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Stability of traveling waves of the Keller–Segel system with logarithmic sensitivity

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Proceeding with a series of works (Refs. 12, 23–25) by the authors, this paper establishes the nonlinear asymptotic stability of traveling wave solutions of the Keller–Segel system with nonzero chemical diffusion and linear consumption rate, where the right asymptotic state of cell density is vacuum (zero) and the initial value is a perturbation with zero integral from the spatially shifted traveling wave. The main challenge of the problem is various singularities caused by the logarithmic sensitivity and the vacuum asymptotic state, which are overcome by a Hopf–Cole type transformation and the weighted energy estimates with an unbounded weight function introduced in the paper.

Keywords: Chemotaxis; traveling waves; vacuum state; nonlinear stability; weighted energy estimates.

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1. Introduction

There are various wave propagating phenomena driven by chemotaxis which, unlike diffusion, directs the motion of species up or down a chemical concentration gradient. Examples include the propagation of traveling band of bacterial toward the oxygen,[1,2] the outward propagation of concentric ring waves by *E. coli*,[3–5] the spiral wave patterns during the aggregation of *Dictyostelium discoideum*[10,17] and the migration of *Myxococcus xanthus* as traveling waves in the early stage of starvation-induced fruiting body development.[43] The mathematical study of chemotactic traveling waves was started by Keller and Segel in the '70s[14,15] where they proposed the following model:

$$\begin{cases} u_t = [Du_x - \chi u(\log c)_x]_x, \\ c_t = \varepsilon c_{xx} - uc^m, \end{cases} \tag{1.1}$$

to describe the propagation of traveling bands of chemotactic bacteria with a constant speed observed in the celebrated experiment of Adler,[1,2] where $u(x,t)$ denotes the bacterial density and $c(x,t)$ the oxygen concentration. $D > 0$ and $\varepsilon \geq 0$ are the bacterial and chemical diffusion coefficients, and $\chi > 0$ is called the chemotactic coefficient.

When $0 \leq m < 1$, Keller and Segel[16] showed that model (1.1) with $\varepsilon = 0$ can generate the traveling bands, whose speeds were in satisfactory agreement with experimental observation of Refs. 1 and 2. Subsequently, a sequence of rigorous works on various aspects of traveling wave solutions of (1.1) with $\varepsilon \geq 0$ had been carried out, cf. Refs. 28, 29, 32, 33, 35, 36 and references therein. When $m = 1$, the model (1.1) was used by Nossal[31] to describe the chemotactic boundary formation by bacterial population in response to the substrate consisting of nutrients if $\varepsilon = 0$, and by Rosen[33,34] to show the phenomenological theory for the chemotaxis and consumption of oxygen by motile aerobic bacteria if $\varepsilon > 0$. Recently the structure of chemotaxis model (1.1) with $\varepsilon \geq 0$ have been advocated to describe the directed movement of endothelial cells toward the signaling molecule vascular endothelial growth factor (VEGF) during the initiation of angiogenesis,[6–8,18,40] where u denotes the density of endothelial cells and c stands for the concentration of VEGF. When $m > 1$, the model (1.1) does not admit traveling wave solutions (e.g. see Refs. 36 and 41), and the global solutions of (1.1) with other forms of chemotactic sensitivity function were studied in Refs. 7–9, 20 and 39 for both bounded and unbounded domains.

The existence of traveling wave solutions of the Keller–Segel model (1.1) has been extensively studied as aforementioned, the stability of traveling wave solutions remains open for quite a long time due to the singular logarithmic sensitivity $\log c$. The linear instability of traveling wave solutions to (1.1) in certain functional spaces was first obtained in Ref. 29 for a special case $m = 0$. The linear stability/instability of traveling wave solutions for $m \neq 0$ still remains open. The nonlinear stability of traveling wave solutions to (1.1) is more challenging and remains open for an even longer time until recently the last two authors proved the nonlinear stability of traveling waves in Refs. 23 and 24 for $\varepsilon = 0$ and in Ref. 25 for $\varepsilon > 0$ small with $m = 1$, $u_+ > 0$, where the initial data were prescribed as

$$(u(x,0), c(x,0)) = (u_0(x), c_0(x)) \to (u_\pm, c_\pm) \quad \text{as } x \to \pm\infty. \tag{1.2}$$

Since u and c in (1.1) represent the biological particle densities, our attention will be restricted to the biologically relevant regime in which $u_\pm \geq 0$ and $c_\pm \geq 0$.

Studying the nonlinear stability of traveling wave solutions of (1.1) directly remains very challenging due to the logarithmic singularity in the first equation. The works in Refs. 23–25 adopt the Hopf–Cole type transformation in Refs. 19 and 42

$$v = -(\log c)_x = -\frac{c_x}{c} \tag{1.3}$$

and transform the system (1.1) and (1.2) with $m = 1$ into a system of viscous conservation laws

$$\begin{cases} u_t - \chi(uv)_x = Du_{xx}, \\ v_t + (\varepsilon v^2 - u)_x = \varepsilon v_{xx}, \end{cases} \tag{1.4}$$

with initial data

$$(u, v)(x,0) = (u_0, v_0)(x) \to (u_\pm, v_\pm) \quad \text{as } x \to \pm\infty, \tag{1.5}$$

where $v_0(x) = -(\log c_0)_x = -\frac{c_{0x}}{c_0}$. After the transformation, the logarithmic singularity in (1.1) does not appear in (1.4), which provides an opportunity for analysis. The transformation (1.3) was first applied in Ref. 19 to obtain system (1.4) with $\varepsilon = 0$ and later was further employed to obtain the model (1.4) with $\varepsilon > 0$ in Ref. 25 to study the existence and stability of traveling wave solutions where $\varepsilon > 0$ was assumed to be small. Recently the last author of this paper establishes the existence of traveling wave solutions of (1.1) for any large $\varepsilon > 0$ and $u_+ = 0$ in Ref. 40. Apparently the transformation (1.3) is not helpful for the Keller–Segel model (1.1) with $m \neq 1$. The results of Refs. 23–25 essentially assumed that the right asymptotic state of cell density $u(x,t)$ is strictly positive, i.e. $u_+ > 0$, due to technical difficulties for the case $u_+ = 0$ which leads to singularities in the energy estimates. For the stability in the case of $u_+ = 0$, the recent work[12] overcame the singularities by using the weighted energy estimates to establish the nonlinear stability of traveling wave solutions in the inviscid case $\varepsilon = 0$.

The purpose of this paper is to establish the same stability result as that in Ref. 12 for the viscous case $\varepsilon > 0$. It turns out that the nonlinear stability analysis for $\varepsilon > 0$ requires a lot of additional effort in the energy estimates for the diffusion coefficient ε generates a nonlinear convection term in the hyperbolic system (1.4). Thanks to the transformation (1.3), the nonlinear stability of traveling wave solutions of (1.1) can be proved for $\varepsilon > 0$ small. Although the smallness of ε does not seem to be necessary as illustrated by our numerical simulation in Sec. 4, the stability of traveling wave solutions for large $\varepsilon > 0$ has to be left as an open problem due to the technical difficulties. Novel ideas are anticipated to attack such a problem. Moreover, we give the regularity and asymptotic behavior of the chemical concentration $c(x,t)$ and present the numerical simulations of the stability of traveling wave solutions $(U, \mathcal{C})$, where the results for c are new comparing with those in the previous papers.[12,23–25] Finally we mention that as $\varepsilon = 0$, the initial-boundary value problem of the model (1.4) has been studied in Refs. 22 and 44, and the Cauchy problem was investigated in Refs. 11, 20 and 45.

The rest of this paper is arranged as follows. In Sec. 2, we state our main results on the nonlinear stability of (U, V) and then on $\mathcal{C}$. In Sec. 3, we show the details of weighted energy estimates and prove our main results. In Sec. 4, we show the numerical simulations to verify our analytical results and make predictions for the remaining open questions.

2. Preliminaries and Main Results

A traveling wave solution of (1.1) in $(x,t) \in \mathbb{R} \times [0, \infty)$ is a particular non-constant solution in the form

$$u(x,t) = U(z), \quad c(x,t) = \mathcal{C}(z), \quad z = x - st, \tag{2.1}$$

with $U, \mathcal{C} \in C^\infty(\mathbb{R})$ satisfying boundary conditions

$$U(\pm\infty) = u_\pm, \quad \mathcal{C}(\pm\infty) = c_\pm, \quad U'(\pm\infty) = \mathcal{C}'(\pm\infty) = 0, \tag{2.2}$$

where s is the wave speed assumed to be non-negative without loss of generality, z is called the wave variable, the prime $'$ means the differentiation in z, u_-/u_+ and c_-/c_+ are called left/right end states of u and c, respectively, describing the asymptotic behavior of traveling wave solutions as $t \to +\infty/-\infty$. When $u_+ > 0$, the existence of traveling wave solutions of (1.1) with $m = 1$ was shown in Ref. 26 and elaborated in Ref. 41. The nonlinear stability of traveling wave solutions with $u_+ > 0$ was established in Ref. 25. When $u_+ = 0$ and $\varepsilon > 0$, the existence of traveling wave solutions with asymptotic decay rates at far field to (1.1), (2.1), (2.2) with $m = 1$ has been established previously in Ref. 40. This paper will be able to investigate the nonlinear stability of traveling wave solutions for this case. To proceed, we first quote the existence results in Ref. 40 for later use.

Proposition 2.1. (Theorem 3.3 and Theorem 4.1 in Ref. 40) *Let $m = 1$, $\varepsilon > 0$ and $u_+ = 0$. Then the system* (1.1) *has a unique* (*up to a translation*) *monotone traveling wave solution* $(U, \mathcal{C})(z)$ *satisfying $c_+ > 0, c_- = 0$ and $U'(z) < 0, \mathcal{C}'(z) > 0$ for all $z \in \mathbb{R}$, where the wave speed is uniquely determined by*

$$s = \chi\sqrt{\frac{u_-}{\chi + \varepsilon}}. \tag{2.3}$$

Moreover, the solution component $U(z)$ has the following asymptotic behavior:

$$\begin{aligned} U(z)^{\frac{D}{\chi}} - \rho^{\frac{D}{\chi}} &\sim Ce^{\lambda z}, \quad \text{as } z \to -\infty, \\ U(z) &\sim Ce^{-\frac{s}{D}z}, \quad \text{as } z \to \infty, \end{aligned} \tag{2.4}$$

where C is a generic positive constant and

$$\lambda = -\frac{s}{\chi} + \frac{-s + \sqrt{s^2 + 4\varepsilon\rho r}}{2\varepsilon} > 0, \quad \rho = \frac{s^2}{\chi^2}(\varepsilon + \chi), \quad r = \frac{\chi}{D} + 1. \tag{2.5}$$

Next we are devoted to investigating the stability of traveling wave solutions obtained in Proposition 2.1. Our plan is to study the traveling wave solutions of the transformed system (1.4) first and then transfer the results back to the original system (1.1).

A traveling wave solution of (1.4) is a solution in the form

$$(u, v)(x, t) = (U, V)(z), \quad z = x - st,$$

satisfying equations

$$\begin{cases} -sU_z - \chi(UV)_z = DU_{zz}, \\ -sV_z + (\varepsilon V^2 - U)_z = \varepsilon V_{zz}, \end{cases} \tag{2.6}$$

with boundary conditions

$$U(\pm\infty) = u_\pm, \quad V(\pm\infty) = v_\pm, \quad U'(\pm\infty) = V'(\pm\infty) = 0. \tag{2.7}$$

Integrating (2.6) with respect to z and using (2.7) yield

$$\begin{cases} DU_z = -sU - \chi UV + \varrho_1, \\ \varepsilon V_z = -sV + \varepsilon V^2 - U + \varrho_2, \end{cases} \tag{2.8}$$

where

$$\begin{cases} \varrho_1 = su_- + \chi u_- v_- = su_+ + \chi u_+ v_+, \\ \varrho_2 = sv_- - \varepsilon(v_-)^2 + u_- = sv_+ - \varepsilon(v_+)^2 + u_+. \end{cases}$$

The speed s is determined by the Rankine–Hugoniot condition

$$\begin{cases} -s(u_+ - u_-) - \chi(u_+v_+ - u_-v_-) = 0, \\ -s(v_+ - v_-) + [\varepsilon(v_+)^2 - u_+ - \varepsilon(v_-)^2 + u_-] = 0. \end{cases} \tag{2.9}$$

Note that the transformation (1.3) and the results in Proposition 2.1 require that $V(z) = -\frac{\mathcal{C}_z}{\mathcal{C}} < 0$. Since $c_+ > 0$, it follows that

$$v_+ = 0. \tag{2.10}$$

Moreover, $v_+ = u_+ = 0$ indicates that $\varrho_1 = \varrho_2 = 0$ and

$$\begin{cases} s + \chi v_- = 0, \\ u_- = (\chi + \varepsilon)v_-^2, \end{cases} \tag{2.11}$$

by which we have $\rho = u_-$. The Hopf–Cole transformation (1.3) yields that

$$V(z) = -\mathcal{C}_z(z)/\mathcal{C}(z),$$

which, in combination with Proposition 2.1 and the transformation (1.3), gives the existence of traveling wave solution $(U, V)(z)$ to the transformed system (1.4) as follows.

