

BOUNDEDNESS, STABILIZATION, AND PATTERN FORMATION DRIVEN BY DENSITY-SUPPRESSED MOTILITY*

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Abstract. We are concerned with the following density-suppressed motility model: $u_t = \Delta(\gamma(v)u) + \mu u(1 - u)$; $v_t = \Delta v + u - v$, in a bounded smooth domain $\Omega \subset \mathbb{R}^2$ with homogeneous Neumann boundary conditions, where the motility function $\gamma(v) \in C^3([0, \infty))$, $\gamma(v) > 0$, $\gamma'(v) < 0$ for all $v \geq 0$, $\lim_{v \rightarrow \infty} \gamma(v) = 0$, and $\lim_{v \rightarrow \infty} \frac{\gamma'(v)}{\gamma(v)}$ exists. The model is proposed to advocate a new possible mechanism: density-suppressed motility can induce spatio-temporal pattern formation through self-trapping. The major technical difficulty in the analysis of above density-suppressed motility model is the possible degeneracy of diffusion from the condition $\lim_{v \rightarrow \infty} \gamma(v) = 0$. In this paper, by treating the motility function $\gamma(v)$ as a weight function and employing the method of weighted energy estimates, we derive the a priori L^∞ -bound of v to rule out the degeneracy and establish the global existence of classical solutions of the above problem with a uniform-in-time bound. Furthermore, we show if $\mu > \frac{K_0}{16}$ with $K_0 = \max_{0 \leq v \leq \infty} \frac{|\gamma'(v)|^2}{\gamma(v)}$, the constant steady state $(1, 1)$ is globally asymptotically stable and, hence, pattern formation does not exist. For small $\mu > 0$, we perform numerical simulations to illustrate aggregation patterns and wave propagation formed by the model.

Key words. density-suppressed motility, degeneracy, large time behavior, pattern formation

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1. Introduction. It is well known that reaction-diffusion models can generate a wide variety of exquisite spatio-temporal patterns arising in embryogenesis and development due to Turing instability [11]. In addition, colonies of bacteria and eukaryotes can also generate rich and complex patterns where these patterns typically result from coordinated cell movement, growth, and differentiation that often involve the detection and processing of extracellular signals and can be described by the Keller–Segel-type models in the population level [21, Chapter 5]. In many instances, the models invoke nonlinear cell diffusion which increases with respect to the local cell density. However the opposite case of density suppressing motility was also able to produce spatio-temporal patterns through a “self-trapping” mechanism. This mechanism was recently introduced into the bacterium *E. coli* in the experiment reported in [14] by a synthetic biology approach and stripe pattern formation was found, where bacteria excrete a small signaling molecule acyl-homoserine lactone (AHL) such that at low AHL levels, the bacteria undergo run-and-tumble random motion and are motile, while at high AHL levels, the bacteria tumble incessantly and become immotile as

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a result of a vanishing macroscopic motility. Though the experimental observations can be described by a mathematical model of three-component equations in the paper [14], the mechanism of pattern formation remains unclear and hence a simplified two-component model was proposed in a later work [6] to capture the essential features underlying the stripe pattern formation driven by the density-suppressed motility. The model in [6] reads as

$$(1.1) \quad \begin{cases} u_t = \Delta(\gamma(v)u) + \mu u(1 - u/K), \\ v_t = d\Delta v + \alpha u - \beta v, \end{cases}$$

where $u(x, t)$ and $v(x, t)$ represent the densities of bacteria and AHL, respectively. The first equation describes logistic bacterial growth saturated at density K with intrinsic rate μ , where the bacterial diffusion rate depends on a motility function $\gamma(v)$ which takes into account the repressive effect of AHL concentration on cell (bacterium) motility by assuming $\gamma'(v) < 0$.

If we expand the Laplacian term in the first equation, the system (1.1) can be rewritten as

$$(1.2) \quad \begin{cases} u_t = \nabla \cdot (\gamma(v)\nabla u + u\gamma'(v)\nabla v) + \mu u(1 - u/K), \\ v_t = d\Delta v + \alpha u - \beta v, \end{cases}$$

which is a special Keller–Segel-type chemotaxis model proposed in [10] if the ratio of effective body length of cell/bacteria is zero, namely, the cells do not sense the concentration between receptors; see details in [10] or in [38]. The interplay of diffusion, chemotaxis, and logistic growth has been an interesting topic and was extensively studied in the literature (see [4] and reference therein). For instance, for the following classical chemotaxis-growth system

$$(1.3) \quad \begin{cases} u_t = \nabla \cdot (\nabla u - \chi u \nabla v) + \mu u(1 - u/K), \\ v_t = d\Delta v + \alpha u - \beta v, \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) with smooth boundary and homogeneous Neumann boundary conditions, it was proved that there exists a constant $\mu_* \geq 0$ (equality holds if $n = 2$) such that if $\mu > \mu_*$, the global classical solutions exist for all $n \geq 2$ (cf. [37, 16, 31, 30]), which is quite different from the case $\mu = 0$ for which solutions may blow up in finite time in two or higher dimensions (see [7, 33, 4] for details). However, we should point out that chemotactic collapses (i.e., blowup) are still possible in the presence of logistic-like growth. Indeed, if the logistic source $\mu u(1 - u/K)$ in system (1.3) is replaced by $\lambda u - \mu u^\zeta$ with $\lambda \geq 0$ and $1 < \zeta < \frac{3}{2} + \frac{1}{2n-2}$, Winkler [32] has proved that there exist some initial data such that the corresponding solutions blow up in finite time in an n -dimensional ball ($n \geq 5$) for the simplified parabolic-elliptic chemotaxis-growth model. Besides numerous results on the existence and boundedness, the chemotaxis-growth system (1.3) possesses quite a large variety of dynamical properties. Specifically, the global solution will converge to the nontrivial spatially homogeneous equilibria if the logistic term is suitably strong (i.e., μ_* is suitably large); see [19, 34, 3]. Moreover, the spontaneous emergence of patterns [30, 13], the transient growth phenomena [18, 35], and chaotic behavior [22] were also studied by delicate analysis or numerical experiments.

The distinctive feature of the model (1.2) is that the cell diffusion depends on the chemical concentration v through the motility function $\gamma(v)$. Hence the possible

degeneracy (i.e., when $\gamma(v)$ touches zero) may bring considerable challenges to the analysis. To our knowledge, the results of the Keller–Segel model where the diffusion depends on chemical concentration are very limited. For the system (1.1), not many rigorous mathematical results have been available either. When $\gamma(v)$ is a piecewise decreasing function, formal analysis of (1.1) has been performed in [6] to explain the mechanism of stripe formation, and the dynamics of the interface where $\gamma(v)$ jumps was recently studied in [24]. The purpose of this paper is to derive some qualitative behaviors for the system (1.1) with a smooth motility function $\gamma(v)$ in a bounded domain with Neumann boundary conditions. That is we shall consider the following initial boundary value problem:

$$(1.4) \quad \begin{cases} u_t = \Delta(\gamma(v)u) + \mu u(1 - u), & x \in \Omega, \quad t > 0, \\ v_t = \Delta v + u - v, & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), & x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary and ν denotes the outward unit normal vector on $\partial\Omega$. In the problem (1.4), we have assumed $d = \alpha = \beta = K = 1$ for simplicity since the specific values of these parameter are not of importance to our analysis. As mentioned above, $\min_{v \in [0, \infty]} \gamma(v) = 0$ is possible under the biological assumption $\gamma'(v) < 0$, and hence degeneracy may occur. This will hinder the understanding of global dynamics of solutions to (1.4). Therefore how to deal with the degeneracy will be the first issue for the analytical studies. There are some results recently obtained for the case $\mu = 0$ (namely, bacteria has no growth). First, in [39], the authors considered a particular form of $\gamma(v)$ as follows:

$$(1.5) \quad \gamma(v) = \frac{c_0}{v^k}, \quad c_0 > 0, k > 0.$$

Since it can be shown directly from the second equation of (1.4) that v has a positive lower bound, then $\gamma(v)$ defined above will have no singularity. Hence the main concern is the degeneracy of diffusion. In [39], Yoon and Kim used a step-function approximation approach for the motility function $\gamma(v)$ to resolve this difficulty and showed that if $c_0 > 0$ is small, the problem (1.4) with $\mu = 0$ admits a global classical solution which is uniformly bounded in time. Furthermore by assuming that $\gamma(v)$ has a positive lower and upper bound (i.e., $\gamma_1 \leq \gamma(v) \leq \gamma_2$ for all $v \geq 0$, where γ_1, γ_2 are two positive constants), Tao and Winkler [29] recently established the existence of global classical solutions in two dimensions and global weak solutions in three dimensions for (1.4) with $\mu = 0$. To our knowledge, no much rigorous results for the system (1.4) with $\mu > 0$ have been available to date except some formal analysis work in [6, 24]. The goal of this paper is to establish the global existence and asymptotic behavior of solutions to (1.4) with $\mu > 0$ which is the original model constructed in [6]. The idea here is not to use the skilled approximation technique as in [39] or the direct lower-upper bound assumption in [29], but to develop a new estimate idea to resolve the degeneracy of diffusion in (1.4). For definiteness, we impose the following assumptions on the motility function $\gamma(v)$:

$$(H) \quad \gamma(v) \in C^3([0, \infty)), \gamma(v) > 0, \text{ and } \gamma'(v) < 0 \text{ on } [0, \infty), \lim_{v \rightarrow \infty} \gamma(v) = 0, \text{ and } \lim_{v \rightarrow \infty} \frac{\gamma'(v)}{\gamma(v)} \text{ exists.}$$

We note there are many candidates for $\gamma(v)$ to satisfy the hypothesis (H), for instance,

$$\gamma(v) = be^{-av}, \quad \gamma(v) = \frac{a}{(1 + e^{bv})^m}, \quad \gamma(v) = \frac{a}{(1 + bv)^m}, \quad \text{or} \quad \gamma(v) = 1 - \frac{v}{\sqrt{1 + v^2}},$$

where a, b, m are positive constants. In particular, the Hill function chosen in [14] fulfills the hypothesis (H). Since $\gamma(v) \rightarrow 0$ as $v \rightarrow \infty$, deriving the a priori L^∞ -bound for v to rule out the degeneracy is a key to understanding the behavior of (1.4). The new idea in this paper is to treat $\gamma(v)$ as a weight function in the energy estimates and control the nonlinear advection (i.e., chemotaxis-like cross-diffusion) term using the weighted diffusive dissipation along with the logistic damping. Then we derive the L^2 -boundedness of u (see section 3.1) and obtain the uniform-in-time boundedness of v with the help of parabolic regularity theory applied to the second equation of (1.4), which eliminate the degeneracy. Then we apply the Moser iteration method to show the boundedness of u and the existence of global classical solutions (section 3.2). Furthermore we find the large time behavior of solutions by constructing a Lyapunov functional. Our main results are stated as follows.

