where* $(\tilde{\chi}_1, \tilde{\chi}_2)$ *can be arbitrarily chosen such that* $\min\{\tilde{\chi}_1, \tilde{\chi}_2\} \geq 1$.

The last theorem is concerned with the convergence of traveling wave solutions as $\varepsilon \to 0$. Since the wave speed determines the wave profile, one needs to guarantee the convergence of wave speed with respect to ε first. Fortunately if we integrate the third equation of (2.2) over $(-\infty, \infty)$, we find the wave speed as

$$c = (N_1 + N_2)/m$$

which is a constant depending only on cell mass N_1, N_2 and m, but independent of ε. This allows the possibility to prove the ε-convergence of traveling wave solutions. Indeed, we have the following precise result:

Theorem 2.3. *Let traveling wave solutions of (1.2) be denoted by $(U_\varepsilon, V_\varepsilon, W_\varepsilon)$ for $\varepsilon > 0$ and by (U_0, V_0, W_0) for $\varepsilon = 0$. Then for each $z \in \mathbb{R}$, it follows that*

$$|(U_\varepsilon, V_\varepsilon, W_\varepsilon) - (U_0, V_0, W_0)| = \mathcal{O}(\varepsilon), \text{ as } \varepsilon \to 0.$$

3. **Key theorems.** In this section, we shall establish several theorems that are essential to prove our main results.

Proposition 1. *Consider the problem*

$$\begin{cases} \varepsilon p'' + cp' - \alpha e^{-kz} p^\lambda = 0, \\ p(-\infty) = 0, \ p(\infty) = \omega > 0 \end{cases} \tag{3.1}$$

with $\varepsilon > 0$, where $p = p(z), z \in \mathbb{R}$, $\alpha, k, \lambda > 0$ and $c \geq 0$. Then

(i) If $\lambda < 1$, (3.1) does not have a solution;

(ii) If $\lambda > 1$, (3.1) has a unique solution for each $c > 0$ such that $p'(z) > 0$ and

$$p(z) = \begin{cases} \left[\left(\frac{\eta}{\alpha}\right)^{1/(\lambda-1)} - Ce^{\gamma z}\right] e^{-\mu z}, & \text{as } z \to -\infty, \\ \omega - Ce^{-\frac{c}{\varepsilon} z}, & \text{as } z \to \infty \end{cases} \tag{3.2}$$

where

$$\mu = \frac{k}{1-\lambda} < 0, \ \eta = \varepsilon\mu^2 - c\mu > 0, \ \gamma = \mu + \frac{-c + \sqrt{c^2 + 4(\varepsilon\mu^2 - c\mu)\lambda}}{2} > 0.$$

(iii) If $\lambda = 1$, then for each $c > 0$ (3.1) has a unique solution satisfying $p'(z) > 0$ and

$$p(z) \sim e^{(\frac{k}{4} - \frac{c}{2\varepsilon})z - \frac{2\sqrt{\alpha}}{c\sqrt{\varepsilon}} e^{-\frac{k}{2} z}}, \text{ as } z \to -\infty,$$
$$p(z) - \omega \sim e^{-\frac{c}{\varepsilon} z}, \text{ as } z \to \infty.$$

The most of results in Proposition 3.1 has been proved in [22, 30, 6]. The full proof is length and we present it in Appendix for completeness. Next we employ Proposition 1 to establish the following theorem which is the key to the proof of our main results in the next section.

Theorem 3.1. *Consider the problem*

$$\begin{cases} \varepsilon\phi''(z) + c\phi'(z) = c_1 e^{-sz}\phi^{\lambda_1} + c_2 e^{-sz}\phi^{\lambda_2}, \\ \phi(-\infty) = 0, \ \phi(\infty) = m > 0 \end{cases} \tag{3.3}$$

with $\varepsilon > 0$, where $c_1, c_2, s > 0$ are constants and $\lambda_1, \lambda_2 > 0$ are parameters. If $F(z, \phi) \in L^1(\mathbb{R})$, where $F(z, \phi) = c_1 e^{-sz}\phi^{\lambda_1} + c_2 e^{-sz}\phi^{\lambda_2}$, then we have

(i) If $\min\{\lambda_1, \lambda_2\} < 1$, the problem (3.3) has no solution.

(ii) If $\min\{\lambda_1, \lambda_2\} \geq 1$, the problem (3.3) has a unique solution with $\phi' > 0$. Moreover if $\lambda_1 \geq \lambda_2 \geq 1$, then the asymptotic behavior of the solution satisfies

(a) If $\lambda_2 > 1$, then

$$\phi(z) \sim \begin{cases} (A_1 + A_2 e^{\gamma_2 z}) e^{-\mu_2 z}, \text{ as } z \to -\infty, \\ m - A_3 e^{-\frac{c}{\varepsilon} z}, \text{ as } z \to \infty. \end{cases} \tag{3.4}$$

where $A_i > 0$ $(i = 1, 2, 3)$ are constants and

$$\mu_2 = \frac{s}{1-\lambda_2} < 0, \ \gamma_2 = \mu_2 + \frac{-c + \sqrt{c^2 + 4(\mu_2^2 - c\mu_2)\lambda_2}}{2} > 0.$$

(b) If $\lambda_2 = 1$, then

$$\begin{aligned} &\phi(z) \sim e^{(\frac{s}{4}-\frac{c}{2\varepsilon})z-\frac{2\sqrt{\tilde{c}}}{c\sqrt{\varepsilon}}e^{-\frac{s}{2}z}}, \text{ as } z \to -\infty, \\ &\phi(z) - m \sim e^{-\frac{c}{\varepsilon}z}, \text{ as } z \to \infty \end{aligned} \tag{3.5}$$

where $\tilde{c}$ is a positive constant such that $c_2 \le \tilde{c} \le c_2 + c_1 m^{\lambda_1-\lambda_2}$. If $\lambda_2 \ge \lambda_1 \ge 1$, the above decay rates hold true by interchanging λ_1 and λ_2.

Proof. (i) Without loss of generality, we assume that $0 < \lambda_1 < 1$. Then we prove the theorem by contradiction. Suppose that (3.3) has a solution ϕ. Then by the maximum principle, $\phi > 0$ in $\mathbb{R}$. Hence

$$\begin{cases} \varepsilon\phi'' + c\phi' \ge c_1 e^{-sz}\phi^{\lambda_1}, \ z \in \mathbb{R}, \\ \phi(-\infty) = 0, \ \phi(\infty) = m > 0 \end{cases} \tag{3.6}$$

which asserts that ϕ is a sub-solution of

$$\begin{cases} \varepsilon\varphi'' + c\varphi' = c_1 e^{-sz}\varphi^{\lambda_1}, \ z \in \mathbb{R}, \\ \varphi(-\infty) = 0, \ \varphi(\infty) = m > 0. \end{cases} \tag{3.7}$$

To construct a super-solution of (3.7), we set

$$\psi(z) = m\xi(z+K) \text{ for } z \in \mathbb{R}$$

where K is a nonzero constant to be determined and ξ is the positive solution of

$$\begin{cases} \varepsilon\xi'' + c\xi' = c_1 e^{-sz}\xi^2, \ z \in \mathbb{R}, \\ \xi(-\infty) = 0, \ \xi(\infty) = 1. \end{cases} \tag{3.8}$$

Note that the existence of monotone solutions to (3.8) has been established in [30], see also Proposition 1 (ii). Hence $0 < \xi < 1$ in $\mathbb{R}$ due to $\xi' > 0$. It is then easy to check from (3.8) that

$$\begin{aligned} \varepsilon\psi''(z) + c\psi'(z) &= \frac{c_1}{m} e^{-s(z+K)}\psi^2(z) \\ &= c_1 m e^{-s(z+K)}\xi^2(z+K) \\ &\le c_1 m e^{-s(z+K)}\xi^{\lambda_1}(z+K) \\ &= c_1 m^{1-\lambda_1} e^{-s(z+K)}\psi^{\lambda_1}(z) \\ &= c_1 e^{-sz}\psi^{\lambda_1}(z) \end{aligned} \tag{3.9}$$

for all $z \in \mathbb{R}$, provided that K satisfies $e^{sK} = m^{1-\lambda_1}$, namely $K = \frac{(1-\lambda_1)\ln m}{s}$. Hence ψ satisfies

$$\begin{cases} \varepsilon\psi'' + c\psi' \le c_1 e^{-sz}\psi^{\lambda_1} \text{ in } \mathbb{R}, \\ \psi(-\infty) = 0, \ \psi(\infty) = m \end{cases} \tag{3.10}$$

which indicates that ψ is a super-solution of (3.7). Now we claim that $\phi \le \psi$ in $\mathbb{R}$. Indeed, let $\eta = \phi - \psi$. Subtracting (3.10) from (3.6), we obtain