Proposition 2.2. *Let $m = 1$, $\varepsilon > 0$ and $u_+ = v_+ = 0$. Then system* (1.4) *has a unique* (*up to a translation*) *monotone traveling wave $(U, V)(x - st)$ satisfying $U_z < 0$ and $V_z > 0$ with wave speed $s = -\chi v_-$. Moreover, the traveling wave solution component U has the following asymptotic behavior*

$$\begin{aligned} U(z) - u_- &\sim Ce^{\lambda z}, \quad \text{as } z \to -\infty, \\ U(z) &\sim Ce^{-\frac{s}{D}z}, \quad \text{as } z \to \infty, \end{aligned} \tag{2.12}$$

where C is a generic positive constant.

Next we proceed to investigate the nonlinear asymptotic stability of traveling wave solutions to the transformed system (1.4) with initial condition (1.5). The approach employed in Ref. 25 for $u_+ > 0$ cannot be applied for $u_+ = 0$ directly due to the singularity in the estimates. In this paper, we shall introduce an unbounded weight function to overcome the singularity and apply the weighted energy estimates to prove the nonlinear stability of traveling wave solutions for $u_+ = 0$. The weight function we choose is the following

$$w(z) = 1 + e^{\eta z} \quad \text{with } \eta := \frac{s}{D}, \quad z \in \mathbb{R}. \tag{2.13}$$

In what follows, $H^k_w(\Omega)$ denotes the space of measurable functions f so that $\sqrt{w}\partial_x^j f \in L^2$ for $0 \le j \le k$ with norm $\|f\|_{H^k_w(\Omega)} := (\sum_{j=0}^k \int w(x)|\partial_x^j f|^2 dx)^{\frac{1}{2}}$. For simplicity, the convention $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_k := \|\cdot\|_{H^k(\Omega)}$ and $\|\cdot\|_{k,w} := \|\cdot\|_{H^k_w(\Omega)}$ will be used.

Then the nonlinear stability of traveling wave solutions to (1.4) and (1.5) is as follows.

Theorem 2.1. *Let $u_+ = v_+ = 0$, and $(U, V)(z)$ be the solution obtained in Proposition* 2.2. *Let $\varepsilon > 0$ be suitably small. Assume that there exists a constant x_0 such that the initial perturbation from the spatially shifted traveling waves with shift x_0 is of integral zero, namely $\phi_0(\infty) = \psi_0(\infty) = 0$, where*

$$(\phi_0, \psi_0)(x) := \int_{-\infty}^{x} (u_0(y) - U(y + x_0), v_0(y) - V(y + x_0))dy.$$

Then there exists a constant $\delta_0 > 0$, such that if $\|u_0 - U\|_{1,w} + \|v_0 - V\|_{1,w} + \|\phi_0\|_w + \|\psi_0\| \le \delta_0$, the system (1.4) *has a unique solution $(u, v)(x, t)$ satisfying*

$$(u - U, v - V) \in C([0, \infty), H^1_w) \cap L^2((0, \infty), H^2_w)$$

and the following asymptotic stability:

$$\sup_{x\in\mathbb{R}} |(u,v)(x,t) - (U,V)(x+x_0-st)| \to 0, \quad \text{as } t\to\infty.$$

Finally, transferring the results for the transformed system (1.4) back to the original chemotaxis model (1.1), we obtain the following theorem.

Theorem 2.2. *Let* $m = 1, u_+ = c_- = 0$ *and* $\mathcal{C}$ *be the traveling wave solution obtained in Proposition* 2.1. *If* $\varepsilon > 0$ *is small, then there exists a constant* $\varepsilon_0 > 0$ *such that if* $\|u_0 - U\|_{1,w} + \|(\ln c_0)_x - (\ln \mathcal{C})_x\|_{1,w} + \|\phi_0\|_w + \|\psi_0\| \le \varepsilon_0$ *and* $\phi_0(\infty) = \psi_0(\infty) = 0$, *where*

$$\phi_0(x) = \int_{-\infty}^{x} (u_0(y) - U(y+x_0))dy, \quad \psi_0(x) = -\ln c_0(x) + \ln \mathcal{C}(x+x_0),$$

the Cauchy problem (1.1) *and* (1.2) *has a unique global solution* $(u,c)(x,t)$ *with*

$$(u-U, c_x/c - \mathcal{C}_x/\mathcal{C}) \in C([0,\infty); H^1_w) \cap L^2((0,\infty); H^2_w),$$

or furthermore

$$c - \mathcal{C} \in C([0,\infty); H^1) \cap L^2((0,\infty); H^2), \quad \text{if } c_0(x) - \mathcal{C}(x+x_0) \in H^1 \tag{2.14}$$

and the following asymptotic stability

$$\sup_{x\in\mathbb{R}} |(u,c)(x,t) - (U,\mathcal{C})(x+x_0-st)| \to 0, \quad \text{as } t\to\infty.$$

Finally we remark that regularity result for $c - C$ given by (2.14) is a new result compared to the previous results in Ref. 12 where $\varepsilon = 0$. We prove the convergence of $c - \mathcal{C}$ as a direct consequence of (2.14), which is also different from the proof in Ref. 12 (see details in Remark 3.1). Particularly we find that the weighted function is not needed for the chemical concentration $c(x,t)$.

3. Nonlinear Asymptotic Stability

3.1. *Energy estimates*

The weighted energy method has been used for the stability of viscous shock waves of systems of hyperbolic conservation laws and their variants (e.g. see Refs. 13, 21 and 30). To make this method available for the system (1.4) with vacuum asymptotic state, we have to overcome the technical difficulties arising from the singularities caused by vacuum and the nonlinearities caused by the advection term $\varepsilon(v^2)_x$ in (1.4).

In this subsection, we establish some *a priori* estimates in order to prove the nonlinear stability of the traveling wave solution of (1.4) and (1.5) with $\chi > 0, u_+ = 0$. The main result is that the solution of (1.4) with data (1.5), which is a small perturbation with zero integral from the traveling wave solution $(U,V)(x-st)$, approaches this traveling wave solution $(U,V)(x-st)$, translated properly by an amount x_0, i.e.

$$\sup_{x\in\mathbb{R}} |(u,v)(x,t)-(U,V)(x+x_0-st)| \to 0, \quad \text{as } t\to+\infty,$$

where x_0 satisfies the following identity derived from the conservation of mass principle

$$\int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x)-U(x) \\ v_0(x)-V(x) \end{pmatrix} dx = x_0 \begin{pmatrix} u_+-u_- \\ v_+-v_- \end{pmatrix} + \beta r_1(u_-,v_-),$$

where $r_1(u_-,v_-)$ denotes the first right eigenvector of the Jacobian matrix of (1.4) in the absence of viscous terms evaluated at (u_-,v_-), see details in Ref. 37. The coefficient β yields the diffusion wave in general.[37] Both β and x_0 will be uniquely determined by the initial data (u_0,v_0) in (1.5). For the stability of small-amplitude shock waves of conservation laws with diffusion wave, i.e. $\beta\neq 0$, we refer to Refs. 27 and 38 for details. In this paper, we do not consider diffusion waves, i.e. assuming $\beta=0$ and we consider the stability of large-amplitude waves. Then by the conservation laws (1.4), we derive that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \begin{pmatrix} u(x,t)-U(x+x_0-st) \\ v(x,t)-V(x+x_0-st) \end{pmatrix} dx \\
&\quad= \int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x)-U(x+x_0) \\ v_0(x)-V(x+x_0) \end{pmatrix} dx \\
&\quad= \int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x)-U(x) \\ v_0(x)-V(x) \end{pmatrix} dx + \int_{-\infty}^{+\infty} \begin{pmatrix} U(x)-U(x+x_0) \\ V(x)-V(x+x_0) \end{pmatrix} dx \\
&\quad= \int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x)-U(x) \\ v_0(x)-V(x) \end{pmatrix} dx - x_0 \begin{pmatrix} u_+-u_- \\ v_+-v_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{aligned} \tag{3.1}$$

By assuming $\beta=0$, the initial perturbation is a spatially shifted traveling wave with a shift x_0 such that the following integral is zero:

$$\int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x)-U(x+x_0) \\ v_0(x)-V(x+x_0) \end{pmatrix} dx = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.2}$$

Then we employ the anti-derivative to decompose the solution as

$$(u,v)(x,t) = (U,V)(x+x_0-st) + (\phi_z,\psi_z)(z,t), \tag{3.3}$$

where $z=x-st$. Then

$$\begin{aligned}
(\phi(z,t),\psi(z,t)) = \int_{-\infty}^{z} (&u(y,t)-U(y+x_0-st), v(y,t) \\
&-V(y+x_0-st))dy,
\end{aligned} \tag{3.4}$$

for all $z \in \mathbb{R}$ and $t \geq 0$. It then follows from (3.1) that

$$\phi(\pm\infty, t) = \psi(\pm\infty, t) = 0, \quad \text{for all } t > 0. \tag{3.5}$$

The initial data of (ϕ, ψ) is thus given by

$$(\phi_0, \psi_0)(z) = \int_{-\infty}^{z} (u_0(y) - U(y + x_0), v_0(y) - V(y + x_0))dy, \tag{3.6}$$

which satisfies $(\phi_0, \psi_0)(\pm\infty) = 0$ by the assumption (3.2).

Substituting (3.3) into (1.4), integrating the resultant equation with respect to z and using (2.6), we get

$$\begin{cases} \phi_t = D\phi_{zz} + (s + \chi V)\phi_z + \chi U\psi_z + \chi\phi_z\psi_z, \\ \psi_t = \varepsilon\psi_{zz} + (s - 2\varepsilon V)\psi_z + \phi_z - \varepsilon\psi_z^2. \end{cases} \tag{3.7}$$

We look for solutions to the reformulated system (3.7) in the following solution space

$$\begin{aligned} X(0,T) := \{(\phi(z,t), \psi(z,t)) : \phi \in C([0,T]; H_w^2), \phi_z \in L^2((0,T); H_w^2), \\ \psi \in C([0,T]; H^2), \psi_z \in C([0,T]; H_w^1) \cap L^2((0,T); H_w^2)\}, \end{aligned}$$

where the weight function w is given by (2.13). Define

$$N(t) := \sup_{\tau \in [0,t]} (\|\phi(\cdot, \tau)\|_{2,w} + \|\psi(\cdot, \tau)\|_2 + \|\psi_z(\cdot, \tau)\|_{1,w}). \tag{3.8}$$

By the Sobolev embedding theorem, it holds

$$\sup_{\tau \in [0,t]} \{\|\phi(\cdot, \tau)\|_{L^\infty}, \|\phi_z(\cdot, \tau)\|_{L^\infty}, \|\psi(\cdot, \tau)\|_{L^\infty}, \|\psi_z(\cdot, \tau)\|_{L^\infty}\} \leq N(t). \tag{3.9}$$

For system (3.7), we have the following results.

Proposition 3.1. *If $\varepsilon > 0$ is small, there exists a positive constant δ_0, such that if $N(0) \leq \delta_0$, then the Cauchy problem* (3.7) *has a unique global solution $(\phi, \psi) \in X(0, +\infty)$ satisfying*

$$\begin{aligned} \|\phi\|_{2,w}^2 + \|\psi\|_2^2 + \|\psi_z\|_{1,w}^2 + \int_0^t (\|\phi_z(\cdot, \tau)\|_{2,w}^2 + \|\psi_z(\cdot, \tau)\|_{2,w}^2)d\tau \\ \leq C(\|\phi_0\|_{2,w}^2 + \|\psi_0\|_2^2 + \|\psi_{0z}\|_{1,w}^2) \leq CN^2(0), \end{aligned} \tag{3.10}$$

for all $t \in [0, +\infty)$. Moreover, it holds that

$$\sup_{z \in \mathbb{R}} |(\phi_z, \psi_z)(z,t)| \to 0, \quad \text{as } t \to \infty. \tag{3.11}$$

In view of (3.3), Theorem 2.1 is a consequence of Proposition 3.1. To prove Proposition 3.1, by a standard method, we can first prove the local existence of a unique solution to system (3.7) given below; and then by the standard continuation process, the global existence of (ϕ, ψ) follows directly from the following *a priori* estimates.