THEOREM 1.1. *Let Ω be a bounded domain in \mathbb{R}^2 with smooth boundary and the hypothesis (H) holds. Suppose that $(u_0, v_0) \in [W^{1,\infty}(\Omega)]^2$ with $u_0, v_0 \geq 0 (\neq 0)$. Then the problem (1.4) has a unique nonnegative global classical solution $(u, v) \in [C^0([0, \infty) \times \bar{\Omega}) \cap C^{2,1}((0, \infty) \times \bar{\Omega}) \cap L^\infty_{loc}([0, \infty); W^{1,\infty}(\Omega))]^2$ satisfying*

$$(1.6) \quad \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C \quad \text{for all } t > 0,$$

where $C > 0$ is a constant independent of t . Furthermore, if $\mu > \frac{K_0}{16}$ with $K_0 = \max_{0 \leq v \leq \infty} \frac{|\gamma'(v)|^2}{\gamma(v)}$, the constant steady state (1, 1) is globally asymptotically stable in the sense that

$$\lim_{t \rightarrow \infty} (\|u(\cdot, t) - 1\|_{L^\infty(\Omega)} + \|v(\cdot, t) - 1\|_{L^\infty(\Omega)}) = 0.$$

The hypothesis (H) for the motility function $\gamma(v)$ generalizes the form (1.5). In this paper by taking advantage of the logistic damping property, we not only establish the global boundedness of solutions without the smallness assumption imposed in [39] or the lower-upper boundedness condition in [29], but also assert no pattern formation arises for large intrinsic growth rate $\mu > 0$. For small $\mu > 0$, we use numerical simulations to illustrate that the system (1.4) is capable of forming various complex aggregation patterns, and of generating wavefront patterns with different values of μ .

2. Preliminaries: Local existence and some inequalities. In what follows, without confusion, we shall abbreviate $\int_\Omega f dx$ as $\int_\Omega f$ and $\|f\|_{L^2(\Omega)}$ as $\|f\|_{L^2}$ for simplicity. Moreover, we shall use $c_i (i = 1, 2, 3, \dots)$ to denote a generic constant which may vary in context. We first give the existence of local solutions of (1.4) by the Schauder fixed point theorem along with the parabolic regularity theory based on some ideas in [26, Lemma 2.1]. In fact, one can also use the abstract theory (cf. [2, 36, 29, 39]) to show the existence of local solutions.

LEMMA 2.1 (local existence). *Let Ω be a bounded domain in \mathbb{R}^2 with smooth boundary and the hypothesis (H) holds. Assume $(u_0, v_0) \in [W^{1,\infty}(\Omega)]^2$ with $u_0, v_0 \geq 0 (\neq 0)$. Then there exists $T_{max} \in (0, \infty]$ such that the problem (1.4) has a unique classical solution*

$$(u, v) \in [C^0([0, T_{max}) \times \bar{\Omega}) \cap C^{2,1}((0, T_{max}) \times \bar{\Omega}) \cap L^\infty_{loc}([0, T_{max}); W^{1,\infty}(\Omega))]^2$$

satisfying $u, v \geq 0$ for all $t > 0$. Moreover, we have

$$(2.1) \quad \text{either } T_{max} = \infty \text{ or } \limsup_{t \nearrow T_{max}} (\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{W^{1,\infty}}) = \infty.$$

Proof. The proof consists of two steps.

(i) *Existence.* Define $R := \|u_0\|_{L^\infty} + \|v_0\|_{W^{1,\infty}} + 1$. With this R and $T \in (0, 1)$ small to be specified below, we consider a closed bounded convex subset in Banach space $X := C^0(\bar{\Omega} \times [0, T])$:

$$S_T = \{\tilde{u} \in X : \|\tilde{u}(\cdot, t)\|_{L^\infty} \leq R \text{ a.e. in } \Omega \times (0, T)\}$$

and define a mapping $\Phi : S_T \rightarrow S_T$ such that for given $\tilde{u} \in S_T$, $\Phi(\tilde{u}) = u$, where u is the solution to

$$(2.2) \quad \begin{cases} u_t = \nabla \cdot (\bar{\gamma}(v)\nabla u) + \nabla \cdot (\bar{\gamma}'(v)u\nabla v) + \mu u(1 - \tilde{u}), & x \in \Omega, \quad t \in (0, T), \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \quad t \in (0, T), \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases}$$

where v is a solution of

$$(2.3) \quad \begin{cases} v_t - \Delta v + v = \tilde{u}, & x \in \Omega, \quad t \in (0, T), \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t \in (0, T), \\ v(x, 0) = v_0(x), & x \in \Omega, \end{cases}$$

and $\bar{\gamma}(v) \in C^3(\mathbb{R})$ satisfies

$$(2.4) \quad \bar{\gamma}(v) := \begin{cases} \gamma(0) & \text{if } v < 0, \\ \gamma(v) & \text{if } 0 \leq v \leq R, \\ \gamma(R) & \text{if } v > R. \end{cases}$$

Next, we will use the Schauder fixed point theorem to show that Φ has a fixed point u for small $T > 0$. Without confusion, in the proof of Lemma 2.1, we shall use $c_i > 0$ ($i = 1, 2, 3, \dots$) to denote a generic constant, which depends on $\|u_0\|_{L^\infty}$ and $\|v_0\|_{W^{1,\infty}}$ only. Using the standard L^p and Schauder theory of the linear parabolic equation [17], from (2.3) we have

$$(2.5) \quad \|v(\cdot, t)\|_{C^{\theta_1, \frac{\theta_1}{2}}(\bar{\Omega} \times [0, T])} \leq c_1 \text{ for some } \theta_1 \in (0, 1),$$

and $\nabla v \in L^\infty(\Omega \times (0, T))$, which entails

$$(2.6) \quad \|\bar{\gamma}'(v)\nabla v\|_{L^\infty} \leq c_2 \text{ for all } t \in (0, T).$$

From the hypothesis (H) and the definition of $\bar{\gamma}(v)$ in (2.4), we derive that $\bar{\gamma}(v)$ has a positive lower bound depending on R , which implies (2.2) is uniformly parabolic. With the fact $\bar{\gamma}'(v)\nabla v \in L^\infty(\Omega \times (0, T))$ and $\mu(1 - \tilde{u}) \in L^\infty(\Omega \times (0, T))$, we may apply the parabolic regularity results [17, Theorem V1.1] to show that

$$(2.7) \quad \|u(\cdot, t)\|_{C^{\theta_2, \frac{\theta_2}{2}}(\bar{\Omega} \times [0, T])} \leq c_3$$

for some $\theta_2 \in (0, 1)$. Using a nice idea in [26], from (2.5) and (2.7) one has

$$\|v(\cdot, t)\|_{L^\infty} \leq \|v_0\|_{L^\infty} + c_1 t^{\frac{\theta_1}{2}} \text{ for all } t \in (0, T)$$

and

$$\|u(\cdot, t)\|_{L^\infty} \leq \|u_0\|_{L^\infty} + c_3 t^{\frac{\theta_2}{2}} \quad \text{for all } t \in (0, T),$$

which implies $\|v(\cdot, t)\|_{L^\infty(\Omega \times (0, T))} \leq R$ and

$$\|u(\cdot, t)\|_{L^\infty(\Omega \times (0, T))} \leq R$$

by fixing

$$T = t_0 < \min \left\{ \left(\frac{1}{c_1} \right)^{\frac{2}{\theta_1}}, \left(\frac{1}{c_3} \right)^{\frac{2}{\theta_2}} \right\}.$$

Hence $u \in S_T$ and Φ maps S_T into itself for sufficiently small T .

Next, we show Φ is continuous and $\Phi(S_T)$ is relatively compact in X . Let $\tilde{u}_i \in X$ and $u_i = \Phi(\tilde{u}_i)$ for $i = 1, 2$, and v_i satisfy

$$(2.8) \quad \begin{cases} v_{it} - \Delta v_i + v_i = \tilde{u}_i, & x \in \Omega, \quad t \in (0, T), \\ \frac{\partial v_i}{\partial \nu} = 0, & x \in \partial\Omega, \quad t \in (0, T), \\ v_i(x, 0) = v_0(x), & x \in \Omega. \end{cases}$$

Then noting (2.7) and using the classical parabolic regularity [17] again, we obtain $u_i, v_i \in C^{2+\theta_3, 1+\frac{\theta_3}{2}}(\bar{\Omega} \times [\eta, T])$ for all $\eta \in (0, T)$ and certain $\theta_3 \in (0, 1)$. Letting $w = \Phi(\tilde{u}_1) - \Phi(\tilde{u}_2) = u_1 - u_2$, then from (2.2) one has

$$(2.9) \quad \begin{cases} w_t = \nabla \cdot (\bar{\gamma}(v_1)\nabla w) + \bar{\gamma}'(v_1)\nabla v_1 \cdot \nabla w + f_1(x, t)w + f_2(x, t), & x \in \Omega, \quad t \in (0, T), \\ \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \quad t \in (0, T), \\ w(x, 0) = 0, & x \in \Omega, \end{cases}$$

where

$$f_1(x, t) = \bar{\gamma}''(v_1)|\nabla v_1|^2 + \bar{\gamma}'(v_1)\Delta v_1 + \mu(1 - \tilde{u}_1)$$

and

$$f_2(x, t) = \nabla \cdot ([\gamma(v_1) - \gamma(v_2)]\nabla u_2) + \nabla \cdot [(\bar{\gamma}'(v_1) - \bar{\gamma}'(v_2))u_2\nabla v_1] \\ + \nabla \cdot [\bar{\gamma}'(v_2)u_2(\nabla v_1 - \nabla v_2)] + \mu u_2(\tilde{u}_2 - \tilde{u}_1).$$

Noting $f_1(x, t), \bar{\gamma}'(v_1)\nabla v_1 \in L^\infty(\Omega \times (0, T))$ due to the parabolic regularity and applying the L^p -theory to (2.9), we derive

$$(2.10) \quad \begin{aligned} \|w\|_{W^{2,1,p}(\Omega \times (0, T))} &\leq c_4 \|f_2\|_{L^p(\Omega \times (0, T))} \\ &\leq c_5 (\|v_1 - v_2\|_{W^{2,p}(\Omega \times (0, T))} + \|\tilde{u}_1 - \tilde{u}_2\|_{L^p(\Omega \times (0, T))}) \\ &\leq c_6 \|\tilde{u}_1 - \tilde{u}_2\|_{C^0(\bar{\Omega} \times [0, T])}, \end{aligned}$$

where we have used (2.8) to obtain

$$\|v_1 - v_2\|_{W^{2,p}(\Omega \times (0, T))} \leq c_7 \|\tilde{u}_1 - \tilde{u}_2\|_{L^p(\Omega \times (0, T))} \leq c_8 \|\tilde{u}_1 - \tilde{u}_2\|_{C^0(\bar{\Omega} \times [0, T])}.$$

Due to the embedding theorem, for large p , there exists some $\theta_4 \in (0, 1)$ such that

$$\|w\|_{C^0(\bar{\Omega} \times [0, T])} \leq c_9 \|w\|_{C^{\theta_4, \frac{\theta_4}{2}}(\bar{\Omega} \times [0, T])} \leq c_{10} \|w\|_{W^{2,1,p}(\Omega \times (0, T))},$$

which, combined with (2.10), gives

$$\|\Phi(\tilde{u}_1) - \Phi(\tilde{u}_2)\|_{C^0(\bar{\Omega} \times [0, T])} = \|w\|_{C^0(\bar{\Omega} \times [0, T])} \leq c_6 c_{10} \|\tilde{u}_1 - \tilde{u}_2\|_{C^0(\bar{\Omega} \times [0, T])},$$

and hence Φ is continuous. Moreover, for any $\tilde{u} \in S_T$, there exists a constant $c_{11} > 0$ such that $\|\Phi(\tilde{u})\|_{C^{\theta_5, \frac{\theta_5}{2}}(\bar{\Omega} \times [0, T])} \leq c_{11}$ for some $\theta_5 \in (0, 1)$, hence $\Phi(S_T)$ is relatively compact in X by the compact embedding theorem. Then by the Schauder fixed point theorem, Φ has a fixed point in X denoted by u . Further replacing \tilde{u} by u , we obtain the existence of solution v for (2.3). The nonnegativity of u, v directly follows from the maximum principle. Hence one has $0 \leq v \leq R$ and then $\bar{\gamma}(v) = \gamma(v)$ by (2.4), which along with the parabolic regularity theorem implies this fixed point (u, v) is a local classical solution of (1.4). The conclusion (2.1) follows from the fact that the above choice of T depends on $\|u_0\|_{L^\infty}$ and $\|v_0\|_{W^{1, \infty}}$ only.

