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Boundedness, blowup and critical mass phenomenon in competing chemotaxis

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Abstract

We consider the following attraction–repulsion Keller–Segel system:

$$\begin{cases} u_t = \Delta u - \nabla\cdot(\chi u\nabla v) + \nabla\cdot(\xi u\nabla w), & x\in\Omega,\ t>0, \\ v_t = \Delta v + \alpha u - \beta v, & x\in\Omega,\ t>0, \\ 0 = \Delta w + \gamma u - \delta w, & x\in\Omega,\ t>0, \\ u(x,0)=u_0(x),\ v(x,0)=v_0(x), & x\in\Omega, \end{cases}$$

with homogeneous Neumann boundary conditions in a bounded domain $\Omega\subset\mathbb{R}^2$ with smooth boundary. The system models the chemotactic interactions between one species (denoted by u) and two competing chemicals (denoted by v and w), which has important applications in Alzheimer's disease. Here all parameters χ, ξ, α, β, γ and δ are positive. By constructing a Lyapunov functional, we establish the global existence of uniformly-in-time bounded classical solutions with large initial data if the repulsion dominates or cancels attraction (i.e., $\xi\gamma\ge\alpha\chi$). If the attraction dominates (i.e., $\xi\gamma<\alpha\chi$), a critical mass phenomenon is found. Specifically speaking, we find a critical mass $m_*=\frac{4\pi}{\alpha\chi-\xi\gamma}$ such that the solution exists globally with uniform-in-time bound if $M<m_*$ and blows up if $M>m_*$ and $M\notin\{\frac{4\pi m}{\theta} : m\in\mathbb{N}^+\}$ where $\mathbb{N}^+$ denotes the set of positive integers and $M=\int_\Omega u_0dx$ the initial cell mass.

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1. Introduction

This paper is concerned with the initial–boundary value problem of the following attraction–repulsion chemotaxis system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi u \nabla v) + \nabla \cdot (\xi u \nabla w), & x \in \Omega,\ t > 0, \\ \tau_1 v_t = \Delta v + \alpha u - \beta v, & x \in \Omega,\ t > 0, \\ \tau_2 w_t = \Delta w + \gamma u - \delta w, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ \tau_1 v(x,0) = \tau_1 v_0(x),\ \tau_2 w(x,0) = \tau_2 w_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ and ν denotes the outward normal vector of $\partial\Omega$. The model (1.1) was proposed in [28] to describe the aggregation of *Microglia* in the central nervous system in Alzheimer's disease due to the interaction of chemoattractant (β-amyloid) and chemorepellent (TNF-α), where $u(x,t)$, $v(x,t)$ and $w(x,t)$ in the model (1.1) denote the concentrations of *Microglia*, chemoattractant and chemorepellent which are produced by *Microglia*, respectively. The positive parameters χ and ξ are called the chemotactic coefficients, and $\chi, \beta, \gamma, \delta > 0$ are chemical production and degradation rates. τ_1, τ_2 are constants equal to 0 or 1 justifying whether the change of chemicals is stationary or dynamical in time. The model (1.1) was also a particularized system introduced in the paper [33] to model the quorum sensing effect in the chemotactic movement.

Well-known as the Keller–Segel model (see [23]), the prototype of classical attractive chemotaxis model reads as

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi u \nabla v), \\ \tau_1 v_t = \Delta v + \alpha u - \beta v. \end{cases} \tag{1.2}$$

One prominent property of the Keller–Segel model (1.2) is the existence of a Lyapunov functional which continuously stimulates a vast amount of mathematical studies on various aspects of mathematics such as blowup, boundedness, traveling waves, pattern formations, critical mass phenomenon and critical sensitivity exponents (e.g. see [4,5,15,16,19,29,31,32,37,40,41] and the references therein, and review articles [13,18,39]).

On the other hand, for the classical repulsive chemotaxis model which reads as follows:

$$\begin{cases} u_t = \Delta u + \nabla \cdot (\xi u \nabla w), \\ \tau_2 w_t = \Delta w + \gamma u - \delta w, \end{cases}$$

a Lyapunov functional different from that of the attractive Keller–Segel model was found in [6], which led to the global existence of classical solutions in two dimensions and weak solutions in three and four dimensions. The results on the repulsive Keller–Segel model are very limited and a further study on such model was recently given in [36].

Mathematically the three-component attraction–repulsion chemotaxis system (1.1) modeling the aggregation of *Microglia* is a coupled attractive and repulsive Keller–Segel model, and hence is referred to as the attraction–repulsion Keller–Segel (abbreviated as ARKS) model. It is hard to analyze in general due to the complicated interactions between three species u, v and w, and the difficulty of constructing a Lyapunov functional. A few known results are the following. In one dimension, the stationary solutions and time-asymptotic behavior of solutions were established in [21,26], and the time-periodic orbits were found recently in [27] by employing the local and global Hopf bifurcation theory. The traveling wave solutions of an attraction–repulsion chemotaxis system with a volume-filling effect were investigated in [34]. The multi-dimensional analysis was recently given by Tao and Wang [38] where the competing effects of blowup from the attraction and smoothing from the repulsion were untangled. It mainly dealt with a special scenario $\beta = \delta$ (i.e., two competing chemical signals have the same death rates) for which the system (1.1) can be formally transformed into the classical Keller–Segel model and hence the methods based on the Lyapunov functional can be employed. It was found in [38] that the solution behavior of the ARKS model was essentially determined by the competition of attraction and repulsion which is characterized by the sign of $\chi\alpha - \xi\gamma$. For the convenience of statement, we call the number

$$\theta = \chi\alpha - \xi\gamma$$

the competition index in this paper and the biological interpretation of the sign of θ is as follows:

- $\theta < 0 \Leftrightarrow$ repulsion dominates;
- $\theta = 0 \Leftrightarrow$ repulsion balances/cancels attraction;
- $\theta > 0 \Leftrightarrow$ attraction dominates.

For the case $\beta = \delta$, the main results of [38] asserted that: (1) if $\theta \le 0$, then the ARKS model (1.1) has a unique classical global solution which converges to a unique constant steady state asymptotically in time for both $\tau_1 = \tau_2 = 0$ and $\tau_1 = \tau_2 = 1$; (2) if $\theta > 0$, the solution may blow up in finite time in two dimensions if the cell mass is larger than a threshold number for $\tau_1 = \tau_2 = 0$. For the case $\beta \neq \delta$, it was proved in [38] that the classical solutions of (1.1) with $\theta \le 0$ exist with large data if $\tau_1 = \tau_2 = 0$ or with small data if $\tau_1 = \tau_2 = 1$, where the solution bound is independent of time in the former case and dependent on time in the latter case.