Proposition 3.2. (Local existence) *For any $\delta_1 > 0$, there exists a positive constant T_0 depending on δ_1, such that if $(\phi_0, \psi_0) \in H_w^2 \times H_w^2$ and $N(0) \leq \delta_1$, then* (3.7) *has a unique solution $(\phi, \psi) \in X(0, T_0)$ satisfying $N(t) \leq 2N(0)$ for any $t \in [0, T_0]$.*

Proposition 3.3. (*A priori* estimate) *Suppose that $(\phi, \psi) \in X(0, T)$ is a solution to* (3.7) *obtained in Proposition* 3.2 *for some positive T. If $\varepsilon > 0$ is small, then there exists a positive constant $\delta_2 > 0$ independent of T, such that if $N(t) \leq \delta_2$ for any $t \in [0, T]$, then the solution (ϕ, ψ) of* (3.7) *satisfies* (3.10) *for any $t \in [0, T]$.*

Now we present a result which will essentially be employed in the L^2 estimates of (ϕ, ψ). However, it is unnecessary if $\varepsilon = 0$.

Lemma 3.1. *Let (U, V) be a traveling wave solution of* (1.4) *obtained in Proposition* 2.2. *Then there is a constant C_0 independent of $\varepsilon > 0$ such that*

$$|V_z| \leq C_0. \tag{3.12}$$

Proof. First we multiply the second equation of (2.6) by V_z and integrate the results over $(-\infty, z)$ to obtain that

$$-s \int_{-\infty}^{z} V_z^2 dz + \int_{-\infty}^{z} (\varepsilon V^2 - U)_z V_z dz = \int_{-\infty}^{z} \varepsilon V_{zz} V_z dz = \frac{1}{2} \varepsilon V_z^2.$$

Noticing that $\int_{-\infty}^{z} (\varepsilon V^2 - U)_z V_z dz = 2\varepsilon \int_{-\infty}^{z} V V_z^2 dz - \int_{-\infty}^{z} U_z V_z dz$, and $V \leq 0$, we have

$$\begin{aligned} s \int_{-\infty}^{z} V_z^2 dz + \frac{1}{2} \varepsilon V_z^2 + 2\varepsilon \int_{-\infty}^{z} |V| V_z^2 dz &= - \int_{-\infty}^{z} U_z V_z dz \\ &\leq \frac{s}{2} \int_{-\infty}^{z} V_z^2 dz + \frac{1}{2s} \int_{-\infty}^{z} U_z^2 dz, \end{aligned}$$

where we have used the Cauchy–Schwarz inequality. Then it follows that

$$\int_{-\infty}^{z} V_z^2 dz + \frac{\varepsilon}{s} V_z^2 + \frac{4\varepsilon}{s} \int_{-\infty}^{z} |V| V_z^2 dz \leq \frac{1}{s^2} \int_{-\infty}^{z} U_z^2 dz. \tag{3.13}$$

From the first equation of (2.8), and noting that $\varrho_1 = 0$ and $V \leq 0$, $U \geq 0$, we have

$$|U_z| = -U_z = \frac{s}{D} U + \frac{\chi}{D} UV \leq \frac{s}{D} U \leq \frac{s u_-}{D}, \tag{3.14}$$

which entails that

$$\int_{-\infty}^{z} U_z^2 dz \leq -\|U_z\|_{L^\infty} \int_{-\infty}^{z} U_z dz \leq -\frac{su_-}{D} \int_{-\infty}^{z} U_z dz \leq \frac{su_-^2}{D}. \tag{3.15}$$

Next we split into two cases $\varepsilon \geq 1$ and $\varepsilon < 1$ to examine the boundedness of $|V_z|$. When $\varepsilon \geq 1$, we obtain from (3.13) and (3.15) that

$$V_z^2 \leq \frac{1}{\varepsilon s} \int_{-\infty}^{z} U_z^2 dz \leq \frac{u_-^2}{D} := C_1.$$

Hence $|V_z| \leq \sqrt{C_1}$ as $\varepsilon \geq 1$. We proceed to estimate $|V_z|$ as $\varepsilon < 1$. To this end, we multiply the second equation of (2.6) by V_{zz} and integrate the results over $(-\infty, z)$ to obtain

$$\begin{aligned}
&\frac{s}{2}V_z^2 + \varepsilon \int_{-\infty}^{z} V_{zz}^2 dz \\
&\quad = \int_{-\infty}^{z} (\varepsilon V^2 - U)_z V_{zz} dz \\
&\quad = \varepsilon \int_{-\infty}^{z} 2\, V V_z V_{zz} dz - \int_{-\infty}^{z} U_z V_{zz} dz \\
&\quad = \varepsilon \int_{-\infty}^{z} V (V_z^2)_z dz + \int_{-\infty}^{z} \left[-\frac{s}{D} U_z - \frac{\chi}{D} (UV)_z \right] V_z dz - U_z V_z \\
&\quad = \varepsilon\, V V_z^2 - \varepsilon \int_{-\infty}^{z} V_z^3 dz - \frac{s}{D} \int_{-\infty}^{z} U_z V_z dz - \frac{\chi}{D} \int_{-\infty}^{z} (U V_z^2 + V U_z V_z) dz - U_z V_z \\
&\quad \leq -\frac{s}{D} \int_{-\infty}^{z} U_z V_z + \frac{s}{4} V_z^2 + \frac{U_z^2}{s},
\end{aligned} \tag{3.16}$$

where we have used the first equation of (2.8), Cauchy–Schwarz inequality and the fact that $U \geq 0$, $V \leq 0, V_z \geq 0$ and $U_z \leq 0$. From (3.14) and (3.16), we have

$$V_z^2 \leq -\frac{4}{D} \int_{-\infty}^{z} U_z V_z dz + \frac{4u_-^2}{D^2}. \tag{3.17}$$

On the other hand, by Cauchy–Schwarz inequality and (3.13) as well as (3.15), one has

$$\begin{aligned}
-\frac{4}{D} \int_{-\infty}^{z} U_z V_z dz &\leq \frac{4}{2D} \int_{-\infty}^{z} (U_z^2 + V_z^2) dz \\
&\leq \frac{2}{D} \left(1 + \frac{1}{s^2} \right) \int_{-\infty}^{z} U_z^2 dz \\
&\leq \frac{2s}{D^2} \left(1 + \frac{1}{s^2} \right) u_-^2.
\end{aligned} \tag{3.18}$$

Then the combination of (3.17), (3.18) and (2.3) yields that

$$V_z^2 \le \frac{4}{D^2}\left(\frac{s}{2}+\frac{1}{2s}+1\right)u_-^2 = \frac{4}{D^2}\left(\frac{\chi}{2}\sqrt{\frac{u_-}{\chi+\varepsilon}}+\frac{1}{2\chi}\sqrt{\frac{\chi+\varepsilon}{u_-}}+1\right)u_-^2$$
$$\le \frac{4}{D^2}\left(\frac{1}{2}\sqrt{\chi u_-}+\frac{1}{2\chi}\sqrt{\frac{\chi+1}{u_-}}+1\right)u_-^2 =: C_2. \quad (3.19)$$

Then taking $C_0 = \sqrt{C_1}+\sqrt{C_2}$, we complete the proof. □

The proof of Proposition 3.3 is based on the following series of lemmas. Without loss of generality, we assume that $N(t) < 1$. In what follows, for notational simplicity, $\int_{-\infty}^{\infty} f(x,t)dx$ and $\int_0^t\int_{-\infty}^{\infty} f(x,\tau)dxd\tau$ will be abbreviated as $\int f(x,t)$ and $\int_0^t\int f(x,\tau)$.

Lemma 3.2. *Let the assumptions in Proposition* 3.3 *hold, then there exists a constant* $C > 0$ *such that*

$$\|\phi\|_w^2 + \|\psi\|^2 + D\int_0^t \|\phi_z(\cdot,\tau)\|_w^2 d\tau + \varepsilon\int_0^t \|\psi_z(\cdot,\tau)\|^2 d\tau + D\int_0^t \|U_z(\cdot)\phi(\cdot,\tau)\|^2 d\tau$$
$$\le C\left(\|\phi_0\|_w^2 + \|\psi_0\|^2 + N(t)\int_0^t\int w\psi_z^2\right). \quad (3.20)$$

Proof. Multiplying the first equation of (3.7) by ϕ/U and the second one by $\chi\psi$, and adding them, we obtain

$$\frac{1}{2}\left(\frac{\phi^2}{U}+\chi\psi^2\right)_t - \left[\frac{D\phi\phi_z}{U}+\frac{(s+\chi V)\phi^2}{2U}+\varepsilon\chi\psi\psi_z+\chi\left(\frac{s}{2}-\varepsilon V\right)\psi^2+\chi\phi\psi\right]_z$$
$$+\frac{D\phi_z^2}{U}+\varepsilon\chi\psi_z^2+\frac{\phi^2}{2}\left(\frac{s+\chi V}{U}\right)_z$$
$$=\frac{DU_z}{U^2}\phi\phi_z+\varepsilon\chi V_z\psi^2+\frac{\chi\phi\phi_z\psi_z}{U}-\varepsilon\chi\psi\psi_z^2. \quad (3.21)$$

By the first equation of (2.8), a direct calculation gives

$$\left(\frac{s+\chi V}{U}\right)_z = \frac{\chi V_z}{U} - \frac{(s+\chi V)U_z}{U^2} = \frac{\chi V_z}{U}+\frac{DU_z^2}{U^3}, \quad (3.22)$$

with $V_z > 0$ owing to Proposition 2.2. By Young's inequality, it holds that

$$\frac{DU_z}{U^2}\phi\phi_z \le \frac{3D\phi_z^2}{4U}+\frac{DU_z^2\phi^2}{3U^3} \quad \text{and}$$
$$\frac{\chi\phi\phi_z\psi_z}{U} \le \frac{D}{8U}\|\phi(\cdot,t)\|_{L^\infty}\phi_z^2 + \frac{2\chi^2}{DU}\|\phi(\cdot,t)\|_{L^\infty}\psi_z^2. \quad (3.23)$$

Substituting (3.22) and (3.23) into (3.21) and integrating the resultant inequality over $\mathbb{R} \times [0,t]$, we get

$$\begin{aligned}\frac{1}{2}\int\left(\frac{\phi^2}{U}+\chi\psi^2\right)+\frac{D}{4}\int_0^t\int\frac{\phi_z^2}{U}+\varepsilon\chi\int_0^t\int\psi_z^2+\frac{\chi}{2}\int_0^t\int\frac{V_z\phi^2}{U}+\frac{D}{6}\int_0^t\int\frac{U_z^2\phi^2}{U^3}\\ \le \varepsilon\chi\int_0^t\int V_z\psi^2+\frac{DN(t)}{8}\int_0^t\int\frac{\phi_z^2}{U}+\frac{2\chi^2N(t)}{D}\int_0^t\int\frac{\psi_z^2}{U}+\varepsilon\chi N(t)\int_0^t\int\psi_z^2\\ +\frac{1}{2}\int\left(\frac{\phi_0^2}{U}+\chi\psi_0^2\right),\end{aligned}$$

where we have used the fact that $\|\phi(\cdot,t)\|_{L^\infty}\le N(t)<1$ and $\|\psi(\cdot,t)\|_{L^\infty}\le N(t)<1$. Thus, using $\psi_z^2\le u_-\psi_z^2/U$, we derive

$$\begin{aligned}\int\left(\frac{\phi^2}{U}+\chi\psi^2\right)+\frac{D}{4}\int_0^t\int\frac{\phi_z^2}{U}+2\varepsilon\chi\int_0^t\int\psi_z^2+\chi\int_0^t\int\frac{V_z\phi^2}{U}+\frac{D}{3}\int_0^t\int\frac{U_z^2\phi^2}{U^3}\\ \le 2\varepsilon\chi\int_0^t\int V_z\psi^2+CN(t)\int_0^t\int\frac{\psi_z^2}{U}+C\int\left(\frac{\phi_0^2}{U}+\chi\psi_0^2\right).\end{aligned}\tag{3.24}$$

We next estimate the term $\int_0^t\int V_z\psi^2$. Multiplying the first equation of (3.7) by $V\phi/U$ and the second one by $\chi V\psi$ and adding them to get

$$\begin{aligned}&\frac{1}{2}\left[\frac{V}{U}\phi^2+\chi V\psi^2\right]_t+\frac{DV}{U}\phi_z^2+\varepsilon\chi V\psi_z^2\\ &\quad+\frac{\phi^2}{2}\left[\frac{1}{U}(s+\chi V)V\right]_z+\frac{\chi\psi^2}{2}[(s-2\varepsilon V)V]_z\\ &=F(z,t)_z-D\left(\frac{V}{U}\right)_z\phi\phi_z-\chi V_z\phi\psi-\varepsilon\chi V_z\psi\psi_z+\frac{\chi V}{U}\phi\phi_z\psi_z-\varepsilon\chi V\psi\psi_z^2,\end{aligned}\tag{3.25}$$

where $F(z,t)=V\left[\frac{D\phi\phi_z}{U}+\frac{(s+\chi V)\phi^2}{2U}+\varepsilon\chi\psi\psi_z+\chi(\frac{s}{2}-\varepsilon V)\psi^2+\chi\phi\psi\right]$.