(ii) *Uniqueness.* Proceeding as in [31, 30, 8], we shall show the uniqueness of local solutions. For given $T > 0$, suppose (u_1, v_1) and (u_2, v_2) are two solutions of (1.4) in $\Omega \times (0, T)$. We fix $T_0 \in (0, T)$ and let U and V be defined by

$$U := u_1 - u_2 \quad \text{and} \quad V := v_1 - v_2$$

and, hence,

$$(2.11) \quad U(0, x) = 0 \quad \text{and} \quad V(0, x) = 0.$$

Using the regularity of solution $(u_i, v_i) (i = 1, 2)$ and the assumptions on $\gamma(v)$ in (H), we have

$$(2.12) \quad \|u_1\|_{W^{1, \infty}} + \|u_2\|_{W^{1, \infty}} + \|v_1\|_{W^{1, \infty}} + \|v_2\|_{W^{1, \infty}} \leq c_{12}$$

and

$$(2.13) \quad \gamma(v_1) \geq c_{13} > 0, \quad |\gamma'(v_1)| + |\gamma'(v_2)| \leq c_{14}, \quad \text{and} \quad |\gamma(v_1) - \gamma(v_2)| + |\gamma'(v_1) - \gamma'(v_2)| \leq c_{15}|V|$$

for all $t \in (0, T_0)$. Moreover, from system (1.4), one has

$$(2.14) \quad \begin{aligned} U_t &= \nabla \cdot (\gamma(v_1) \nabla U) + \nabla \cdot [(\gamma(v_1) - \gamma(v_2)) \nabla u_2] \\ &\quad + \nabla \cdot [\gamma'(v_1) u_1 \nabla v_1 - \gamma'(v_2) u_2 \nabla v_2] + \mu(1 - u_1 - u_2)U \end{aligned}$$

and

$$(2.15) \quad V_t = \Delta V + U - V.$$

Multiplying (2.14) by U , and integrating it over Ω by parts, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} U^2 + \int_{\Omega} \gamma(v_1) |\nabla U|^2 &= - \int_{\Omega} (\gamma(v_1) - \gamma(v_2)) \nabla u_2 \cdot \nabla U + \mu \int_{\Omega} (1 - u_1 - u_2) U^2 \\ &\quad - \int_{\Omega} (\gamma'(v_1) u_1 \nabla v_1 - \gamma'(v_2) u_2 \nabla v_2) \cdot \nabla U \\ &= I_1 + I_2 + I_3, \end{aligned}$$

which, together with the fact $\gamma(v_1) \geq c_{13} > 0$ in (2.13), gives

$$(2.16) \quad \frac{1}{2} \frac{d}{dt} \int_{\Omega} U^2 + c_{13} \int_{\Omega} |\nabla U|^2 \leq I_1 + I_2 + I_3 \quad \text{for all } t \in (0, T).$$

Next, using the Hölder inequality, Young’s inequality, (2.12), and (2.13), for all $t \in (0, T)$ we have

$$\begin{aligned}
 (2.17) \quad I_1 &= - \int_{\Omega} (\gamma(v_1) - \gamma(v_2)) \nabla u_2 \cdot \nabla U \\
 &\leq c_{12} \int_{\Omega} |\gamma(v_1) - \gamma(v_2)| |\nabla U| \\
 &\leq c_{12} c_{15} \int_{\Omega} |V| |\nabla U| \\
 &\leq \frac{c_{13}}{8} \|\nabla U\|_{L^2}^2 + \frac{2c_{12}^2 c_{15}^2}{c_{13}} \|V\|_{L^2}^2
 \end{aligned}$$

and

$$(2.18) \quad I_2 = \mu \int_{\Omega} (1 - u_1 - u_2) U^2 \leq \mu(1 + \|u_1\|_{L^\infty} + \|u_2\|_{L^\infty}) \|U\|_{L^2}^2 \leq \mu(1 + c_{12}) \|U\|_{L^2}^2$$

as well as

$$\begin{aligned}
 (2.19) \quad I_3 &= - \int_{\Omega} (\gamma'(v_1) u_1 \nabla v_1 - \gamma'(v_2) u_2 \nabla v_2) \cdot \nabla U \\
 &= - \int_{\Omega} \gamma'(v_1) u_1 \nabla V \cdot \nabla U - \int_{\Omega} (\gamma'(v_1) - \gamma'(v_2)) u_1 \nabla v_2 \cdot \nabla U - \int_{\Omega} \gamma'(v_2) U \nabla v_2 \cdot \nabla U \\
 &\leq c_{12} c_{14} \int_{\Omega} |\nabla V| |\nabla U| + c_{12}^2 c_{15} \int_{\Omega} |V| |\nabla U| + c_{12} c_{14} \int_{\Omega} |U| |\nabla U| \\
 &\leq \frac{3c_{13}}{8} \|\nabla U\|_{L^2}^2 + \frac{2c_{12}^2 c_{14}^2}{c_{13}} (\|U\|_{L^2}^2 + \|\nabla V\|_{L^2}^2) + \frac{2c_{12}^4 c_{15}^2}{c_{13}} \|V\|_{L^2}^2.
 \end{aligned}$$

Substituting (2.17)–(2.19) into (2.16), then for all $t \in (0, T)$ one has

$$(2.20) \quad \frac{d}{dt} \|U\|_{L^2}^2 + c_{13} \|\nabla U\|_{L^2}^2 \leq \frac{4c_{12}^2 c_{14}^2 + 2\mu(1 + c_{12})c_{13}}{c_{13}} (\|U\|_{L^2}^2 + \|\nabla V\|_{L^2}^2) + \frac{4c_{12}^2 c_{15}^2 (c_{12}^2 + 1)}{c_{13}} \|V\|_{L^2}^2.$$

Testing (2.15) against V_t , we obtain

$$\frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} |V|^2 + \int_{\Omega} |\nabla V|^2 \right) + \int_{\Omega} V_t^2 \leq \frac{1}{2} \int_{\Omega} V_t^2 + \frac{1}{2} \int_{\Omega} U^2 \text{ for all } t \in (0, T_0)$$

and, hence,

$$(2.21) \quad \frac{d}{dt} (\|V\|_{L^2}^2 + \|\nabla V\|_{L^2}^2) + \|V_t\|_{L^2}^2 \leq \|U\|_{L^2}^2 \text{ for all } t \in (0, T_0).$$

Combining (2.20) and (2.21) and defining $c_{16} := \frac{4c_{12}^2 c_{14}^2 + 2\mu(1 + c_{12})c_{13} + c_{13}}{c_{13}} + \frac{4c_{12}^2 c_{15}^2 (c_{12}^2 + 1)}{c_{13}}$, one has

$$\frac{d}{dt} (\|U\|_{L^2}^2 + \|V\|_{L^2}^2 + \|\nabla V\|_{L^2}^2) \leq c_{16} (\|U\|_{L^2}^2 + \|V\|_{L^2}^2 + \|\nabla V\|_{L^2}^2) \text{ for all } t \in (0, T_0),$$

which, together with Gronwall's inequality and (2.11), implies $U \equiv 0$ and $V \equiv 0$ in $(0, T)$ for $T_0 \in (0, T)$ is arbitrary. This proves the uniqueness of solutions and completes the proof. \square

Next, we prove some basic properties of solutions to the system (1.4).

LEMMA 2.2. *Let (u, v) be the solution of system (1.4). Then there exist two positive constants m_* and C such that*

$$(2.22) \quad \int_{\Omega} u \leq m_* := \max \left\{ \int_{\Omega} u_0, |\Omega| \right\} \quad \text{for all } t \in (0, T_{max}),$$

and

$$(2.23) \quad \int_t^{t+\tau} \int_{\Omega} u^2 \leq C \quad \text{for all } t \in (0, \tilde{T}_{max}),$$

where

$$(2.24) \quad \tau := \min \left\{ 1, \frac{1}{2} T_{max} \right\} \quad \text{and} \quad \tilde{T}_{max} := \begin{cases} T_{max} - \tau & \text{if } T_{max} < \infty, \\ \infty & \text{if } T_{max} = \infty. \end{cases}$$

Proof. Integrating the first equation of system (1.4) over Ω , we have

$$(2.25) \quad \frac{d}{dt} \int_{\Omega} u + \mu \int_{\Omega} u^2 = \mu \int_{\Omega} u \quad \text{for all } t \in (0, T_{max}).$$

On the other hand, using the Cauchy–Schwarz inequality, we have $\int_{\Omega} u^2 \geq \frac{1}{|\Omega|} \left(\int_{\Omega} u \right)^2$, which combined with (2.25) gives

$$(2.26) \quad \frac{d}{dt} \int_{\Omega} u \leq \mu \int_{\Omega} u - \frac{\mu}{|\Omega|} \left(\int_{\Omega} u \right)^2 \quad \text{for all } t \in (0, T_{max}).$$

Then applying the ODE comparison to (2.26), we obtain (2.22). Integrating (2.25) over $(t, t + \tau)$ and using (2.22), one has (2.23). \square

Then we can deduce the following result as a consequence of Lemma 2.2.

LEMMA 2.3. *Let (u, v) be the solution of system (1.4). Then there exists a constant $C > 0$ independent of t such that*

$$(2.27) \quad \|\nabla v\|_{L^2} \leq C \quad \text{for all } t \in (0, T_{max})$$

and

$$(2.28) \quad \int_t^{t+\tau} \int_{\Omega} |\Delta v|^2 \leq C \quad \text{for all } t \in (0, \tilde{T}_{max}),$$

where τ and \tilde{T}_{max} are defined by (2.24).

