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THE KELLER-SEGEL SYSTEM WITH LOGISTIC GROWTH AND SIGNAL-DEPENDENT MOTILITY

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Abstract. The paper is concerned with the following chemotaxis system with nonlinear motility functions

$$\begin{cases} u_t = \nabla\cdot(\gamma(v)\nabla u - u\chi(v)\nabla v) + \mu u(1-u), & x\in\Omega,\ t>0,\\ 0=\Delta v + u - v, & x\in\Omega,\ t>0,\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{$*$}$$

subject to homogeneous Neumann boundary conditions in a bounded domain $\Omega\subset\mathbb{R}^2$ with smooth boundary, where the motility functions $\gamma(v)$ and $\chi(v)$ satisfy the following conditions

- $(\gamma,\chi)\in[C^2[0,\infty)]^2$ with $\gamma(v)>0$ and $\frac{|\chi(v)|^2}{\gamma(v)}$ is bounded for all $v\geq 0$.

By employing the method of energy estimates, we establish the existence of globally bounded solutions of $(*)$ with $\mu>0$ for any $u_0\in W^{1,\infty}(\Omega)$ with $u_0\geq(\not\equiv)0$. Then based on a Lyapunov function, we show that all solutions (u,v) of $(*)$ will exponentially converge to the unique constant steady state $(1,1)$ provided $\mu>\frac{K_0}{16}$ with $K_0=\max\limits_{0\leq v\leq\infty}\frac{|\chi(v)|^2}{\gamma(v)}$.

1. **Introduction and main results.** In this paper, we consider the following chemotaxis model with density-dependent motilities

$$\begin{cases} u_t = \nabla\cdot(\gamma(v)\nabla u - u\chi(v)\nabla v) + \mu u(1-u), & x\in\Omega,\quad t>0,\\ \tau v_t = \Delta v + u - v, & x\in\Omega,\quad t>0,\\ \frac{\partial u}{\partial\nu}=\frac{\partial v}{\partial\nu}=0, & x\in\partial\Omega, t>0,\\ u(x,0)=u_0(x), v(x,0)=v_0(x) & x\in\Omega, \end{cases} \tag{1}$$

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where $\Omega \subset \mathbb{R}^n (n \geq 2)$ is a bounded domain with smooth boundary, $u(x,t)$ denotes the cell density and $v(x,t)$ is the chemical concentration, $\mu \geq 0$ and $\tau = \{0,1\}$. The prominent feature of (1) compared to the classical chemotaxis model is that both the undirected motility (diffusion) and directed motility (chemotaxis) of cells depend on the chemical concentration. The system (1) has several important applications. When $\mu = 0$, the system (1) has been firstly derived by Keller and Segel in [11] to describe the aggregation phase of amoeba cells in response to the chemical signal cAMP emitted by themselves, where the motility functions $\gamma(v) > 0$ and $\chi(v)$ are correlated by the following proportionality relation

$$\chi(v) = (\alpha - 1)\gamma'(v), \tag{2}$$

with α denoting the ratio of effective body length to step size, and $\gamma'(v) < 0$ (resp. $>$ 0) if the diffusive motility decreases (resp. increases) with respect to the chemical concentration. As mentioned in [11], although the motility coefficient $\gamma(v)$ is positive, the chemotactic motility coefficient $\chi(v)$ may be positive or negative depending on the signs of $(\alpha - 1)$ and $\gamma'(v)$.

When both $\gamma(v)$ and $\chi(v)$ are constant, (1) is called the minimal chemotaxis system which has been extensively studied in the literature from various aspects including boundedness, blow-up, large-time behavior and pattern formation of solutions (cf. [35, 20, 31, 29, 14, 33, 4, 21, 16, 29, 12, 13] and reference therein). When $\gamma(v)$ is constant and $\chi(v) = 1/v$, the system (1) with $\mu = 0$ has been studied recently in a number of interesting works (see [8, 32] and references therein). However, if $\gamma(v)$ is non-constant, the results of (1) are very limited. The few existing results are mainly focused on the special case $\chi(v) = -\gamma'(v)$, (i.e. $\alpha = 0$), which reduces the system (1) to

$$\begin{cases} u_t = \Delta(\gamma(v)u) + \mu u(1-u), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v + u - v, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x) & x \in \Omega. \end{cases} \tag{3}$$

Essentially (3) with $\mu > 0$ has been used in [5] to justify that the bacterial motion with density-suppressed motility (i.e., $\gamma'(v) < 0$) can produce the stripe pattern formation observed in the experiment of [15]. Several results on the reduced system (3) are then available as will be recalled below.

When $\mu = 0$ (no cell growth), it was proved in [36] that the system (3) with $\tau = 1$ and $\gamma(v) = c_0/v^k (k > 0)$ admits global classical solutions in any dimensions for small constant $c_0 > 0$. Recently, the smallness assumptions of c_0 was removed in [3] for the parabolic-elliptic case of (3) (i.e., $\tau = 0$) for $0 < k < \frac{2}{n-2}$. Moreover, based on the phase plane analysis and bifurcation analysis, the existence and analytical approximation of non-constant stationary were established in one dimension [34]. By assuming that $\gamma(v)$ has positive lower and upper bounds (i.e. $\delta_1 \leq \gamma(v) \leq \delta_2$ for some positive constants δ_1, δ_2), the global classical solution in two dimensions and global weak solution in three dimensions of (3) with $\mu = 0$ were obtained in [28]. Recently, it is proved in [10, 6] that if $\gamma(v) = e^{-\chi v}$ there exists a critical mass $m_* = \frac{4\pi}{\chi}$ such that the solution of (3) with $\mu = 0$ exists globally with uniform-in-time bound if $\int_\Omega u_0 dx < m_*$ and blows up if $\int_\Omega u_0 dx > m_*$. Turning to the case $\mu > 0$, there are several results below. When $\gamma(v)$ is a decreasing step-wise constant function, the dynamics of discontinuity interface of solutions was studied in [24] in one dimension. In two dimensional spaces, the global boundedness of solutions of

(3) with $\tau = 1$ was established in [9] under the following hypotheses on the motility function $\gamma(v)$:

(H0) $\gamma(v) \in C^3([0,\infty)), \gamma(v) > 0$ and $\gamma'(v) < 0$ on $[0,\infty)$, $\lim\limits_{v\to\infty}\gamma(v) = 0$, $\lim\limits_{v\to\infty}\frac{\gamma'(v)}{\gamma(v)}$ exists.

It was further shown in [9] that the constant steady state $(1,1)$ is globally asymptotically stable provided $\mu > \frac{K_0}{16}$ with $K_0 = \max\limits_{0\le v\le\infty}\frac{|\gamma'(v)|^2}{\gamma(v)}$. Similar results have been extended to higher dimensions ($n \geq 3$) in [30] for large $\mu > 0$. The existence/nonexistence of nonconstant steady states of (3) was recently studied in [17]. Moveover, the global existence of solutions of (3) with $\tau = 0$ was obtained in [7] without the condition "$\lim\limits_{v\to\infty}\frac{\gamma'(v)}{\gamma(v)}$ exists" in (H0).

In summary, for the chemotaxis system (1)-(2) with density-dependent motility, the results are available only for the special case $\alpha = 0$ with various hypotheses on the motility function $\gamma(v)$ as recalled above for (3). Therefore there are various interesting questions remaining open. The following questions comprise the motivation of this paper.