Clearly the results for the cases $\beta \neq \delta$ or $\tau_1 + \tau_2 = 1$ (i.e. $\tau_1 = 1$, $\tau_2 = 0$ or $\tau_1 = 0$, $\tau_2 = 1$) or both were left open in [38]. Recently some of these open questions are solved. When $\beta \neq \delta$ and $\theta > 0$, the blowup of solutions was proved in [9] for $\tau_1 = \tau_2 = 0$. When $\beta \neq \delta$ and $\theta < 0$, the global classical solutions with uniform-in-time bound were established in [25] for $\tau_1 = \tau_2 = 1$. So far, all the results are obtained either for $\tau_1 + \tau_2 = 0$ or for $\tau_1 + \tau_2 = 2$, where the dual gradient in the first equation of the ARKS model can be reduced to a single gradient with a transformation (see the details in [38]). Up to date, the result $\tau_1 + \tau_2 = 1$ completely remains open. The main difficulty of such problem lies in their irreducibility to a two-component classical chemotaxis model even for the simplified case $\beta = \delta$ such that conventional methods and techniques can be utilized as done in [38]. The purpose of this paper will be to make a substantial step forward towards one of these open questions mentioned above, and hope our results may shed lights on the studies of remaining cases. Specifically we shall consider the case $\tau_1 = 1$, $\tau_2 = 0$ for all $\chi, \xi, \alpha, \beta, \gamma, \delta > 0$ and $\theta \in \mathbb{R}$. In particular, our results will include the case $\beta \neq \delta$ which also remains as one of the afore-mentioned open questions except for $\tau_1 = \tau_2 = 0$. A key element in our analysis is a Lyapunov functional that we find for the irreducible three-component ARKS model (1.1), which enables us to study the boundedness of solutions and the critical mass phenomenon. The main results are stated as follows.

Theorem 1.1. *Assume that* $0 \le (u_0, v_0) \in [W^{1,\infty}(\Omega)]^2$ *and* $\chi, \xi, \alpha, \beta, \gamma, \delta > 0$. *Then if* $\theta \le 0$ *(repulsion dominates or balances attraction), there exists a unique triple* (u, v, w) *of nonnegative functions in* $C(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty))$ *which solves* (1.1) *with* $\tau_1 = 1$ *and* $\tau_2 = 0$ *classically such that*

$$\|u(\cdot,t)\|_{L^\infty} \le C$$

where C *is a constant independent of* t.

Remark 1.1. For the case $\tau_1 = 1$, $\tau_2 = 0$, the ARKS model (1.1) is irreducible to a two-component chemotaxis model. Here we succeed in finding a Lyapunov functional to prove the uniform-in-time boundedness of solutions, which was not found in [38]. As we know, it is the first result that presents a Lyapunov functional for an irreducible three component attraction–repulsion chemotaxis model. However it still remains unknown if there is a Lyapunov functional for the case $\tau_1 = \tau_2 = 1$ or $\tau_1 = 0$ and $\tau_2 = 1$ if $\beta \ne \delta$.

Theorem 1.2. *Let the assumptions in Theorem 1.1 hold and let* $M = \int_\Omega u_0(x)dx$. *If* $\theta > 0$ *(attraction dominates), then the following two conclusions hold:*

(i) If $M < \frac{4\pi}{\theta}$, *then the system (1.1) with* $\tau_1 = 1$ *and* $\tau_2 = 0$ *admits a unique classical solution* $(u, v, w) \in C(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty))$ *such that* $\|u(\cdot,t)\|_{L^\infty} \le C$ *for a constant* C *independent of* t.
(ii) If $M > \frac{4\pi}{\theta}$ *and* $M \notin \{\frac{4\pi m}{\theta} : m \in \mathbb{N}^+\}$ *where* $\mathbb{N}^+$ *denotes the set of positive integers, then there exist initial data such that the solutions of* (1.1) *with* $\tau_1 = 1$ *and* $\tau_2 = 0$ *blow up in finite or infinite time.*

The results in Theorem 1.1 and Theorem 1.2 cover the situation $\beta \ne \delta$, which was left in [38] as a major open question. Our results in this paper, together with the previous results in [9,25,38], show that solution behaviors of time-dependent ARKS model, including boundedness, blowup and critical mass, are independent of the values of parameters β and δ (they only rely on the sign of $\theta = \chi\alpha - \xi\gamma$). It seems that $\beta = \delta$ and $\beta \ne \delta$ make no difference to the time-dependent solutions. It turns out this is only partially true. It was shown in [27] that the time-periodic solution of the system (1.1) is impossible for $\beta = \delta$, however, it does occur for $\beta \ne \delta$. We also point out that the critical mass phenomenon for the three-component chemotaxis model with two species and one signal was studied in [8,20], which is apparently different from the ARKS model (1.1) which contains one species and two signals.

Our results in Theorem 1.1 and Theorem 1.2 show that the ARKS model (1.1) admits globally bounded solution if the repulsion dominates (i.e. $\theta \le 0$), but has a critical mass phenomenon if attraction dominates (i.e. $\theta > 0$). Since blowup is generally not accepted as an interpretation for the aggregation process and it is unknown if the existing globally bounded solution (including the case $\theta \le 0$ and subcritical case for $\theta > 0$) approaches a constant asymptotically, the critical mass phenomenon is insufficient to indicate the pattern formation. The numerical simulations performed in [22,38] have shown that the above-mentioned global solutions actually converge to constant asymptotically. Hence the ARKS model (1.1) appears to be inadequate to explain the aggregation phase of *Microglia* in Alzheimer's disease from the results obtained in this paper together with previously existing results in [9,25,38]. But the existence of critical mass phenomenon strongly indicates that the ARKS model (1.1) may provide a useful basic PDE framework to model the aggregates of *Microglia* resulting from the interaction of attraction and repulsion. Hence to understand the complete dynamics and the validity of the model, further mathematical study is demanded and new modeling ideas might be needed in order to fully interpret the aggregation phase occurring in Alzheimer's disease. We are currently working on such issue in a separate paper [22].

2. Basic inequalities

For reader's convenience, we present a few known inequalities which will be frequently used in the paper.

Lemma 2.1. *(See [24].) Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Assume there is a constant $C > 0$ such that*

$$\|u\|_{L^s} \le C, \quad \text{for all } t \in (0, T).$$

If $v_0 \in W^{1,\infty}(\Omega)$, then there exists some constant C_q such that for every $t \in (0, T)$ and $1 \le s < n$, the solution of the problem

$$v_t = \Delta v + \alpha u - \beta v \text{ in } \Omega, \ \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega$$

satisfies

$$\|v\|_{W^{1,q}} \le C_q \tag{2.1}$$

for all $q < \frac{ns}{n-s}$. If $s = n$, then (2.1) is true for all $q < \infty$, and if $s > n$, then (2.1) is true with $q = \infty$.