Proof. We multiply the second equation of system (1.4) by $-\Delta v$, and integrate the result by parts to have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla v|^2 + \int_{\Omega} |\Delta v|^2 + \int_{\Omega} |\nabla v|^2 &= - \int_{\Omega} u \Delta v \\ &\leq \frac{1}{2} \int_{\Omega} |\Delta v|^2 + \frac{1}{2} \int_{\Omega} u^2 \quad \text{for all } t \in (0, T_{max}), \end{aligned}$$

which yields

$$(2.29) \quad \frac{d}{dt} \int_{\Omega} |\nabla v|^2 + \int_{\Omega} |\Delta v|^2 + 2 \int_{\Omega} |\nabla v|^2 \leq \int_{\Omega} u^2 \text{ for all } t \in (0, T_{max}).$$

Multiplying (2.29) by μ , and adding the result to (2.25), then for all $t \in (0, T_{max})$ one has

$$(2.30) \quad \frac{d}{dt} \left(\int_{\Omega} u + \mu \int_{\Omega} |\nabla v|^2 \right) + \int_{\Omega} u + 2\mu \int_{\Omega} |\nabla v|^2 + \mu \int_{\Omega} |\Delta v|^2 \leq (\mu + 1)m_*.$$

Let $y(t) := \int_{\Omega} u + \mu \int_{\Omega} |\nabla v|^2$. Then we obtain from (2.30) that

$$y'(t) + y(t) \leq (\mu + 1)m_* \text{ for all } t \in (0, T_{max}),$$

which, applied to Gronwall's inequality, gives (2.27). Furthermore, (2.28) is obtained by integrating (2.30) over $(t, t + \tau)$. \square

LEMMA 2.4 (see [12]). *Let Ω be a bounded domain in \mathbb{R}^n with smooth boundary. Let $T \in (0, \infty]$ and suppose that $z \in C^0(\bar{\Omega} \times [0, T]) \cap C^{2,1}(\bar{\Omega} \times (0, T))$ is a solution of*

$$\begin{cases} z_t = \Delta z - z + g, & x \in \Omega, t \in (0, T), \\ \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0, T), \end{cases}$$

where $g \in L^\infty((0, T); L^p(\Omega))$. Then there exists a constant $C > 0$ such that

$$\|z(\cdot, t)\|_{W^{1,r}} \leq C$$

with

$$r \in \begin{cases} [1, \frac{np}{n-p}) & \text{if } p \leq n, \\ [1, \infty] & \text{if } p > n. \end{cases}$$

LEMMA 2.5. *Let Ω be a bounded domain in \mathbb{R}^2 with smooth boundary. Then, for any $\varphi \in W^{2,2}(\Omega)$ satisfying $\frac{\partial \varphi}{\partial \nu}|_{\partial\Omega} = 0$, there exists a positive constant C depending only on Ω such that*

$$(2.31) \quad \|\nabla \varphi\|_{L^4} \leq C(\|\Delta \varphi\|_{L^2}^{\frac{1}{2}} \|\nabla \varphi\|_{L^2}^{\frac{1}{2}} + \|\nabla \varphi\|_{L^2}).$$

Proof. Using the Gagliardo–Nirenberg inequality, we have

$$(2.32) \quad \|\nabla \varphi\|_{L^4} \leq c_1 \|D\nabla \varphi\|_{L^2}^{\frac{1}{2}} \|\nabla \varphi\|_{L^2}^{\frac{1}{2}} + c_2 \|\nabla \varphi\|_{L^2},$$

where $|D\nabla \varphi| = (\sum_{|i|=1} |D^i \nabla \varphi|^2)^{\frac{1}{2}}$ and i is a multi-index of orders. On the other hand, it holds that

$$(2.33) \quad \|D\nabla \varphi\|_{L^2} \leq c_3 \|\nabla \varphi\|_{H^1}.$$

Under the homogeneous Neumann boundary condition, $\frac{\partial \varphi}{\partial \nu}|_{\partial\Omega} = 0$, it follows from [5, Lemma 1] that $\|\nabla \varphi\|_{H^1} \leq c_4 \|\Delta \varphi\|_{L^2}$, which applied to (2.33) gives

$$(2.34) \quad \|D\nabla \varphi\|_{L^2} \leq c_3 c_4 \|\Delta \varphi\|_{L^2}.$$

Then substituting (2.34) into (2.32), one gets (2.31) directly. \square

3. Boundedness of solutions. In this section, we are devoted to studying the existence of global classical solutions for system (1.4). To extend the local solution established in Lemma 2.1 to a global one, it suffices to derive some a priori estimates. The conventional method for chemotaxis models with linear diffusion is to use the logistic damping and diffusive dissipation to control the chemotactic advection to get the boundedness of an entropy-like term $\int_{\Omega} |u \ln u|$ based on which boundedness of $\|u(\cdot, t)\|_{L^2}$ can be attained. Then one can proceed to use a bootstrap argument (e.g., see [16]) or boundedness criterion (cf. [4]) to derive the boundedness of $\|u(\cdot, t)\|_{L^\infty}$. However, this approach is not applicable to the current system (1.4) directly since the advantage of diffusive dissipation is lost due to the possible degeneracy of diffusion. To overcome this major obstacle, we employ the L^2 energy estimate directly by treating $\gamma(v)$ as a weight function to gain a weighted diffusive dissipation. This enables us to control the chemotactic advection term and derive a Gronwall-type inequality:

$$\frac{d}{dt} \|u\|_{L^2}^2 \leq c_1 \|u\|_{L^2}^2 \|\Delta v\|_{L^2}^2 + c_2,$$

which yields the uniform-in-time bound of $\|u(\cdot, t)\|_{L^2}$ by using the boundedness of $\int_t^{t+\tau} \int_{\Omega} u^2$ and $\int_t^{t+\tau} \int_{\Omega} |\Delta v|^2$ proved in Lemma 2.2 and Lemma 2.3. Once the bound of $\|u(\cdot, t)\|_{L^2}$ is attained, one can show $\|v(\cdot, t)\|_{L^\infty}$ is uniformly bounded based on the parabolic regularity (see Lemma 2.4), which rules out the possibility of degeneracy. After that we use the bootstrap argument the same as in the chemotaxis model with linear diffusion to obtain the boundedness of $\|u(\cdot, t)\|_{L^\infty}$, which along with Lemma 2.1 proves the existence of bounded global classical solutions to (1.4).

3.1. L^2 -estimate. In this subsection, we will show the boundedness of $\|u(\cdot, t)\|_{L^2}$.

LEMMA 3.1. *Let Ω be a bounded domain in \mathbb{R}^2 with smooth boundary and the hypothesis (H) holds. If (u, v) is a solution of system (1.4), then there exists a constant $C > 0$ such that*

$$(3.1) \quad \|u(\cdot, t)\|_{L^2} \leq C \quad \text{for all } t \in (0, T_{max}).$$

Proof. Multiplying the first equation of system (1.4) by u and integrating the result by parts, and using the Hölder inequality and Young's inequality, we end up with

$$(3.2) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 + \int_{\Omega} \gamma(v) |\nabla u|^2 + \mu \int_{\Omega} u^3 \\ &= - \int_{\Omega} \gamma'(v) u \nabla u \cdot \nabla v + \mu \int_{\Omega} u^2 \\ &\leq \left(\int_{\Omega} \gamma(v) |\nabla u|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega} \frac{|\gamma'(v)|^2}{\gamma(v)} u^2 |\nabla v|^2 \right)^{\frac{1}{2}} + \mu |\Omega|^{\frac{1}{3}} \left(\int_{\Omega} u^3 \right)^{\frac{2}{3}} \\ &\leq \frac{1}{2} \int_{\Omega} \gamma(v) |\nabla u|^2 + \frac{1}{2} \int_{\Omega} \frac{|\gamma'(v)|^2}{\gamma(v)} u^2 |\nabla v|^2 + \frac{\mu}{2} \int_{\Omega} u^3 + \frac{16\mu|\Omega|}{27} \quad \text{for all } t \in (0, T_{max}), \end{aligned}$$

which gives

$$(3.3) \quad \frac{d}{dt} \int_{\Omega} u^2 + \int_{\Omega} \gamma(v) |\nabla u|^2 + \mu \int_{\Omega} u^3 \leq \int_{\Omega} \frac{|\gamma'(v)|^2}{\gamma(v)} u^2 |\nabla v|^2 + c_1 \quad \text{for all } t \in (0, T_{max}),$$

where $c_1 := \frac{32\mu|\Omega|}{27}$. On the other hand, we have

$$\gamma^{\frac{1}{2}}(v)\nabla u = \nabla(\gamma^{\frac{1}{2}}(v)u) - \frac{1}{2} \frac{\gamma'(v)}{\gamma^{\frac{1}{2}}(v)}u\nabla v \text{ for all } t \in (0, T_{max}),$$

which, combined with the fact $|X - Y|^2 \geq \frac{1}{2}X^2 - Y^2$, gives

$$\begin{aligned} (3.4) \quad \gamma(v)|\nabla u|^2 &= |\gamma^{\frac{1}{2}}(v)\nabla u|^2 = \left| \nabla(\gamma^{\frac{1}{2}}(v)u) - \frac{1}{2} \frac{\gamma'(v)}{\gamma^{\frac{1}{2}}(v)}u\nabla v \right|^2 \\ &\geq \frac{1}{2}|\nabla(\gamma^{\frac{1}{2}}(v)u)|^2 - \frac{1}{4} \frac{|\gamma'(v)|^2}{\gamma(v)}u^2|\nabla v|^2 \text{ for all } t \in (0, T_{max}). \end{aligned}$$

Substituting (3.4) into (3.3) gives

$$(3.5) \quad \frac{d}{dt} \int_{\Omega} u^2 + \frac{1}{2} \int_{\Omega} |\nabla(\gamma^{\frac{1}{2}}(v)u)|^2 + \mu \int_{\Omega} u^3 \leq \frac{5}{4} \int_{\Omega} \frac{|\gamma'(v)|^2}{\gamma(v)}u^2|\nabla v|^2 + c_1 \text{ for all } t \in (0, T_{max}).$$

From hypothesis (H), we can find a constant $K_1 > 0$ such that

$$(3.6) \quad \frac{|\gamma'(v)|}{\gamma(v)} \leq K_1 \text{ for all } v \geq 0 \text{ and } t \in (0, T_{max}).$$

Using (3.6) and the Hölder inequality, we derive from (3.5)

$$\begin{aligned} (3.7) \quad &\frac{d}{dt} \int_{\Omega} u^2 + \frac{1}{2} \int_{\Omega} |\nabla(\gamma^{\frac{1}{2}}(v)u)|^2 + \mu \int_{\Omega} u^3 \\ &\leq \frac{5}{4} \int_{\Omega} \frac{|\gamma'(v)|^2}{\gamma^2(v)}(\gamma^{\frac{1}{2}}(v)u)^2|\nabla v|^2 + c_1 \\ &\leq \frac{5K_1^2}{4} \int_{\Omega} |\gamma^{\frac{1}{2}}(v)u|^2|\nabla v|^2 + c_1 \\ &\leq \frac{5K_1^2}{4} \left(\int_{\Omega} |\gamma^{\frac{1}{2}}(v)u|^4 \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla v|^4 \right)^{\frac{1}{2}} + c_1 \text{ for all } t \in (0, T_{max}). \end{aligned}$$