$$\varepsilon\eta'' + c\eta' \ge c_1 e^{-sz}(\phi^{\lambda_1} - \psi^{\lambda_1}) = q(z)\eta = c_1 e^{-sz}\left(\frac{\phi^{\lambda_1} - \psi^{\lambda_1}}{\phi - \psi}\right)\eta \text{ in } \mathbb{R}$$

where $q(z) = c_1 e^{-sz}\frac{\phi^{\lambda_1}-\psi^{\lambda_1}}{\phi-\psi} = c_1\lambda_1 e^{-sz}\tilde{\xi}^{\lambda_1-1} \ge 0$ in $\mathbb{R}$ and $\tilde{\xi}$ is between ϕ and ψ. Then by the maximum principle, we have

$$\max_{|z|\le R} \eta(z) \le \max_{|z|=R} \eta(z) = \max_{|z|=R} (\phi(z) - \psi(z)) \to 0 \text{ as } R \to \infty$$

where we have used the fact that $\phi(-\infty) = \psi(-\infty) = 0$ and $\phi(\infty) = \psi(\infty) = m > 0$. This proves our claim. Then by the comparison principle of elliptic equations, (3.7) has a solution φ such that $\phi \le \varphi \le \psi$. This contradicts Theorem 1 which says that (3.7) does not have a solution if $\lambda_1 < 1$. By contradiction, the proof of (i) is completed.

(ii) We assume that $\lambda_1 \ge \lambda_2 \ge 1$ without loss of generality. Let $\bar{\phi}$ be a positive solution of

$$\begin{cases} \varepsilon\phi'' + c\phi' = c_1 e^{-sz}\phi^{\lambda_1}, \ z \in \mathbb{R}, \\ \phi(-\infty) = 0, \ \phi(\infty) = m > 0. \end{cases} \tag{3.11}$$

Since $\varepsilon\phi'' + c\phi' = c_1 e^{-sz}\phi^{\lambda_1} < c_1 e^{-sz}\phi^{\lambda_1} + c_2 e^{-sz}\phi^{\lambda_2}$, $\bar{\phi}$ is a super-solution of (3.3). Next we shall construct a nonzero sub-solution of (4.5). To this end, let $\varphi(z)$ be a solution of

$$\begin{cases} \varepsilon\varphi'' + c\varphi' = \delta e^{-sz}\varphi^{\lambda_2}, \ z \in \mathbb{R}, \\ \varphi(-\infty) = 0, \ \varphi(\infty) = m \end{cases} \tag{3.12}$$

where δ is a constant such that $c_2 < \delta < c_1 m^{\lambda_1-\lambda_2} + c_2$.

We remark that the existence of unique solution to (3.12) has been shown in [30, 22] for $\lambda_2 > 1$, and in [15] for $\lambda_2 = 1$, see also Proposition 1. Now we assume that

$$\underline{\phi} = \sigma\varphi \tag{3.13}$$

where $0 < \sigma < 1$ is to be determined later. We shall show that for appropriate small σ, $\underline{\phi}$ is a sub-solution of (3.3). To this end, we first substitute (3.13) into (3.12) and obtain

$$\begin{cases} \varepsilon\underline{\phi}'' + c\underline{\phi}' = \delta\sigma e^{-sz}\varphi^{\lambda_2}, \ z \in \mathbb{R}, \\ \underline{\phi}(-\infty) = 0, \ \underline{\phi}(\infty) = \sigma m. \end{cases} \tag{3.14}$$

Then to prove that $\underline{\phi}$ is a sub-solution of (3.3), it suffices to require that

$$\delta\sigma e^{-sz}\varphi^{\lambda_2} > c_1 e^{-sz}\sigma^{\lambda_1}\varphi^{\lambda_1} + c_2 e^{-sz}\sigma^{\lambda_2}\varphi^{\lambda_2} \text{ for all } z \in \mathbb{R}$$

which, upon cancelation, is equivalent to

$$c_1\sigma^{\lambda_1-1}\varphi^{\lambda_1-\lambda_2} + c_2\sigma^{\lambda_2-1} < \delta, \text{ for all } z \in \mathbb{R}. \tag{3.15}$$

Noticing that $0 < \sigma < 1$, $\lambda_1 - \lambda_2 \ge 0$ and φ is a solution of (3.12) with $\varphi'(z) > 0$ and hence $\varphi < m$, we have $c_1\sigma^{\lambda_1-1}\varphi^{\lambda_1-\lambda_2} + c_2\sigma^{\lambda_2-1} < c_1\sigma^{\lambda_2-1}m^{\lambda_1-\lambda_2} + c_2\sigma^{\lambda_2-1}$. Hence if we choose σ such that

$$\sigma < \min\left\{1, \left(\frac{\delta}{c_1 m^{\lambda_1-\lambda_2} + c_2}\right)^{\frac{1}{\lambda_2-1}}\right\}, \tag{3.16}$$

then (3.15) is ensured, which shows that $\underline{\phi}$ is a sub-solution of (3.3). Next we show that $\underline{\phi} \le \bar{\phi}$ for all $z \in \mathbb{R}$. To this end, let $\zeta = \underline{\phi} - \bar{\phi}$. Noting that with (3.16), $\underline{\phi}$ satisfies from (3.14)

$$\begin{cases} \varepsilon\underline{\phi}'' + c\underline{\phi}' > c_1 e^{-sz}\underline{\phi}^{\lambda_1} + c_2 e^{-sz}\underline{\phi}^{\lambda_2}, \ z \in \mathbb{R}, \\ \underline{\phi}(-\infty) = 0, \ \underline{\phi}(\infty) = \sigma m. \end{cases} \tag{3.17}$$

Then ζ satisfies that

$$\varepsilon\zeta'' + c\zeta' \ge c_1 e^{-sz}(\underline{\phi}^{\lambda_1} - \bar{\phi}^{\lambda_1}) + c_2 e^{-sz}\underline{\phi}^{\lambda_2} \ge c_1 e^{-sz}(\underline{\phi}^{\lambda_1} - \bar{\phi}^{\lambda_1}) = c_1 e^{-sz}\frac{\underline{\phi}^{\lambda_1} - \bar{\phi}^{\lambda_1}}{\underline{\phi} - \bar{\phi}}\zeta.$$

Since $\frac{\underline{\phi}^{\lambda_1}-\bar{\phi}^{\lambda_1}}{\underline{\phi}-\bar{\phi}} \geq 0$ and $\underline{\phi}(-\infty) = 0 = \bar{\phi}(-\infty)$, $(\underline{\phi} - \bar{\phi})(\infty) = (\sigma - 1)m < 0$, it follows from the maximum principle that

$$\max_{|z|\leq R} \zeta(z) \leq \max_{|z|=R} \zeta(z) = \max_{|z|=R} (\underline{\phi}(z) - \bar{\phi}(z)) \leq 0 \text{ as } R \to \infty$$

which implies that $\underline{\phi}(z) \leq \bar{\phi}(z)$ for all $z \in \mathbb{R}$. By the comparison principle, we obtain a solution ϕ to (3.3) such that $\underline{\phi} \leq \phi \leq \bar{\phi}$.

For the uniqueness, we assume that ϕ and ψ are the solutions of (3.3), i.e.

$$\varepsilon\phi'' + c\phi' = c_1 e^{-sz}\phi^{\lambda_1} + c_2 e^{-sz}\phi^{\lambda_2} \text{ in } \mathbb{R} \text{ and } \phi(-\infty) = 0,\ \phi(\infty) = m > 0$$

and

$$\varepsilon\psi'' + c\psi' = c_1 e^{-sz}\psi^{\lambda_1} + c_2 e^{-sz}\psi^{\lambda_2} \text{ in } \mathbb{R} \text{ and } \psi(-\infty) = 0,\ \psi(\infty) = m > 0.$$

Then defining $\eta = \phi - \psi$, we have

$$\eta'' + c\eta' = H(z)\eta,$$

where $H(z) = c_1 e^{-sz}\frac{\phi^{\lambda_1}-\psi^{\lambda_1}}{\phi-\psi} + c_2 e^{-sz}\frac{\phi^{\lambda_2}-\psi^{\lambda_2}}{\phi-\psi} \geq 0$. Hence by the maximum principle again, we have

$$\max_{|z|\leq R} |\eta(z)| \leq \max_{|z|=R} |\eta(z)| \to 0 \text{ as } R \to +\infty$$

which implies that $\eta \equiv 0$ for all $z \in \mathbb{R}$. Therefore $\phi \equiv \psi$ in $\mathbb{R}$ and the uniqueness is proved. Next we prove the monotonicity of the traveling wave solution ϕ. To this end, we first show $\phi'(-\infty) = 0$. Indeed integrating the first equation of (3.3) over (y, z), we have

$$0 = \varepsilon(\phi'(y) - \phi'(z)) + c(\phi(y) - \phi(z)) + \int_y^z F(z, \phi)dz$$

which, along with the assumption $F(z, \phi) \in L^1(\mathbb{R})$, indicates that $\phi'(\pm\infty)$ exists. Since $\phi(\pm\infty)$ exists, we obtain that $\phi'(\pm\infty) = 0$. Then we can rewrite the equation (3.3) as

$$(e^{\frac{c}{\varepsilon}z}\phi')' = \frac{c_1}{\varepsilon}e^{(\frac{c}{\varepsilon}-s)z}\phi^{\lambda_1} + \frac{c_2}{\varepsilon}e^{(\frac{c}{\varepsilon}-s)z}\phi^{\lambda_2}.$$