Lemma 2.2 *(Trudinger–Moser inequality). (See [30].) Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary. Then for any $\varepsilon > 0$ there exist a constant C_ε depending on ε and Ω such that*

$$\int_\Omega \exp|u| dx \le C_\varepsilon \exp\left\{\left(\frac{1}{8\pi} + \varepsilon\right)\|\nabla u\|_{L^2}^2 + \frac{1}{|\Omega|}\|u\|_{L^1}\right\}. \tag{2.2}$$

Lemma 2.3. *(See [10].) Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Assume $1 \le p < n$ and $u \in W^{1,p}(\Omega)$. Then $u \in L^{p^*}(\Omega)$ with the estimate*

$$\|u\|_{L^{p^*}} \le C\|u\|_{W^{1,p}}, \tag{2.3}$$

where $p^* = \frac{np}{n-p}$ and the constant C depends only on p, n and Ω.

Lemma 2.4. *(See [30].) Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary. Then for any $\varepsilon > 0$, there exists a positive constant C_ε such that*

$$\|u\|_{L^3} \le \varepsilon \|\nabla u\|_{L^2}^{\frac{2}{3}} \|u \ln u\|_{L^1}^{\frac{1}{3}} + C_\varepsilon(\|u \ln u\|_{L^1} + \|u\|_{L^1}^{\frac{1}{3}}). \tag{2.4}$$

Lemma 2.5 *(Gagliardo–Nirenberg inequality). (See [11].) Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Let l and k be any integers satisfying $0 \le l < k$, and let $1 \le q, r \le \infty$, and $p \in \mathbb{R}^+$, $\frac{l}{k} \le a \le 1$ such that*

$$\frac{1}{p} - \frac{l}{n} = a\left(\frac{1}{q} - \frac{k}{n}\right) + (1-a)\frac{1}{r}. \tag{2.5}$$

Then, for any $u \in W^{k,q}(\Omega) \cap L^r(\Omega)$, there exists a constant c depending only on Ω, q, k, r and n such that:

$$\|D^l u\|_{L^p} \le c(\|D^k u\|_{L^q}^a \|u\|_{L^r}^{1-a} + \|u\|_{L^r}), \tag{2.6}$$

with the following exception: if $1 < q < \infty$ and $k - l - \frac{n}{q}$ is a nonnegative integer, then (2.6) holds only for a satisfying $\frac{l}{k} \le a < 1$.

3. Preliminaries on boundedness

With $\tau_1 = 1$ and $\tau_2 = 0$, the system (1.1) becomes the following one:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi u \nabla v) + \nabla \cdot (\xi u \nabla w), & x \in \Omega,\ t > 0, \\ v_t = \Delta v + \alpha u - \beta v, & x \in \Omega,\ t > 0, \\ 0 = \Delta w + \gamma u - \delta w, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{3.1}$$

The local existence theorem of (3.1) can be proved by the fixed point theorem and maximum principle along the same line shown in [38].

Lemma 3.1. *Assume that $0 \le (u_0, v_0) \in [W^{1,\infty}(\Omega)]^2$. Then there exist $T_{max} \in (0, \infty]$ and a unique triple (u, v, w) of nonnegative functions from $C(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}))$ solving (3.1) classically in $\Omega \times (0, T_{max})$. Moreover $u > 0$ in $\Omega \times (0, T_{max})$ and*

$$\text{if } T_{max} < \infty, \text{ then } \|u(\cdot,t)\|_{L^\infty} \to \infty \text{ as } t \nearrow T_{max}. \tag{3.2}$$

By the blowup criterion given in Lemma 3.1, it suffices to derive $\|u(\cdot,t)\|_{L^\infty} < \infty$ for all $t > 0$ to obtain the global-in-time solutions. In this section, we will present the basic framework used in this paper to derive the boundedness of solutions of system (3.1). We first notice that L^1-norm of the solutions of (3.1) is bounded by integrating equations of (3.1) over Ω.

Lemma 3.2. *The solution (u, v, w) of (3.1) satisfies the following properties*

$$\|u(\cdot,t)\|_{L^1} = \|u_0\|_{L^1} :\equiv M, \tag{3.3}$$

$$\|v(\cdot,t)\|_{L^1} = \frac{\alpha}{\beta}\|u_0\|_{L^1} - \left(\frac{\alpha}{\beta}\|u_0\|_{L^1} - \|v_0\|_{L^1}\right)e^{-\beta t}, \tag{3.4}$$

$$\|w(\cdot,t)\|_{L^1} = \frac{\gamma}{\delta}\|u_0\|_{L^1}. \tag{3.5}$$

Next we give a lemma concerning the uniform-in-time bound of $\|u\|_{L^2}$ irrespective of the sign of $\theta = \chi\alpha - \xi\gamma$. This result will be essentially used to prove the boundedness of solutions for both $\theta \le 0$ and $\theta > 0$. In the sequel, we use C_i or c_i, $i = 1, 2, 3, \cdots$, to denote generic constants which may vary in the context.

Lemma 3.3. *If we can find a constant $C_1 > 0$ such that the solution of (3.1) satisfies*

$$\|u \ln u\|_{L^1} + \int_0^t \|v_t(\tau)\|_{L^2}^2 d\tau \le C_1, \tag{3.6}$$

then there exists a constant $C_2 > 0$ such that the solution of (3.1) satisfies

$$\|u\|_{L^2} \le C_2. \tag{3.7}$$

Proof. Multiplying the first equation of (3.1) by u, integrating the result with respect to x, and using the second and third equation of (3.1), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega |\nabla u|^2 dx \\
&= \frac{\chi}{2}\int_\Omega \nabla u^2 \cdot \nabla v dx - \frac{\xi}{2}\int_\Omega \nabla u^2 \cdot \nabla w dx \\
&= -\frac{\chi}{2}\int_\Omega u^2(v_t - \alpha u + \beta v) dx + \frac{\xi}{2}\int_\Omega u^2(\delta w - \gamma u) dx \\
&= \frac{\chi\alpha - \xi\gamma}{2}\int_\Omega u^3 dx + \frac{\xi\delta}{2}\int_\Omega u^2 w dx - \frac{\chi}{2}\int_\Omega u^2 v_t dx - \frac{\chi\beta}{2}\int_\Omega u^2 v dx \\
&\le \frac{\theta}{2}\int_\Omega u^3 dx + \frac{\xi\delta}{2}\int_\Omega u^2 w dx - \frac{\chi}{2}\int_\Omega u^2 v_t dx,
\end{aligned}$$

which yields

$$\frac{d}{dt}\int_\Omega u^2 dx + 2\int_\Omega |\nabla u|^2 dx \le \xi\delta \int_\Omega u^2 w dx - \chi \int_\Omega u^2 v_t dx + |\theta| \int_\Omega u^3 dx. \tag{3.8}$$

Next, we estimate the first term on the right-hand side in (3.8). By the Young's inequality:

$$ab \le \varepsilon a^q + (\varepsilon q)^{-r/q} r^{-1} b^r \text{ for any } a, b \ge 0,\ \varepsilon > 0,\ q,\ r > 0,\ \frac{1}{q} + \frac{1}{r} = 1, \tag{3.9}$$

we have

$$\xi\delta\int_\Omega u^2 w dx \le \frac{1}{2}\int_\Omega u^3 dx + \frac{16}{27}(\xi\delta)^3\int_\Omega w^3 dx. \tag{3.10}$$