On one hand, the Gagliardo–Nirenberg inequality along with the fact $\gamma(v) \leq \gamma(0) = c_2$ due to $\gamma(v) \in C^3([0, \infty))$ gives

$$\begin{aligned} (3.8) \quad &\left(\int_{\Omega} |\gamma^{\frac{1}{2}}(v)u|^4 \right)^{\frac{1}{2}} = \|\gamma^{\frac{1}{2}}(v)u\|_{L^4}^2 \\ &\leq c_3 \left(\|\nabla(\gamma^{\frac{1}{2}}(v)u)\|_{L^2} \|\gamma^{\frac{1}{2}}(v)u\|_{L^2} + \|\gamma^{\frac{1}{2}}(v)u\|_{L^2}^2 \right) \\ &\leq c_4 (\|\nabla(\gamma^{\frac{1}{2}}(v)u)\|_{L^2} \|u\|_{L^2} + \|u\|_{L^2}^2) \text{ for all } t \in (0, T_{max}), \end{aligned}$$

where $c_4 := (c_2^{\frac{1}{2}} + c_2)c_3$. Using Lemma 2.5 and the fact $\|\nabla v\|_{L^2} \leq c_5$ (see (2.27)), we obtain

$$\begin{aligned} (3.9) \quad &\left(\int_{\Omega} |\nabla v|^4 \right)^{\frac{1}{2}} = \|\nabla v\|_{L^4}^2 \leq c_6 (\|\Delta v\|_{L^2} \|\nabla v\|_{L^2} + \|\nabla v\|_{L^2}^2) \\ &\leq c_7 (\|\Delta v\|_{L^2} + 1) \text{ for all } t \in (0, T_{max}) \end{aligned}$$

with $c_7 := (c_5 + c_5^2)c_6$. Combining (3.8) and (3.9) and using Young’s inequality, one derives

$$\begin{aligned}
 & \frac{5K_1^2}{4} \left(\int_{\Omega} |\gamma^{\frac{1}{2}}(v)u|^4 \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla v|^4 \right)^{\frac{1}{2}} \\
 & \leq \frac{5K_1^2 c_4 c_7}{4} \left(\|\nabla(\gamma^{\frac{1}{2}}(v)u)\|_{L^2} \|u\|_{L^2} + \|u\|_{L^2}^2 \right) (\|\Delta v\|_{L^2} + 1) \\
 (3.10) \quad & = \frac{5K_1^2 c_4 c_7}{4} \|\nabla(\gamma^{\frac{1}{2}}(v)u)\|_{L^2} \|u\|_{L^2} \|\Delta v\|_{L^2} + \frac{5K_1^2 c_4 c_7}{4} \|\nabla(\gamma^{\frac{1}{2}}(v)u)\|_{L^2} \|u\|_{L^2} \\
 & \quad + \frac{5K_1^2 c_4 c_7}{4} \|u\|_{L^2}^2 \|\Delta v\|_{L^2} + \frac{5K_1^2 c_4 c_7}{4} \|u\|_{L^2}^2 \\
 & \leq \frac{1}{2} \|\nabla(\gamma^{\frac{1}{2}}(v)u)\|_{L^2}^2 + c_8 \|u\|_{L^2}^2 + c_9 \|u\|_{L^2}^2 \|\Delta v\|_{L^2}^2 \quad \text{for all } t \in (0, T_{max}),
 \end{aligned}$$

where $c_8 := \frac{(5K_1^2 c_4 c_7 + 2)^2}{16}$ and $c_9 := \frac{25K_1^4 c_4^2 c_7^2}{8}$. Substituting (3.10) into (3.7), using the estimate $c_8 \|u\|_{L^2}^2 \leq \mu \int_{\Omega} u^3 + \frac{4c_8^3 |\Omega|}{27\mu^2}$ and letting $c_{10} := c_1 + \frac{4c_8^3 |\Omega|}{27\mu^2}$, we obtain

$$(3.11) \quad \frac{d}{dt} \|u\|_{L^2}^2 \leq c_9 \|u\|_{L^2}^2 \|\Delta v\|_{L^2}^2 + c_{10} \quad \text{for all } t \in (0, T_{max}).$$

Next, we shall use (2.23) and (2.28) to obtain (3.1) based on the inequality (3.11). In fact, for any $t \in (0, T_{max})$ and in both cases $t \in (0, \tau)$ and $t \geq \tau$ with $\tau = \min\{1, \frac{1}{2}T_{max}\}$, from (2.23) we can find $t_0 = t_0(t) \in ((t - \tau)_+, t)$ such that $t_0 \geq 0$ and

$$(3.12) \quad \int_{\Omega} u^2(x, t_0) \leq c_{11}.$$

On the other hand, from (2.28) in Lemma 2.3, we can find a constant $c_{12} > 0$ such that

$$(3.13) \quad \int_{t_0}^{t_0 + \tau} \int_{\Omega} |\Delta v(x, s)|^2 \leq c_{12} \quad \text{for all } t_0 \in (0, \tilde{T}_{max}).$$

Then integrating (3.11) over (t_0, t) , and using (3.12) and (3.13) and noting that $t \leq t_0 + \tau \leq t_0 + 1$, we derive

$$\begin{aligned}
 \|u(\cdot, t)\|_{L^2}^2 & \leq \|u(\cdot, t_0)\|_{L^2}^2 \cdot e^{c_9 \int_{t_0}^t \|\Delta v(\cdot, s)\|_{L^2}^2 ds} + c_{10} \int_{t_0}^t e^{c_9 \int_s^t \|\Delta v(\cdot, \sigma)\|_{L^2}^2 d\sigma} ds \\
 & \leq c_{11} e^{c_9 c_{12}} + c_{10} e^{c_9 c_{12}} \quad \text{for all } t \in (0, T_{max}),
 \end{aligned}$$

which yields (3.1) and thereby completes the proof. □

3.2. L^∞ -estimate and global existence. In this subsection, we will derive the boundedness of $\|u(\cdot, t)\|_{L^\infty}$. To this end, we first show the uniform-in-time bound of $\|v(\cdot, t)\|_{L^\infty}$ to exclude the possibility of degeneracy. Then based on the boundedness of $\|u(\cdot, t)\|_{L^2}$, we can use a similar argument as in Lemma 3.1 to derive a bound on u in $L^\infty((0, T_{max}); L^p(\Omega))$ for arbitrary $p \geq 2$, which implies the boundedness of $\|\nabla v(\cdot, t)\|_{L^\infty}$ due to Lemma 2.4. With this in hand, the boundedness of $\|u(\cdot, t)\|_{L^\infty}$ can be established by the well-known Moser iteration procedure (cf. [1] or [25, 39, 27]).

LEMMA 3.2. *Suppose that the conditions in Lemma 3.1 hold. Then the solution of system (1.4) satisfies*

$$(3.14) \quad \|u(\cdot, t)\|_{L^\infty} \leq C \quad \text{for all } t \in (0, T_{max}),$$

where the constant $C > 0$ independent of t .

Proof. Applying Lemma 2.4 and noting the fact $\|u(\cdot, t)\|_{L^2} \leq c_1$ in Lemma 3.1, from the second equation of (1.4) we have

$$(3.15) \quad \|v(\cdot, t)\|_{W^{1,4}} \leq c_2 \quad \text{for all } t \in (0, T_{max}),$$

which, combined with the Sobolev inequality, gives

$$(3.16) \quad \|v(\cdot, t)\|_{L^\infty} \leq K_* \quad \text{for all } t \in (0, T_{max}).$$

Using the hypothesis (H) and (3.16), we obtain

$$(3.17) \quad \gamma(v) \geq \gamma(K_*) > 0 \quad \text{and} \quad |\gamma'(v)| \leq c_3 \quad \text{for all } t \in (0, T_{max}).$$

Next, using u^{p-1} with $p \geq 2$ as a test function for the first equation in (1.4), and integrating the resulting equation by parts, we obtain

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p + (p-1) \int_{\Omega} \gamma(v) u^{p-2} |\nabla u|^2 + \mu \int_{\Omega} u^{p+1} \\ &= -(p-1) \int_{\Omega} \gamma'(v) u^{p-1} \nabla u \cdot \nabla v + \mu \int_{\Omega} u^p \quad \text{for all } t \in (0, T_{max}), \end{aligned}$$

which, together with (3.17), the Hölder inequality, and Young's inequality, gives

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p + (p-1) \gamma(K_*) \int_{\Omega} u^{p-2} |\nabla u|^2 + \mu \int_{\Omega} u^p + \mu \int_{\Omega} u^{p+1} \\ (3.18) \quad & \leq c_3 (p-1) \int_{\Omega} u^{p-1} |\nabla u| |\nabla v| + 2\mu \int_{\Omega} u^p \\ & \leq \frac{(p-1)\gamma(K_*)}{2} \int_{\Omega} u^{p-2} |\nabla u|^2 + \frac{c_3^2(p-1)}{2\gamma(K_*)} \int_{\Omega} u^p |\nabla v|^2 + 2\mu \int_{\Omega} u^p \\ & \leq \frac{(p-1)\gamma(K_*)}{2} \int_{\Omega} u^{p-2} |\nabla u|^2 + \frac{c_3^2(p-1)}{2\gamma(K_*)} \int_{\Omega} u^p |\nabla v|^2 + \mu \int_{\Omega} u^{p+1} + c_4 \end{aligned}$$

for all $t \in (0, T_{max})$ and for all $p \geq 2$, where $c_4 := (\frac{2p}{p+1})^p \cdot \frac{2\mu|\Omega|}{p+1}$. This along with the fact $\frac{p(p-1)\gamma(K_*)}{2} \int_{\Omega} u^{p-2} |\nabla u|^2 = \frac{2(p-1)\gamma(K_*)}{p} \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2$ implies

$$(3.19) \quad \frac{d}{dt} \int_{\Omega} u^p + p\mu \int_{\Omega} u^p + \frac{2(p-1)\gamma(K_*)}{p} \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 \leq \frac{c_3^2 p(p-1)}{2\gamma(K_*)} \int_{\Omega} u^p |\nabla v|^2 + c_4 p$$

for all $t \in (0, T_{max})$. Using the Hölder inequality and the Gagliardo–Nirenberg inequality, and noting (3.15) and the fact $\|u^{\frac{p}{2}}(\cdot, t)\|_{L^{\frac{4}{p}}} = \|u(\cdot, t)\|_{L^2}^{\frac{p}{2}} \leq c_1^{\frac{p}{2}}$, then for all $t \in (0, T_{max})$ one has

$$\begin{aligned} (3.20) \quad & \frac{c_3^2 p(p-1)}{2\gamma(K_*)} \int_{\Omega} u^p |\nabla v|^2 \leq \frac{c_3^2 p(p-1)}{2\gamma(K_*)} \left(\int_{\Omega} u^{2p} \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla v|^4 \right)^{\frac{1}{2}} \\ & \leq \frac{c_2^2 c_3^2 p(p-1)}{2\gamma(K_*)} \|u^{\frac{p}{2}}\|_{L^4}^2 \\ & \leq c_5 (\|\nabla u^{\frac{p}{2}}\|_{L^2}^{2(1-\frac{1}{p})} \|u^{\frac{p}{2}}\|_{L^{\frac{4}{p}}}^{\frac{2}{p}} + \|u^{\frac{p}{2}}\|_{L^{\frac{4}{p}}}^2) \\ & \leq c_5 c_1 \|\nabla u^{\frac{p}{2}}\|_{L^2}^{2(1-\frac{1}{p})} + c_5 c_1^p \\ & \leq \frac{2(p-1)\gamma(K_*)}{p} \|\nabla u^{\frac{p}{2}}\|_{L^2}^2 + \frac{2\gamma(K_*)}{p} \left(\frac{c_1 c_5}{2\gamma(K_*)} \right)^p + c_5 c_1^p. \end{aligned}$$

Substituting (3.20) into (3.19), and letting $c_6 := c_4 p + \frac{2\gamma(K_*)}{p} \left(\frac{c_1 c_5}{2\gamma(K_*)} \right)^p + c_5 c_1^p$, we obtain

$$\frac{d}{dt} \int_{\Omega} u^p + p\mu \int_{\Omega} u^p \leq c_6 \quad \text{for all } t \in (0, T_{max}),$$

which, combined with Gronwall's inequality, yields

$$(3.21) \quad \|u(\cdot, t)\|_{L^p}^p \leq e^{-p\mu t} \|u_0\|_{L^p}^p + \frac{c_6}{p\mu} (1 - e^{-p\mu t}) \leq \|u_0\|_{L^p}^p + \frac{c_6}{p\mu} \quad \text{for all } t \in (0, T_{max}).$$

Then choosing $p = 4$ in (3.21) and using Lemma 2.4, one can find a constant $c_7 > 0$ independent of p such that $\|\nabla v(\cdot, t)\|_{L^\infty} \leq c_7$. Then applying the Moser iteration procedure (cf. [1] or [25, 27, 39]), one has (3.14). Hence the proof of Lemma 3.2 is completed. \square

Hence, we obtain the following results on the global existence of solutions.