It is easy to compute that

$$\begin{aligned}\left[\frac{1}{U}(s+\chi V)V\right]_z&=\frac{V_z}{U}(2\chi V+s)+\frac{DVU_z^2}{U^3}\\ &=\frac{\chi V_z(V-v_-)}{U}+\frac{\chi V_zV}{U}+\frac{DVU_z^2}{U^3},\end{aligned}\tag{3.26}$$

where we have used the fact that $s=-\chi v_-$ and $s+\chi V=-\frac{DU_z}{U}$. Similarly,

$$\frac{\chi}{2}[(s-2\varepsilon V)V]_z=\chi V_z\left(\frac{s}{2}-2\varepsilon V\right)=\chi V_z\left(\frac{s}{2}+2\varepsilon|V|\right).\tag{3.27}$$

Noticing that $v_- \le V \le 0$, by Young's inequality, we have

$$\begin{aligned}\left|D\left(\frac{V}{U}\right)_z \phi\phi_z\right| &= \left|\frac{DV_z}{U}\phi\phi_z - \frac{DVU_z}{U^2}\phi\phi_z\right| \\ &\le \frac{DV_z}{2U}\phi^2 + \frac{DV_z}{2U}\phi_z^2 + \frac{D|V|}{2U}\phi_z^2 + \frac{D|V|U_z^2}{2U^3}\phi^2, \end{aligned} \tag{3.28}$$

$$|\chi V_z\phi\psi| \le \frac{\chi V_z\phi^2}{s} + \frac{\chi s}{4}V_z\psi^2 \quad \text{and} \quad |\varepsilon\chi V_z\psi\psi_z| \le \frac{\chi\varepsilon}{2}(\psi_z^2 + V_z^2\psi^2). \tag{3.29}$$

Substituting (3.26)–(3.29) into (3.25) and applying the fact $V_z \le V_z u_-/U$, we obtain

$$\begin{aligned}&\chi\left(\frac{s}{2} + 2\varepsilon|V|\right)V_z\psi^2 + \frac{\chi V_z(V - v_-)}{2U}\phi^2 \\ &\quad\le F_z(z,t) + \frac{1}{2}|V|\left(\frac{\phi^2}{U} + \chi\psi^2\right)_t + \frac{D}{2}(V_z + 3|V|)\frac{\phi_z^2}{U} + \left(\frac{\chi\varepsilon}{2} + \varepsilon\chi|V|\right)\psi_z^2 \\ &\qquad + \left(\frac{\chi|V|}{2} + \frac{D}{2} + \frac{\chi u_-}{s}\right)\frac{V_z\phi^2}{U} + D|V|\frac{U_z^2\phi^2}{U^3} + \left(\frac{\chi s}{4} + \frac{\chi\varepsilon V_z}{2}\right)V_z\psi^2 \\ &\qquad + \frac{\chi V}{U}\phi\phi_z\psi_z - \varepsilon\chi V\psi\psi_z^2. \end{aligned} \tag{3.30}$$

Then rearranging the above inequality and integrating the result yield that

$$\begin{aligned}&\chi\int_0^t\int\left(\frac{s}{4} + 2\varepsilon|V|\right)V_z\psi^2 + \chi\int_0^t\int\frac{V_z(V - v_-)}{2U}\phi^2 \\ &\quad\le \frac{1}{2}\int|V|\left(\frac{\phi^2}{U} + \chi\psi^2 - \frac{\phi_0^2}{U} - \chi\psi_0^2\right) + \frac{D}{2}(\|V_z\|_\infty + 3\|V\|_\infty)\int_0^t\int\frac{\phi_z^2}{U} \\ &\qquad + \left(\frac{1}{2} + \|V\|_\infty\right)\varepsilon\chi\int_0^t\int\psi_z^2 + \left(\frac{\chi\|V\|_\infty}{2} + \frac{D}{2} + \frac{\chi u_-}{s}\right)\int_0^t\int\frac{V_z\phi^2}{U} \\ &\qquad + D\|V\|_\infty\int_0^t\int\frac{U_z^2\phi^2}{U^3} + \frac{\chi\varepsilon}{2}\|V_z\|_\infty\int_0^t\int V_z\psi^2 \\ &\qquad + \int_0^t\int\left(\frac{\chi|V\phi\phi_z\psi_z|}{U} + \varepsilon\chi|V\psi|\psi_z^2\right). \end{aligned} \tag{3.31}$$

Using (3.24), it follows that

$$\begin{aligned}&\int_0^t\int\left(\frac{\chi s}{4} + 2\chi\varepsilon|V|\right)V_z\psi^2 \\ &\quad\le 2\zeta\chi\varepsilon\int_0^t\int V_z\psi^2 + C\zeta\int\left(\frac{\phi_0^2}{U} + \psi_0^2\right) + C\zeta N(t)\int_0^t\int\frac{\psi_z^2}{U} \\ &\qquad + \int_0^t\int\left(\frac{\chi|V\phi\phi_z\psi_z|}{U} + \varepsilon\chi|V\psi|\psi_z^2\right), \end{aligned} \tag{3.32}$$

with

$$\zeta = \frac{21}{2}\|V\|_\infty + \frac{9}{4}\|V_z\|_\infty + \frac{1}{4} + \frac{D}{2\chi} + \frac{u_-}{s} \le \frac{21|v_-|}{2} + \frac{9C_0}{4} + \frac{1}{4} + \frac{D}{2\chi} + \frac{u_-}{s} = \tilde{\zeta},$$

where we have used the fact $\|V\|_{L^\infty} \le -v_-$ and Lemma 3.1. Therefore

$$\begin{aligned}
&\int_0^t \int \left(\frac{\chi s}{4} + 2\chi\varepsilon|V| - 2\tilde{\zeta}\chi\varepsilon\right) V_z \psi^2 \\
&\quad \le C\tilde{\zeta} \int \left(\frac{\phi_0^2}{U} + \psi_0^2\right) + C\tilde{\zeta} N(t) \int_0^t \int \frac{\psi_z^2}{U} \\
&\qquad + \int_0^t \int \left(\frac{\chi|V\phi\phi_z\psi_z|}{U} + \varepsilon\chi|V\psi|\psi_z^2\right). \qquad (3.33)
\end{aligned}$$

Then choosing ε small enough such that

$$\varepsilon < \frac{s}{8\tilde{\zeta}},$$

we have

$$\begin{aligned}
\int_0^t \int V_z \psi^2 &\le C \int \left(\frac{\phi_0^2}{U} + \psi_0^2\right) + CN(t) \int_0^t \int \frac{\psi_z^2}{U} \\
&\quad + \int_0^t \int \left(\frac{\chi|V\phi\phi_z\psi_z|}{U} + \varepsilon\chi|V\psi|\psi_z^2\right) \\
&\le C \int \left(\frac{\phi_0^2}{U} + \psi_0^2\right) + CN(t) \int_0^t \int \frac{\psi_z^2}{U} + CN(t) \int_0^t \int \frac{\phi_z^2}{U}, \qquad (3.34)
\end{aligned}$$

where we have used the fact $\|\phi(\cdot,t)\|_\infty \le N(t)$ and the following estimates derived from the Cauchy–Schwarz inequality

$$\frac{\chi|V\phi\phi_z\psi_z|}{U} \le CN(t)\left(\frac{\phi_z^2}{U} + \frac{\psi_z^2}{U}\right), \quad \varepsilon\chi|V\psi|\psi_z^2 \le Cu_- N(t)\frac{\psi_z^2}{U}.$$

Substituting (3.34) into (3.24), we obtain

$$\begin{aligned}
&\int \left(\frac{\phi^2}{U} + \chi\psi^2\right) + \left(\frac{D}{4} - CN(t)\right) \int_0^t \int \frac{\phi_z^2}{U} + \varepsilon \int_0^t \int \psi_z^2 + \frac{D}{3} \int_0^t \int \frac{U_z^2 \phi^2}{U^3} \\
&\quad \le C \int \left(\frac{\phi_0^2}{U} + \psi_0^2\right) + CN(t) \int_0^t \int \frac{\psi_z^2}{U}.
\end{aligned}$$

Thus, when $N(t)$ is small enough, we get

$$\begin{aligned}
&\int \left(\frac{\phi^2}{U} + \chi\psi^2\right) + D \int_0^t \int \frac{\phi_z^2}{U} + \varepsilon \int_0^t \int \psi_z^2 + \frac{D}{3} \int_0^t \int \frac{U_z^2 \phi^2}{U^3} \\
&\quad \le C \int \left(\frac{\phi_0^2}{U} + \psi_0^2\right) + CN(t) \int_0^t \int \frac{\psi_z^2}{U}. \qquad (3.35)
\end{aligned}$$

To complete the proof, we now bound $\frac{1}{U}$ in terms of weight function w defined in (2.13). By (2.12), there exists a constant $M > 0$ such that for any $z \geq M$, $\frac{1}{U} \sim Ce^{\frac{s}{D}z}$. By the definition of weight function w in (2.13), one can find two constants $\nu > \mu > 0$ such that

$$\mu w \leq \frac{1}{U} \leq \nu w, \quad \text{for any } z \geq M.$$

When $z < M$, because $\frac{1}{U}$ is monotone increasing in $(-\infty, \infty)$ and $1 < w(z) \leq 2e^{\eta M}$, we have

$$\frac{w}{2u_- e^{\eta M}} \leq \frac{1}{u_-} \leq \frac{1}{U} \leq \frac{1}{U(M)} \leq \frac{w}{U(M)}, \quad \text{for any } z < M.$$

In all, one can find two constants $C_2 > C_1 > 0$ such that

$$C_1 w(z) \leq \frac{1}{U(z)} \leq C_2 w(z), \quad \text{for all } z \in \mathbb{R}. \tag{3.36}$$

The desired inequality (3.20) follows from (3.35) and (3.36). □

Lemma 3.3. *Let the assumptions in Proposition* 3.3 *hold*, *then there exists a constant* $C > 0$ *such that*

$$\|\phi_z\|_w^2 + \|\psi_z\|^2 + \|\psi_z\|_w^2 + D\int_0^t \|\phi_{zz}(\cdot,\tau)\|_w^2 d\tau + \int_0^t \|\psi_z(\cdot,\tau)\|_w^2 d\tau$$
$$+ \varepsilon \int_0^t \|\psi_{zz}(\cdot,\tau)\|_w^2 d\tau \leq C(\|\phi_0\|_{1,w}^2 + \|\psi_0\|_1^2 + \|\psi_{0z}\|_w^2). \tag{3.37}$$

Proof. Differentiating (3.7) with respect to z yields

$$\begin{cases} \phi_{zt} = D\phi_{zzz} + (s+\chi V)\phi_{zz} + \chi V_z\phi_z + \chi U_z\psi_z + \chi U\psi_{zz} + \chi(\phi_z\psi_z)_z, \\ \psi_{zt} = \varepsilon\psi_{zzz} + s\psi_{zz} - 2\varepsilon(V\psi_z)_z + \phi_{zz} - \varepsilon(\psi_z^2)_z. \end{cases} \tag{3.38}$$

Multiplying the first equation of (3.38) by ϕ_z/U and the second by $\chi\psi_z$, integrating the resultant equations with respect to z and adding them, noting that

$$\frac{D\phi_{zzz}\phi_z}{U} = D\left(\frac{\phi_{zz}\phi_z}{U}\right)_z - \frac{D\phi_{zz}^2}{U} - \left[\frac{D\phi_z^2}{2}\left(\frac{1}{U}\right)_z\right]_z + \frac{\phi_z^2}{2}\left(\frac{D}{U}\right)_{zz},$$

$$\frac{(s+\chi V)\phi_{zz}\phi_z}{U} = \left(\frac{(s+\chi V)\phi_z^2}{2U}\right)_z - \frac{\phi_z^2}{2}\left(\frac{s+\chi V}{U}\right)_z,$$

$$\frac{\chi(\phi_z\psi_z)_z\phi_z}{U} = \left(\frac{\chi\phi_z^2\psi_z}{U}\right)_z - \frac{\chi\phi_z\psi_z\phi_{zz}}{U} + \frac{\chi U_z\phi_z^2\psi_z}{U^2},$$

$$\chi[s\psi_{zz} - 2\varepsilon(V\psi_z)_z]\psi_z = \chi\left[\left(\frac{s}{2} - \varepsilon V\right)\psi_z^2\right]_z - \varepsilon\chi V_z\psi_z^2 \quad \text{and}$$

$$-\varepsilon\chi(\psi_z^2)_z\psi_z = -\frac{2\varepsilon\chi}{3}(\psi_z^3)_z,$$

we have

$$\frac{1}{2}\frac{d}{dt}\int\left(\frac{\phi_z^2}{U}+\chi\psi_z^2\right)+D\int\frac{\phi_{zz}^2}{U}+\varepsilon\chi\int\psi_{zz}^2+\varepsilon\chi\int V_z\psi_z^2$$
$$=\frac{1}{2}\int\phi_z^2\left[\left(\frac{D}{U}\right)_{zz}-\left(\frac{s+\chi V}{U}\right)_z\right]+\chi\int\frac{V_z\phi_z^2}{U}+\chi\int\frac{U_z\psi_z\phi_z}{U}$$
$$-\chi\int\left(\frac{\phi_{zz}\phi_z\psi_z}{U}-\frac{U_z\phi_z^2\psi_z}{U^2}\right). \tag{3.39}$$