The combination of (3.8) and (3.10) yields that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^2 dx + 2\int_\Omega |\nabla u|^2 dx\\ &\le \frac{1+2|\theta|}{2}\int_\Omega u^3 dx + \frac{16}{27}(\xi\delta)^3\int_\Omega w^3 dx - \chi\int_\Omega u^2 v_t dx.\end{aligned} \tag{3.11}$$

To estimate the term $\int_\Omega w^3 dx$, we apply the Agmon–Douglis–Nirenberg L^p-estimates [1,2] to the following linear elliptic equations with zero Neumann boundary conditions:

$$\begin{cases} -\Delta w + \delta w = \gamma u, & \text{in } \Omega\\ \frac{\partial w}{\partial \nu} = 0, & \text{on } \partial\Omega \end{cases}$$

where $\delta > 0$, and find a constant c_1 such that

$$\|w(\cdot,t)\|_{W^{2,p}} \le c_1 \|u(\cdot,t)\|_{L^p}. \tag{3.12}$$

Specially, we choose $p = 2$ in (3.12) to obtain

$$\|w(\cdot,t)\|_{W^{2,2}} \le c_1 \|u(\cdot,t)\|_{L^2}. \tag{3.13}$$

The by the Sobolev embedding inequality, Hölder inequality and interpolation inequality $\|u\|_{L^2} \le \|u\|_{L^1}^{\frac{1}{4}}\|u\|_{L^3}^{\frac{3}{4}} = M^{\frac{1}{4}}\|u\|_{L^3}^{\frac{3}{4}}$, we have

$$\|w\|_{L^3}^3 \le c_2\|w\|_{W^{2,2}}^3 \le c_3\|u\|_{L^2}^3 \le c_3|M|^{\frac{3}{4}}\|u\|_{L^3}^{\frac{9}{4}}$$

which, combined with the Young's inequality, yields a constant $c_4 > 0$ such that

$$\frac{16}{27}(\xi\delta)^3\int_\Omega w^3 dx \le \frac{1}{2}\|u\|_{L^3}^3 + c_4. \tag{3.14}$$

Inserting (3.14) into (3.11), we obtain that

$$\frac{d}{dt}\int_\Omega u^2 dx + 2\int_\Omega |\nabla u|^2 dx \le (1+|\theta|)\int_\Omega u^3 dx - \chi\int_\Omega u^2 v_t dx + c_4. \tag{3.15}$$

Furthermore by Hölder and Gagliardo–Nirenberg inequalities, we have

$$
\begin{aligned}
-\chi\int_\Omega u^2 v_t dx &\le \chi\left\|v_t\right\|_{L^2}\left\|u\right\|_{L^4}^2\\
&\le c_5\left\|v_t\right\|_{L^2}\left(\|\nabla u\|_{L^2}^{\frac12}\|u\|_{L^2}^{\frac12}+\|u\|_{L^2}\right)^2\\
&\le 2c_5\left\|v_t\right\|_{L^2}\left(\|\nabla u\|_{L^2}\left\|u\right\|_{L^2}+\|u\|_{L^2}^2\right)\\
&\le \|\nabla u\|_{L^2}^2+\left(c_5^2\left\|v_t\right\|_{L^2}^2+2c_5\left\|v_t\right\|_{L^2}\right)\left\|u\right\|_{L^2}^2. \qquad (3.16)
\end{aligned}
$$

Collecting (3.15) and (3.16) with (2.4), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left\|u\right\|_{L^2}^2+\left\|\nabla u\right\|_{L^2}^2\\
&\le(1+|\theta|)\|u\|_{L^3}^3+\left(c_5^2\left\|v_t\right\|_{L^2}^2+2c_5\left\|v_t\right\|_{L^2}\right)\left\|u\right\|_{L^2}^2+c_4\\
&\le\varepsilon^3(1+|\theta|)\left\|\nabla u\right\|_{L^2}^2\left\|u\ln u\right\|_{L^1}+c_6(\|u\ln u\|_{L^1}^3+\|u\|_{L^1})\\
&\quad+(c_5^2\left\|v_t\right\|_{L^2}^2+2c_5\left\|v_t\right\|_{L^2})\left\|u\right\|_{L^2}^2+c_4. \qquad (3.17)
\end{aligned}
$$

Using the facts $\|u\ln u\|_{L^1}\le c_7$ from the condition in Lemma 3.3 and $\|u\|_{L^1}=M$, we let ε be small enough such that $\varepsilon^3(1+|\theta|)\left\|u\ln u\right\|_{L^1}<\frac12$, and have from (3.17)

$$
\frac{d}{dt}\left\|u\right\|_{L^2}^2+\frac12\|\nabla u\|_{L^2}^2\le\left(c_5^2\left\|v_t\right\|_{L^2}^2+2c_5\left\|v_t\right\|_{L^2}\right)\left\|u\right\|_{L^2}^2+c_8. \qquad (3.18)
$$

On the other hand the Gagliardo–Nirenberg inequality and Cauchy–Schwarz inequality with the fact $\|u\|_{L^1}=M$ yield

$$
\|u\|_{L^2}^2\le c_9(\|\nabla u\|_{L^2}\|u\|_{L^1}+\|u\|_{L^1}^2)\le\frac12\|\nabla u\|_{L^2}^2+c_{10}. \qquad (3.19)
$$

Then adding (3.18) and (3.19), and using the Young's inequality, we can find two constants $c_{11}:=c_8+c_{10}$ and $c_{12}:=3c_5^2$ such that

$$
\begin{aligned}
\frac{d}{dt}\left\|u\right\|_{L^2}^2+\left\|u\right\|_{L^2}^2&\le\left(c_5^2\left\|v_t\right\|_{L^2}^2+2c_5\left\|v_t\right\|_{L^2}\right)\left\|u\right\|_{L^2}^2+c_8+c_{10}\\
&\le\left(c_5^2\left\|v_t\right\|_{L^2}^2+2c_5^2\left\|v_t\right\|_{L^2}^2+\frac12\right)\left\|u\right\|_{L^2}^2+c_{11}\\
&=c_{12}\left\|v_t\right\|_{L^2}^2\left\|u\right\|_{L^2}^2+\frac12\left\|u\right\|_{L^2}^2+c_{11},
\end{aligned}
$$

which yields

$$
\frac{d}{dt}\left\|u\right\|_{L^2}^2+\left(\frac12-c_{12}\left\|v_t\right\|_{L^2}^2\right)\left\|u\right\|_{L^2}^2\le c_{11}. \qquad (3.20)
$$