(Q1) So far no results of (1)-(2) are available for $\alpha \neq 0$ in the prescribed proportionality relation (2). Furthermore as remarked in [11], the prescribed proportionality (2) between the motility functions $\gamma(v)$ and $\chi(v)$ is derived based on assumption that the cell step size is constant and the total step frequency is solely determined by the mean concentration of the chemical. However, $\chi(v)$ would no longer be simply proportional to $\gamma'(v)$ if both step size and total step frequency were permitted to vary with the chemical concentration. Hence, it would be meaningful and interesting to study the system (1) with more general $\gamma(v)$ and $\chi(v)$ beyond the proportionality (2).

(Q2) The existing results recalled above are mostly restricted to the case $\gamma'(v) < 0$ or some special form of $\chi(v)$ (cf. [3, 9, 30, 10]). However, as discussed in [11, Section 3], the cell motion may be more vigorous at high concentrations than at low concentrations, which motives us to study the case $\gamma'(v) > 0$ or even non-monotone $\gamma(v)$ so that the analytical results can cover more possible applications.

Inspired by the above mentioned questions, in this paper we shall develop some first-hand results on the global boundedness and large time behavior of solutions to the system (1) with general motility functions $\gamma(v)$ and $\chi(v)$. Specifically we consider (1) with $\tau = 0$

$$\begin{cases} u_t = \nabla\cdot(\gamma(v)\nabla u - u\chi(v)\nabla v) + \mu u(1-u), & x\in\Omega, \quad t>0,\\ 0 = \Delta v + u - v, & x\in\Omega, \quad t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega, t>0,\\ u(x,0) = u_0(x), v(x,0) = v_0(x) & x\in\Omega, \end{cases} \tag{4}$$

under the following assumptions on $\gamma(v)$ and $\chi(v)$:

(H1) $(\gamma,\chi) \in [C^2[0,\infty)]^2$ with $\gamma(v) > 0$ and $\frac{|\chi(v)|^2}{\gamma(v)}$ is bounded for all $v \geq 0$.

The main results of this paper are the following.

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary and the hypotheses (H1) hold. Suppose that $u_0 \in W^{1,\infty}(\Omega)$ with $u_0 \geq 0 (\not\equiv 0)$. Then the problem (4) has a unique global classical solution $(u,v) \in [C([0,\infty)\times\bar{\Omega}) \cap C^{2,1}((0,\infty)\times$*

$\bar{\Omega})] \times C^{2,1}((0,\infty)\times\bar{\Omega})$ *satisfying* $u, v > 0$ *for all* $t > 0$ *and*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C_1 \quad \text{for all } t > 0,$$

where $C_1 > 0$ *is a constant independent of* t. *Furthermore, if* $\mu > \frac{K_0}{16}$ *with* $K_0 = \max\limits_{0\le v\le\infty} \frac{|\chi(v)|^2}{\gamma(v)}$, *then there exist two positive constants* C_2 *and* δ *such that*

$$\|u(\cdot,t)-1\|_{L^\infty(\Omega)} + \|v(\cdot,t)-1\|_{L^\infty(\Omega)} \le C_2 e^{-\delta t}.$$

The results in Theorem 1.1 not only address the questions raised in (Q1) and (Q2), but also improve the existing results on the specialized system (3) where $\chi(v) = -\gamma'(v)$. Indeed with $\alpha = 0$ in (2) with $\gamma'(v) < 0$, one can check that "$\lim\limits_{v\to\infty} \frac{\gamma'(v)}{\gamma(v)}$ exists" in (H0) is a stronger condition than "$\frac{|\chi(v)|^2}{\gamma(v)}$ is bounded for all $v \ge 0$" in (H1). For example, if $\gamma(v) = e^{-e^v}$, then $\lim\limits_{v\to\infty} \frac{\gamma'(v)}{\gamma(v)} = -\infty$ but $\frac{|\chi(v)|^2}{\gamma(v)} = \frac{|\gamma'(v)|^2}{\gamma(v)} = e^{(2v-e^v)} \le e^{2(\ln 2-1)}$ for any $v \ge 0$. We remark the same results of (3) with $\tau = 0$ as in [9] for $\tau = 1$ are obtained in [7] without the condition "$\lim\limits_{v\to\infty} \frac{\gamma'(v)}{\gamma(v)}$ exists" in (H0), where the methods developed therein essentially rely on the monotonicity of $\gamma(v)$ and the proportionality relation $\chi(v) = -\gamma'(v)$ and hence are inapplicable to our present problem where we consider more general $\gamma(v)$ and $\chi(v)$ without such restrictions.

2. **Local existence and preliminaries.**

In what follows, without confusion, we shall abbreviate $\int_\Omega f dx$ as $\int_\Omega f$ and $\|f\|_{L^2(\Omega)}$ as $\|f\|_{L^2}$ for simplicity. Moreover, we shall use $c_i (i = 1, 2, 3, \cdots)$ to denote a generic constant which may vary in the context. The existence of local solutions of (4) can be proved by Schauder fixed point theorem as illustrated in [9, Lemma 2.1] for the system (3) with $\tau = 1$, we omit the details for brevity.

Lemma 2.1 (Local existence)**.** *Let* Ω *be a bounded domain in* $\mathbb{R}^2$ *with smooth boundary and the hypothesis (H) hold. Assume* $u_0 \in W^{1,\infty}(\Omega)$ *with* $u_0 \ge 0 (\not\equiv 0)$. *Then there exists* $T_{max} \in (0,\infty]$ *such that the problem* (4) *has a unique classical solution* $(u,v) \in [C([0,\infty)\times\bar{\Omega}) \cap C^{2,1}((0,\infty)\times\bar{\Omega})] \times C^{2,1}((0,\infty)\times\bar{\Omega})$ *satisfying* $u, v > 0$ *for all* $t > 0$. *Moreover, we have*

$$\text{Either} \quad T_{max} = \infty, \ \text{or} \quad \limsup_{t\nearrow T_{max}} (\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{W^{1,\infty}}) = \infty.$$

Lemma 2.2. *Let* (u,v) *be the solution of system* (4). *Then it holds that*

$$\int_\Omega u \le m_* := \max\{\|u_0\|_{L^1}, |\Omega|\}, \quad \text{for all} \quad t \in (0, T_{max}). \tag{5}$$

Proof. We integrate the first equation of (4) over Ω to have

$$\frac{d}{dt}\int_\Omega u + \mu\int_\Omega u^2 = \mu\int_\Omega u, \quad \text{for all} \quad t \in (0, T_{max}),$$

which, together with $\int_\Omega u^2 \ge \frac{1}{|\Omega|}\left(\int_\Omega u\right)^2$, gives

$$\frac{d}{dt}\int_\Omega u \le \mu\int_\Omega u - \frac{\mu}{|\Omega|}\left(\int_\Omega u\right)^2, \quad \text{for all} \quad t \in (0, T_{max})$$

and hence (5) follows. ∎

3. **Proof of Theorem 1.1.** In this section, we shall prove Theorem 1.1. First, we show the global existence of uniformly-in-time bounded solutions.