LEMMA 3.3. *Let Ω be a bounded domain in \mathbb{R}^2 with smooth boundary and the hypothesis (H) holds. Assume $(u_0, v_0) \in [W^{1,\infty}(\Omega)]^2$ with $u_0, v_0 \geq 0$ ($\neq 0$). Then the problem (1.4) has a unique global classical solution*

$$(u, v) \in [C^0([0, \infty) \times \bar{\Omega}) \cap C^{2,1}((0, \infty) \times \bar{\Omega}) \cap L_{loc}^\infty([0, \infty), W^{1,\infty}(\Omega))]^2$$

satisfying (1.6).

Proof. From Lemma 3.2 and Lemma 2.4, we have $\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{W^{1,\infty}} \leq c_1$. Then using Lemma 2.1, we obtain Lemma 3.3 directly. \square

4. Large time behavior. For the chemotaxis-growth model (1.3) there exists a constant $\mu_* > 0$ such that if $\mu > \mu_*$, the solutions will converge to the nontrivial spatially homogeneous equilibria [19, 34, 3]. The system (1.4) captures some similar features of the chemotaxis-growth model (1.3). Based on some ideas in [3, 28], below we will exploit the large time behavior of solutions for system (1.4) by constructing a Lyapunov functional. Suppose that

$$(4.1) \quad K_0 = \max_{0 \leq v \leq \infty} \frac{|\gamma'(v)|^2}{\gamma(v)}.$$

Note that the finiteness of K_0 has been guaranteed by hypothesis (H). Next we shall show that if $\mu > \frac{K_0}{16}$, the constant steady state $(1, 1)$ is globally asymptotically stable with L^∞ -norm, which implies the system (1.4) has no pattern formation for large $\mu > 0$.

4.1. Convergence of solutions. Motivated by some ideas from [28, 9], we will construct a Lyapunov functional to study the convergence of solutions.

LEMMA 4.1. *Let (u, v) be the solution of system (1.4) obtained in Lemma 3.3 and K_0 defined by (4.1). Suppose that*

$$(4.2) \quad \mu > \frac{K_0}{16},$$

then there exist two positive constants δ, α_0 such that for all $t > 0$, the nonnegative function

$$(4.3) \quad \mathcal{E}(t) := \int_{\Omega} (u - 1 - \ln u) + \frac{\delta}{2} \int_{\Omega} (v - 1)^2$$

satisfies

$$(4.4) \quad \mathcal{E}'(t) \leq -\mathcal{F}(t) \quad \text{for all } t \in (0, T_{max}),$$

where

$$\mathcal{F}(t) := \alpha_0 \cdot \left\{ \int_{\Omega} (u - 1)^2 + \int_{\Omega} (v - 1)^2 \right\}.$$

Proof. Multiplying the first equation of system (1.4) by $\frac{u-1}{u}$ and integrating the result over Ω , we have

$$(4.5) \quad \begin{aligned} \frac{d}{dt} \int_{\Omega} (u - 1 - \ln u) &= - \int_{\Omega} \nabla \left(\frac{u-1}{u} \right) \cdot [\gamma(v)\nabla u + \gamma'(v)u\nabla v] - \mu \int_{\Omega} (u - 1)^2 \\ &= - \int_{\Omega} \gamma(v) \frac{|\nabla u|^2}{u^2} - \int_{\Omega} \gamma'(v) \frac{\nabla u \cdot \nabla v}{u} - \mu \int_{\Omega} (u - 1)^2 \end{aligned}$$

for all $t \in (0, T_{max})$. We proceed to multiply the second equation of system (1.4) by $v - 1$ and integrate the result to obtain

$$(4.6) \quad \frac{1}{2} \frac{d}{dt} \int_{\Omega} (v-1)^2 = - \int_{\Omega} |\nabla v|^2 - \int_{\Omega} (v-1)^2 + \int_{\Omega} (u-1)(v-1) \quad \text{for all } t \in (0, T_{max}).$$

Now multiplying (4.6) by a positive constant δ to be determined later and adding the result to (4.5), for all $t \in (0, T_{max})$ we get

$$(4.7) \quad \begin{aligned} &\frac{d}{dt} \int_{\Omega} \left[(u - 1 - \ln u) + \frac{\delta}{2}(v - 1)^2 \right] \\ &= - \int_{\Omega} \left(\gamma(v) \frac{|\nabla u|^2}{u^2} + \delta |\nabla v|^2 + \gamma'(v) \frac{\nabla u \cdot \nabla v}{u} \right) \\ &\quad + \int_{\Omega} [-\mu(u - 1)^2 - \delta(v - 1)^2 + \delta(u - 1)(v - 1)] \\ &=: J_1 + J_2. \end{aligned}$$

For J_1 , we can rewrite it as

$$J_1 = -\Theta_1^T A_1 \Theta_1, \quad \Theta_1 = \begin{bmatrix} \nabla u \\ \nabla v \end{bmatrix}, \quad A_1 = \begin{bmatrix} \frac{\gamma(v)}{u^2} & \frac{\gamma'(v)}{2u} \\ \frac{\gamma'(v)}{2u} & \delta \end{bmatrix},$$

where Θ_1^T denotes the transpose of Θ_1 . One can check that A_1 is nonnegative definite and, hence, $J_1 \leq 0$ if and only if

$$(4.8) \quad \delta \geq \max_{0 \leq v \leq \infty} \frac{|\gamma'(v)|^2}{4\gamma(v)} = \frac{K_0}{4}.$$

Similarly, we can rearrange J_2 as

$$J_2 = -\Theta_2^T A_2 \Theta_2, \quad \Theta_2 = \begin{bmatrix} u - 1 \\ v - 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} \mu & -\frac{\delta}{2} \\ -\frac{\delta}{2} & \delta \end{bmatrix}.$$

Hence J_2 is positive definite if and only if

$$(4.9) \quad \mu > \frac{\delta}{4}.$$

Therefore under the assumption (4.2), we can find a positive constant δ fulfilling (4.8) and thus (4.9) hold. Since A_1 is nonnegative definite and A_2 is positive definite, using (4.7) and the definition of $\mathcal{E}(t)$ defined in (4.3), we find a constant $\alpha_0 > 0$ such that (4.4) holds.

Finally we show the nonnegativity of $\mathcal{E}(t)$. In fact, let $\phi(u) := u - 1 - \ln u, u > 0$. Then it is easy to check that $\phi'(u) = 1 - \frac{1}{u}$ and $\phi''(u) = \frac{1}{u^2} > 0$ for all $u > 0$, and, hence, $\phi(1) = \phi'(1) = 0$. So it follows that $\min_{u>0} \phi(u) = \phi(1) = 0$, which implies $\mathcal{E}(t) \geq 0$. The proof is completed. \square

LEMMA 4.2. *Let (u, v) be the global classical solution of the system (1.4) and $\mu > \frac{K_0}{16}$. Then the following asymptotics hold:*

$$(4.10) \quad \|u(\cdot, t) - 1\|_{L^\infty} \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

and

$$(4.11) \quad \|v(\cdot, t) - 1\|_{L^\infty} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Proof. Notice that (4.10) follows directly, if one can show

$$(4.12) \quad \|u(\cdot, t) - 1\|_{C^0} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Next, we will prove (4.12) by the argument of contradiction inspired by an idea in the proof of [28, Lemma 3.10]. In fact, if (4.12) is false, then for some $c_1 > 0$, there exist some sequences $(x_j)_{j \in \mathbb{N}} \subset \Omega$ and $(t_j)_{j \in \mathbb{N}} \subset (0, \infty)$ satisfying $t_j \rightarrow \infty$ as $j \rightarrow \infty$ such that

$$|u(x_j, t_j) - 1| \geq c_1 \quad \text{for all } j \in \mathbb{N}.$$

On the other hand, from Lemma 3.3 and the parabolic regularity theory (e.g., see [23, Theorem 1.3] or [28, Lemma 3.2]) we can show that

$$\|u\|_{C^{\sigma, \frac{\sigma}{2}}(\bar{\Omega} \times [t, t+1])} + \|v\|_{C^{2+\sigma, 1+\frac{\sigma}{2}}(\bar{\Omega} \times [t, t+1])} \leq C$$

for all $t \geq 1$ and some $\sigma \in (0, 1)$. Hence $u - 1$ is uniformly continuous in $\Omega \times (1, \infty)$, and there exist $r > 0$ and $T_1 > 0$ such that for any $j \in \mathbb{N}$,

$$(4.13) \quad |u(x, t) - 1| \geq \frac{c_1}{2} \quad \text{for all } x \in B_r(x_j) \cap \Omega \text{ and } t \in (t_j, t_j + T_1).$$

Due to the smoothness of $\partial\Omega$, we can get a constant $c_2 > 0$ such that

$$(4.14) \quad |B_r(x_j) \cap \Omega| \geq c_2 \quad \text{for all } x_j \in \Omega.$$

Using (4.13) and (4.14), then for all $j \in \mathbb{N}$, we have

$$(4.15) \quad \begin{aligned} \int_{t_j}^{t_j+T_1} \int_{\Omega} |u(x, t) - 1|^2 dx dt &\geq \int_{t_j}^{t_j+T_1} \int_{B_r(x_j) \cap \Omega} |u(x, t) - 1|^2 dx dt \\ &\geq \int_{t_j}^{t_j+T_1} |B_r(x_j) \cap \Omega| \cdot \left(\frac{c_1}{2}\right)^2 dt \\ &\geq \frac{c_1^2 c_2 T_1}{4}. \end{aligned}$$

On the other hand, since $\mathcal{E}(t)$ is nonnegative by Lemma 4.1, it follows (4.4) that

$$\int_1^\infty \mathcal{F}(s)ds \leq \mathcal{E}(1) - \mathcal{E}(t) \leq \mathcal{E}(1) < \infty.$$

Using the definition of $\mathcal{F}(t)$ in Lemma 4.1, one obtain

$$(4.16) \quad \int_1^\infty \int_\Omega [(u - 1)^2 + (v - 1)^2] < \infty.$$

From (4.16), we derive that as $j \rightarrow \infty$, $t_j \rightarrow \infty$ and, hence,

$$\int_{t_j}^{t_j+T_1} \int_\Omega (u(x, t) - 1)^2 dxdt \leq \int_{t_j}^\infty \int_\Omega (u(x, t) - 1)^2 dxdt \rightarrow 0 \text{ as } j \rightarrow \infty,$$

which, however, contradicts (4.15). This shows that (4.12) and hence (4.10) hold by argument of contradiction. In a similar way, we can derive (4.11) by using the fact $\int_1^\infty \int_\Omega (v - 1)^2 < \infty$. Hence the proof of Lemma 4.2 is completed. \square

Proof of Theorem 1.1. Theorem 1.1 is a consequence of Lemmas 3.3 and 4.2. \square

5. Simulations and discussions.

5.1. Numerical pattern formation. In this paper, we establish the global existence of solutions to a reaction-diffusion system (1.4) with density-suppressed motility proposed in [6]. When the intrinsic cell growth rate is large, we further show that the homogeneous steady state (1, 1) is globally asymptotically stable. Our results complement the existing results in [29, 39] where the cell growth was not considered. The mathematical study of density-suppressed motility models is still in its infant stage, and there are many interesting open questions. In particular, whether the model can generate pattern formation remains unsolved in our present work. In this section, we shall use numerical simulations to illustrate that the system (1.4) can generate numerous patterns. This suggests that the density-suppressed motility is a complement to the existing mechanisms driving pattern formation such as diffusion-driven (or Turing) or chemotaxis-driven instability (cf. [21]). We also provide some useful clues for further studies based on the numerical findings.