By the Gronwall's inequality, it follows that

$$\|u\|_{L^2}^2 \le \|u_0\|_{L^2}^2 e^{-\int_0^t (1/2 - c_{12}\|v_t(\tau)\|_{L^2}^2) d\tau} + c_{11} \int_0^t e^{-\int_s^t (1/2 - c_{12}\|v_t(\tau)\|_{L^2}^2) d\tau} ds.$$

With (3.6), a simple calculation yields a constant c_{13} such that $\|u\|_{L^2}^2 \le c_{13}$. The proof of this lemma is completed. □

Lemma 3.4. *If (3.7) holds, then there exists a constant C independent of t such that the solution (u, v, w) of (3.1) satisfies*

$$\|(\nabla v, \nabla w)\|_{L^\infty} \le C. \tag{3.21}$$

Proof. First, the combination of (3.12) and (3.7) generates a constant $c_1 > 0$ such that

$$\|w\|_{W^{2,2}} \le c_1. \tag{3.22}$$

Using the Gagliardo–Nirenberg inequality, (3.5) and (3.22), one can find two constants $c_2, c_3 > 0$ such that

$$\|\nabla w\|_{L^4} \le c_2 \left(\|D^2 w\|_{L^2}^{\frac{5}{6}} \|w\|_{L^1}^{\frac{1}{6}} + \|w\|_{L^1} \right) \le c_2 \left(c_1^{\frac{5}{6}} \left(\frac{\gamma}{\delta}\right)^{\frac{1}{6}} M^{\frac{1}{6}} + \frac{\gamma M}{\delta} \right) = c_3. \tag{3.23}$$

Furthermore, from Lemma 2.1 and (3.7), we obtain a constant $c_4 > 0$ such that

$$\|\nabla v\|_{L^4} \le c_4. \tag{3.24}$$

Next, we will prove (3.21) by using (3.23) and (3.24). Multiplying the first equation of (3.1) by u^2 to get that

$$\begin{aligned}
&\frac{1}{3}\frac{d}{dt}\int_\Omega u^3 dx + \frac{8}{9}\int_\Omega |\nabla u^{\frac{3}{2}}|^2 dx \\
&= 2\chi \int_\Omega u^2 \nabla u \cdot \nabla v dx - 2\xi \int_\Omega u^2 \nabla u \cdot \nabla w dx \\
&\le \frac{4\chi}{3} \int_\Omega \left| u^{\frac{3}{2}} \nabla u^{\frac{3}{2}} \cdot \nabla v \right| dx + \frac{4\xi}{3} \int_\Omega \left| u^{\frac{3}{2}} \nabla u^{\frac{3}{2}} \cdot \nabla w \right| dx.
\end{aligned} \tag{3.25}$$

Applying Hölder inequality and the Gagliardo–Nirenberg inequality, and using (3.23) and (3.24), we have

$$
\begin{aligned}
&\frac{4\chi}{3}\int_\Omega \left|u^{\frac{3}{2}}\nabla u^{\frac{3}{2}}\cdot\nabla v\right|dx+\frac{4\xi}{3}\int_\Omega \left|u^{\frac{3}{2}}\nabla u^{\frac{3}{2}}\cdot\nabla w\right|dx\\
&\le \frac{1}{9}\int_\Omega |\nabla u^{\frac{3}{2}}|^2dx+4\chi^2\int_\Omega u^3|\nabla v|^2dx+\frac{2}{9}\int_\Omega |\nabla u^{\frac{3}{2}}|^2dx+2\xi^2\int_\Omega u^3|\nabla w|^2dx\\
&\le \frac{1}{3}\|\nabla u^{\frac{3}{2}}\|_{L^2}^2+4\chi^2\|u^{\frac{3}{2}}\|_{L^4}^2\|\nabla v\|_{L^4}^2+2\xi^2\|u^{\frac{3}{2}}\|_{L^4}^2\|\nabla w\|_{L^4}^2\\
&= \frac{1}{3}\|\nabla u^{\frac{3}{2}}\|_{L^2}^2+\left(4c_4^2\chi^2+2c_3^2\xi^2\right)\|u^{\frac{3}{2}}\|_{L^4}^2\\
&\le \frac{1}{3}\|\nabla u^{\frac{3}{2}}\|_{L^2}^2+c_5\left(\|\nabla u^{\frac{3}{2}}\|_{L^2}^{\frac{4}{3}}\|u^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^{\frac{2}{3}}+\|u^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^2\right)\\
&\le \frac{1}{3}\|\nabla u^{\frac{3}{2}}\|_{L^2}^2+c_5c_6^{\frac{2}{3}}\|\nabla u^{\frac{3}{2}}\|_{L^2}^{\frac{4}{3}}+c_2c_6^2\\
&\le \frac{5}{9}\|\nabla u^{\frac{3}{2}}\|_{L^2}^2+c_7,
\end{aligned}
\tag{3.26}
$$

where we have used the inequality $\|u^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}=\|u^2\|_{L^2}^{\frac{3}{4}}\le c_6$, and the following estimate

$$
c_5c_6^{\frac{2}{3}}\|\nabla u^{\frac{3}{2}}\|_{L^2}^{\frac{4}{3}}+c_2c_6^2\le\frac{2}{9}\|\nabla u^{\frac{3}{2}}\|_{L^2}^2+c_7.
$$

Substituting (3.26) into (3.25), we have that

$$
\frac{d}{dt}\int_\Omega u^3dx+\|\nabla u^{\frac{3}{2}}\|_{L^2}^2\le 3c_7. \tag{3.27}
$$

Furthermore the Gagliardo–Nirenberg inequality gives

$$
\|u^{\frac{3}{2}}\|_{L^2}\le c_8\left(\|\nabla u^{\frac{3}{2}}\|_{L^2}^{\frac{1}{3}}\|u^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^{\frac{2}{3}}+\|u^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}\right)\le c_8\left(\|\nabla u^{\frac{3}{2}}\|_{L^2}^{\frac{1}{3}}c_6^{\frac{2}{3}}+c_6\right)
$$

by which we find two constant $c_9, c_{10}>0$ by using (3.7) such that

$$
\int_\Omega u^3dx=\|u^{\frac{3}{2}}\|_{L^2}^6\le\frac{1}{c_9}\|\nabla u^{\frac{3}{2}}\|_{L^2}^2+c_{10}. \tag{3.28}
$$

Inserting (3.28) into (3.27), we have

$$
\frac{d}{dt}\|u\|_{L^3}^3+c_9\|u\|_{L^3}^3\le 3c_7+c_9c_{10}=c_{11},
$$

which, along with Gronwall's inequality, implies

$$
\|u\|_{L^3}^3\le e^{-c_9t}\|u_0\|_{L^3}^3+\frac{c_{11}}{c_9}\le c_{12}. \tag{3.29}
$$

Then using Lemma 2.1 and (3.29), we can find a constant $c_{13} > 0$ such that

$$\|\nabla v\|_{L^\infty} \le c_{13}. \tag{3.30}$$

Furthermore, from (3.12) and (3.29), one has $\|w(\cdot,t)\|_{W^{2,3}} \le c_{14}$ which, along with the Sobolev embedding theorem, asserts that $\|\nabla w\|_{L^\infty} \le c_{14}$. This, combined with (3.30), completes the proof of the lemma. □

Next we shall show that (3.6) is a sufficient condition to ensure the global boundedness of solutions of (3.1). To this end we cite the following known result (see [14], Lemma 1) which was proved based on the iteration method (e.g., see [3]).