3.1. **Boundedness of solutions.**

Lemma 3.1. *Suppose the conditions in Theorem 1.1 hold. Then there exists a constant $C > 0$ independent of t such that*

$$\|u \ln u\|_{L^1} \leq C \text{ for all } \; t \in (0, T_{max}) \tag{6}$$

and

$$\|\nabla v\|_{L^2} \leq C \;\text{ for all }\; t \in (0, T_{max}). \tag{7}$$

Proof. Multiplying the first equation of (4) by $\ln u$, and integrating the result by part, one has

$$\begin{aligned} \frac{d}{dt}\left(\int_\Omega u \ln u - \int_\Omega u\right) + \int_\Omega \gamma(v)\frac{|\nabla u|^2}{u} = &\int_\Omega \chi(v)\nabla v \cdot \nabla u + \mu \int_\Omega u \ln u \\ &- \mu \int_\Omega u^2 \ln u. \end{aligned} \tag{8}$$

From the assumptions in (H1), we can find a constant $K > 0$ such that

$$\frac{|\chi(v)|^2}{\gamma(v)} \leq K \;\text{ for all }\; v \geq 0. \tag{9}$$

Using the Cauchy-Schwarz inequality and (9), we have

$$\begin{aligned} \int_\Omega \chi(v)\nabla v \cdot \nabla u &\leq \frac{1}{2}\int_\Omega \gamma(v)\frac{|\nabla u|^2}{u} + \frac{1}{2}\int_\Omega \frac{|\chi(v)|^2}{\gamma(v)}|\nabla v|^2 u \\ &\leq \frac{1}{2}\int_\Omega \gamma(v)\frac{|\nabla u|^2}{u} + \frac{K}{2}\|\nabla v\|_{L^4}^2 \|u\|_{L^2}, \end{aligned}$$

which, substituted into (8), gives

$$\begin{aligned} &\frac{d}{dt}\left(\int_\Omega u \ln u - \int_\Omega u\right) + \frac{1}{2}\int_\Omega \gamma(v)\frac{|\nabla u|^2}{u} \\ &\leq \frac{K}{2}\|\nabla v\|_{L^4}^2 \|u\|_{L^2} + \mu \int_\Omega u \ln u - \mu \int_\Omega u^2 \ln u. \end{aligned} \tag{10}$$

Applying the Agmon-Douglis-Nirenberg L^p estimates (cf. [1, 2]) to the second equation of (4) with homogeneous Neumann boundary conditions, we know that for all $p > 1$, there exists a constant $c_1 > 0$ such that

$$\|v(\cdot, t)\|_{W^{2,p}} \leq c_1 \|u(\cdot, t)\|_{L^p}. \tag{11}$$

The Sobolev embedding theorem yields $\|\nabla v\|_{L^4} \leq c_2 \|v\|_{W^{2,\frac{4}{3}}}$ in two dimensions (i.e. $n = 2$), which together with (11) implies

$$\|\nabla v\|_{L^4}^2 \leq c_2^2 \|v\|_{W^{2,\frac{4}{3}}}^2 \leq c_3 \|u\|_{L^{\frac{4}{3}}}^2. \tag{12}$$

On the other hand, using the L^p-interpolation inequality and the fact $\|u(\cdot, t)\|_{L^1} \leq m_*$ (see Lemma 2.2), we have

$$\|u\|_{L^{\frac{4}{3}}}^2 \leq \|u\|_{L^2}\|u\|_{L^1} \leq m_* \|u\|_{L^2}. \tag{13}$$

We substitute (12) and (13) into (10) to obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega u\ln u-\int_\Omega u\right)+\frac{1}{2}\int_\Omega \gamma(v)\frac{|\nabla u|^2}{u}+\left(\int_\Omega u\ln u-\int_\Omega u\right)\\
&\le \frac{Kc_3m_*}{2}\|u\|_{L^2}^2+(\mu+1)\int_\Omega u\ln u-\mu\int_\Omega u^2\ln u-\int_\Omega u\\
&\le \frac{Kc_3m_*}{2}\|u\|_{L^2}^2+(\mu+1)\int_\Omega u\ln u-\mu\int_\Omega u^2\ln u\\
&\le c_4,
\end{aligned}\tag{14}$$

where we have used the facts (see [25, Lemma 3.1]): Let $\mu>0$ and $A\ge 0$, then there exists a constant $L:=L(\mu,A)>0$ such that

$$(1+\mu)z\ln z+Az^2-\mu z^2\ln z\le L,\quad \text{for all}\quad z>0.$$

Hence from (14), we obtain

$$\frac{d}{dt}\left(\int_\Omega u\ln u-\int_\Omega u\right)+\int_\Omega u\ln u-\int_\Omega u\le c_5,$$

which gives $\int_\Omega u\ln u-\int_\Omega u\le c_6$ and then

$$\int_\Omega u\ln u\le c_6+\int_\Omega u\le c_7. \tag{15}$$

Since $u\ln u\ge -\frac{1}{e}$, from (15) we derive

$$\int_\Omega |u\ln u|\le \int_\Omega u\ln u+\frac{2|\Omega|}{e}\le c_8,$$

which yields (6). Finally (7) is a consequence of [25, Lemma A.4]) applied to the second equation of (4). □

Next, we will show that there exists some $p>1$ close to 1 such that $\int_\Omega u^p$ is uniformly bounded in time.

Lemma 3.2. *Suppose the conditions in Theorem 1.1 hold. Then there exists $p>1$ close to 1 such that*

$$\|u(\cdot,t)\|_{L^p}\le C,\ \text{for all}\quad t\in(0,T_{max}), \tag{16}$$

where $C>0$ is a constant independent of t.

Proof. We multiply the first equation of (4) by u^{p-1} to obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p+(p-1)\int_\Omega \gamma(v)u^{p-2}|\nabla u|^2\\
&=(p-1)\int_\Omega \chi(v)u^{p-1}\nabla u\cdot\nabla v+\mu\int_\Omega u^p-\mu\int_\Omega u^{p+1}.
\end{aligned}\tag{17}$$

The Cauchy-Schwarz inequality and (9) allow us to have

$$\begin{aligned}
&(p-1)\int_\Omega \chi(v)u^{p-1}\nabla u\cdot\nabla v\\
&\le \frac{p-1}{2}\int_\Omega \gamma(v)u^{p-2}|\nabla u|^2+\frac{p-1}{2}\int_\Omega \frac{|\chi(v)|^2}{\gamma(v)}u^p|\nabla v|^2\\
&\le \frac{p-1}{2}\int_\Omega \gamma(v)u^{p-2}|\nabla u|^2+\frac{(p-1)K}{2}\int_\Omega u^p|\nabla v|^2.
\end{aligned}\tag{18}$$

Using Gagliardo-Nirenberg inequality, (7) and (11), one has

$$\begin{aligned}\int_\Omega u^p|\nabla v|^2 &\le \|u\|_{L^{p+1}}^p\|\nabla v\|_{L^{2(p+1)}}^2 \le c_1\|u\|_{L^{p+1}}^p\|v\|_{W^{2,p+1}}\|\nabla v\|_{L^2}\\ &\le c_2\|u\|_{L^{p+1}}^p\|v\|_{W^{2,p+1}} \le c_3\|u\|_{L^{p+1}}^{p+1}.\end{aligned} \tag{19}$$

Then we can substitute (18) and (19) into (17) to obtain

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{(p-1)}{2}\int_\Omega \gamma(v)u^{p-2}|\nabla u|^2\\ &\le \frac{(p-1)Kc_3}{2}\int_\Omega u^{p+1} + \mu\int_\Omega u^p - \mu\int_\Omega u^{p+1}.\end{aligned} \tag{20}$$

Using the Hölder inequality and Cauchy-Schwarz inequality, one can show that

$$(1+\mu)\int_\Omega u^p \le (1+\mu)|\Omega|^{\frac{1}{p+1}}\left(\int_\Omega u^{p+1}\right)^{\frac{p}{p+1}} \le \frac{\mu}{2}\int_\Omega u^{p+1} + c_4. \tag{21}$$

Moreover, we can choose $p = 1+\epsilon > 1$ satisfying $\frac{\epsilon K c_3}{2} < \frac{\mu}{2}$ to derive that

$$\frac{(p-1)Kc_3}{2}\int_\Omega u^{p+1} \le \frac{\mu}{2}\int_\Omega u^{p+1}. \tag{22}$$

Then the combination of (21), (22) and (20) gives

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p \le c_4. \tag{23}$$

Applying Gronwall's inequality to (23), we have (16) for some $p > 1$ close to 1. $\square$

Next, we will show $\|v(\cdot,t)\|_{L^\infty}$ is uniformly bounded in time, which rules out the possibility of degeneracy.