Then we integrate it over $(-\infty, z]$ and obtain that

$$\phi' = \frac{1}{\varepsilon}e^{-\frac{c}{\varepsilon}z}\left[c_1\int_{-\infty}^z e^{(\frac{c}{\varepsilon}-s)\xi}\phi^{\lambda_1}(\xi)d\xi + c_2\int_{-\infty}^z e^{(\frac{c}{\varepsilon}-s)\xi}\phi^{\lambda_2}(\xi)d\xi\right]$$

which asserts that $\phi' > 0$ since $\phi > 0$ in $\mathbb{R}$ and $c_1, c_2 > 0$.

To finish the proof of Theorem 3.1 (ii), it remains to derive the asymptotic decay rates announced. We consider the case $\lambda_1 \geq \lambda_2 \geq 1$. Since $\phi'(z) > 0$, we have from (3.3) that $0 < \phi(z) < m$ for all $z \in \mathbb{R}$. Then we have

$$\begin{aligned}\varepsilon\phi'' + c\phi' &= e^{-sz}\phi^{\lambda_2}(c_1\phi^{\lambda_1-\lambda_2} + c_2)\\ &\leq (c_2 + c_1 m^{\lambda_1-\lambda_2})e^{-sz}\phi^{\lambda_2}.\end{aligned}$$

On the other hand, it has that

$$\varepsilon\phi'' + c\phi' = c_1 e^{-sz}\phi^{\lambda_1} + c_2 e^{-sz}\phi^{\lambda_2} \geq c_2 e^{-sz}\phi^{\lambda_2}.$$

Therefore it follows that

$$\varepsilon\phi'' + c\phi' \sim \tilde{c}e^{-sz}\phi^{\lambda_2} \text{ for all } z \in \mathbb{R} \tag{3.18}$$

where $c_2 \le \tilde{c} \le c_2 + c_1 m^{\lambda_1-\lambda_2}$. Then the asymptotic behavior (6.8) follows directly from (3.18) by Proposition 1 (ii). When $\lambda_2 = 1$, we obtain (3.5) from (3.18) with Proposition 1 (iii). The proof of Theorem 3.1 (ii) is completed. □

4. **Existence of traveling wave solutions to system (1.2).** In this section, we shall apply Theorem 3.1 to establish traveling wave solutions to the system (1.2). To this end, we rewrite (2.2) with $\phi(w)=\ln w$ as

$$\begin{cases} -cU' = U'' - \chi_1(U\frac{W'}{W})' + \beta(U\frac{V'}{V})', \\ -cV' = V'' - \chi_2(V\frac{W'}{W})' + \beta(V\frac{U'}{U})', \\ -cW' = \varepsilon W'' - (U+V). \end{cases} \tag{4.1}$$

As the analysis in section 2, the existence of traveling wave solutions to (1.2) is equivalent to the existence of solutions of (4.1) subject to the boundary condition (2.4). Solving the first two equations of (4.1) and using the boundary condition (2.4), we have

$$U(z) = C_1 e^{-cz} W^{\chi_1} V^{-\beta},\ V(z) = C_2 e^{-cz} W^{\chi_2} U^{-\beta} \tag{4.2}$$

if the integration constant is zero, where $C_1, C_2 > 0$ are constants of integration. In the following, we shall prove Theorem 2.1 and Theorem 2.2.

4.1. **Proof of Theorem 2.1.** Theorem 2.2 gives the existence and non-existence of traveling wave solutions for $\beta^2 = 1$. Next we consider the case $\beta^2 \neq 1$ for which we first solve (4.2) to obtain that

$$U(z) = C_3 e^{-sz} W^{\lambda_1},\ V(z) = C_4 e^{-sz} W^{\lambda_2} \tag{4.3}$$

with $C_3 = \left(\frac{C_2^\beta}{C_1}\right)^{\frac{1}{\beta^2-1}}, C_4 = C_3\left(\frac{C_1}{C_2}\right)^{\frac{1}{\beta-1}}$ and $s = \frac{c}{\beta+1}, \lambda_1 = \frac{\beta\chi_2-\chi_1}{\beta^2-1}, \lambda_2 = \frac{\beta\chi_1-\chi_2}{\beta^2-1}$.

If $\min\{\lambda_1,\lambda_2\} \le 0$, it is clear that $U(-\infty) = \infty$ if $\lambda_1 \le 0$ and $V(-\infty) = \infty$ if $\lambda_2 \le 0$ since $W(-\infty) = 0$. Thus a necessary condition for (U, V) in (4.3) to be a traveling wave solution component satisfying (2.4) is

$$\lambda_1 = \frac{\beta\chi_2 - \chi_1}{\beta^2 - 1} > 0,\ \lambda_2 = \frac{\beta\chi_1 - \chi_2}{\beta^2 - 1} > 0. \tag{4.4}$$

Substituting (4.3) into the third equation of (4.1) and using (2.4), we obtain the following problem

$$\begin{cases} \varepsilon W'' + cW' = C_3 e^{-sz} W^{\lambda_1} + C_4 e^{-sz} W^{\lambda_2}, \\ W(-\infty) = 0,\ W(\infty) = m > 0. \end{cases} \tag{4.5}$$

We note from (4.3) and (2.3) that $F(z, W) = C_3 e^{-sz} W^{\lambda_1} + C_4 e^{-sz} W^{\lambda_2} \in L^1(\mathbb{R})$. It is helpful to give a remark before proceeding.

Remark 2. If $s < 0$, then the evaluation of equation (4.5) at $z = \infty$ will contradict the boundary condition $W(\infty) = m > 0$. Hence a necessary condition for (4.5) to have a solution is $s = \frac{c}{\beta+1} > 0$, which is equivalent to $\beta > -1$ due to $c > 0$.

Now we are ready to prove Theorem 2.1. We split our analysis to the case $\varepsilon > 0$ and $\varepsilon = 0$.

Zero diffusion $\varepsilon = 0$. With $\varepsilon = 0$, (4.5) becomes

$$W' = c_3 e^{-sz} W^{\lambda_1} + c_4 e^{-sz} W^{\lambda_2},\ W(-\infty) = 0,\ W(\infty) = m > 0. \tag{4.6}$$

where $c_3 = C_3/c > 0, c_4 = C_4/c > 0$. We first have the following non-existence result.

Lemma 4.1. *If* $\min\{\lambda_1, \lambda_2\} < 1$, *then there is no solution to (4.6).*

Proof. Without loss of generality, we assume $\lambda_2 < 1$. Supposing that (4.6) has a solution $W(z)$, we have from (4.6) that

$$W' > c_4 e^{-sz} W^{\lambda_2} > 0. \tag{4.7}$$

Since $W(-\infty) = 0$, it follows that $W(z) > 0$ for all $z \in \mathbb{R}$. Now using the boundary condition in (4.6) to solve (4.7), one has

$$W^{1-\lambda_2} < m^{1-\lambda_2} - \frac{c_4(1-\lambda_2)}{s} e^{-sz}.$$

Clearly, if $z < 0$ and $|z|$ is sufficiently large such that $z < \tilde{z} = -\frac{1}{s} \ln \frac{s m^{1-\lambda_2}}{c_4(1-\lambda_2)}$, above inequality implies that $W^{1-\lambda_2} < 0$ which contradicts the fact that $W(z) > 0$ for all $z \in \mathbb{R}$. Then the proof is finished. $\square$

Next, we shall prove the following result.