Lemma 3.5. *Let the components of the vector field $\Phi : \Omega \times (0,\infty) \to \mathbb{R}^n$ be uniformly bounded, and let $u_0 \in L^\infty(\Omega) \cap L^1(\Omega)$ with $u_0 \ge 0$. If $u \in C(\bar{\Omega} \times [0,T)) \cap C^{2,1}(\bar{\Omega} \times (0,T))$ is a solution of the following initial–boundary value problem:*

$$\begin{cases} u_t = \nabla \cdot (\nabla u - u\Phi), & x \in \Omega,\ t > 0, \\ (\nabla u - u\Phi) \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega \end{cases}$$

then there exists a constant $c > 0$, only depending on $\|\Phi\|_{L^\infty(\Omega)}$, $\|u_0\|_{L^1(\Omega)}$ and $\|u_0\|_{L^\infty(\Omega)}$, such that

$$\|u(t)\|_{L^\infty(\Omega)} \le c \qquad \text{for all } t \in (0,T).$$

Then the following lemma concerning the global existence of classical solutions of (3.1) with uniform-in-time bound can be proved.

Lemma 3.6. *Assume that $0 \le (u_0, v_0) \in [W^{1,\infty}(\Omega)]^2$. If (3.6) holds, then there exists a unique triple (u, v, w) of nonnegative functions belonging to $C(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty))$ which solves (3.1) classically such that $\|u(\cdot,t)\|_{L^\infty} \le C$, where C is a constant independent of t.*

Proof. If (3.6) holds, then from Lemma 3.3, we can find a constant $c_1 > 0$ such that $\|u\|_{L^2} \le c_1$. Then using Lemma 3.4, we can find a constant $c_2 > 0$ such that

$$\|(\nabla v, \nabla w)\|_{L^\infty} \le c_2. \tag{3.31}$$

Now we write the first equation of (3.1) as $u_t = \nabla \cdot (\nabla u - u\Phi)$ with $\Phi = \chi \nabla v - \xi \nabla w$. Note that the zero Neumann boundary condition implies the zero-flux boundary condition in Lemma 3.5. Then the application of Lemma 3.5 with (3.31) produces a constant $c_3 > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty} \le c_3 \quad \text{for all } t \in (0,T). \tag{3.32}$$

Thus the assertion of Lemma 3.6 is an immediate consequence of (3.32) and Lemma 3.1. □

From Lemma 3.6, we see that it suffices to prove (3.6) to obtain the global existence of classical solutions of (3.1). In the subsequent sections, we shall show that (3.6) indeed holds either for $\theta \le 0$ or for $\theta > 0$ and $M \le \frac{4\pi}{\theta}$.

4. Boundedness for $\theta \le 0$

In this section, we are devoted to proving Theorem 1.1. Although the ARKS model (3.1) is irreducible to the classical two-component chemotaxis model, we are fortunately able to find a Lyapunov functional:

$$F(u,v,w) = \int_\Omega u\ln u dx + \frac{\chi}{2\alpha}\int_\Omega (\beta v^2 + |\nabla v|^2)dx + \frac{\xi}{2\gamma}\int_\Omega (\delta w^2 + |\nabla w|^2)dx - \chi\int_\Omega uvdx. \tag{4.1}$$

Lemma 4.1. *Let $F(u,v,w)$ be defined in (4.1). Then the solutions of (3.1) satisfy*

$$\frac{d}{dt}F(u,v,w) + G(u,v,w) = 0, \tag{4.2}$$

where

$$G(u,v,w) = \frac{\chi}{\alpha}\int_\Omega v_t^2 dx + \int_\Omega u|\nabla(\ln u - \chi v + \xi w)|^2 dx. \tag{4.3}$$

Proof. Multiplying the first equation of (3.1) by $\ln u - \chi v + \xi w$ and integrating the result with respect to x over Ω, we have

$$\begin{aligned}\int_\Omega u_t(\ln u - \chi v + \xi w)dx &= \int_\Omega \nabla\cdot(\nabla u - \chi u\nabla v + \xi u\nabla w)(\ln u - \chi v + \xi w)dx \\ &= -\int_\Omega u|\nabla(\ln u - \chi v + \xi w)|^2 dx. \end{aligned}\tag{4.4}$$

Using the fact that $\int_\Omega u_t dx = 0$, we have

$$\begin{aligned}&\int_\Omega u_t(\ln u - \chi v + \xi w)dx \\ &= \frac{d}{dt}\int_\Omega u\ln u dx - \chi\frac{d}{dt}\int_\Omega uvdx + \chi\int_\Omega uv_t dx + \xi\int_\Omega u_t w dx. \end{aligned}\tag{4.5}$$

From the second equation of (3.1), one has $u = \frac{1}{\alpha}v_t - \frac{1}{\alpha}\Delta v + \frac{\beta}{\alpha}v$, which gives

$$\int_\Omega uv_t dx = \frac{1}{\alpha}\int_\Omega v_t^2 dx + \frac{1}{2\alpha}\frac{d}{dt}\int_\Omega |\nabla v|^2 dx + \frac{\beta}{2\alpha}\frac{d}{dt}\int_\Omega v^2 dx. \tag{4.6}$$

Similarly, from the third equation of (3.1), we can derive that

$$\int_\Omega u_t w dx = \frac{\delta}{2\gamma}\frac{d}{dt}\int_\Omega w^2 dx + \frac{1}{2\gamma}\frac{d}{dt}\int_\Omega |\nabla w|^2 dx. \tag{4.7}$$

The combination of (4.5), (4.6) and (4.7) leads to

$$\begin{aligned}
&\int_\Omega u_t(\ln u - \chi v + \xi w)dx \\
&= \frac{d}{dt}\int_\Omega \left(u\ln u - \chi uv + \frac{\beta\chi}{2\alpha}v^2 + \frac{\chi}{2\alpha}|\nabla v|^2 + \frac{\xi\delta}{2\gamma}w^2 + \frac{\xi}{2\gamma}|\nabla w|^2\right)dx + \frac{\chi}{\alpha}\int_\Omega v_t^2 dx,
\end{aligned}$$

which together with (4.4) leads to (4.1). The proof of Lemma 4.1 is completed. □

Next, we will prove Theorem 1.1 by using the Lyapunov functional (4.1) for the case $\theta \le 0$.