Lemma 3.3. *Suppose the conditions in Theorem 1.1 hold. Then there exists a constant $K_1 > 0$ such that*

$$\|v(\cdot,t)\|_{L^\infty} \le K_1, \text{ for all } \ t\in(0,T_{max}) \tag{24}$$

and

$$0 < \gamma_1 \le \gamma(v) \le \gamma_2. \tag{25}$$

Proof. From Lemma 3.2, we can find a constant $c_1 > 0$ such that $\|u(\cdot,t)\|_{L^p} \le c_1$ for some $p > 1$. Then applying the elliptic regularity estimate to the second equation of (4), one has $\|v(\cdot,t)\|_{W^{2,p}} \le c_2\|u(\cdot,t)\|_{L^p} \le c_1c_2$, which along with the Sobolev inequality give (24). Then since $0 < \gamma(v) \in C^2([0,\infty))$, we can find two positive constants γ_1 and γ_2 such that (25) holds. $\square$

Lemma 3.4. *Suppose the conditions in Theorem 1.1 hold. Then there exists a constant $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^2} \le C, \text{ for all } \ t\in(0,T_{max}). \tag{26}$$

Proof. Multiplying the first equation of (4) by u and integrating the result by parts, using Cauchy-Schwarz inequality and (9), we end up with

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega u^2+\int_\Omega \gamma(v)|\nabla u|^2+\mu\int_\Omega u^3\\
&=\int_\Omega \chi(v)u\nabla u\cdot\nabla v+\mu\int_\Omega u^2\\
&=\frac{1}{2}\int_\Omega \gamma(v)|\nabla u|^2+\frac{1}{2}\int_\Omega \frac{|\chi(v)|^2}{\gamma(v)}u^2|\nabla v|^2+\mu\int_\Omega u^2\\
&\le \frac{1}{2}\int_\Omega \gamma(v)|\nabla u|^2+\frac{K}{2}\int_\Omega u^2|\nabla v|^2+\mu\int_\Omega u^2,
\end{aligned}$$

which, combined with (25), gives

$$\frac{d}{dt}\int_\Omega u^2+\gamma_1\int_\Omega|\nabla u|^2+2\mu\int_\Omega u^3\le K\int_\Omega u^2|\nabla v|^2+2\mu\int_\Omega u^2. \tag{27}$$

We differentiate the second equation of system (4) and multiply the result by $2\nabla v$ to obtain

$$\begin{aligned}
0&=2\nabla v\cdot\nabla\Delta v+2\nabla v\cdot\nabla u-2|\nabla v|^2\\
&=\Delta|\nabla v|^2-2|D^2v|^2+2\nabla v\cdot\nabla u-2|\nabla v|^2,
\end{aligned}\tag{28}$$

where we have used the identity $\Delta|\nabla v|^2=2\nabla v\cdot\nabla\Delta v+2|D^2v|^2$. Then multiplying (28) by $|\nabla v|^2$ and integrating the results, we have

$$\begin{aligned}
&\int_\Omega|\nabla|\nabla v|^2|^2+2\int_\Omega|\nabla v|^2|D^2v|^2+2\int_\Omega|\nabla v|^4\\
&=\int_{\partial\Omega}|\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}dS+2\int_\Omega|\nabla v|^2\nabla v\cdot\nabla u\\
&=\int_{\partial\Omega}|\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}dS-2\int_\Omega u\Delta v|\nabla v|^2-2\int_\Omega u\nabla(|\nabla v|^2)\cdot\nabla v\\
&\le\int_{\partial\Omega}|\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}dS+2\int_\Omega u\left(|\Delta v||\nabla v|^2+|\nabla|\nabla v|^2||\nabla v|\right).
\end{aligned}\tag{29}$$

With the inequality $\frac{\partial|\nabla v|^2}{\partial\nu}\le 2\lambda|\nabla v|^2$ on $\partial\Omega$ (see [18, Lemma 4.2]) and the following trace inequality [23, Remark 52.9] for any $\varepsilon>0$:

$$\|\varphi\|_{L^2(\partial\Omega)}\le\varepsilon\|\nabla\varphi\|_{L^2(\Omega)}+C_\varepsilon\|\varphi\|_{L^2(\Omega)},$$

we have

$$\int_{\partial\Omega}|\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}dS\le 2\lambda\||\nabla v|^2\|^2_{L^2(\partial\Omega)}\le\frac{1}{4}\int_\Omega|\nabla|\nabla v|^2|^2+c_1\||\nabla v|^2\|^2_{L^2}. \tag{30}$$

By the Gagliardo-Nirenberg inequality and the fact $\||\nabla v|^2\|_{L^1}=\|\nabla v\|^2_{L^2}\le c_2$(see Lemma 3.1), we have

$$\begin{aligned}
c_1\||\nabla v|^2\|^2_{L^2}&\le c_3\|\nabla|\nabla v|^2\|_{L^2}\||\nabla v|^2\|_{L^1}+c_3\||\nabla v|^2\|^2_{L^1}\\
&\le\frac{1}{4}\int_\Omega|\nabla|\nabla v|^2|^2+c_4.
\end{aligned}\tag{31}$$

Then the combination of (31) and (30) gives

$$\int_{\partial\Omega}|\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}dS\le\frac{1}{2}\int_\Omega|\nabla|\nabla v|^2|^2+c_4. \tag{32}$$

Next, we will estimate the last term on the right of (29). To this end, we use the Young's inequality and the facts $|\Delta v| \leq \sqrt{2}|D^2 v|$ and $\nabla|\nabla v|^2 = 2D^2 v \cdot \nabla v$ to derive

$$\begin{aligned}
&2\int_\Omega u\left(|\Delta v||\nabla v|^2 + \left|\nabla|\nabla v|^2\right||\nabla v|\right)\\
&\leq 2\sqrt{2}\int_\Omega u|\nabla v|^2|D^2 v| + 4\int_\Omega u|\nabla v|^2|D^2 v|\\
&\leq 2(\sqrt{2}+2)\int_\Omega u|\nabla v|^2|D^2 v|\\
&\leq 2\int_\Omega |\nabla v|^2|D^2 v|^2 + \frac{(2+\sqrt{2})^2}{2}\int_\Omega u^2|\nabla v|^2.
\end{aligned} \tag{33}$$

Substituting (32) and (33) into (29), one has

$$\int_\Omega |\nabla|\nabla v|^2|^2 + 4\int_\Omega |\nabla v|^4 \leq (2+\sqrt{2})^2\int_\Omega u^2|\nabla v|^2 + 2c_4. \tag{34}$$

Combining (27) and (34) and using the Young's inequality, we can find some $\zeta > 0$ such that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u^2 + \gamma_1\int_\Omega |\nabla u|^2 + 2\mu\int_\Omega u^3 + \int_\Omega |\nabla|\nabla v|^2|^2 + 4\int_\Omega |\nabla v|^4\\
&\leq [K + (2+\sqrt{2})^2]\int_\Omega u^2|\nabla v|^2 + 2\mu\int_\Omega u^2 + 2c_4\\
&\leq [K + (2+\sqrt{2})^2]\|u\|_{L^3}^2\|\nabla v\|_{L^6}^2 + 2\mu|\Omega|^{\frac{1}{3}}\|u\|_{L^3}^2 + 2c_4\\
&\leq c_5\|u\|_{L^3}^3 + \zeta\|\nabla v\|_{L^6}^6 + \mu\|u\|_{L^3}^3 + c_6.
\end{aligned} \tag{35}$$