Owing to the first equation of (2.6) and the fact that $u_+ = v_+ = 0$, a simple calculation gives

$$\left(\frac{D}{U}\right)_{zz}-\left(\frac{s+\chi V}{U}\right)_z=-\frac{2U_z}{U^3}(s+\chi v_+)u_+=0. \tag{3.40}$$

Using the first equation of (2.8) and $V\le 0$, we get $|\frac{DU_z}{U}|=-\frac{DU_z}{U}=s+\chi V\le s$. Thus, by Cauchy–Schwarz inequality and $\|\psi_z(\cdot,t)\|_{L^\infty}\le N(t)<1$, we have

$$\begin{aligned}&\int\left|\frac{U_z\psi_z\phi_z}{U}\right|\le\frac{s}{D}\int|\psi_z\phi_z|\le C\int U\psi_z^2+C\int\frac{\phi_z^2}{U}\quad\text{and}\\&\int\left|\frac{U_z\phi_z^2\psi_z}{U^2}\right|\le CN(t)\int\frac{\phi_z^2}{U}.\end{aligned} \tag{3.41}$$

Substituting (3.40) and (3.41) into (3.39), integrating the equation over $(0,t)$, we obtain

$$\frac{1}{2}\int\left(\frac{\phi_z^2}{U}+\psi_z^2\right)+D\int_0^t\int\frac{\phi_{zz}^2}{U}+\varepsilon\chi\int_0^t\int\psi_{zz}^2+\varepsilon\chi\int_0^t\int V_z\psi_z^2$$
$$\le\frac{1}{2}\int\left(\frac{\phi_{0z}^2}{U}+\psi_{0z}^2\right)+C\int_0^t\int\frac{\phi_z^2}{U}+C\int_0^t\int U\psi_z^2$$
$$+\frac{DN(t)}{2}\int_0^t\int\frac{\phi_{zz}^2}{U}+CN(t)\int_0^t\int\frac{\phi_z^2}{U}.$$

This inequality in combination with (3.20) and the fact that $V_z>0$ leads to

$$\int\left(\frac{\phi_z^2}{U}+\psi_z^2\right)+D\int_0^t\int\frac{\phi_{zz}^2}{U}+\varepsilon\int_0^t\int\psi_{zz}^2$$
$$\le C\left(\|\psi_0\|_1^2+\|\phi_0\|_{1,w}^2+\int_0^t\int U\psi_z^2+N(t)\int_0^t\int\frac{\psi_z^2}{U}\right). \tag{3.42}$$

We proceed to estimate the term $\int_0^t\int U\psi_z^2$. To achieve this, we multiply the first equation of (3.7) by ψ_z to get

$$\chi U\psi_z^2=\phi_t\psi_z-D\phi_{zz}\psi_z-s\phi_z\psi_z-\chi V\phi_z\psi_z-\chi\phi_z\psi_z^2. \tag{3.43}$$

Owing to the second equation of (3.38),

$$\begin{aligned}\phi_t\psi_z &= (\phi\psi_z)_t - \phi\psi_{zt} = (\phi\psi_z)_t - \phi[\varepsilon\psi_{zzz} + s\psi_{zz} - 2\varepsilon(V\psi_z)_z + \phi_{zz} - \varepsilon(\psi_z^2)_z] \\ &= (\phi\psi_z)_t - \varepsilon(\phi\psi_{zz})_z + \varepsilon\phi_z\psi_{zz} - s(\phi\psi_z)_z + s\phi_z\psi_z - (\phi\phi_z)_z + \phi_z^2 \\ &\quad + 2\varepsilon(\phi V\psi_z)_z - 2\varepsilon V\phi_z\psi_z + \varepsilon(\phi\psi_z^2)_z - \varepsilon\phi_z\psi_z^2 \end{aligned} \tag{3.44}$$

and

$$\begin{aligned} -D\phi_{zz}\psi_z &= D\psi_z[-\psi_{zt} + \varepsilon\psi_{zzz} + s\psi_{zz} - 2\varepsilon(V\psi_z)_z - \varepsilon(\psi_z^2)_z] \\ &= D\left[-\frac{1}{2}(\psi_z^2)_t - \varepsilon\psi_{zz}^2 + \varepsilon(\psi_z\psi_{zz})_z + \frac{s}{2}(\psi_z^2)_z \right. \\ &\quad \left. - \varepsilon V_z\psi_z^2 - \varepsilon(V\psi_z^2)_z - \frac{2\varepsilon}{3}(\psi_z^3)_z\right]. \end{aligned} \tag{3.45}$$

Substituting (3.44) and (3.45) into (3.43) and integrating the resultant equation over $\mathbb{R} \times [0, t]$, we derive

$$\begin{aligned} &\frac{D}{2}\int \psi_z^2 + \varepsilon D\int_0^t\int \psi_{zz}^2 + \chi\int_0^t\int U\psi_z^2 + \varepsilon D\int_0^t\int V_z\psi_z^2 \\ &\le \frac{D}{2}\int \psi_{0z}^2 + \int \phi\psi_z - \int \phi_0\psi_{0z} + \frac{\varepsilon D}{2}\int_0^t\int \psi_{zz}^2 + \left(\frac{\varepsilon}{2D} + 1\right)\int_0^t\int \phi_z^2 \\ &\quad + \frac{\chi}{2}\int_0^t\int U\psi_z^2 + \left(\frac{\varepsilon^2}{\chi} + \chi\right)\int_0^t\int \frac{V^2\phi_z^2}{U} + (\chi+\varepsilon)N(t)\int_0^t\int \psi_z^2, \end{aligned} \tag{3.46}$$

where we have used the Young's inequality and the fact that $\|\phi_z(\cdot, t)\|_{L^\infty} \le N(t)$. Noting $V_z > 0$, $\phi_z^2 \le \frac{u_-}{U}\phi_z^2$ and $|V| \le -v_-$, it then follows from (3.46) and (3.20) that

$$\begin{aligned} &\int \psi_z^2 + \varepsilon\int_0^t\int \psi_{zz}^2 + \int_0^t\int U\psi_z^2 \\ &\quad \le C\left(\int \psi_{0z}^2 + \int \phi_0^2 + \int \phi^2 + \int_0^t\int \frac{\phi_z^2}{U} + N(t)\int_0^t\int \psi_z^2\right) \\ &\quad \le C\left(\|\psi_0\|_1^2 + \|\phi_0\|_w^2 + N(t)\int_0^t\int \frac{\psi_z^2}{U}\right), \end{aligned} \tag{3.47}$$

which in combination with (3.42) gives that

$$\begin{aligned} &\int\left(\frac{\phi_z^2}{U} + \psi_z^2\right) + \int_0^t\int \frac{\phi_{zz}^2}{U} + \varepsilon\int_0^t\int \psi_{zz}^2 \\ &\quad \le C\left(\|\psi_0\|_1^2 + \|\phi_0\|_{1,w}^2 + N(t)\int_0^t\int \frac{\psi_z^2}{U}\right). \end{aligned} \tag{3.48}$$

The fact that U is monotone decreasing in $(-\infty, \infty)$ implies $U(0) < U(z) < u_-$ for $z \in (-\infty, 0)$. Noting $1 < w(z) < 2$ for $z \in (-\infty, 0)$, we have $U(z) > U(0) >$

$\frac{U(0)}{2}w(z)$ for $z \in (-\infty, 0)$. Thus, one can deduce from (3.47) and (3.36) that

$$\int_{-\infty}^{0} w\psi_z^2 + \varepsilon \int_0^t \int_{-\infty}^{0} w\psi_{zz}^2 + \int_0^t \int_{-\infty}^{0} w\psi_z^2$$
$$\leq C\left(\|\psi_0\|_1^2 + \|\phi_0\|_w^2 + N(t)\int_0^t \int w\psi_z^2\right). \tag{3.49}$$

Now it is left to estimate $\int_0^t \int w\psi_z^2$. Multiplying the second equation of (3.38) by $e^{\eta z}\psi_z$, we derive

$$\left(\frac{e^{\eta z}}{2}\psi_z^2\right)_t + \left[\frac{s\eta}{2} + \varepsilon V_z - \varepsilon\eta V\right] e^{\eta z}\psi_z^2 + \varepsilon e^{\eta z}\psi_{zz}^2$$
$$= \left[\varepsilon\psi_{zz}e^{\eta z}\psi_z + \left(\frac{s}{2} - \varepsilon V\right) e^{\eta z}\psi_z^2 - \frac{2\varepsilon}{3}\psi_z^3 e^{\eta z}\right]_z$$
$$- \varepsilon\eta e^{\eta z}\psi_z\psi_{zz} + e^{\eta z}\psi_z\phi_{zz} + \frac{2\varepsilon\eta}{3}e^{\eta z}\psi_z^3. \tag{3.50}$$

By Young's inequality,

$$|\varepsilon\eta e^{\eta z}\psi_z\psi_{zz}| \leq \frac{\varepsilon}{2}e^{\eta z}\psi_{zz}^2 + \frac{\varepsilon\eta^2}{2}e^{\eta z}\psi_z^2 \quad \text{and} \quad |e^{\eta z}\psi_z\phi_{zz}| \leq \frac{s\eta}{4}e^{\eta z}\psi_z^2 + \frac{1}{s\eta}e^{\eta z}\phi_{zz}^2.$$

Thus, integrating (3.50) over $\mathbb{R} \times [0, t]$ and using (3.48), we get

$$\int e^{\eta z}\psi_z^2 + \int_0^t \int \left[\frac{s\eta}{2} + 2\varepsilon V_z - 2\eta\varepsilon V - \varepsilon\eta^2\right] e^{\eta z}\psi_z^2 + \varepsilon \int_0^t \int e^{\eta z}\psi_{zz}^2$$
$$\leq \int e^{\eta z}\psi_{0z}^2 + \frac{2}{s\eta}\int_0^t \int e^{\eta z}\phi_{zz}^2 + \frac{4\varepsilon\eta N(t)}{3}\int_0^t \int e^{\eta z}\psi_z^2$$
$$\leq C\left(\|\phi_0\|_{1,w}^2 + \|\psi_0\|_1^2 + \|\psi_{0z}\|_w^2 + N(t)\int_0^t \int w\psi_z^2\right), \tag{3.51}$$

where we have used $\|\psi_z(\cdot, t)\|_{L^\infty} \leq N(t)$ in the first inequality and the fact that $e^{\eta z} \leq w \leq \frac{1}{C_1 U}$ for $z \in \mathbb{R}$ by (3.36) in the second inequality. When $\varepsilon > 0$ is small enough so that $\varepsilon \leq \frac{s}{2\eta} = \frac{D}{2}$, it follows from (3.51) that

$$\int_0^{+\infty} e^{\eta z}\psi_z^2 + \int_0^t \int_0^{+\infty} e^{\eta z}\psi_z^2 + \varepsilon \int_0^t \int_0^{+\infty} e^{\eta z}\psi_{zz}^2$$
$$\leq C\left(\|\phi_0\|_{1,w}^2 + \|\psi_0\|_1^2 + \|\psi_{0z}\|_w^2 + N(t)\int_0^t \int w\psi_z^2\right). \tag{3.52}$$

Recalling from (2.13) that $w = 1 + e^{\eta z}$, it holds that $e^{\eta z} \geq \frac{w(z)}{2}$ for $z \in [0, +\infty)$. Then it follows from (3.52) that

$$\int_0^{+\infty} w\psi_z^2 + \int_0^t \int_0^{+\infty} w\psi_z^2 + \varepsilon \int_0^t \int_0^{+\infty} w\psi_{zz}^2$$
$$\leq C\left(\|\phi_0\|_{1,w}^2 + \|\psi_0\|_1^2 + \|\psi_{0z}\|_w^2 + N(t)\int_0^t \int w\psi_z^2\right),$$

which in combination with (3.49) gives

$$\begin{aligned}\int w\psi_z^2 + \int_0^t\int w\psi_z^2 + \varepsilon\int_0^t\int w\psi_{zz}^2 \\ \leq C\left(\|\phi_0\|_{1,w}^2 + \|\psi_0\|_1^2 + \|\psi_{0z}\|_w^2 + N(t)\int_0^t\int w\psi_z^2\right).\end{aligned}$$

Thus

$$\int w\psi_z^2 + (1 - CN(t))\int_0^t\int w\psi_z^2 + \varepsilon\int_0^t\int w\psi_{zz}^2 \leq C(\|\phi_0\|_{1,w}^2 + \|\psi_0\|_1^2 + \|\psi_{0z}\|_w^2).$$

Choosing $N(t)$ small enough, we have

$$\int w\psi_z^2 + \int_0^t\int w\psi_z^2 + \varepsilon\int_0^t\int w\psi_{zz}^2 \leq C(\|\phi_0\|_{1,w}^2 + \|\psi_0\|_1^2 + \|\psi_{0z}\|_w^2). \tag{3.53}$$

Therefore, by (3.20), (3.48) and (3.53), we derive (3.37). □

We next give the estimates of the second-order derivatives of (ϕ, ψ).