Lemma 4.2. *If* $\min\{\lambda_1, \lambda_2\} \geq 1$, *then (4.6) has a unique solution.*

Proof. We proceed with two cases: $\lambda_1 = \lambda_2$ and $\lambda_1 \neq \lambda_2$.
Case 1. $\lambda_1 = \lambda_2$. In this case, the problem (4.6) becomes

$$W' = (c_3 + c_4) e^{-sz} W^{\lambda_2},\ W(-\infty) = 0,\ W(\infty) = m > 0.$$

Solving above problem directly yields the unique solution as

$$W(z) = \begin{cases} \left(m^{1-\lambda_2} + \frac{(c_3+c_4)(\lambda_2-1)}{s} e^{-sz}\right)^{\frac{1}{1-\lambda_2}}, & \lambda_1 = \lambda_2 > 1, \\ m e^{-\frac{c_3+c_4}{s} e^{-sz}}, & \lambda_1 = \lambda_2 = 1 \end{cases} \tag{4.8}$$

which gives (2.5) in Theorem 2.1.
Case 2. $\lambda_1 \neq \lambda_2$. Without loss of generality, we assume that $\lambda_1 > \lambda_2$. We first notice that $W(z) < m$ for all $z \in R$ since $W' > 0$ and $W(\infty) = m$. Hence $W^{\lambda_1} < W^{\lambda_2} m^{\lambda_1 - \lambda_2}$ and we have from (4.6) that

$$c_4 e^{-sz} W^{\lambda_2} < W' < (c_3 m^{\lambda_1-\lambda_2} + c_4) e^{-sz} W^{\lambda_2} = c_5 e^{-sz} W^{\lambda_2} \tag{4.9}$$

where $c_5 = c_3 m^{\lambda_1 - \lambda_2} + c_4$. Solving above inequality with boundary conditions in (4.6), we obtain that $W(z)$ is bounded such that $W_1(z) \leq W(z) \leq W_2(z)$, where W_1 is the solution of $W' = c_5 e^{-sz} W^{\lambda_2}$, $W(-\infty) = 0$, $W(\infty) = m > 0$ and W_2 is the solution of $W' = c_4 e^{-sz} W^{\lambda_2}$, $W(-\infty) = 0$, $W(\infty) = m > 0$. This shows that (4.6) has a solution. For the asymptotic behavior, we see from (4.9) that

$$W' \sim C e^{-sz} W^{\lambda_2}$$

where $c_4 \leq C \leq c_5$. This gives that

$$W(z) \sim \left(m^{1-\lambda_2} + \frac{C(\lambda_2 - 1)}{s} e^{-sz}\right)^{\frac{1}{1-\lambda_2}},\ \text{as } z \to \pm\infty \tag{4.10}$$

which yields (2.6) in Theorem 2.1. $\square$

Then the proof of Theorem 2.1 (i) is completed. Next we consider the case $\varepsilon > 0$.

Nonzero diffusion $\varepsilon > 0$. When $\varepsilon > 0$, the existence of solution $W(z)$ of (4.5) with $W' > 0$ follows directly from Theorem 3.1 with Remark 2. Once we obtain W, we can substitute W back into (4.3) to obtain the solution component (U, V) of the system (4.1). Now we need to verify that the obtained solution (U, V, W) fulfills the boundary condition (2.4). First the boundary condition for $W(z)$ has been verified from (4.5) directly. Furthermore from (4.3), it follows that $U(+\infty) = V(+\infty) = 0$ since $s > 0$. Hence it remains to check if $U(-\infty) = V(-\infty) = 0$ which is not obvious yet. Under the assumption that $\lambda_1 \geq \lambda_2 \geq 1$, we obtain from Theorem 3.1 (ii) that

$$W(z) \sim \begin{cases} (A_1 + A_2 e^{\gamma_2 z}) e^{-\mu_2 z}, & \lambda_2 > 1, \\ e^{(\frac{s}{4} - \frac{c}{2\varepsilon})z - \frac{2\sqrt{\tilde{c}}}{c\sqrt{\varepsilon}} e^{-\frac{s}{2}z}}, & \lambda_2 = 1, \end{cases} \quad \text{as } z \to -\infty$$

where the constants $A_1, A_2, \gamma_2, \mu_2$ are given in Theorem 3.1 (ii) and $\tilde{c}$ is an arbitrary constant such that $C_4 \leq \tilde{c} \leq C_4 + C_3 m^{\lambda_1 - \lambda_2}$. Therefore one has from (4.3) that

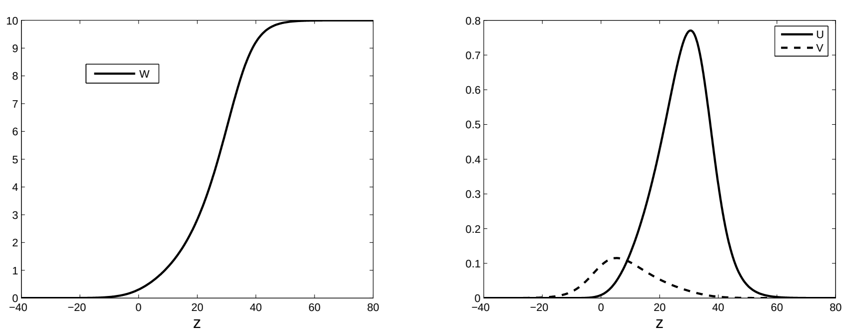
$$U(z) = C_3 e^{-sz} W^{\lambda_1} \sim \begin{cases} (A_1 + A_2 e^{\gamma_2 z}) e^{-\frac{s(1+\lambda_1-\lambda_2)}{1-\lambda_2} z}, & \lambda_2 > 1, \\ e^{[\lambda_1(\frac{s}{4} - \frac{c}{2\varepsilon}) - s]z - \frac{2\lambda_1\sqrt{\tilde{c}}}{c\sqrt{\varepsilon}} e^{-\frac{s}{2}z}}, & \lambda_2 = 1, \end{cases} \quad \text{as } z \to -\infty$$

and

$$V(z) = C_4 e^{-sz} W^{\lambda_2} \sim \begin{cases} (A_1 + A_2 e^{\gamma_2 z}) e^{-\frac{s}{1-\lambda_2} z}, & \lambda_2 > 1, \\ e^{[\lambda_2(\frac{s}{4} - \frac{c}{2\varepsilon}) - s]z - \frac{2\lambda_2\sqrt{\tilde{c}}}{c\sqrt{\varepsilon}} e^{-\frac{s}{2}z}}, & \lambda_2 = 1, \end{cases} \quad \text{as } z \to -\infty.$$

Noticing that $\frac{s(1+\lambda_1-\lambda_2)}{1-\lambda_2} < 0$ and $\frac{s}{1-\lambda_2} < 0$ due to $\lambda_1 \geq \lambda_2 > 1$, one immediately obtains that $U(-\infty) = V(-\infty) = 0$, which is consistent with the boundary condition (2.4).

Fig. 2 (a) shows a numerical solution of (4.5) and Fig. 2 (b) plots the numerical solution U and V which are obtained from W shown in Fig. 2 (a) by (4.3). From the simulation, we see that W is a wavefront and U, V are pulsating waves, as expected.



(a) (b)

FIGURE 2. An illustration of numerical traveling wave solutions of system (1.2), where $\varepsilon = 0.1, c = 2, s = 1/4, \lambda_1 = 4, \lambda_2 = 2, C_3 = C_4 = 1, m = 10$. The solution W is solved from (4.5), U and V are obtained from (4.3).

4.2. **Proof of Theorem 2.2.** (i) If $\beta = -1$, we can derive from (4.2) that $C_1C_2W^{\chi_1+\chi_2}(z) = e^{2cz}$, which does not hold for $\chi_1 + \chi_2 = 0$. When $\chi_1 + \chi_2 \neq 0$, we have that $W(-\infty) = \infty$ if $\chi_1 + \chi_2 < 0$ and $W(\infty) = \infty$ if $\chi_1 + \chi_2 > 0$. Hence (4.1) does not possess a traveling wave solution for any $\chi_1, \chi_2 \in \mathbb{R}$ when $\beta = -1$.

(ii) If $\beta = 1$ and $\chi_1 \neq \chi_2$, we solve (4.2) and have $W(z) = \left(\frac{C_1}{C_2}\right)^{\frac{1}{\chi_2-\chi_1}}$ is a constant, which leads to from the third equation of (4.1) that $U + V \equiv 0$. This implies that $U = V \equiv 0$ for all $z \in \mathbb{R}$ since $U, V \geq 0$. Therefore (4.1) does not have a solution for $\beta = 1, \chi_1 \neq \chi_2$.