With the boundedness of $\|u\|_{L^1}$ and $\|u\ln u\|_{L^1}$ and the inequality in [19, Lemma 3.5], we can choose ε small enough to obtain

$$\|u\|_{L^3}^3 \leq \varepsilon\|\nabla u\|_{L^2}^2\|u\ln u\|_{L^1} + C_\varepsilon(\|u\ln u\|_{L^1}^3 + \|u\|_{L^1}) \leq \frac{\gamma_1}{c_5}\|\nabla u\|_{L^2}^2 + c_7. \tag{36}$$

On the other hand, using the Gagliardo-Nirenberg inequality, we can derive that

$$\begin{aligned}
\|\nabla v\|_{L^6}^6 = \||\nabla v|^2\|_{L^3}^3 &\leq c_8(\|\nabla|\nabla v|^2\|_{L^2}^2\||\nabla v|^2\|_{L^1} + \||\nabla v|^2\|_{L^1}^3)\\
&\leq c_8c_2\|\nabla|\nabla v|^2\|_{L^2}^2 + c_8c_2^3.
\end{aligned} \tag{37}$$

Substituting (36) and (37) into (35), and choosing $\zeta = \frac{1}{c_2c_8}$, we end up with $\frac{d}{dt}\int_\Omega u^2 + \mu\int_\Omega u^3 \leq c_{11}$ which along with the Young inequality: $\int_\Omega u^2 \leq \mu\int_\Omega u^3 + c_{12}$ yields

$$\frac{d}{dt}\int_\Omega u^2 + \int_\Omega u^2 \leq c_{11} + c_{12}.$$

This gives (26) with the help of Gronwall's inequality. □

Next, we shall show the boundedness of $\|u(\cdot,t)\|_{L^\infty}$. To this end, we first improve the regularity of v. More precisely, we have the following results.

Lemma 3.5. *Suppose the conditions in Theorem 1.1 hold. Then we have*

$$\|\nabla v(\cdot,t)\|_{L^\infty} \leq C, \text{ for all } \ t \in (0, T_{max}), \tag{38}$$

where $C > 0$ is a constant independent of t.

Proof. Using (11) and the fact $\|u(\cdot,t)\|_{L^2} \le c_1$, we can derive that $\|v(\cdot,t)\|_{W^{2,2}} \le c_2\|u(\cdot,t)\|_{L^2} \le c_1c_2$, which by the Sobolev embedding theorem $(n=2)$ gives

$$\|\nabla v\|_{L^4} \le c_3. \tag{39}$$

Then multiplying the first equation of (4) by u^2 and integrating it over Ω by parts, one obtains

$$\begin{aligned}
&\frac{1}{3}\frac{d}{dt}\int_\Omega u^3 + 2\int_\Omega \gamma(v)u|\nabla u|^2 + \mu\int_\Omega u^4\\
&= 2\int_\Omega u^2\chi(v)\nabla u\cdot\nabla v + \mu\int_\Omega u^3\\
&\le \int_\Omega \gamma(v)u|\nabla u|^2 + \int_\Omega \frac{|\chi(v)|^2}{\gamma(v)}u^3|\nabla v|^2 + \frac{\mu}{2}\int_\Omega u^4 + c_4,
\end{aligned}$$

which subject to the facts (9) and (39) gives rise to

$$\begin{aligned}
\frac{1}{3}\frac{d}{dt}\int_\Omega u^3 + \frac{4\gamma_1}{9}\int_\Omega |\nabla u^{\frac{3}{2}}|^2 + \frac{\mu}{2}\int_\Omega u^4 &\le K\int_\Omega u^3|\nabla v|^2 + c_4\\
&\le K\|u\|_{L^6}^3\|\nabla v\|_{L^4}^2 + c_4\\
&\le c_3^2K\|u\|_{L^6}^3 + c_4.
\end{aligned} \tag{40}$$

Using the Gagliardo-Nirenberg inequality with the fact $\|u^{\frac{3}{2}}\|_{L^{\frac{4}{3}}} = \|u\|_{L^2}^{\frac{3}{2}} \le c_5$, we can show that

$$\begin{aligned}
c_3^2K\|u\|_{L^6}^3 = c_3^2K\|u^{\frac{3}{2}}\|_{L^4}^2 &\le c_6\left(\|\nabla u^{\frac{3}{2}}\|_{L^2}^{\frac{4}{3}}\|u^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^{\frac{2}{3}} + \|u^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^2\right)\\
&\le c_7\|\nabla u^{\frac{3}{2}}\|_{L^2}^{\frac{4}{3}} + c_7\\
&\le \frac{4\gamma_1}{9}\int_\Omega |\nabla u^{\frac{3}{2}}|^2 + c_8.
\end{aligned} \tag{41}$$

On the other hand, using the Hölder inequality and Young inequality, one has

$$\int_\Omega u^3 \le |\Omega|^{\frac{1}{4}}\left(\int_\Omega u^4\right)^{\frac{3}{4}} \le \frac{\mu}{2}\int_\Omega u^4 + c_9. \tag{42}$$

Substituting (41) and (42) into (40) gives

$$\frac{1}{3}\frac{d}{dt}\int_\Omega u^3 + \int_\Omega u^3 \le c_{10},$$

which along with the Gronwall's inequality gives

$$\|u(\cdot,t)\|_{L^3} \le c_{11}. \tag{43}$$

Using the elliptic regularity (11) and Sobolev embedding theorem again, from (43) we derive

$$\|\nabla v\|_{L^\infty} \le c_{12}\|v\|_{W^{2,3}} \le c_{13}\|u\|_{L^3} \le c_{11}c_{13}.$$

This finishes the proof. ∎

Lemma 3.6. *Suppose the conditions in Theorem 1.1 hold. Then the solution of* (4) *satisfies*

$$\|u(\cdot,t)\|_{L^\infty} \le C, \text{ for all } t\in(0,T_{max}), \tag{44}$$

where the constant $C>0$ independent of t.

Proof. Multiplying the first equation of (4) by $u^{p-1}(p \geq 2)$ and integrating it by parts over Ω, and using (38) and Young's inequality, we can find a constant $c_1 > 0$ independent of p such that

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + (p-1)\int_\Omega \gamma(v)u^{p-2}|\nabla u|^2 + \mu\int_\Omega u^{p+1} \\
&= (p-1)\int_\Omega \chi(v)u^{p-1}\nabla u\cdot\nabla v + \mu\int_\Omega u^p \\
&\leq c_1(p-1)\int_\Omega |\chi(v)|u^{p-1}|\nabla u| + \mu(p-1)\int_\Omega u^p \\
&\leq \frac{p-1}{2}\int_\Omega \gamma(v)u^{p-2}|\nabla u|^2 + \left(\frac{c_1^2K}{2}+\mu\right)(p-1)\int_\Omega u^p,
\end{aligned}
\tag{45}
$$

which, together with the fact $\gamma(v) \geq \gamma_1 > 0$ in (25), gives a positive constant $c_2 = \frac{c_1^2K}{2} + \mu + 1$ such that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega u^p + p(p-1)\int_\Omega u^p + \frac{2(p-1)\gamma_1}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \\
&\leq c_2 p(p-1)\int_\Omega u^p \\
&\leq \frac{2(p-1)\gamma_1}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + c_3 p(p-1)(1+p^2)\left(\int_\Omega u^{\frac{p}{2}}\right)^2,
\end{aligned}
\tag{46}
$$

where the last inequality is obtained based on the following inequality (see [26])

$$
\|f\|_{L^2}^2 \leq \varepsilon\|\nabla f\|_{L^2}^2 + c_4(1+\varepsilon^{-1})\|f\|_{L^1}^2, \quad \text{for any} \quad \varepsilon > 0.
$$