Lemma 3.4. *Let the assumptions in Proposition* 3.3 *hold, then there exists a constant* $C > 0$ *such that*

$$\begin{aligned}&\|\phi_{zz}\|_w^2 + \|\psi_{zz}\|^2 + D\int_0^t \|\phi_{zzz}(\cdot,\tau)\|_w^2 d\tau + \int_0^t \|\psi_{zz}(\cdot,\tau)\|_w^2 d\tau + \varepsilon\int_0^t \|\psi_{zzz}(\cdot,\tau)\|^2 d\tau \\ &\quad\leq C(\|\phi_0\|_{2,w}^2 + \|\psi_0\|_2^2 + \|\psi_{0,z}\|_{1,w}^2).\end{aligned} \tag{3.54}$$

Proof. Differentiating (3.38) with respect to z gives

$$\begin{cases}\phi_{zzt} = D\phi_{zzzz} + (s+\chi V)\phi_{zzz} + \chi V_z\phi_{zz} + \chi(V_z\phi_z)_z + \chi(U_z\psi_z)_z \\ \qquad\qquad + \chi U_z\psi_{zz} + \chi U\psi_{zzz} + \chi(\phi_z\psi_z)_{zz}, \\ \psi_{zzt} = \varepsilon\psi_{zzzz} + s\psi_{zzz} - 2\varepsilon(V\psi_z)_{zz} + \phi_{zzz} - \varepsilon(\psi_z^2)_{zz}.\end{cases} \tag{3.55}$$

Multiplying the first equation of (3.55) by ϕ_{zz}/U and the second by $\chi\psi_{zz}$ and using

$$\begin{aligned}D\phi_{zzzz}\cdot\frac{\phi_{zz}}{U} &= D\left(\phi_{zzz}\cdot\frac{\phi_{zz}}{U}\right)_z - \frac{D\phi_{zzz}^2}{U} \\ &\quad - \frac{D}{2}\left(\phi_{zz}^2\left(\frac{1}{U}\right)_z\right)_z + \frac{D}{2}\phi_{zz}^2\left(\frac{1}{U}\right)_{zz}, \\ (s+\chi V)\phi_{zzz}\cdot\frac{\phi_{zz}}{U} &= \left[\frac{(s+\chi V)}{2U}\phi_{zz}^2\right]_z - \frac{\phi_{zz}^2}{2}\left(\frac{s+\chi V}{U}\right)_z, \\ [s\psi_{zzz} - 2\varepsilon(V\psi_z)_{zz}]\psi_{zz} &= \left[\frac{s}{2}(\psi_{zz}^2) - 2\varepsilon(V\psi_z)_z\psi_{zz}\right]_z + 2\varepsilon V_z\psi_z\psi_{zzz} \\ &\quad + \varepsilon(V\psi_{zz}^2)_z - \varepsilon V_z\psi_{zz}^2, \\ -\varepsilon(\psi_z^2)_{zz}\psi_{zz} &= -\varepsilon[(\psi_z^2)_z\psi_{zz}]_z + 2\varepsilon\psi_z\psi_{zz}\psi_{zzz},\end{aligned}$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\int\left(\frac{\phi_{zz}^2}{U}+\psi_{zz}^2\right)+D\int\frac{\phi_{zzz}^2}{U}+\varepsilon\chi\int\psi_{zzz}^2+\varepsilon\chi\int V_z\psi_{zz}^2$$
$$=\chi\bigg(\int\frac{V_z\phi_{zz}^2}{U}+\int\frac{(V_z\phi_z)_z\phi_{zz}}{U}+\int\frac{(U_z\psi_z)_z\phi_{zz}}{U}+\int\frac{U_z\psi_{zz}\phi_{zz}}{U}$$
$$+\int\frac{(\phi_z\psi_z)_{zz}\phi_{zz}}{U}+2\varepsilon\int V_z\psi_z\psi_{zzz}+2\varepsilon\int\psi_z\psi_{zz}\psi_{zzz}\bigg), \tag{3.56}$$

where we have used (3.40). Because $|U_z/U|\le C$, $|V_z|\le C$, $\|\psi_z(\cdot,t)\|_{L^\infty}\le N(t)$ and $\|\phi_z(\cdot,t)\|_{L^\infty}\le N(t)$ for any $t\in[0,T]$, we get by Cauchy–Schwarz inequality

$$\chi\int\frac{(V_z\phi_z)_z\phi_{zz}}{U}=-\chi\int\frac{V_z\phi_z\phi_{zzz}}{U}+\chi\int\frac{V_zU_z\phi_z\phi_{zz}}{U^2}$$
$$\le\frac{D}{4}\int\frac{\phi_{zzz}^2}{U}+\frac{C\chi^2}{D}\int\frac{\phi_z^2}{U}+D\int\frac{\phi_{zz}^2}{U}+\frac{C\chi^2}{D}\int\frac{\phi_z^2}{U},$$
$$\chi\int\frac{(U_z\psi_z)_z\phi_{zz}}{U}=-\chi\int\frac{U_z\psi_z\phi_{zzz}}{U}+\chi\int\frac{U_z^2\psi_z\phi_{zz}}{U^2}$$
$$\le\frac{D}{4}\int\frac{\phi_{zzz}^2}{U}+\frac{C\chi^2}{D}\int U\psi_z^2+D\int\frac{\phi_{zz}^2}{U}+\frac{C\chi^2}{D}\int U\psi_z^2,$$
$$\chi\int\frac{(\phi_z\psi_z)_{zz}\phi_{zz}}{U}=-\int\frac{(\phi_z\psi_z)_z\phi_{zzz}}{U}+\int\frac{(\phi_z\psi_z)_z\phi_{zz}U_z}{U^2}$$
$$\le\frac{DN(t)}{4}\int\frac{\phi_{zzz}^2}{U}+\frac{2\chi^2N(t)}{D}\int\frac{\phi_{zz}^2}{U}+\frac{2\chi^2N(t)}{D}\int\frac{\psi_{zz}^2}{U}.$$

Integrating (3.56) over $(0,t)$ and using (3.37), we have

$$\int\left(\frac{\phi_{zz}^2}{U}+\psi_{zz}^2\right)+D\int_0^t\int\frac{\phi_{zzz}^2}{U}+\varepsilon\int_0^t\int\psi_{zzz}^2$$
$$\le C\left(\|\phi_0\|_{2,w}^2+\|\psi_{0z}\|_w^2+\|\psi_0\|_2^2+N(t)\int_0^t\int\frac{\psi_{zz}^2}{U}+\int_0^t\int U\psi_{zz}^2\right). \tag{3.57}$$

We next estimate $\int_0^t\int U\psi_{zz}^2$. Multiplying the first equation of (3.38) by ψ_{zz}, we have

$$\chi U\psi_{zz}^2=\phi_{zt}\psi_{zz}-[D\phi_{zzz}+s\phi_{zz}+\chi V\phi_{zz}+\chi V_z\phi_z+\chi U_z\psi_z+\chi(\phi_z\psi_z)_z]\psi_{zz}.$$

Noting that by the second equation of (3.55), it has that

$$\phi_{zt}\psi_{zz}=(\phi_z\psi_{zz})_t-\phi_z\psi_{zzt}$$
$$=(\phi_z\psi_{zz})_t-\phi_z[\varepsilon\psi_{zzzz}+s\psi_{zzz}+\phi_{zzz}-2\varepsilon(V\psi_z)_{zz}-\varepsilon(\psi_z^2)_{zz}]$$

$$
\begin{aligned}
&= (\phi_z\psi_{zz})_t - \varepsilon(\phi_z\psi_{zzz})_z + \varepsilon\phi_{zz}\psi_{zzz} - s(\phi_z\psi_{zz})_z + s\phi_{zz}\psi_{zz}\\
&\quad - (\phi_z\phi_{zz})_z + \phi_{zz}^2 + 2\varepsilon(\phi_z(V\psi_z)_z)_z - 2\varepsilon\phi_{zz}(V\psi_z)_z\\
&\quad + \varepsilon(\phi_z(\psi_z^2)_z)_z - \varepsilon\phi_{zz}(\psi_z^2)_z,\\
-D\phi_{zzz}\psi_{zz} &= D[-\psi_{zzt} + \varepsilon\psi_{zzzz} + s\psi_{zzz} - 2\varepsilon(V\psi_z)_{zz} - \varepsilon(\psi_z^2)_{zz}]\psi_{zz}\\
&= D\bigg\{-\frac{1}{2}(\psi_{zz}^2)_t + \varepsilon(\psi_{zz}\psi_{zzz})_z - \varepsilon\psi_{zzz}^2 + \frac{s}{2}(\psi_{zz}^2)_z - 2\varepsilon[\psi_{zz}(V\psi_z)_z]_z\\
&\qquad + 2\varepsilon\psi_{zzz}(V\psi_z)_z - \varepsilon[\psi_{zz}(\psi_z^2)_z]_z + \varepsilon\psi_{zzz}(\psi_z^2)_z\bigg\}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\frac{D}{2}\frac{d}{dt}\int\psi_{zz}^2 + \chi\int U\psi_{zz}^2 + \varepsilon D\int\psi_{zzz}^2\\
&\quad = \frac{d}{dt}\int\phi_z\psi_{zz} + \varepsilon\int\phi_{zz}\psi_{zzz} + \int\phi_{zz}^2 - 2\varepsilon\int\phi_{zz}(V\psi_z)_z\\
&\qquad - \varepsilon\int\phi_{zz}(\psi_z^2)_z + 2\varepsilon D\int\psi_{zzz}(V\psi_z)_z + \varepsilon D\int\psi_{zzz}(\psi_z^2)_z\\
&\qquad - \chi\int[V\phi_{zz} + V_z\phi_z + U_z\psi_z + (\phi_z\psi_z)_z]\psi_{zz}.
\end{aligned}
\tag{3.58}
$$

Note that the Cauchy–Schwarz inequality gives,

$$
\begin{aligned}
&\chi\int(V\phi_{zz} + V_z\phi_z + U_z\psi_z)\psi_{zz}\\
&\quad \le \frac{\chi}{4}\int U\psi_{zz}^2 + C\chi\left(\int\frac{V^2\phi_{zz}^2}{U} + \int\frac{V_z^2\phi_z^2}{U} + \int\frac{U_z^2\psi_z^2}{U}\right),\\
&\int(\phi_z\psi_z)_z\psi_{zz} \le N(t)\int\psi_{zz}^2 + N(t)\int\frac{\phi_{zz}^2}{U} + \frac{N(t)}{4}\int U\psi_{zz}^2.
\end{aligned}
$$

Thus, integrating (3.58) over $[0,t]$, using the Cauchy–Schwarz inequality and choosing $N(t)$ small enough, we have

$$
\begin{aligned}
&\frac{1}{2}\int\psi_{zz}^2 + \int_0^t\int U\psi_{zz}^2 + \varepsilon\int_0^t\int\psi_{zzz}^2\\
&\quad \le C\left(\|\psi_0\|_2^2 + \|\psi_{0z}\|_w^2 + \|\phi_0\|_{2,w}^2 + N(t)\int_0^t\int\frac{\psi_{zz}^2}{U}\right).
\end{aligned}
\tag{3.59}
$$

Substituting (3.59) into (3.57) leads to

$$
\begin{aligned}
&\int\left(\frac{\phi_{zz}^2}{U} + \psi_{zz}^2\right) + \int_0^t\int\frac{\phi_{zzz}^2}{U} + \varepsilon\int_0^t\int\psi_{zzz}^2\\
&\quad \le C\bigg(\|\phi_0\|_{2,w}^2 + \|\psi_{0z}\|_w^2 + \|\psi_0\|_2^2 + N(t)\int_0^t\int\frac{\psi_{zz}^2}{U}\bigg).
\end{aligned}
\tag{3.60}
$$

Applying the same argument in deriving (3.49), we get from (3.59) that

$$\int_{-\infty}^{0} w\psi_{zz}^2 + \int_0^t \int_{-\infty}^{0} w\psi_{zz}^2 + \varepsilon \int_0^t \int_{-\infty}^{0} w\psi_{zzz}^2 \\ \leq C\left(\|\psi_0\|_2^2 + \|\psi_{0z}\|_w^2 + \|\phi_0\|_{2,w}^2 + N(t) \int_0^t \int w\psi_{zz}^2 \right). \tag{3.61}$$