(iii) For the case $\beta = 1, \chi_1 = \chi_2$, we derive from (4.2) that $C_1 = C_2 =: C^2$ with $C > 0$ and

$$(UV)(z) = C^2 e^{-cz} W^{\chi} \tag{4.11}$$

where $\chi =: \chi_1 = \chi_2$. We may write (4.11) as

$$U(z) = Ce^{-\tilde{c}_1 z} W^{\tilde{\chi}_1}, V(z) = Ce^{-\tilde{c}_2 z} W^{\tilde{\chi}_2} \tag{4.12}$$

where $c = \tilde{c}_1 + \tilde{c}_2, \chi = \tilde{\chi}_1 + \tilde{\chi}_2$. It can be easily seen that if either $\tilde{c}_1 \leq 0$ or $\tilde{c}_2 \leq 0$, the boundary condition of U or V as $z \to \infty$ will be violated. Hence we assume that $\tilde{c}_1 > 0, \tilde{c}_2 > 0$. Then substituting (4.12) into the third equation of (4.1) and using the boundary condition (2.4), we obtain

$$\begin{cases} \varepsilon W'' + cW' = Ce^{-\tilde{c_1}z}W^{\tilde{\chi}_1} + Ce^{-\tilde{c_2}z}W^{\tilde{\chi}_2}, \\ W(-\infty) = 0, \ W(\infty) = m > 0. \end{cases} \tag{4.13}$$

If $\min\{\tilde{\chi}_1, \tilde{\chi}_2\} < 1$ and thus $\chi = \tilde{\chi}_1 + \tilde{\chi}_2 < 2$, the existence of solutions to (4.13) is ruled out by Theorem 3.1 (i). Hence we consider the case $\min\{\tilde{\chi}_1, \tilde{\chi}_2\} \geq 1$ which implies that $\chi \geq 2$. Under this condition, if we consider a special case $\tilde{c_1} = \tilde{c_2} = \frac{c}{2}$, we can apply Theorem 3.1 to conclude that (4.13) has a solution. Particularly if $\chi = \chi_1 = \chi_2 = 2$, then $\tilde{\chi}_1 = \tilde{\chi}_2 = 1$ and the asymptotic behavior of the traveling wave solution for (4.13) is given by Proposition 1 (iii), which gives (2.10) and moreover (2.9) was implied by (4.13). However when $\chi = \chi_1 = \chi_2 > 2$, for each combination of $\tilde{\chi}_1 \geq 1$ and $\tilde{\chi}_2 \geq 1$ such that $\tilde{\chi}_1 + \tilde{\chi}_2 = \chi > 2$, (1.2) has a solution with corresponding asymptotic decay rates as given by Theorem 3.1, which finally leads to the asymptotic behavior as announced in the Theorem. Once we obtain W, the substitution of W into (4.12) gives the existence and asymptotical behavior of U and V. Hence the proof of Theorem 2.2 (iii) is finished.

4.3. **Parameter regimes for traveling waves.** From the results derived in the preceding subsection, we know that under assumption $\beta > -1$ and $\beta \neq 1$, a sufficient and necessary condition for the existence of traveling wave solutions to (1.2) is $\min\{\lambda_1, \lambda_2\} \geq 1$. In this section, we shall show that the set

$$I = \{(\chi_1, \chi_2) \mid \min\{\lambda_1, \lambda_2\} \geq 1\} \tag{4.14}$$

is not empty, where λ_1, λ_2 are defined in terms of χ_1, χ_2 in (4.4). That is we show for any given $\beta > -1$ and $\beta \neq 1$, there exist $\chi_1, \chi_2 \in \mathbb{R}$ such that $(\chi_1, \chi_2) \in I$. From (4.4), we see that $\min\{\lambda_1, \lambda_2\} \geq 1$ is equivalent to

$$\frac{\beta\chi_2 - \chi_1}{\beta^2 - 1} \geq 1, \ \frac{\beta\chi_1 - \chi_2}{\beta^2 - 1} \geq 1. \tag{4.15}$$

Then we have the following cases to solve (4.15).

Case 1. If $\beta > 1$, then $\beta^2 - 1 > 0$. We solve (4.15) and obtain that

$$I = \{(\chi_1, \chi_2) \mid \chi_1/\beta + (\beta^2 - 1)/\beta \leq \chi_2 \leq \beta\chi_1 - (\beta^2 - 1)\}.$$

An illustration of such region I is plotted in Fig. 1 (a), where we choose $\beta = 2$.
Case 2. If $0 < \beta < 1$, then $\beta^2 - 1 < 0$. Then solving (4.15) gives

$$I = \{(\chi_1, \chi_2) \mid \beta\chi_1 - (\beta^2 - 1) \le \chi_2 \le \chi_1/\beta + (\beta^2 - 1)/\beta\}.$$

A plot of such region I is given in Fig. 1 (b), where $\beta = 1/2$.
Case 3. If $\beta = 0$, then (4.15) directly gives

$$I = \{(\chi_1, \chi_2) \mid \chi_1 \ge 1, \ \chi_2 \ge 1\}$$

whose numerical graph is given in Fig. 1 (c).
Case 4. If $-1 < \beta < 0$, then $\beta^2 - 1 < 0$. Then we solve (4.15) and have

$$I = \{(\chi_1, \chi_2) \mid \chi_2 \ge \max\{\beta\chi_1 - (\beta^2 - 1), \chi_1/\beta + (\beta^2 - 1)/\beta\}\}.$$

A numerical plot of such region I is given in Fig. 1 (d), where $\beta = -1/2$.

Thus, if (χ_1, χ_2) lies in the region I as plotted in Fig. 1, the existence of traveling wave solutions of (1.2) can be guaranteed.

5. **Diffusion limit.** In this section, we show the traveling wave solutions are convergent with respect to the chemical diffusion coefficient ε based on the ideas of [23, 30]. We first establish the ε-convergence for the solution component W as follows.

Lemma 5.1. *Let W_ε and W_0 be the traveling wave solution of (4.5) for $\varepsilon > 0$ and for $\varepsilon = 0$, respectively. Then it follows that*

$$|W_\varepsilon - W_0| = \mathcal{O}(\varepsilon) \text{ as } \varepsilon \to 0.$$

Proof. Denote the wave speed for $\varepsilon > 0$ by c_ε and c_0 for $\varepsilon = 0$. Then it is important to notice that $c_\varepsilon = c_0 = (N_1 + N_2)/m$ by integrating the third equation of (4.1). For convenience, we denote $h(z, W) = C_3 e^{-sz} W^{\lambda_1} + C_4 e^{-sz} W^{\lambda_2}$ and $\tilde{W} = W_\varepsilon - W_0$. Then by (4.5) and (2.4), we can derive that $\tilde{W}$ satisfies

$$\varepsilon\tilde{W}'' + c_0\tilde{W}' + \varepsilon W_0'' = h(z, W_\varepsilon) - h(z, W_0) \tag{5.1}$$

subject to

$$\tilde{W}(\pm\infty) = \tilde{W}'(\pm\infty) = 0. \tag{5.2}$$

By the mean value theorem, we find two positive numbers ξ_1 and ξ_2 between W_0 and W_ε such that

$$\begin{aligned} h(z, W_\varepsilon) - h(z, W_0) &= e^{-kz}[C_3(W_\varepsilon^{\lambda_1} - W_0^{\lambda_1}) + C_4(W_\varepsilon^{\lambda_2} - W_0^{\lambda_2})] \\ &= e^{-kz}[C_3\lambda_1\xi_1^{\lambda_1 - 1}(W_\varepsilon - W_0) + C_4\lambda_2\xi_2^{\lambda_2 - 1}(W_\varepsilon - W_0)] \\ &= e^{-kz}(C_3\lambda_1\xi_1^{\lambda_1 - 1} + C_4\lambda_2\xi_2^{\lambda_2 - 1})\tilde{W}. \end{aligned}$$

Noticing that $C_3\lambda_1\xi_1^{\lambda_1 - 1} + C_4\lambda_2\xi_2^{\lambda_2} > 0$, we multiply (5.1) by $\tilde{W}$ and obtain

$$\varepsilon\tilde{W}\tilde{W}'' + c_0\tilde{W}\tilde{W}' + \varepsilon W_0''\tilde{W} = (C_3\lambda_1\xi_1^{\lambda_1 - 1} + C_4\lambda_2\xi_2^{\lambda_2 - 1})\tilde{W}^2 \ge 0. \tag{5.3}$$

Hence integrating (5.3) on both sides over (z, ∞), yields

$$I = \int_z^\infty (\varepsilon\tilde{W}'' + c_0\tilde{W}')\tilde{W}\,dy + \varepsilon\int_z^\infty \tilde{W}W_0''\,dy \ge 0 \tag{5.4}$$

which, along with the boundary condition (5.2), yields

$$\begin{aligned} I &= -\varepsilon \tilde{W}'\tilde{W} - \frac{c_0}{2}\tilde{W}^2 - \varepsilon \int_z^\infty |\tilde{W}'|^2 dy + \varepsilon \int_z^\infty \tilde{W} W_0'' dy \\ &\le -\frac{\varepsilon}{2}(\tilde{W}^2)' - \frac{c_0}{2}\tilde{W}^2 + \varepsilon \int_z^\infty \tilde{W} W_0'' dy. \end{aligned} \tag{5.5}$$