The inequality (46) can be rewritten as

$$
\frac{d}{dt}\int_\Omega u^p + p(p-1)\int_\Omega u^p \leq c_3 p(p-1)(1+p^2)\left(\int_\Omega u^{\frac{p}{2}}\right)^2,
$$

which, combined with the fact $(1+p^2) \leq (1+p)^2$, gives

$$
\frac{d}{dt}\left(e^{p(p-1)t}\int_\Omega u^p\right) \leq c_3 e^{p(p-1)t}p(p-1)(1+p)^2\left(\int_\Omega u^{\frac{p}{2}}\right)^2. \tag{47}
$$

We integrate (47) over $[0,t]$ for $0 < t < T_{max}$ to obtain

$$
\int_\Omega u^p \leq \int_\Omega u_0^p + c_3(1+p)^2 \sup_{0\leq t\leq T_{max}}\left(\int_\Omega u^{\frac{p}{2}}\right)^2. \tag{48}
$$

Define

$$
N(p) := \max\left\{\|u_0\|_{L^\infty}, \sup_{0\leq t\leq T_{max}}\left(\int_\Omega u^p\right)^{\frac{1}{p}}\right\}. \tag{49}
$$

Then, we can derive from (48) and (49) that

$$
N(p) \leq [c_4(1+p)^2]^{\frac{1}{p}} N(\frac{p}{2}) \ \text{ for } \ p \geq 2.
$$

Taking $p = 2^j, j = 1, 2, \cdots$, one obtains

$$\begin{aligned}
N(2^j) &\le c_4^{2^{-j}}(1+2^j)^{2^{-j+1}} N(2^{j-1}) \\
&\ \ \vdots \\
&\le c_4^{2^{-j}+\cdots+2^{-1}}(1+2^j)^{2^{-j+1}} \cdots (1+2)N(1) \\
&\le c_4[2^{j2^{-j+1}}(2^{-j}+1)^{2^{-j+1}}] \cdots [2(2^{-1}+1)]N(1) \\
&\le c_4 2^{2[j2^{-j}+(j-1)2^{-(j-1)}+\cdots+2^{-1}]} \cdot 2^{2[2^{-j}+2^{-(j-1)}+\cdots+2^{-1}]} N(1) \\
&\le c_4 2^6 N(1).
\end{aligned}$$

If $\|u(\cdot,t)\|_{L^\infty} \le \|u_0\|_{L^\infty}$, the proof is then finished. Otherwise sending $j \to \infty$ and using the boundedness of $\|u\|_{L^1}$, we have

$$\|u(\cdot,t)\|_{L^\infty} \le c_4 2^6 N(1) \le c_4 2^6 \max\{\|u_0\|_{L^\infty}, \|u_0\|_{L^1}\} \le c_5,$$

which gives (44). ☐

Lemma 3.7. *Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary and the hypothesis (H1) hold. Suppose that $u_0 \in W^{1,\infty}(\Omega)$ with $u_0 \ge 0 (\not\equiv 0)$. Then the problem (4) has a unique solution $(u,v) \in [C^0([0,\infty) \times \bar{\Omega}) \cap C^{2,1}((0,\infty) \times \bar{\Omega})] \times C^{2,1}((0,\infty) \times \bar{\Omega})$, which satisfies*

$$\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{W^{1,\infty}} \le C.$$

Proof. From Lemma 3.6, we can find a constant $c_1 > 0$ such that $\|u(\cdot,t)\|_{L^\infty} \le c_1$. Then using the elliptic regularity, from the second equation of (4) one obtains $\|v(\cdot,t)\|_{W^{1,\infty}} \le c_2$. By Lemma 2.1, the existence of global classical solutions follows immediately. ☐

3.2. **Large time behavior.** In this section, we will study the large time behavior of solution for the system (4). Let

$$K_0 = \max_{0 \le v \le \infty} \frac{|\chi(v)|^2}{\gamma(v)} \tag{50}$$

and

$$\mathcal{E}(t) := \int_\Omega (u - 1 - \ln u). \tag{51}$$

Then based on some ideas in [9, 27], we shall show that the constant steady state $(1,1)$ is globally asymptotically stable by showing $\mathcal{E}(t)$ is a Lyapunov functional under the conditions $\mu > \frac{K_0}{16}$. More precisely, we have the following result.

Lemma 3.8. *Suppose (u,v) is the solution of (4) obtained in Lemma 3.7. Let K_0 and $\mathcal{E}(t)$ be defined by (50) and (51), respectively. Then we have the following results:*

(1) $\mathcal{E}(t) \ge 0$ for any $t > 0$;

(2) If $\mu > \frac{K_0}{16}$, then there exists a positive constant β such that for all $t > 0$

$$\mathcal{E}'(t) \le -\mathcal{F}(t), \tag{52}$$

where

$$\mathcal{F}(t) := \beta \cdot \Big\{ \int_\Omega (u-1)^2 + \int_\Omega (v-1)^2 \Big\}.$$

Proof. First, we will show the non-negativity of $\mathcal{E}(t)$. In fact, letting $\phi(u) := u - 1 - \ln u, u > 0$ and noting that $\phi(1) = \phi'(1) = 0$, and applying the Taylor's formula to $\phi(u)$ at $u = 1$ gives

$$\phi(u) = \frac{1}{2}\phi''(\tilde{u})(u-1)^2 = \frac{1}{2\tilde{u}^2}(u-1)^2 \geq 0, \tag{53}$$

where $\tilde{u}$ is between 1 and u, which implies $\mathcal{E}(t) \geq 0$.

Next, we show (52) hold. In fact, using the first equation of (4), we have

$$\begin{aligned}\mathcal{E}'(t) &= \frac{d}{dt}\int_\Omega (u - 1 - \ln u)\\ &= -\int_\Omega \nabla\left(\frac{u-1}{u}\right)\cdot[\gamma(v)\nabla u - \chi(v)u\nabla v] - \mu\int_\Omega (u-1)^2\\ &= -\int_\Omega \gamma(v)\frac{|\nabla u|^2}{u^2} + \int_\Omega \chi(v)\frac{\nabla u\cdot\nabla v}{u} - \mu\int_\Omega (u-1)^2.\end{aligned} \tag{54}$$

On the other hand, we multiply the second equation of system (4) by $v - 1$ and integrate it by parts to obtain

$$0 = -\int_\Omega |\nabla v|^2 - \int_\Omega (v-1)^2 + \int_\Omega (u-1)(v-1). \tag{55}$$

Multiplying (55) by a constant $\delta > 0$ and adding the result to (54), we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega (u - 1 - \ln u) = &\underbrace{-\int_\Omega \gamma(v)\frac{|\nabla u|^2}{u^2} - \delta\int_\Omega |\nabla v|^2 + \int_\Omega \chi(v)\frac{\nabla u\cdot\nabla v}{u}}_{I_1}\\ &\underbrace{-\mu\int_\Omega (u-1)^2 - \delta\int_\Omega (v-1)^2 + \delta\int_\Omega (u-1)(v-1)}_{I_2}.\end{aligned} \tag{56}$$

For I_1, we can rewrite it as

$$I_1 = -\Theta_1^T A_1 \Theta_1,\ \Theta_1 = \begin{pmatrix} \nabla u \\ \nabla v \end{pmatrix},\quad A_1 = \begin{pmatrix} \frac{\gamma(v)}{u^2} & -\frac{\chi(v)}{2u} \\ -\frac{\chi(v)}{2u} & \delta \end{pmatrix}$$

where Θ_1^T denotes the transpose of Θ_1. One can check that A_1 is non-negative definite if and only if

$$\delta \geq \max_{0\leq v\leq\infty}\frac{|\chi(v)|^2}{4\gamma(v)} = \frac{K_0}{4}. \tag{57}$$

Similarly, we can also rewrite I_2 as

$$I_2 = -\Theta_2^T A_2 \Theta_2,\ \Theta_2 = \begin{pmatrix} u-1 \\ v-1 \end{pmatrix},\quad A_2 = \begin{pmatrix} \mu & \frac{\delta}{2} \\ \frac{\delta}{2} & \delta \end{pmatrix}.$$

A_2 is positive definite if and only if

$$\mu > \frac{\delta}{4}. \tag{58}$$

Hence, we can always find a positive constant δ such that (57) and(58) hold provided $\mu > \frac{K_0}{16}$. Since A_1 is non-negative definite and A_2 is positive definite, then from (56), we can find a constant $\beta > 0$ such that (52) holds. $\square$

Next, we will use (52) to show the convergence of solution (u, v, w) in L^∞-norm. Before that, we first improve the regularity of solutions (u, v).