We proceed to estimate $\int_0^t \int w\psi_{zz}^2$. Multiplying the second equation of (3.55) by $e^{\eta z}\psi_{zz}$, as in (3.50), we get

$$\left(\frac{e^{\eta z}}{2}\psi_{zz}^2\right)_t + \frac{s\eta e^{\eta z}}{2}\psi_{zz}^2 + \varepsilon e^{\eta z}\psi_{zzz}^2 \\ = \left[\varepsilon\psi_{zzz}e^{\eta z}\psi_{zz} + \frac{s}{2}e^{\eta z}\psi_{zz}^2 - 2\varepsilon(V\psi_z)_z e^{\eta z}\psi_{zz} - \varepsilon(\psi_z^2)_z e^{\eta z}\psi_{zz}\right]_z \\ - \varepsilon\eta e^{\eta z}\psi_{zz}\psi_{zzz} + 2\varepsilon(V\psi_z)_z e^{\eta z}\psi_{zzz} + 2\varepsilon(V\psi_z)_z \eta e^{\eta z}\psi_{zz} + e^{\eta z}\phi_{zzz}\psi_{zz} \\ + \varepsilon(\psi_z^2)_z \eta e^{\eta z}\psi_{zz} + \varepsilon(\psi_z^2)_z e^{\eta z}\psi_{zzz}. \tag{3.62}$$

By Young's inequality, it follows that

$$|-\varepsilon\eta e^{\eta z}\psi_{zz}\psi_{zzz} + 2\varepsilon(V\psi_z)_z e^{\eta z}\psi_{zzz}| \leq \frac{\varepsilon}{4}e^{\eta z}\psi_{zzz}^2 + C\varepsilon e^{\eta z}(\psi_z^2 + \psi_{zz}^2), \\ |e^{\eta z}\phi_{zzz}\psi_{zz}| \leq e^{\eta z}\phi_{zzz}^2 + Ce^{\eta z}\psi_{zz}.$$

Thus, integrating (3.62) over $\mathbb{R} \times [0, t]$, by (3.36) we have

$$\int e^{\eta z}\psi_{zz}^2 + s\eta \int_0^t \int e^{\eta z}\psi_{zz}^2 + \frac{3\varepsilon}{2}\int_0^t \int e^{\eta z}\psi_{zzz}^2 \\ \leq \int e^{\eta z}\psi_{0zz}^2 + C\varepsilon \int_0^t \int e^{\eta z}\psi_{zz}^2 + C\varepsilon \int_0^t \int e^{\eta z}\psi_z^2 \\ + 2\int_0^t \int e^{\eta z}\phi_{zzz}^2 + \frac{\varepsilon N(t)}{2}\int_0^t \int e^{\eta z}\psi_{zzz}^2 \\ \leq C\left(\|\phi_0\|_{2,w}^2 + \|\psi_0\|_2^2 + \|\psi_{0z}\|_{1,w}^2 + N(t)\int_0^t \int w\psi_{zz}^2 \right), \tag{3.63}$$

where we have used $|V_z| \leq C$, $|V| \leq C$ and $\|\psi_z(\cdot, t)\|_{L^\infty} \leq N(t)$ in the first inequality, and (3.37) and (3.60) in the second inequality. It follows from (3.63) that

$$\int_0^{+\infty} e^{\eta z}\psi_{zz}^2 + \int_0^t \int_0^{+\infty} e^{\eta z}\psi_{zz}^2 + \varepsilon \int_0^t \int_0^{+\infty} e^{\eta z}\psi_{zzz}^2 \\ \leq C\left(\|\phi_0\|_{2,w}^2 + \|\psi_0\|_2^2 + \|\psi_{0z}\|_{1,w}^2 + N(t)\int_0^t \int w\psi_{zz}^2 \right). \tag{3.64}$$

Noting $e^{\eta w} \geq \frac{w(z)}{2}$ for $z \in [0, +\infty)$, we get from (3.64) that

$$\int_0^{+\infty} w\psi_{zz}^2 + \int_0^t \int_0^{+\infty} w\psi_{zz}^2 + \varepsilon \int_0^t \int_0^{+\infty} w\psi_{zzz}^2$$
$$\leq C\left(\|\phi_0\|_{2,w}^2 + \|\psi_0\|_2^2 + \|\psi_{0z}\|_{1,w}^2 + N(t)\int_0^t \int w\psi_{zz}^2\right).$$

This inequality, in combination with (3.61), yields

$$\int w\psi_{zz}^2 + \int_0^t \int w\psi_{zz}^2 + \varepsilon \int_0^t \int w\psi_{zzz}^2$$
$$\leq C\left(\|\phi_0\|_{2,w}^2 + \|\psi_0\|_2^2 + \|\psi_{0z}\|_{1,w}^2 + N(t)\int_0^t \int w\psi_{zz}^2\right).$$

Choosing $N(t)$ small enough, we have

$$\int w\psi_{zz}^2 + \int_0^t \int w\psi_{zz}^2 + \varepsilon \int_0^t \int w\psi_{zzz}^2 \leq C(\|\phi_0\|_{2,w}^2 + \|\psi_0\|_2^2 + \|\psi_{0z}\|_{1,w}^2). \quad (3.65)$$

Therefore, the desired (3.54) follows from (3.20), (3.37), (3.57) and (3.65). □

Proof. (of Proposition 3.1) From global estimate (3.10), we have

$$\|\phi_z(\cdot, t), \psi_z(\cdot, t)\|_{1,w} \to 0, \quad \text{as } t \to +\infty.$$

Hence, for all $z \in \mathbb{R}$,

$$\phi_z^2(z,t) = 2\int_{-\infty}^z \phi_z \phi_{zz}(y,t)dy$$
$$\leq 2\left(\int_{-\infty}^{\infty} \phi_z^2 dy\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \phi_{zz}^2 dy\right)^{\frac{1}{2}}$$
$$\leq \|\phi_z(\cdot, t)\|_{1,w}^2 \to 0, \quad \text{as } t \to +\infty.$$

Applying the same procedure to ψ_z leads to

$$\psi_z(z,t) \to 0, \quad \text{as } t \to +\infty, \quad \text{for all } z \in \mathbb{R}.$$

Hence (3.11) is proved. □

3.2. *Proof of main results*

Theorem 2.1 is a direct consequence of Proposition 3.1 where the global estimate (3.10) guarantees that $N(t)$ is small for all $t > 0$ under the assumption that $N(0)$ is small.

It only remains to prove Theorem 2.2. Noticing that u in the original system (1.1), (1.4) remains the same as one in the transformed system, we only need to derive the results for $c(x, t)$.

First as a consequence of Theorem 2.1 and the transformation (1.3), we have

$$(u-U, c_x/c - \mathcal{C}_x/\mathcal{C}) \in C([0,\infty); H_w^1) \cap L^2([0,\infty); H_w^2).$$

Next we proceed to derive the results for $c-\mathcal{C}$, namely (2.14) in Theorem 2.2. To this end, we note that $c(x,t)$ and $\mathcal{C}(z)$ satisfy equations

$$c_t = \varepsilon c_{xx} - uc, \quad 0 = \varepsilon \mathcal{C}_{zz} - U\mathcal{C} + s\mathcal{C}_z,$$

where $z = x - st$ with $s > 0$. By defining $\theta(z,t) = c(x,t) - \mathcal{C}(x+x_0-st)$, we derive from the above two equations that

$$\theta_t = \varepsilon\theta_{zz} - (U+\phi_z)\theta - \mathcal{C}\phi_z + s\theta_z. \tag{3.66}$$

We first derive the L^2-estimates for θ. To this end, we multiply Eq. (3.66) by θ and integrate the result with respect to z, and obtain

$$\frac{1}{2}\frac{d}{dt}\int\theta^2 + \varepsilon\int\theta_z^2 + \int U\theta^2 = -\int\phi_z\theta^2 - \int\mathcal{C}\phi_z\theta. \tag{3.67}$$

Next we will use the technique of smallness of θ, as done for ϕ and ψ above. Indeed from the transformation (1.3) and decomposition (3.3), we have

$$\frac{c_0(x)}{\mathcal{C}(x+x_0)} = e^{\int_x^\infty (v_0(\xi)-V(\xi+x_0))d\xi} = e^{-\psi_0(x)},$$

where we have used (3.6) and the fact $\psi_0(\pm\infty)=0$ to derive that $\int_x^\infty(v_0(\xi)-V(\xi+x_0))d\xi = -\int_{-x}^\infty(v_0(\xi)-V(\xi+x_0))d\xi = -\psi_0(x)$. Note that $\mathcal{C}(x+x_0)$ is a traveling wave solution bounded by $c_+>0$. Then by the Taylor expansion, we have

$$\begin{aligned}\theta_0(x) = \theta(x,0) = c_0(x) - \mathcal{C}(x+x_0) &= \mathcal{C}(x+x_0)e^{-\psi_0(x)} - \mathcal{C}(x+x_0)\\ &= \mathcal{C}(x+x_0)(e^{-\psi_0(x)}-1)\\ &= \mathcal{C}(x+x_0)\psi_0\sum_{n=1}^\infty\frac{(-1)^n}{n!}\psi_0^{n-1}.\end{aligned} \tag{3.68}$$

Since we have assumed that $\|\psi_0\|_{L^\infty}$ is small, see (3.9), the series $\sum_{n=1}^\infty\frac{(-1)^n}{n!}\psi_0^{n-1}$ is convergent. Noticing that $0\le\mathcal{C}\le c_+$ for $x\in\mathbb{R}$, then by the assumption of Theorem 2.2 and (3.68), we have $\|\theta_0\|_1 \le \delta_0$ for some small constant $\delta_0>0$. Now we set $M(t) = \sup_{\tau\in[0,t]}\|\theta(\cdot,\tau)\|_1$, which gives $M(0)\le\delta_0$. We assume that $M(t)\le\delta_1$ where $1>\delta_1>0$. Then we employ the Cauchy–Schwarz inequality to have: $-\phi_z\theta^2 \le \frac{1}{2}M(t)U\theta^2 + \frac{1}{2}M(t)\frac{\phi_z^2}{U}$ and $\mathcal{C}\phi_z\theta \le c_+|\phi_z\theta| \le \frac{1}{2}U\theta^2 + \frac{c_+^2}{2}\frac{\phi_z^2}{U}$, which in combination with (3.67) yield that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int\theta^2 + \varepsilon\int\theta_z^2 + \frac{1}{2}(1-M(t))\int U\theta^2 &\le \left(\frac{c_+^2}{2} + M(t)\right)\int\frac{\phi_z^2}{U}\\ &\le C\int w\phi_z^2.\end{aligned} \tag{3.69}$$

From the inequality (3.20), (3.53) and noting that $M(t) < 1$, it follows that

$$\int_0^t \|\phi_z(\cdot,\tau)\|^2 d\tau \le \int_0^t \|\phi_z(\cdot,\tau)\|_w^2 d\tau \le C(\|\phi_0\|_{1,w}^2 + \|\psi_0\|_1^2 + \|\psi_{0z}\|_w^2). \tag{3.70}$$

Then integrating the inequality (3.69) in t and using (3.70), we end up with

$$\|\theta\|^2 + \varepsilon \int_0^t \|\theta_z\|^2 + \int_0^t \int U\theta^2 \le C(\|\phi_0\|_{1,w}^2 + \|\psi_0\|_1^2 + \|\psi_{0z}\|_w^2 + \|\theta_0\|^2). \tag{3.71}$$

We proceed to estimate the first-order derivative of θ. Multiplying (3.66) by $-\theta_{zz}$ and integrating the resulting equation with respect to z, we get

$$\frac{1}{2}\frac{d}{dt}\int \theta_z^2 + \varepsilon \int \theta_{zz}^2 = \int U\theta\theta_{zz} + \int \phi_z\theta\theta_{zz} + \int \mathcal{C}\phi_z\theta_{zz}. \tag{3.72}$$

By the Cauchy–Schwarz inequality, we have

$$U\theta\theta_{zz} \le \frac{\varepsilon}{4u_-}U\theta_{zz}^2 + \frac{u_-}{\varepsilon}U\theta^2 \le \frac{\varepsilon}{4}\theta_{zz}^2 + \frac{u_-}{\varepsilon}U\theta^2,$$

$$\phi_z\theta\theta_{zz} \le M(t)|\phi_z\theta_{zz}| \le \frac{\varepsilon}{4}\theta_{zz}^2 + \frac{M^2(t)}{\varepsilon}\phi_z^2,$$

$$\mathcal{C}\phi_z\theta_{zz} \le c_+|\phi_z\theta_{zz}| \le \frac{\varepsilon}{4}\theta_{zz}^2 + \frac{c_+^2}{\varepsilon}\phi_z^2.$$

Then applying the above inequalities into (3.72) yields

$$\frac{1}{2}\frac{d}{dt}\int \theta_z^2 + \frac{\varepsilon}{4}\int \theta_{zz}^2 \le \frac{u_-}{\varepsilon}\int U\theta^2 + \frac{1}{\varepsilon}(M^2(t) + c_+^2)\int \phi_z^2,$$

which, upon the integration with respect to t, gives

$$\|\theta_z\|^2 + \varepsilon \int_0^t \|\theta_{zz}\|^2 \le C(\|\phi_0\|_{1,w}^2 + \|\psi_0\|_1^2 + \|\psi_{0z}\|_w^2 + \|\theta_0\|_1^2), \tag{3.73}$$

where the assumption $M(t) \le \delta_1 < 1$, (3.70) and (3.71) have been used. Then the combination of (3.71) and (3.73) gives (2.14) in Theorem 2.2.