Proof of Theorem 1.1. From Lemma 3.6, Theorem 1.1 can be proved directly if (3.6) holds. Next, we will show if $\theta \le 0$, (3.6) actually holds. First we rewrite the third equation of (3.1) as

$$u = \frac{\delta}{\gamma}w - \frac{1}{\gamma}\Delta w. \tag{4.8}$$

Then using (4.8) and the Cauchy–Schwarz inequality, one can derive that

$$\begin{aligned}
\chi\int_\Omega uv dx &= \frac{\chi\delta}{\gamma}\int_\Omega vw dx + \frac{\chi}{\gamma}\int_\Omega \nabla w\cdot\nabla v dx \\
&\le \frac{\chi\delta}{\gamma}\left(\frac{\xi}{2\chi}\int_\Omega w^2 dx + \frac{\chi}{2\xi}\int_\Omega v^2 dx\right) + \frac{\chi}{\gamma}\left(\frac{\xi}{2\chi}\int_\Omega |\nabla w|^2 dx + \frac{\chi}{2\xi}\int_\Omega |\nabla v|^2 dx\right) \\
&= \frac{\xi\delta}{2\gamma}\int_\Omega w^2 dx + \frac{\chi^2\delta}{2\xi\gamma}\int_\Omega v^2 dx + \frac{\xi}{2\gamma}\int_\Omega |\nabla w|^2 dx + \frac{\chi^2}{2\xi\gamma}\int_\Omega |\nabla v|^2 dx.
\end{aligned} \tag{4.9}$$

Substituting (4.9) into (4.1), we have

$$\begin{aligned}
F(u,v,w) &\ge \int_\Omega u\ln u dx + \left(\frac{\beta\chi}{2\alpha} - \frac{\chi^2\delta}{2\xi\gamma}\right)\int_\Omega v^2 dx + \left(\frac{\chi}{2\alpha} - \frac{\chi^2}{2\xi\gamma}\right)\int_\Omega |\nabla v|^2 dx \\
&= \int_\Omega u\ln u dx + \frac{\chi(\xi\gamma\beta - \chi\alpha\delta)}{2\alpha\xi\gamma}\int_\Omega v^2 dx + \frac{\chi(\xi\gamma - \chi\alpha)}{2\xi\gamma\alpha}\int_\Omega |\nabla v|^2 dx.
\end{aligned} \tag{4.10}$$

Integrating (4.2) with respect to t and using (4.10), we have

$$\int_\Omega u\ln u dx + \frac{\chi(\xi\gamma-\chi\alpha)}{2\xi\gamma\alpha}\int_\Omega |\nabla v|^2 dx + \frac{\chi}{\alpha}\int_0^t\int_\Omega v_t^2 dxd\tau$$
$$+\int_0^t\int_\Omega u|\nabla(\ln u-\chi v+\xi w)|^2 dxd\tau \le F(u_0,v_0) + \frac{\chi|\xi\gamma\beta-\chi\alpha\delta|}{2\alpha\xi\gamma}\int_\Omega v^2 dx. \quad (4.11)$$

To complete the proof of this lemma, it remains to estimate the last term of (4.11). Using Lemma 2.1 and $u\in L^1(\Omega)$, we can find a constant $c_1>0$ such that $\|v\|_{W^{1,p}}\le c_1$ for all $1\le p<2$. Hence using Lemma 2.3 and choosing $p=1$, we obtain

$$\|v\|_{L^2}\le c_2\|v\|_{W^{1,1}}\le c_1c_2. \quad (4.12)$$

Substituting (4.12) into (4.11) and using the condition $\xi\gamma-\chi\alpha\ge 0$, we have

$$\int_\Omega u\ln u dx + \frac{\chi}{\alpha}\int_0^t\int_\Omega v_t^2 dxd\tau + \int_0^t\int_\Omega u|\nabla(\ln u-\chi v+\xi w)|^2 dxd\tau$$
$$\le F(u_0,v_0) + \frac{\chi|\xi\gamma\beta-\chi\alpha\delta|c_1^2c_2^2}{2\alpha\xi\gamma}\le c_3,$$

which implies

$$\int_\Omega u\ln u dx + \frac{\chi}{\alpha}\int_0^t\int_\Omega v_t^2 dxd\tau \le c_3. \quad (4.13)$$

Noticing that $u\ln u > -\frac{1}{e}$ for all $u\ge 0$, it follows from (4.13) that

$$\int_0^t\int_\Omega v_t^2 dxd\tau \le \frac{\alpha}{\chi}\left(c_3+\frac{|\Omega|}{e}\right) \quad (4.14)$$

and

$$\int_\Omega |u\ln u| dx$$
$$=\int_\Omega\left|u\ln u+\frac{1}{e}-\frac{1}{e}\right|dx \le \int_\Omega\left(u\ln u+\frac{1}{e}\right)dx + \int_\Omega \frac{1}{e}dx \le c_3+\frac{2|\Omega|}{e}. \quad (4.15)$$

Then the combination of (4.14) and (4.15) implies (3.6) holds, and hence the assertion of Theorem 1.1 follows from Lemma 3.6. □

5. Critical mass phenomenon for $\theta > 0$

In this section, we will show that if $\theta > 0$, there exists a critical value $m_* = \frac{4\pi}{\theta}$ such that the solution is bounded uniformly in time if $\int_\Omega u_0(x)dx < m_*$ (subcritical mass) and may blow up if $\int_\Omega u_0(x)dx > m_*$ (supercritical mass).

5.1. Boundedness for subcritical mass

Lemma 5.1. *If $\theta > 0$ and $\int_\Omega u_0(x)dx < \frac{4\pi}{\theta}$, then there exists a constant $C > 0$ independent of t such that (3.6) holds.*

Proof. For convenience, we denote $F[t] = F(u, v, w)$. Then from (4.1), we have

$$F[t] = \int_\Omega u \ln u dx - \frac{\theta}{\alpha}\int_\Omega uv dx + \frac{\chi}{2\alpha}\int_\Omega (\beta v^2 + |\nabla v|^2)dx + \frac{\xi}{2\gamma}\int_\Omega (\delta w^2 + |\nabla w|^2)dx - \frac{\xi\gamma}{\alpha}\int_\Omega uv dx. \tag{5.1}$$

Using the third equation of (3.1) and the Cauchy–Schwarz inequality one can derive that

$$\begin{aligned}\frac{\xi\gamma}{\alpha}\int_\Omega uv dx &= \frac{\xi\delta}{\alpha}\int_\Omega vw dx + \frac{\xi}{\alpha}\int_\Omega \nabla w\cdot\nabla v dx \\ &\le \frac{\xi\delta}{2\gamma}\int_\Omega w^2 dx + \frac{\xi\gamma\delta}{2\alpha^2}\int_\Omega v^2 dx + \frac{\xi}{2\gamma}\int_\Omega |\nabla w|^2 dx + \frac{\xi\gamma}{2\alpha^2}\int_\Omega |\nabla v|^2 dx. \end{aligned}\tag{5.2}$$