Lemma 3.9. *There exist $\sigma \in (0,1)$ and $C > 0$ such that*

$$\|u\|_{C^{\sigma,\frac{\sigma}{2}}(\bar{\Omega}\times[t,t+1])} \leq C, \text{ for } all \ \ t \geq 0. \tag{59}$$

Proof. From Lemma 3.7, we can find three positive constants c_1, c_2, c_3 such that $0 < u(x,t) \leq c_1, 0 < v(x,t) \leq c_2$ and $|\nabla v(x,t)| \leq c_3$ for all $x \in \Omega$ and $t \in (0, T_{max})$. The first equation of (4) can be rewritten as

$$u_t = \nabla \cdot A(x,t,\nabla u) + B(x,t) \text{ for all } x \in \Omega \text{ and } \ t \in (0, T_{max}), \tag{60}$$

where

$$A(x,t,\xi) := \gamma(v)\cdot\xi - \chi(v)u\nabla v$$

and

$$B(x,t) := \mu u(\cdot,t)(1 - u(\cdot,t)).$$

Noting the assumptions in (H1) and using the Young's inequality, we can obtain that

$$\begin{aligned} A(x,t,\nabla u)\cdot\nabla u &= \gamma(v)|\nabla u|^2 - \chi(v)u\nabla v\cdot\nabla u \\ &\geq \gamma(v)|\nabla u|^2 - |\chi(v)|u|\nabla v||\nabla u| \\ &\geq \frac{\gamma(v)}{2}|\nabla u|^2 - \frac{|\chi(v)|^2}{2\gamma(v)}u^2|\nabla v|^2 \end{aligned} \tag{61}$$

and

$$|A(x,t,\nabla u)| \leq \gamma_2|\nabla u| + c_4 \text{ for all } x \in \Omega \text{ and } \ t \in (0, T_{max})$$

as well as

$$|B(x,t)| \leq \mu c_1(1 + c_1) \text{ for all } x \in \Omega \text{ and } \ t \in (0, T_{max}). \tag{62}$$

Then (61)-(62) allow us to apply the Hölder regularity for quasilinear parabolic equations [22, Theorem 1.3 and Remark 1.4] to conclude that u satisfies (59). $\square$

Lemma 3.10. *Suppose that $\mu > \frac{K_0}{16}$ and let (u,v) be the global classical solution of the system (4). Then it follows that*

$$\|u(\cdot,t) - 1\|_{L^\infty} \to 0, \quad \text{as} \quad t \to \infty \tag{63}$$

and

$$\|v(\cdot,t) - 1\|_{L^\infty} \to 0, \quad \text{as} \quad t \to \infty. \tag{64}$$

Proof. From Lemma 3.8, we know $\mathcal{E}(t) \geq 0$ for all $t > 0$. Then integrating (52) over $[1,t]$, we have

$$\int_1^t \mathcal{F}(s)ds \leq \mathcal{E}(1) - \mathcal{E}(t) \leq \mathcal{E}(1), \quad \text{for all } t > 1.$$

Using the definition of $\mathcal{F}(t)$, one can derive

$$\int_1^t \int_\Omega \left[(u-1)^2 + (v-1)^2\right] < \infty. \tag{65}$$

Then combining (65) and Lemma 3.9, and using a similar argument as in [9, Lemma 4.2], we obtain (63). On the other hand, from the second equation of (4), we infer that $\psi(x,t) := v(x,t) - 1$ satisfies

$$\begin{cases} -\Delta\psi + \psi = u - 1, & x \in \Omega, t > 0, \\ \frac{\partial\psi}{\partial\nu} = 0, & x \in \Omega, t > 0. \end{cases} \tag{66}$$

Then using the elliptic maximum principle, we obtain from (66) that

$$\|v(\cdot,t)-1\|_{L^\infty} = \|\psi(\cdot,t)\|_{L^\infty} \le \|u(\cdot,t)-1\|_{L^\infty}, \tag{67}$$

which together with (63) gives (64). □

3.3. **Exponential decay.** Next, we shall show the convergence rate is exponential.

Lemma 3.11. *Assume that $\mu > \frac{K_0}{16}$, and suppose (u,v) is the global classical solution of the system (4). Then there exists two positive constants C, δ_* such that for all $t>0$*

$$\|u(\cdot,t)-1\|_{L^2} \le Ce^{-\frac{\delta_*}{2}t}. \tag{68}$$

Proof. From (63), we can get a $t_0>0$ such that for all $t>t_0$

$$\|u(\cdot,t)-1\|_{L^\infty} < \frac{1}{2},$$

which immediately gives

$$u(x,t) \in \left(\frac{1}{2},\frac{3}{2}\right) \quad \text{for all} \quad x\in\Omega \quad \text{and} \quad t>t_0. \tag{69}$$

Then using (53) and (69), we can get two positive constants c_1 and c_2 such that

$$c_1(u-1)^2 \le u-1-\ln u \le c_2(u-1)^2 \quad \text{for all} \quad u \in \left(\frac{1}{2},\frac{3}{2}\right). \tag{70}$$

Hence, using (51) and (70), and choosing $\delta_* = \frac{\beta}{c_2}$, we have for all $t>t_0$ that

$$\mathcal{E}(t) \le c_2 \int_\Omega (u-1)^2 \le \frac{1}{\delta_*}\mathcal{F}(t),$$

which yields

$$\mathcal{F}(t) \ge \delta_* \mathcal{E}(t) \qquad \text{for all } t>t_0. \tag{71}$$

Then the combination of (52) and (71) gives for all $t>t_0$

$$\mathcal{E}'(t) \le -\mathcal{F}(t) \le -\delta_*\mathcal{E}(t),$$

and hence

$$\mathcal{E}(t) \le \mathcal{E}(t_0)e^{-\delta_*(t-t_0)}, \quad \text{for all} \quad t>t_0,$$

which together with the fact $\mathcal{E}(t) \ge c_1\int_\Omega (u-1)^2$ gives (68). Then we finish the proof of Lemma 3.11. □

Next, we shall show the boundedness of $\|\nabla u\|_{L^4}$ to obtain the convergence rate with L^∞-norm. More precisely, we have the following results.