Finally the global estimates (3.71) and (3.73) validate our assumption $M(t) \le \delta_1 < 1$ under the condition that $M(0)$ is small. We then have from the standard argument that $\|\theta_z\| \to 0$ as $t \to \infty$ and $\|\theta(\cdot,t)\|$ is bounded for all $t > 0$. Then for all $z \in \mathbb{R}$, we get

$$\theta^2(x,t) = 2\int_{-\infty}^x \theta\theta_y(y,t)dy \le 2\|\theta(\cdot,t)\| \cdot \|\theta_z(\cdot,t)\| \le C\|\theta_z(\cdot,t)\| \to 0, \quad \text{as } t \to \infty,$$

which implies $|\theta(x,t)| \to 0$ as $t \to \infty$. This completes the proof of Theorem 2.2.

Remark 3.1. Here we prove the convergence of $\mathcal{C}$ as a natural consequence of the global estimates (3.71) and (3.73) as shown above. An alternative approach to prove the convergence of $\mathcal{C}$ is to use the transformation (1.3) and write $|c(x,t) - \mathcal{C}(x + x_0 - st)| = \mathcal{C}(x + x_0 - st)|1 - e^{-\psi(x,t)}|$ and then use the fact $\psi(x,t) \to 0$ as $t \to \infty$ by the global estimate (3.10).

4. Numerical Simulations

In this section, we will use numerical simulations to illustrate the stability of traveling wave solutions of (1.1) and (1.2) for $m = 1$ and small $\varepsilon > 0$ as proved in the paper. On the other hand, for large $\varepsilon > 0$, the numerical simulation will be performed to make the predictions. In our theoretical results, the nonlinear stability of traveling wave solutions requires $\varepsilon > 0$ to be small, which is generally undesirable since $\varepsilon > 0$ is the chemical diffusion coefficient in the original model (1.1) as a stabilizing factor. The numerical simulation indeed shows that the traveling wave solution still remains stable for large $\varepsilon > 0$, which verifies our suspicion.

Numerically solving the system (1.1) is very challenging due to the logarithmic sensitivity. As we know, the numerical solutions of the Keller–Segel model (1.1) have not yet been obtained. Our strategy here is to solve the transformed system (1.4) with the second equation of (1.1). That is we solve the following system of three equations instead of two equations:

$$\begin{cases} u_t - \chi(uv)_x = Du_{xx}, \\ v_t + (\varepsilon v^2 - u)_x = \varepsilon v_{xx}, \\ c_t = \varepsilon c_{xx} - uc, \end{cases} \tag{4.1}$$

with initial data

$$(u, v, c)(x, 0) = (u_0, v_0, c_0)(x) \to (u_\pm, v_\pm, c_\pm), \quad \text{as } x \to \pm\infty, \tag{4.2}$$

where $v_0(x) = -(\log c_0)_x = -\frac{c_{0x}}{c_0}$. Since the traveling wave solutions converge to the asymptotic states exponentially as $x \to \infty$, the domain $\mathbb{R}$ can be effectively approximated by a finite interval with a suitably large length. In simulation, the domain is set as $[0, 400]$ and mesh size is 0.2. The boundary conditions are set as Dirichlet conditions to comply with the initial data. The Matlab PDE solver based on finite difference scheme is implemented to perform the numerical computations. Only the interested biological quantities $u(x, t)$ and $c(x, t)$ will be plotted numerically. We first set an initial traveling wavefront profile $(U, \mathcal{C})(x)$ of the Keller–Segel system (1.1) as

$$\begin{aligned} U(x) = \tilde{u}(x) &= \frac{u_-}{1 + \exp(2(x - 100))}, \\ \mathcal{C}(x) = \tilde{c}(x) &= \frac{c_+}{1 + \exp(-2(x - 100))}. \end{aligned} \tag{4.3}$$

Then by the relation (1.3), the initial traveling wave profile for $V(x)$ is

$$V(x) = -\frac{\mathcal{C}'(x)}{\mathcal{C}(x)} = \frac{-2}{1 + \exp((x - 100))}.$$

Thus the left and right asymptotic states of the traveling wave solution $(U, V, \mathcal{C})$, namely (u_-, v_-, c_-) and (u_+, v_+, c_+), satisfy $u_+ = c_- = v_+ = 0, v_- = -2$ and $u_- = (\varepsilon + \chi)v_-^2 = 4(\varepsilon + \chi)$, where $c_+ > 0$ can be arbitrarily chosen. Next we choose appropriate perturbations. Noting that only the right end state $u_+ = 0$ generates the singularity and hence the exponential weight function is merely imposed at $z = +\infty$ as seen from (2.13). That is the initial value u_0 must exponentially decay at $x = +\infty$ with rate $\eta = \frac{s}{D}$. On the other hand, from the proof for the convergence of $\mathcal{C}$ in Sec. 3.2, we know that weight function is not needed for the chemical concentration $c(x,t)$ at $x = +\infty$. Following these observations, we set the initial value (u_0, c_0) as

$$\begin{aligned} (u_0 - U)(x) &= \frac{\sin x}{((x-100)/10)^2 + 1} \cdot \frac{1}{1 + e^{\eta(x-100)}}, \\ (c_0 - \mathcal{C})(x) &= \frac{\sin x}{((x-100)/10)^2 + 1} \cdot \frac{1}{1 + e^{-2(x-100)}}, \end{aligned} \tag{4.4}$$

such that $(u_0 - U, c_0 - \mathcal{C}) \in H^1_w(\mathbb{R}) \times H^1(\mathbb{R})$ as required by Theorem 2.2, where $\eta = \frac{s}{D}$, $s = -\chi v_-$ and the initial value for $v(x,t)$ is $v_0(x) = -\frac{c_{0x}}{c_0}$. In the numerical simulations, we fix several parameter values: $D = 2$, $\chi = 1$, $c_+ = 1$. Then $s = -\chi v_- = 2$ and $\eta = 1$, and the initial data (u_0, c_0) satisfying (4.3) and (4.4) are plotted in Fig. 1.

We explore the numerical simulations for two cases: small and large $\varepsilon > 0$. First we let $\varepsilon = 0.1$ small and hence $u_- = (\varepsilon + \chi)v_-^2 = 4.4$. With the initial data given in (4.4), we employ the Matlab PDE solver to compute the system (4.1). The resulting numerical solution (u,c) is plotted in Fig. 2. It is observed that the solution (u,c) stabilizes to a traveling wave profile, which verifies our theoretical results. Next we examine the case of large $\varepsilon > 0$. For this scenario, it remains unknown in this paper whether or not the traveling wave solutions is stable since our theoretical result requires the smallness of $\varepsilon > 0$. Hence it is worthwhile to explore this case numerically to foresee the possible outcomes. To this end, we choose $\varepsilon = 5$ and then $u_- = (\varepsilon + \chi)v_-^2 = 24$. The initial value (u_0, c_0) is set as in (4.4). Figure 3 shows the time evolution of numerical solution starting from such an initial value. It is clearly found in the simulation that the solution evolves to a steady traveling

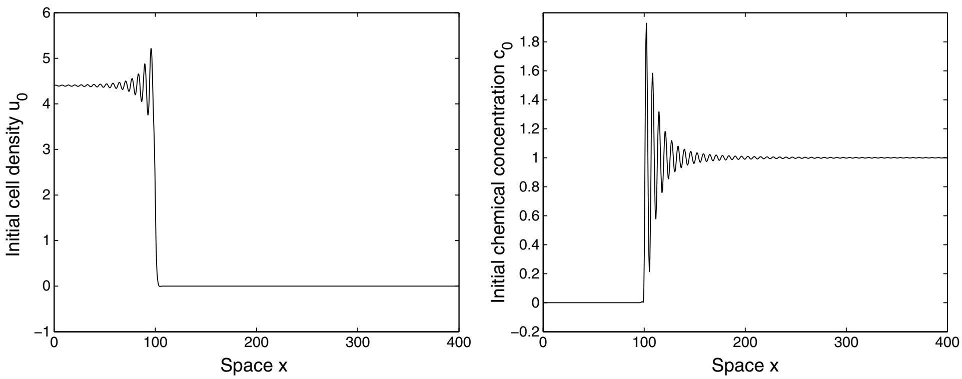

Fig. 1. The numerical plot of initial value (u_0, c_0) given by (4.4) with $\varepsilon = 0.1$, where $u_- = 4.4, c_+ = 1, u_+ = c_- = 0$.

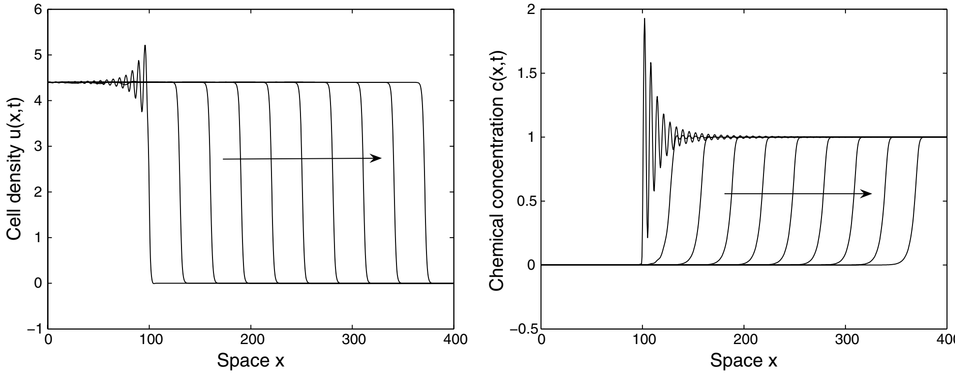

Fig. 2. The numerical illustration of the time evolution of the traveling wave solution (u, c) to the system (4.1) and (4.2) with $u_+ = 0$ for small $\varepsilon > 0$, where the initial value satisfies (4.4) and is plotted in Fig. 1. Other parameter values are $\varepsilon = 0.1, D = 2, \chi = 1, c_+ = 1, u_- = 4.4$. Each curve represents the solution (wave) profile at a certain time starting at $t = 0$ and spaced by $t = 15$. The arrow indicates that the wave propagates from the left to the right.

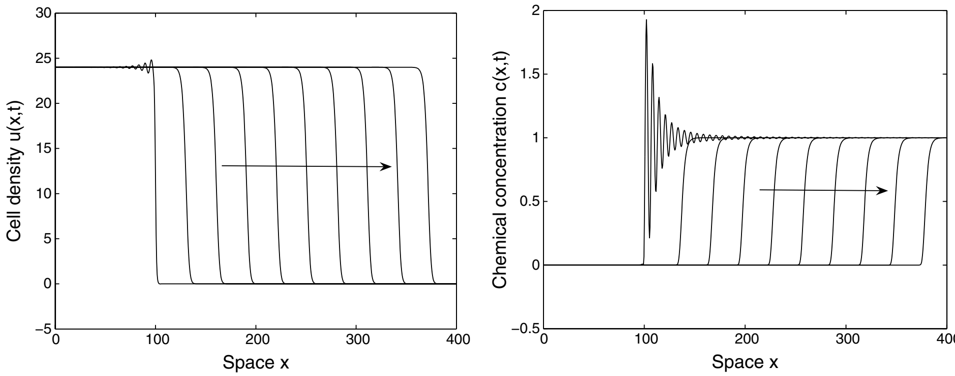

Fig. 3. The numerical simulation of the evolutionary traveling wave solution (u, c) of the system (4.1) and (4.2) with $u_+ = 0$ for large $\varepsilon > 0$, where $\varepsilon = 5$ and other parameter values are the same as those in Fig. 2. Each curve represents the solution (wave) profile at a certain time starting at $t = 0$ and spaced by $t = 15$. The arrow indicates the direction of wave propagation.

wave solution. This implies that traveling wave solution of the system (1.1) for large $\varepsilon > 0$ is still stable. However, the technical difficulty makes us leave this problem open in this paper.

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