This gives

$$\frac{\varepsilon}{2}(\tilde{W}^2)' + \frac{c_0}{2}\tilde{W}^2 \le \varepsilon \int_z^\infty \tilde{W} W_0'' dy. \tag{5.6}$$

Let z_0 be a point at which $|\tilde{W}(z)|$ attains its maximum on $\mathbb{R}$. Then $(\tilde{W}^2)'(z_0) = 2\tilde{W}'(z_0)\tilde{W}(z_0) = 0$, and it follows from (5.6) that

$$\tilde{W}^2(z_0) \le \frac{2\varepsilon}{c_0}\int_{z_0}^\infty |\tilde{W}(z_0)| \cdot |W_0''| dy \le \frac{2\varepsilon}{c_0}|\tilde{W}(z_0)| \int_{z_0}^\infty |W_0''| dy \tag{5.7}$$

which gives that

$$|\tilde{W}(z)| \le |\tilde{W}(z_0)| \le \frac{2\varepsilon}{c_0}\|W_0''\|_{L^1(\mathbb{R})}. \tag{5.8}$$

With (4.8) and (4.10), by simple calculation, we see that W_0'' exponentially decays with respect to z as $|z| \to \infty$, which implies $\|W_0''\|_{L^1(\mathbb{R})} < \infty$. Therefore (5.8) gives rise to

$$|\tilde{W}| = |W_\varepsilon - W_0| = \mathcal{O}(\varepsilon) \text{ as } \varepsilon \to 0 \tag{5.9}$$

which completes the proof. ☐

With Lemma 5.1, we are ready to prove Theorem 2.3.

Proof of Theorem 2.3. Due to Lemma 5.1, it remains to show the ε-convergence for U_ε and V_ε. In fact, by (4.3) and the mean value theorem, we derive that

$$\begin{aligned} U_\varepsilon - U_0 &= C_3 e^{-sz} W_\varepsilon^{\lambda_1} - C_3 e^{-sz} W_0^{\lambda_1} \\ &= C_3 e^{-sz} \zeta^{\lambda_1 - 1}(W_\varepsilon - W_0) \end{aligned}$$

where ζ is between W_0 and W_ε with $0 < \zeta < m$. Hence by (5.9), it has that

$$|U_\varepsilon - U_0| = \mathcal{O}(\varepsilon) \text{ as } \varepsilon \to 0$$

for each $z \in \mathbb{R}$. The same argument applied to V_ε and V_0 yields that

$$|V_\varepsilon - V_0| = \mathcal{O}(\varepsilon) \text{ as } \varepsilon \to 0$$

for all $z \in \mathbb{R}$. The proof of Theorem 2.3 is thus completed. ☐

6. **Appendix.**

In this appendix, we are devoted to presenting the proof for Proposition 1 to make the paper self-contained. To this end, we shall first present two results of [5] that were used in our proof.

Proposition 2. ([5, Chapter IV (Theorem 2)]) *Let A be a constant matrix whose characteristic roots $\lambda_1, \dots, \lambda_n$ are all simple, and let ξ_i be a characteristic vector of A belonging to the characteristic root $\lambda_i (i = 1, \dots, n)$. If $B(t)$ is a continuous matrix defined for $t \ge t_0$ such that*

$$\int_{t_0}^\infty |B(t)| dt < \infty,$$

then the equation

$$\frac{dx}{dt} = [A + B(t)]x$$

has a fundamental system of solutions $x_1(t), \ldots, x_n(t)$ *satisfying for* $t \to \infty$

$$x_k(t) \sim e^{\lambda_k t}\xi_k, \qquad (k = 1, \ldots, n).$$

Proposition 3. *([5, Chapter IV (Theorem 14)]) Let* $f(x)$ *be a positive, twice continuously differentiable function for* $x \geq x_0$ *such that*

$$\int_{x_0}^{\infty} |f^{-\frac{3}{2}} f''| dx < \infty.$$

Then the equation

$$\frac{d^2y}{dx^2} = f(x)y$$

has a fundamental system of solutions satisfying for $x \to \infty$

$$y \sim [f(x)]^{-\frac{1}{4}} \exp\left\{ \pm \int_{x_0}^{x} [f(\xi)]^{\frac{1}{2}} d\xi \right\}.$$

Next we present a well-known result for the traveling wave solutions of the Fisher equation which was summarized in [31]. Consider the Fisher equation

$$u_t = \varepsilon u_{xx} + f(u), \;\; x \in \mathbb{R}, \;\; t \geq 0 \tag{6.1}$$

where the kinetic function $f(u)$ satisfies the following conditions

$$\begin{aligned} &(1)\;\; f(u), f'(u) \in C[0, \infty); \\ &(2)\;\; f(0) = f(a) = 0 \text{ for some } a > 0; \\ &(3)\;\; f(u) > 0 \text{ for all } u \in (0, a) \text{ and } f(u) < 0 \text{ for } u \in (a, \infty); \\ &(4)\;\; f'(0) > 0 \text{ and } f'(a) < 0. \end{aligned} \tag{6.2}$$

A prototypical form of $f(u)$ satisfying condition (6.2) is $f(u) = u(1-u/a)$ where a is called the carrying capacity. The traveling wave solution $u(x,t) = U(x-ct) =: U(z)$ of (6.1) satisfies the equation

$$\varepsilon U'' + cU' + f(U) = 0.$$

Then the following result holds.

Theorem 6.1. *Let (6.2) hold. Then (6.1) has a unique (up to a translation) bounded nonnegative traveling wave solution* $U(z)$ *with* $U' < 0$ *for all* $z \in \mathbb{R}$ *and boundary condition* $U(-\infty) = a, \;\; U(+\infty) = 0$ *if and only if*

$$c \geq \bar{c} = 2\sqrt{\varepsilon f'(0)}. \tag{6.3}$$

Moreover, the traveling wave solution $U(z) = U(x-ct)$ *has the following asymptotic behavior as* $|z| \to \infty$ *for* $c > \bar{c} = 2\sqrt{\varepsilon f'(0)}$*:*

$$U(z) = \begin{cases} a - Ce^{\lambda_+(a)z}, & z \to -\infty \\ Ce^{\lambda_+(0)z}, & z \to \infty. \end{cases}$$

where

$$\lambda_+(\xi) = \frac{-c + \sqrt{c^2 - 4\varepsilon f'(\xi)}}{2\varepsilon}.$$

Proof of Proposition 1. The assertions (i) and (ii) in Proposition 1 have been proved in papers [22, 30] by transforming the equation (3.1) into an equation with constant coefficients via a change of variable. However such transformation only works for $\lambda \neq 1$. When $\lambda = 1$, the existence of solutions to (3.1) in the assertion (iii) has already been given in [6] by applying the result of [15]. For completeness, we sketch those proofs below and present a detailed proof for the decay rates in (iii) which is new. we first consider the case $\lambda \neq 1$ for which we define

$$\mu = \frac{k}{1-\lambda} \tag{6.4}$$

and introduce a new variable $P(z)$ such that

$$p(z) = P(z)e^{-\mu z}. \tag{6.5}$$

Then substituting (6.5) into (3.1) and canceling $e^{-\mu z}$ yield that

$$\varepsilon P'' + sP' + f(P) = 0 \tag{6.6}$$

where

$$f(P) = \eta P - \alpha P^{\lambda} = \eta P(1 - \alpha P^{\lambda-1}/\eta) \tag{6.7}$$

and $s = c - 2\varepsilon\mu,\ \eta = \varepsilon\mu^2 - c\mu$. If $\lambda < 1$, then $f(P)$ is not differentiable at $P = 0$ and there is no traveling wave solution to (6.6) by simple analysis. If $\lambda > 1$, then $\mu < 0$ and $f(P)$ defined in (6.7) satisfies the conditions in (6.2) with $a = (\frac{\eta}{\alpha})^{\frac{1}{\lambda-1}}$ and hence by Theorem 6.1, we know that (6.6) have a unique solution (up to a translation) with $P'(z) < 0$ for all $z \in \mathbb{R}$ and $P(-\infty) = (\frac{\eta}{\alpha})^{\frac{1}{\lambda-1}},\ P(+\infty) = 0$ iff $s \geq 2\sqrt{\varepsilon f'(0)} = 2\sqrt{\varepsilon\eta}$. Moreover the traveling wave solutions have the following asymptotic decay rate for $s > 2\sqrt{\varepsilon\eta}$ (i.e. $c > 0$)

$$\begin{aligned} &P(z) = \left(\frac{\eta}{\alpha}\right)^{\frac{1}{\lambda-1}} - q_1 e^{\lambda_2^+ z},\ \text{as } z \to -\infty, \\ &P(z) = q_2 e^{\lambda_1^+ z},\ \text{as } z \to \infty \end{aligned} \tag{6.8}$$

where q_1, q_2 are positive constants and

$$\lambda_1^+ = \frac{-s + \sqrt{s^2 - 4\varepsilon\eta}}{2\varepsilon} = \frac{(2\varepsilon\mu - c) + \sqrt{c^2}}{2\varepsilon} = \mu$$

and

$$\lambda_2^+ = \frac{-s + \sqrt{s^2 + 4\varepsilon\eta(\lambda-1)}}{2\varepsilon} = \frac{(2\varepsilon\mu - c) + \sqrt{c^2 + 4\varepsilon\eta\lambda}}{2\varepsilon}.$$