Lemma 3.12. *There exists a constant $C>0$ independent of t such that the solution (u,v) of (4) satisfies*

$$\|\nabla u(\cdot,t)\|_{L^4} \le C \quad \text{for all} \quad t\in(0,T_{max}). \tag{72}$$

Proof. Using the first equation of (4), we obtain

$$\begin{aligned} \frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla u|^4 &= \int_\Omega |\nabla u|^2 \nabla u\cdot\nabla u_t \\ &= \int_\Omega |\nabla u|^2\nabla u \cdot \nabla(\nabla\cdot(\gamma(v)\nabla u)) \\ &\quad - \int_\Omega |\nabla u|^2 \nabla u\cdot\nabla(\nabla\cdot(\chi(v)u\nabla v)) + \mu\int_\Omega (1-2u)|\nabla u|^4 \\ &=: J_1+J_2+J_3. \end{aligned} \tag{73}$$

We can estimate the term J_1 as follows:

$$
\begin{aligned}
J_1 &= -\int_\Omega |\nabla u|^2 \Delta u \nabla \cdot (\gamma(v)\nabla u) - \int_\Omega \nabla |\nabla u|^2 \cdot \nabla u \nabla \cdot (\gamma(v)\nabla u) \\
&= \int_\Omega \gamma(v)|\nabla u|^2 \nabla \Delta u \cdot \nabla u - \int_\Omega \gamma'(v)\nabla|\nabla u|^2 \cdot \nabla u \nabla u \cdot \nabla v \\
&= \frac{1}{2}\int_\Omega \gamma(v)|\nabla u|^2 \Delta |\nabla u|^2 - \int_\Omega \gamma(v)|\nabla u|^2 |D^2 u|^2 \\
&\quad - \int_\Omega \gamma'(v)\nabla|\nabla u|^2 \cdot \nabla u \nabla u \cdot \nabla v \\
&\le \frac{1}{2}\int_{\partial\Omega} \gamma(v)|\nabla u|^2 \frac{\partial |\nabla u|^2}{\partial \nu} dS - \frac{1}{2}\int_\Omega \gamma(v)|\nabla|\nabla u|^2|^2 - \int_\Omega \gamma(v)|\nabla u|^2|D^2u|^2 \\
&\quad + \frac{3}{2}\int_\Omega |\gamma'(v)||\nabla|\nabla u|^2||\nabla u|^2|\nabla v|.
\end{aligned}
\tag{74}
$$

Using the boundedness of $\|u\|_{L^\infty}$ and $\|v\|_{W^{1,\infty}}$ obtained in Lemma 3.7 and the assumptions in (H1) as well as the fact $\Delta v = v - u$, we have

$$
\begin{aligned}
\nabla \cdot (\chi(v)u\nabla v) &= \chi'(v)u|\nabla v|^2 + \chi(v)\nabla u \cdot \nabla v + \chi(v)u\Delta v \\
&= \gamma''(v)u|\nabla v|^2 + \chi(v)\nabla u \nabla v + \chi(v)uv - \chi(v)u^2 \\
&\le c_1(1 + |\nabla u|),
\end{aligned}
$$

which substituted into J_2 gives

$$
\begin{aligned}
J_2 &= \int_\Omega \nabla|\nabla u|^2 \cdot \nabla u \nabla \cdot (\chi(v)u\nabla v) + \int_\Omega |\nabla u|^2 \Delta u \nabla \cdot (\chi(v)u\nabla v) \\
&\le c_1 \int_\Omega |\nabla u||\nabla|\nabla u|^2|(1+|\nabla u|) + c_1\int_\Omega |\nabla u|^2|\Delta u|(1+|\nabla u|).
\end{aligned}
\tag{75}
$$

Moreover, the boundedness of $\|u\|_{L^\infty}$ directly gives

$$
J_3 \le c_2 \int_\Omega |\nabla u|^4. \tag{76}
$$

Substituting (74)-(76) into (73), and noting the facts $\gamma(v) \ge \gamma_1 > 0$ and $|\Delta u| \le \sqrt{2}|D^2u|$, we have

$$
\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla u|^4 + \frac{\gamma_1}{2}\int_\Omega |\nabla|\nabla u|^2|^2 + \gamma_1 \int_\Omega |\nabla u|^2|D^2u|^2 \\
&\le \frac{1}{2}\int_{\partial\Omega} \gamma(v)|\nabla u|^2\frac{\partial|\nabla u|^2}{\partial\nu}dS + \frac{3}{2}\int_\Omega |\gamma'(v)||\nabla|\nabla u|^2||\nabla u|^2|\nabla v| \\
&\quad + c_1\int_\Omega |\nabla u||\nabla|\nabla u|^2|(1+|\nabla u|) + c_1\int_\Omega |\nabla u|^2|\Delta u|(1+|\nabla u|) + c_2\int_\Omega |\nabla u|^4 \\
&\le \frac{\gamma_1}{4}\int_\Omega |\nabla|\nabla u|^2|^2 + \frac{\gamma_1}{2}\int_\Omega |\nabla u|^2|D^2u|^2 + c_3\int_\Omega |\nabla u|^4 + c_4,
\end{aligned}
$$

which leads to

$$
\frac{d}{dt}\int_\Omega |\nabla u|^4 + \gamma_1\int_\Omega |\nabla|\nabla u|^2|^2 + 2\gamma_1\int_\Omega |\nabla u|^2|D^2u|^2 \le 4c_3\int_\Omega |\nabla u|^4 + 4c_4. \tag{77}
$$

On the other hand, using the boundedness of $\|u\|_{L^\infty}$ and the fact $|\Delta u| \le \sqrt{2}|D^2u|$ again, we have

$$\begin{aligned}\left(\frac{3}{2}+4c_3\right)\int_\Omega |\nabla u|^4 &= \left(\frac{3}{2}+4c_3\right)\int_\Omega |\nabla u|^2\nabla u\cdot\nabla u\\ &= -\left(\frac{3}{2}+4c_3\right)\int_\Omega u\nabla|\nabla u|^2\cdot\nabla u - \left(\frac{3}{2}+4c_3\right)\int_\Omega u|\nabla u|^2\Delta u\\ &\le \gamma_1\int_\Omega |\nabla|\nabla u|^2|^2 + 2\gamma_1\int_\Omega |\nabla u|^2|D^2u|^2 + c_5\int_\Omega|\nabla u|^2\\ &\le \gamma_1\int_\Omega |\nabla|\nabla u|^2|^2 + 2\gamma_1\int_\Omega |\nabla u|^2|D^2u|^2 + \frac{1}{2}\int_\Omega|\nabla u|^4 + c_6,\end{aligned}$$

which substituted into (77) gives

$$\frac{d}{dt}\int_\Omega |\nabla u|^4 + \int_\Omega |\nabla u|^4 \le c_7. \tag{78}$$

Then applying the Gronwall's inequality to (78) yields (72) and the proof is completed. □

Lemma 3.13. *Suppose $\mu > \frac{K_0}{16}$, and let (u, v) be the global classical solution of the system (4). Then there exists a constants $C > 0$ such that for all $t > 0$*

$$\|u(\cdot,t)-1\|_{L^\infty} \le Ce^{-\frac{\delta_*}{6}t}, \tag{79}$$

and

$$\|v(\cdot,t)-1\|_{L^\infty} \le Ce^{-\frac{\delta_*}{6}t}. \tag{80}$$

Proof. Using the Gagliardo-Nirenberg inequality, (68) and (72), we have

$$\begin{aligned}\|u-1\|_{L^\infty} &\le c_1\|\nabla u\|_{L^4}^{\frac{2}{3}}\|u-1\|_{L^2}^{\frac{1}{3}} + c_1\|u-1\|_{L^2}\\ &\le c_2e^{-\frac{\delta_*}{6}t} + c_2e^{-\frac{\delta_*}{2}t}\\ &\le 2c_2e^{-\frac{\delta_*}{6}t},\end{aligned}$$

which gives (79). (80) follows from (79) due to (67). This competes the proof of Lemma 3.13. □

Proof of Theorem 1.1. Theorem 1.1 is an immediate consequence of Lemma 3.7 and Lemma 3.13. □

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