To start, we first consider the system (1.4) with constant diffusion rate D , namely, $\gamma(v) = D$,

$$(5.1) \quad \begin{cases} u_t = D\Delta u + \mu u(1 - u), & x \in \Omega, t > 0, \\ v_t = \Delta v + u - v, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), & x \in \Omega. \end{cases}$$

The first equation of (5.1) is the well-known Fisher-KPP equation and the constant steady state $u = 1$ is globally asymptotically stable (e.g., see [15]). Using the result (or proof) of Lemma 4.2, we know that the constant steady state (1, 1) is globally asymptotically stable for the system (5.1). This indicates that the system (5.1) is not able to generate pattern formation. Next we consider the system (1.4) with variable coefficient $\gamma(v)$ as a function of v and perform linear stability analysis to show that

pattern formation can be discovered. We start with the system in the absence of spatial components:

$$(5.2) \quad \begin{cases} u_t = \mu u(1-u), \\ v_t = u - v. \end{cases}$$

It can be easily verified that the ODE system (5.2) has two steady states $(0, 0)$ and $(1, 1)$, where the former is an unstable saddle point and the latter is a stable node. Therefore we explore the possible patterns bifurcating from the constant steady state $(1, 1)$. To this end, we linearize the reaction-diffusion system (1.4) at $(1, 1)$ and get the linearized system

$$(5.3) \quad \begin{cases} \Phi_t = A\Delta\Phi + B\Phi, & x \in \Omega, t > 0, \\ (\nu \cdot \nabla)\Phi = 0, & x \in \partial\Omega, t > 0, \\ \Phi(x, 0) = (u_0 - 1, v_0 - 1)^T, & x \in \Omega, \end{cases}$$

where \mathcal{T} denotes the transpose and

$$\Phi = \begin{pmatrix} u - 1 \\ v - 1 \end{pmatrix}, \quad A = \begin{pmatrix} \gamma(1) & \gamma'(1) \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -\mu & 0 \\ 1 & -1 \end{pmatrix}.$$

Let $W_k(x)$ denote the eigenfunction of the following eigenvalue problem:

$$(5.4) \quad \Delta W_k(x) + k^2 W_k(x) = 0, \quad \frac{\partial W_k(x)}{\partial \nu} = 0,$$

where k is called the wavenumber. Since the system (5.3) is linear, we shall look for a solution $\Phi(x, t)$ in the form

$$(5.5) \quad \Phi(x, t) = \sum_{k \geq 0} c_k e^{\lambda t} W_k(x),$$

where the constants c_k are determined by Fourier expansions of the initial conditions in terms of $W_k(x)$ and λ is the temporal growth rate. Substituting (5.5) into (5.3), one has

$$\lambda W_k(x) = -k^2 A W_k(x) + B W_k(x),$$

which implies λ is the eigenvalue of the following matrix

$$M_k = \begin{pmatrix} -\gamma(1)k^2 - \mu & -\gamma'(1)k^2 \\ 1 & -k^2 - 1 \end{pmatrix}.$$

Calculating the eigenvalue of matrix M_k , we get the eigenvalues $\lambda(k^2)$ as functions of the wavenumber k as the roots of

$$\lambda^2 + a(k^2)\lambda + b(k^2) = 0,$$

where

$$a(k^2) = (\gamma(1) + 1)k^2 + \mu + 1, \quad b(k^2) = \gamma(1)k^4 + (\gamma(1) + \gamma'(1) + \mu)k^2 + \mu.$$

If the real part of eigenvalue $\lambda(k^2)$ is positive, the steady state $(1, 1)$ is linearly unstable and the spatial pattern formation may arise. Since $\gamma(v) > 0$ for all $v \geq 0$, one has $a(k^2) = (\gamma(1) + 1)k^2 + \mu + 1 > 0$ for all k . Hence $\text{Re}\lambda(k^2)$ can be positive only if $b(k^2) < 0$ for some $k \neq 0$. Since $\gamma(1), \mu > 0$, then a necessary condition for $b(k^2) < 0$ for some $k \neq 0$ is

$$0 < \mu < -\gamma(1) - \gamma'(1) \text{ and } (\gamma(1) + \gamma'(1) + \mu)^2 - 4\mu\gamma(1) > 0,$$

which is equivalent to

$$(5.6) \quad -\gamma'(1) > \gamma(1) \text{ and } 0 < \mu < \mu_0 = (\sqrt{-\gamma'(1)} - \sqrt{\gamma(1)})^2.$$

The allowable wavenumber k satisfies

$$(5.7) \quad \frac{-\eta - \sqrt{\eta^2 - 4\mu\gamma(1)}}{2\gamma(1)} = k_1^2 < k^2 < k_2^2 = \frac{-\eta + \sqrt{\eta^2 - 4\mu\gamma(1)}}{2\gamma(1)},$$

where $\eta = \gamma(1) + \gamma'(1) + \mu < 0$.

Note the allowable wavenumbers k are discrete in a bounded domain, for instance, if $\Omega = (0, l)$, then $k = \frac{n\pi}{l}$ for $n = 1, 2, \dots$. Hence the condition (5.6) is only necessary because the interval (k_1^2, k_2^2) may not contain the desired discrete number k^2 . However if there is an allowable wavenumber k fulfilling (5.7), the condition (5.6) becomes sufficient. Hence we have the following conclusion.

LEMMA 5.1. *Let $\mu > 0$ and $\gamma(v)$ satisfy the hypothesis (H). If the eigenvalue problem (5.4) admits a wavenumber k satisfying condition (5.7), then the homogeneous steady state $(1, 1)$ of system (1.4) is linearly unstable if and only if (5.6) holds.*

From (5.6), we see that conditions on both density-suppressed motility function $\gamma(v)$ and intrinsic growth rate μ at the homogeneous steady state $(1, 1)$ are required to make pattern formation possible. In particular the second condition in (5.6) says that the intrinsic growth rate μ must be small in order to generate pattern formation. This is consistent with our analytical results given in Theorem 1.1 which asserts that all solutions will converge to the homogeneous steady state $(1, 1)$ and hence no pattern formation arises for (large) $\mu > \frac{K_0}{16}$. If we write the first condition in (5.6) as

$$\left. \frac{\partial}{\partial v} \ln \gamma(v) \right|_{v=1} = \frac{\gamma'(1)}{\gamma(1)} < -1,$$

then it says that the decay of $\gamma(v)$ in v cannot be too slow in order to generate pattern formation. For example, if $\gamma(v) = e^{-\sigma v}$ with $\sigma > 0$, then $\ln e^{-\sigma v} = -\sigma v$ and there is no pattern formation if $\sigma \leq 1$. This implies that not all motility functions $\gamma(v)$ satisfying the hypothesis (H) will produce pattern formation.

We remark that in one dimension $\Omega = (0, l)$, $k = \frac{n\pi}{l}$ for $n = 1, 2, \dots$ and, hence, condition (5.7) becomes

$$(5.8) \quad \frac{k_1 l}{\pi} = n_1 < n < n_2 = \frac{k_2 l}{\pi}.$$

In the rest of this section, we shall implement the MATLAB PDE solver based on a

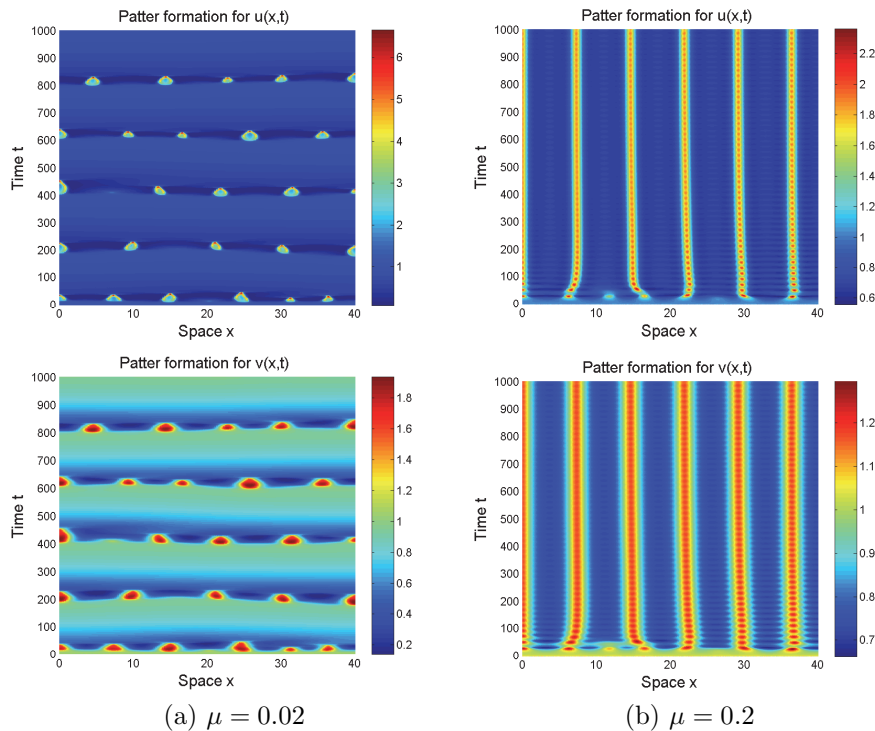


FIG. 1. Numerical simulation of spatio-temporal pattern formation of (1.4) in the interval $[0, 20]$, where the initial data (u_0, v_0) are chosen as a small random perturbation of the constant steady state $(1, 1)$.