Then by (6.5) and (6.8), we have

$$p(z) = \left[\left(\frac{\eta}{\alpha}\right)^{\frac{1}{\lambda-1}} - q_1 e^{\lambda_2^+ z}\right] e^{-\mu z},\ \text{as } z \to -\infty, \tag{6.9}$$

and

$$p(z) = q_2 e^{(\lambda_1^+ - \mu)z} = q_2,\ \text{as } z \to +\infty.$$

From the boundary condition in (3.1), we see that $q_2 = \omega$. But we need to derive the asymptotic decay rate of $p(z)$ as $z \to +\infty$ since $\lambda_1^+ - \mu = 0$. From (3.1), we see that the dynamics of $q(z)$ as $z \to +\infty$ is determined by the equation $\varepsilon p'' + cp' = 0$, which yields that $p(z) - \omega \sim e^{-\frac{c}{\varepsilon}z}$. Noticing that $\mu < 0$, we can directly verify that $s \geq 2\sqrt{\varepsilon\eta} \iff c \geq 0$. But when $c = 0$, the equation (3.1) becomes $\varepsilon p'' = \alpha e^{-kz}p^{\lambda} > 0$, which means that p is convex and can not satisfies the boundary condition in (3.1). Hence the problem (3.1) has a solution iff $c > 0$. To finish the proof of Proposition 1 (ii), it remains to show the monotonicity of p. To this end, we first remark from (6.9) it follows that $p'(-\infty)=0$. Then we integrate the equation (6.16) over $(-\infty,z)$ and obtain

$$p'=\alpha e^{-\frac{c}{\varepsilon}z}\int_{-\infty}^{z}e^{(\frac{c}{\varepsilon}-k)\xi}p(\xi)d\xi>0$$

for all $z\in(-\infty,\infty)$ since $p(\xi)\geq 0$ and $p(\xi)\not\equiv 0$ for all $\xi\in\mathbb{R}$.

Next we proceed to prove Proposition 1 (iii). For the existence, above approach for $\lambda>1$ no longer works for $\lambda=1$ due to the transformation (6.4). In this case, we define a change of independent variable with an idea of [6]

$$\tau=e^{-\frac{c}{\varepsilon}z},$$

namely $\frac{d}{dz}=-\frac{c}{\varepsilon}e^{-\frac{c}{\varepsilon}z}\frac{d}{d\tau}=-\frac{c}{\varepsilon}\tau\frac{d}{d\tau}$. It is clear $\tau\in[0,\infty)$ and

$$\tau=\begin{cases}0, & \text{if } z=+\infty,\\ \infty, & \text{if } z=-\infty.\end{cases}$$

By defining $w(\tau)=p(-\varepsilon\ln\tau/c)$, we obtain from (3.1) that $w''(\tau)-\frac{\alpha\varepsilon}{c^2}e^{(\frac{2c}{\varepsilon}-k)z}w(\tau)^r=0$, which leads to

$$w''(\tau)=\zeta\tau^{\theta}w(\tau)^r,\ w(0)=\omega,\ w(\infty)=0 \tag{6.10}$$

where

$$\zeta=\frac{\alpha\varepsilon}{c^2}>0,\ \theta=\frac{k\varepsilon}{c}-2=\frac{\varepsilon}{d}-2.$$

The equation in (6.10) is a type of linear Emden-Fowler equation [2] and the existence of solutions of (6.10) with condition $w'<0$ was guaranteed by [15], as shown in [6]. We proceed to show the uniqueness of solutions. If we let $p_1(z)$ and $p_2(z)$ be two solutions of (3.1). Then $\tilde{p}(z)=p_1(z)-p_2(z)$ satisfies

$$\begin{cases}\varepsilon\tilde{p}''+c\tilde{p}'-\alpha e^{-kz}\tilde{p}=0,\\ \tilde{p}(-\infty)=0,\ \tilde{p}(\infty)=0.\end{cases} \tag{6.11}$$

Then multiplying the first equation of (6.11) by $\tilde{p}$ and integrating the result over $\mathbb{R}$ yields

$$\int_{\mathbb{R}}(\varepsilon|\tilde{p}'(z)|^2+\alpha e^{-kz}\tilde{p}^2(z))dz=0$$

which implies that $\tilde{p}\equiv 0$ since $\alpha>0$ and hence the uniqueness is obtained. To finish the proof, it remains to derive the asymptotic behavior of the solutions announced in (iii). We first study the asymptotics as $z\to\infty$ by considering the following problem

$$\begin{cases}\varepsilon p''+cp'-\alpha e^{-kz}p=0,\ z\in(0,\infty)\\ p(0)=\varrho,\ p(\infty)=\omega\end{cases} \tag{6.12}$$

where we assume that $p(0)=\varrho$ with $0<\varrho<\omega$.

Denoting $p'=\rho$ and $X=\begin{bmatrix}p\\ \rho\end{bmatrix}$, we can write the first equation of (6.12) as

$$X'=(A+B(z))X \tag{6.13}$$

where

$$A=\begin{bmatrix}0 & 1\\ 0 & -\frac{c}{\varepsilon}\end{bmatrix},\ B(z)=\begin{bmatrix}0 & 0\\ \frac{\alpha e^{-kz}}{\varepsilon} & 0\end{bmatrix}.$$

It is straightforward to obtain the eigenvalues of A as

$$\lambda_1 = -\frac{c}{\varepsilon},\ \lambda_2 = 0 \tag{6.14}$$

with corresponding eigenvectors

$$\mathbf{x}_i = \begin{bmatrix} 1 \\ \lambda_i \end{bmatrix}, i = 1, 2.$$

Considering the fact that

$$\int_0^\infty |B(z)|dz = \frac{\alpha}{\varepsilon}\int_0^\infty e^{-kz}dz = \frac{\alpha}{\varepsilon k} < \infty,$$

then by Proposition 2, the solution of system (6.13) satisfies

$$X \sim c_1 e^{\lambda_1 z}\mathbf{x}_1 + c_2 e^{\lambda_2 z}\mathbf{x}_2,\ \text{as } z \to \infty$$

and hence $p = c_1 e^{-\frac{c}{\varepsilon}z} + c_2$. Applying the boundary condition yields that $c_1 = \varrho - \omega$ and $c_2 = \omega$. Therefore it follows that

$$p - \omega \sim e^{-\frac{c}{\varepsilon}z},\ \text{as } z \to \infty.$$

Next we proceed to explore the asymptotics as $z \to -\infty$ and consider the problem

$$\begin{cases} \varepsilon p'' + cp' - \alpha e^{-kz}p = 0,\ z \in (-\infty, 0) \\ p(-\infty) = 0,\ p(0) = \varrho. \end{cases} \tag{6.15}$$

Note that the coefficient e^{-kz} is singular at $z = -\infty$, it is unfeasible to treat the problem (6.15) directly. Instead we shall transform the problem to a non-singular problem. To this end, we rewrite the differential equation as

$$(e^{\frac{c}{\varepsilon}z}p')' - \frac{\alpha}{\varepsilon}e^{(c/\varepsilon - k)z}p = 0. \tag{6.16}$$

By change of variables

$$\tau = e^{-\frac{c}{\varepsilon}z},\quad q(\tau) = p(-\varepsilon \ln \tau / c) \tag{6.17}$$

we have $p'(z) = -\frac{c}{\varepsilon}e^{-\frac{c}{\varepsilon}z}q'(\tau)$, where $\tau \in [1, \infty)$ and

$$\tau = \begin{cases} 1, & \text{if } z = 0, \\ \infty, & \text{if } z = -\infty. \end{cases} \tag{6.18}$$