finite difference scheme to solve the system (1.4) in an interval $\Omega = [0, l] = [0, 20]$, and numerically illustrate the spatio-temporal patterns formed by (1.4). We set initial data (u_0, v_0) as a small random perturbation of the constant steady state $(1, 1)$, and fix the motility function $\gamma(v)$ as

$$(5.9) \quad \gamma(v) = \frac{1}{1 + e^{8(v-1)}}.$$

Then $\gamma'(v) = -\frac{8e^{8(v-1)}}{(1+e^{8(v-1)})^2}$. It can easily be verified that $2 = -\gamma'(1) > \gamma(1) = 1/2$ and $\mu_0 = (\sqrt{-\gamma'(1)} - \sqrt{\gamma(1)})^2 = 0.5$. By the results of Lemma 5.1, the pattern formation can be expected if $0 < \mu < 0.5$. First we choose $\mu = 0.02$ which is far less than the critical number 0.5. Then it can be explicitly computed that $n_1 = \frac{k_1 l}{\pi} = 0.0647$ and $n_2 = \frac{k_2 l}{\pi} = 10.2650$. Hence there are allowable integers n between n_1 and n_2 to fulfill the condition (5.8) and hence pattern formation is expected to arise from the homogeneous steady state $(1, 1)$ by the result of Lemma 5.1. This is confirmed by our numerical simulation shown in Figure 1(a) where spatio-temporal aggregation patterns are observed but seem to be unstable. We remark that similar so-called chaotic patterns to Figure 1(a) have been numerically found in [24] for a different smoothed-out step function of $\gamma(v)$. Next we choose $\mu = 0.2$ closer to the critical number 0.5, and check that $n_1 = \frac{k_1 l}{\pi} = 0.0000577$ and $n_2 = \frac{k_2 l}{\pi} = 18.8439$, which allow many integers between n_1 and n_2 and pattern formation is therefore expected. We show the numerical simulations in Figure 1(b) where stable spatio-temporal aggregation

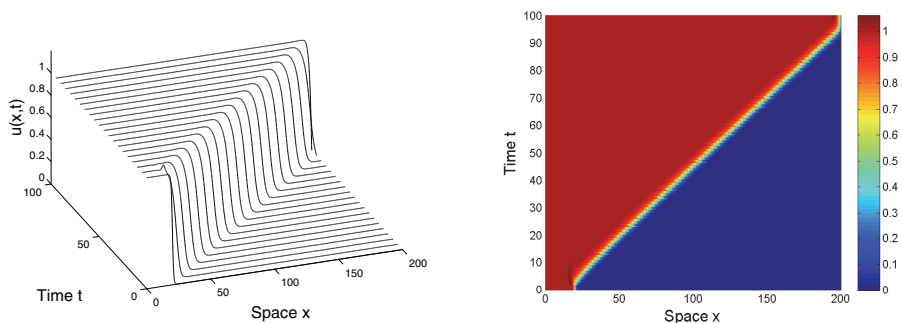


FIG. 2. Numerical simulation of wavefront propagation of the model (1.1) in an interval $[0, 200]$, where the initial data are chosen as in (5.10) and $d = \alpha = \beta = K = \mu = 1$.

patterns are visible. By the simulations, we find the aggregation pattern profiles are quite different between small and large values of μ . Precisely the aggregation patterns appear to be more stable as μ is closer to the critical value μ_0 . When the value of μ crosses over the critical value 0.5, no pattern formation is visible. This complies with our results in Lemma 5.1. Though we do not study the stationary solutions of (1.4), our simulations do indicate that the stationary problem will be rather complicated and its stability depends upon the size of intrinsic growth rate $\mu > 0$.

In addition to aggregation patterns, another type of pattern observed in the experiment of [14] is the wave propagation. Below we shall perform the numerical simulations to illustrate that the density-suppressed motility model is also capable of generating traveling wavefronts. Since traveling wave solutions are usually considered in the whole space, while the numerical simulations can only be done in a finite space, we consider the problem (1.1) in a large interval $[0, 200]$ with Dirichlet boundary conditions to approximate the realistic situation. We choose the same motility function $\gamma(v)$ given by (5.9). To initiate a wavefront, we impose the initial data as

$$(5.10) \quad u_0(x) = v_0(x) = \frac{1}{1 + \exp(2(x - 20))}.$$

We first set $\mu = 1$. With numerical simulations shown in Figure 2, we observe the steady wave propagation where the wave fronts and tails are constant profiles. Next we choose $\mu = 0.2$ and show the numerical simulations in Figure 3 where we still observe the wavefront propagation, but the wave fronts are now connected to oscillating wave tails which are different from those with a larger value of μ shown in Figure 2. The wave pattern in Figure 3 can be viewed as a one-dimensional version of propagating ring patterns observed in the experiment of [14, Figure 1]. The simulations shown in Figures 2 and 3 indicate that the intrinsic growth rate μ plays a role in determining the diversity of wave patterns.

The model (1.4) with $\mu = 0$ is not considered in the present paper, but its global solutions have been obtained in [29, 39] under some conditions mentioned in the introduction. Here we would like to show some numerical simulations for this case and make a comparison with the case $\mu > 0$. In order to compare, we stick with the same motility function $\gamma(v)$ given in (5.9) and the same initial data as chosen in Figure 1. The numerical patterns with $\mu = 0$ are plotted in Figure 4, where we observe the stripe patterns qualitatively similar to the patterns shown in Figure 1(b) for large $\mu > 0$. But we do not find similar patterns to Figure 1(a). This implies

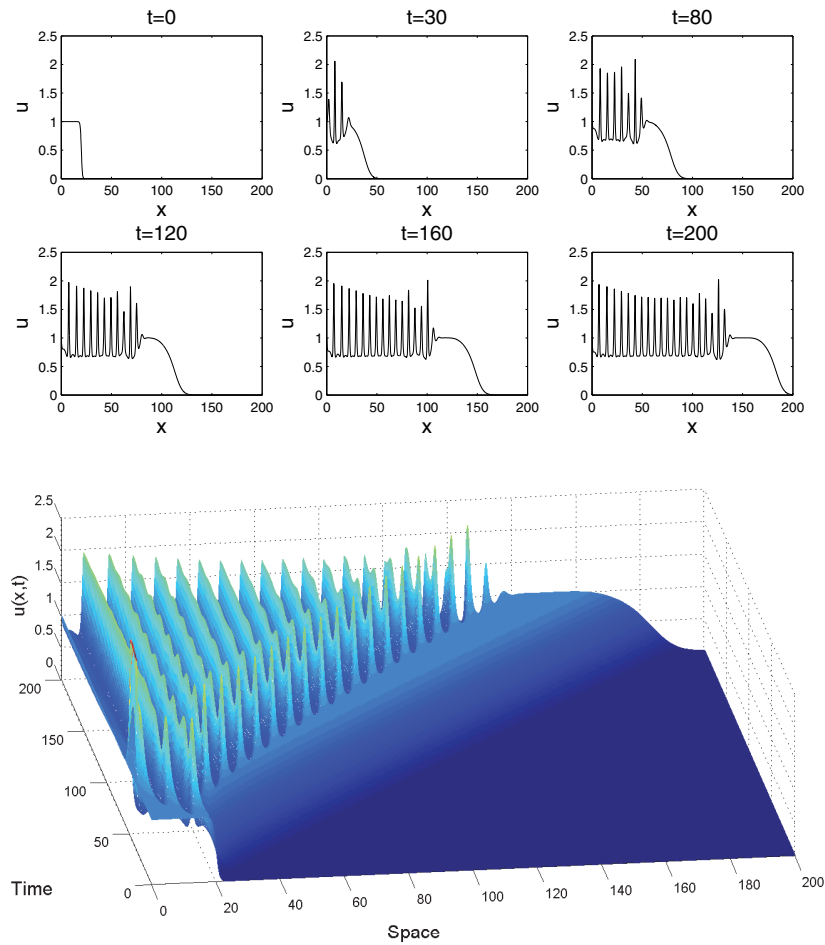


FIG. 3. Numerical simulations of wavefront propagation of the model (1.1) in an interval $[0, 200]$, where the initial data are chosen as in (5.10) and $d = \alpha = \beta = K = 1, \mu = 0.2$. Upper panel: time snapshot of wavefronts; lower panel: landscape view of wavefront propagation.

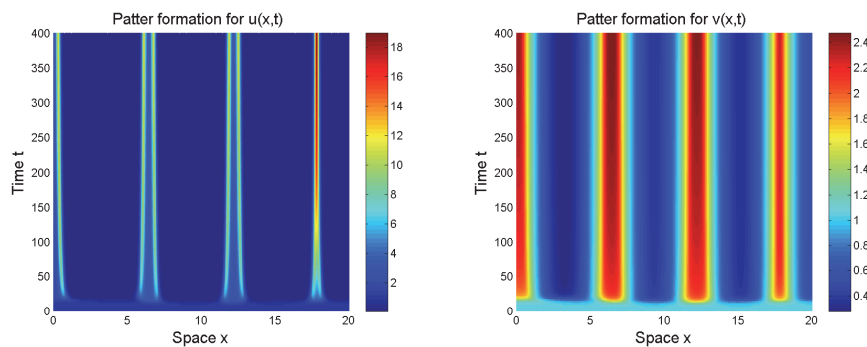


FIG. 4. Numerical simulation of spatio-temporal pattern formation of (1.4) with $\mu = 0$ in the interval $[0, 20]$, where the initial data (u_0, v_0) are chosen as a small random perturbation of $(1, 1)$, and $\gamma(v)$ is given in (5.9).

that the inclusion of cell growth will produce more types of patterns. Furthermore we numerically explore the wave propagation for $\mu = 0$ and other data the same as those in Figures 2 or 3. However, we do not find any type of wave propagation profiles. This indicates that cell growth is an indispensable factor to describe wave propagation phenomena.

5.2. Discussion. Numerical simulations above have demonstrated that the density-suppressed motility model (1.4) with logistic growth can produce rich patterns and dynamics, which leave us many interesting questions to explore. Below we discuss some possible directions but are by no means confined to those.

The first question is the existence and stability of nonconstant steady states (stationary solutions). Theorem 1.1 asserts that nonconstant steady states do not exist for large $\mu > \mu^0 = \frac{K_0}{16}$, where $K_0 = \max_{0 \leq v \leq \infty} \frac{|\gamma'(v)|^2}{\gamma(v)}$. For small $\mu > 0$, as illustrated in Figure 1, patterns are observed and nonconstant stationary solutions exist. How to prove the existence of nonconstant stationary solutions of (1.4) is a challenging question since the corresponding stationary problem is a system with cross diffusion and no mature methods are available, in general. The global bifurcation or degree theory may be useful in such a scenario (cf. [20]). Furthermore whether the constant $\mu^0 = \frac{K_0}{16}$ is a threshold number for the existence/nonexistence of non-constant steady states is not completely known although a partial answer has been given in this paper. To see this, we take (5.9) as an example and explicitly compute that $\mu^0 = \frac{K_0}{16} = 0.59$ which is bigger than the critical value $\mu_0 = 0.5$ for the pattern formation. Therefore we may suspect $\mu^0 = \frac{K_0}{16}$ is not a threshold number for the existence/nonexistence of nonconstant steady states. Hence how to find such a threshold number becomes an intriguing question. Even for the small value $\mu > 0$, we observe the stable and unstable patterns in Figure 1 for different values of μ . Hence the stability of stationary solutions will be another interesting problem to pursue in the future.

The second question is the existence and stability of traveling wave solutions to (1.1). Numerically we have found that system (1.1) admits propagating traveling wavefronts as shown in Figures 2 and 3. The numerical simulations have showed two type of wavefront profiles depending on the value of μ . Indeed the wavefront propagation patterns are one of the main patterns observed in the experiment reported in papers [6, 14]. Hence it is very desirable to undertake a rigorous procedure to justify the existence of traveling wave solutions of (1.1) and further investigate how the density-suppressed motility affects the properties of wave propagation such as the wave speed, stability, and so on. Exploring these questions needs delicate analysis due to strong couplings with nonlinearities in the system.

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