Then the substitution of (6.17) into (6.15) yields

$$\begin{cases} q''(\tau) = \frac{\varepsilon\alpha}{c^2}\tau^\theta q(\tau),\ \tau \in [1, \infty) \\ q(1) = \varrho,\ q(\infty) = 0 \end{cases} \tag{6.19}$$

where

$$\theta = \frac{\varepsilon k}{c} - 2. \tag{6.20}$$

For the convenience to proceed, we let $f(\tau) = \frac{\varepsilon\alpha}{c^2}\tau^\theta$ and shall apply the results in Proposition 3 to derive the asymptotical behavior of solutions to (6.19). To this end, we need to verify $\int_1^\infty |f^{-\frac{3}{2}}f''|d\tau < \infty$. Since $f''(\tau) = \frac{\varepsilon\alpha}{c^2}\theta(\theta-1)\tau^{\theta-2}$ and $-\frac{\theta}{2} - 2 = -\frac{\varepsilon k}{2c} - 1 < -1$, we find that

$$\begin{aligned} \int_1^\infty |f^{-\frac{3}{2}}f''|d\tau &= \frac{\alpha\varepsilon}{c^2}|\theta(\theta-1)|\int_1^\infty |f^{-\frac{3}{2}}\tau^{\theta-2}|d\tau \\ &\le \frac{c}{\sqrt{\varepsilon\alpha}}|\theta(\theta-1)|\int_1^\infty \tau^{-\frac{\theta}{2}-2}d\tau < \infty. \end{aligned}$$

Then by Proposition 3, the solution of system (6.19) has the following asymptotic behavior as $\tau \to \infty$:

$$q(\tau) \sim [f(\tau)]^{-\frac{1}{4}} e^{-\int_1^\tau [f(\xi)]^{\frac{1}{2}} d\xi} = \left(\frac{c^2}{\varepsilon\alpha}\right)^{\frac{1}{4}} \tau^{-\frac{\theta}{4}} e^{-\frac{\sqrt{\varepsilon\alpha}}{c}\int_1^\tau \xi^{\frac{\theta}{2}} d\xi}$$
$$= \left(\frac{c^2}{\varepsilon\alpha}\right)^{\frac{1}{4}} e^{\frac{2\sqrt{\varepsilon\alpha}}{c(\theta+2)}} \tau^{-\frac{\theta}{4}} e^{-\frac{2\sqrt{\varepsilon\alpha}}{c(\theta+2)}\tau^{\frac{\theta}{2}+1}}$$

which along with (6.20) yields

$$q(\tau) \sim \tau^{-\frac{\varepsilon k - 2c}{4c}} e^{-\frac{2\sqrt{\alpha}}{k\sqrt{\varepsilon}}\tau^{\frac{\varepsilon k}{2c}}}.$$

Thanks to (6.17) and (6.18), we have

$$p(z) = q(e^{-\frac{c}{\varepsilon}z}) \sim e^{(\frac{k}{4}-\frac{c}{2\varepsilon})z - \frac{2\sqrt{\alpha}}{k\sqrt{\varepsilon}}e^{-\frac{k}{2}z}}, \text{ as } z \to -\infty.$$

Hence we obtain the asymptotic behavior of solutions as $z \to \pm\infty$ for $\lambda = 1$, and the proof of Proposition 1 is completed. $\square$

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REFERENCES

[1] J. Adler, Chemotaxis in bacteria, *Science*, **153** (1966), 708–716.

[2] R. Bellman, *Stability Theory of Differential Equations*, McGraw-Hill Book Company, New York-Toronto-London, 1953.

[3] F. Berezovskaya, A. Novozhilov and G. Karev, Families of traveling impulse and fronts in some models with cross-diffusion, *Nonlinear Anal. Real World Appl.*, **9** (2008), 1866–1881.

[4] C. Conca, E. Espejo and K. Vilches, Remarks on the blowup and global existence for a two species chemotactic Keller-Segel system in $\mathbb{R}^2$, *Euro. J. Appl. Math.*, **22** (2011), 553–580.

[5] W. Coppel, *Stability and Asymptotic Behavior of Differential Equations*, D. C. Heath and Co., Boston, 1965.

[6] Y. Ebihara, Y. Furusho and T. Nagai, Singular solution of traveling waves in a chemotactic model, *Bull. Kyushu Inst. Tech. Math. Natur. Sci.*, **39** (1992), 29–38.

[7] E. Espejo Arenas, A. Stevens and J. Velázques, Simultaneous finite time blow-up in a two-species model for chemotaxis, *Analysis (Munich)*, **29** (2009), 317–338.

[8] A. Fasano, A. Mancini and M. Primicerio, Equilibrium of two populations subjected to chemotaxis, *Math. Models Methods. Appli. Sci.*, **14** (2004), 503–533.

[9] D. Horstmann, Generalizing the Keller-Segel model: Lyapunov functionals, steady state analysis, and blow-up results for multi-species chemotaxis models in the presence of attraction and repulsion between competitive interacting species, *J. Nonlinear Sci.*, **21** (2011), 231–270.

[10] D. Horstmann and A. Stevens, A constructive approach to traveling waves in chemotaxis, *J. Nonlin. Sci.*, **14** (2004), 1–25.

[11] H.-Y. Jin, J. Li and Z.-A. Wang, Asymptotic stability of traveling waves of a chemotaxis model with singular sensitivity, *J. Differential Equations*, **255** (2013), 193–219.

[12] Y. Kalinin, L. Jiang, Y. Tu and M. Wu, Logarithmic sensing in Escherichia coli bacterial chemotaxis, *Biophysical Journal*, **96** (2009), 2439–2448.

[13] E. Keller and L. Segel, Traveling bands of chemotactic bacteria: A theorectical analysis, *J. Theor. Biol.*, **30** (1971), 235–248.

[14] F. Kelley, K. Dapsis and D. Lauffenburger, Effects of bacterial chemotaxis on dynamics of microbial competition, *Microb. Ecol.*, **16** (1988), 115–131.

[15] I. Kiguradse, On the non-negative non-increasing solutions of non-linear second order differential equations, *Ann. Mat. Pura Appl. (4)*, **81** (1969), 169–191.

[16] D. Lauffenburger, Quantitative studies of bacterial chemotaxis and microbial population dynamics, *Microb. Ecol.*, **22** (1991), 175–185.

[17] D. Le, Coexistence with chemotaxis, *SIAM J. Math. Anal.*, **32** (2000), 504–521.

[18] T. Li and Z.-A. Wang, Nonlinear stability of traveling waves to a hyperbolic-parabolic system modeling chemotaxis, *SIAM J. Appl. Math.*, **70** (2009/10), 1522–1541.

[19] T. Li and Z.-A. Wang, Nonlinear stability of large amplitude viscous shock waves of a hyperbolic-parabolic system arising in chemotaxis, *Math. Models Methods Appl. Sci.*, **20** (2010), 1967–1998.

[20] T. Li and Z.-A. Wang, Asymptotic nonlinear stability of traveling waves to conservation laws arising from chemotaxis, *J. Differential Equations*, **250** (2011), 1310–1333.

[21] M. Luca, A. Chavez-Ross, L. Edelstein-Keshet and A. Mogilner, Chemotactic signaling, microglia, and Alzheimer's disease senile plaques: Is there a connection?, *Bull. Math. Biol.*, **65** (2003), 693–730.

[22] R. Lui and Z.-A. Wang, Traveling wave solutions from microscopic to macroscopic chemotaxis models, *J. Math. Biol.*, **61** (2010), 739–761.

[23] T. Nagai and T. Ikeda, Traveling waves in a chemotaxis model, *J. Math. Biol.*, **30** (1991), 169–184.

[24] G. Odell and E. Keller, Traveling bands of chemotactic bacteria revisited, *J. Theor. Biol.*, **56** (1976), 243–247.

[25] G. Rosen, Steady-state distribution of bacteria chemotactic toward oxygen, *Bull. Math. Biol.*, **40** (1978), 671–674.

[26] G. Rosen, Theoretical significance of the condition $\delta = 2\mu$ in bacterial chemotaxis, *Bull. Math. Biol.*, **45** (1983), 151–153.

[27] G. Rosen and S. Baloga, On the stability of steadily propogating bands of chemotactic bacteria, *Math. Biosci.*, **24** (1975), 273–279.

[28] H. Schwetlick, Traveling waves for chemotaxis systems, *Proc. Appl. Math. Mech.*, **3** (2003), 476–478.

[29] Y. Tao and Z.-A. Wang, Competing effects of attraction vs. repulsion in chemotaxis, *Math. Models Methods Appl. Sci.*, **23** (2013), 1–36.

[30] Z.-A. Wang, Wavefront of an angiogenesis model, *Discrete Contin. Dyn. Syst. Series B*, **17** (2012), 2849–2860.

[31] Z.-A. Wang, Mathematics of traveling waves in chemotaxis–review paper, *Discrete Contin. Dyn. Syst. Series B*, **18** (2013), 601–641.

[32] G. Wolansky, Multi-components chemotaxis system in absence of conflict, *Eur. J. Appl. Math.*, **13** (2002), 641–661.

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