Substituting (5.2) into (5.1), then for any $\eta > 0$ we have

$$\begin{aligned} F[t] &\ge \int_\Omega u\ln u dx - \frac{\theta}{\alpha}\int_\Omega uv dx + \frac{\theta}{2\alpha^2}\int_\Omega |\nabla v|^2 dx + \frac{\chi\alpha\beta - \xi\gamma\delta}{2\alpha^2}\int_\Omega v^2 dx \\ &= \int_\Omega u\ln u dx - \left(\frac{\theta}{\alpha}+\eta\right)\int_\Omega uv dx + \eta\int_\Omega uv dx \\ &\quad + \frac{\theta}{2\alpha^2}\int_\Omega |\nabla v|^2 dx + \frac{\chi\alpha\beta - \xi\gamma\delta}{2\alpha^2}\int_\Omega v^2 dx \\ &\ge -\int_\Omega u\ln\frac{e^{\left(\frac{\theta}{\alpha}+\eta\right)v}}{u}dx + \frac{\theta}{2\alpha^2}\int_\Omega |\nabla v|^2 dx \\ &\quad + \frac{\chi\alpha\beta - \xi\gamma\delta}{2\alpha^2}\int_\Omega v^2 dx + \eta\int_\Omega uv dx. \end{aligned}\tag{5.3}$$

Since $-\ln z$ is a convex function for all $z \geq 0$ and $\int_\Omega \frac{u}{M} dx = 1$, then using the Jensen's inequality, we obtain

$$
\begin{aligned}
-\ln\left(\frac{1}{M}\int_\Omega e^{\left(\frac{\theta}{\alpha}+\eta\right)v} dx\right) &= -\ln\left(\int_\Omega \frac{e^{\left(\frac{\theta}{\alpha}+\eta\right)v}}{u}\frac{u}{M} dx\right) \\
&\leq \int_\Omega \left(-\ln\frac{e^{\left(\frac{\theta}{\alpha}+\eta\right)v}}{u}\right)\frac{u}{M} dx \\
&= -\frac{1}{M}\int_\Omega u \ln\frac{e^{\left(\frac{\theta}{\alpha}+\eta\right)v}}{u} dx.
\end{aligned}
\tag{5.4}
$$

Collecting (5.3) and (5.4), we have

$$
\begin{aligned}
F[t] \geq& -M\ln\left(\frac{1}{M}\int_\Omega e^{\left(\frac{\theta}{\alpha}+\eta\right)v} dx\right) + \frac{\theta}{2\alpha^2}\int_\Omega |\nabla v|^2 dx \\
&+ \frac{\chi\alpha\beta - \xi\gamma\delta}{2\alpha^2}\int_\Omega v^2 dx + \eta\int_\Omega uv dx.
\end{aligned}
\tag{5.5}
$$

Using the Trudinger–Moser inequality (2.2) and the fact $\|v\|_{L^1} \leq c_1$ (see (3.4)), we can obtain two constants $c_2 > 0$ and $c_3 > 0$ depending on ε such that

$$
\begin{aligned}
\int_\Omega e^{\left(\frac{\theta}{\alpha}+\eta\right)v} dx &\leq c_2 e^{\left(\frac{1}{8\pi}+\varepsilon\right)\left(\frac{\theta}{\alpha}+\eta\right)^2\|\nabla v\|_{L^2}^2 + \frac{\frac{\theta}{\alpha}+\eta}{|\Omega|}\|v\|_{L^1}} \\
&\leq c_3 e^{\left(\frac{1}{8\pi}+\varepsilon\right)\left(\frac{\theta}{\alpha}+\eta\right)^2\|\nabla v\|_{L^2}^2}.
\end{aligned}
\tag{5.6}
$$

Substituting (5.6) into (5.5), we can find a constant $c_4 = M\ln\frac{c_3}{M}$ such that

$$
\begin{aligned}
F[t] \geq& \left[\frac{\theta}{2\alpha^2} - \left(\frac{1}{8\pi}+\varepsilon\right)\left(\frac{\theta}{\alpha}+\eta\right)^2 M\right]\int_\Omega |\nabla v|^2 dx \\
&+ \frac{\chi\alpha\beta - \xi\gamma\delta}{2\alpha^2}\int_\Omega v^2 dx + \eta\int_\Omega uv dx - c_4.
\end{aligned}
\tag{5.7}
$$

Since $M = \int_\Omega u_0 dx < \frac{4\pi}{\theta}$, we can choose $\varepsilon > 0$ and $\eta > 0$ small enough such that

$$
\frac{\theta}{2\alpha^2} - \left(\frac{1}{8\pi}+\varepsilon\right)\left(\frac{\theta}{\alpha}+\eta\right)^2 M > 0. \tag{5.8}
$$

Substituting (5.8) into (5.7) and using the fact (4.12), we can find a constant $c_5 > 0$ such that

$$
\begin{aligned}
F[t] &\geq \eta \int_\Omega uv dx + \frac{\chi\alpha\beta - \xi\gamma\delta}{2\alpha^2} \int_\Omega v^2 dx - c_4 \\
&\geq \eta \int_\Omega uv dx - \frac{|\chi\alpha\beta - \xi\gamma\delta|}{2\alpha^2} \int_\Omega v^2 dx - c_4 \\
&\geq \eta \int_\Omega uv dx - c_5,
\end{aligned} \tag{5.9}
$$

which implies

$$
F[t] \geq -c_5. \tag{5.10}
$$

Since $F[t] \leq F[0]$, then from (5.9) we see for any $\eta > 0$ that

$$
\int_\Omega uv dx \leq \frac{F[0] + c_5}{\eta}. \tag{5.11}
$$

Using (4.1) and (5.11) and the fact $F[t] \leq F[0]$, one has

$$
\begin{aligned}
\int_\Omega u \ln u dx &\leq F[t] + \chi \int_\Omega uv dx \\
&\leq F[t] + \chi \int_\Omega uv dx \leq \left(1 + \frac{\chi}{\eta}\right) F[0] + \frac{\chi c_5}{\eta} \leq c_6.
\end{aligned} \tag{5.12}
$$

Noticing again that $u \ln u \geq -\frac{1}{e}$, which along with (5.12) indicates that (see also the proof of (4.15))

$$
\int_\Omega |u \ln u| dx \leq c_6 + \frac{2|\Omega|}{e}. \tag{5.13}
$$

Integrating (4.2) with respect t, we have

$$
\frac{\chi}{\alpha} \int_0^t \int_\Omega v_t^2 dx d\tau + \int_0^t \int_\Omega u |\nabla(\ln u - \chi v + \xi w)|^2 dx d\tau \leq F[0] - F[t] \leq F[0] + c_5. \tag{5.14}
$$

The combination of (5.13) and (5.14) yields (3.6). Then the proof is completed. □

The following lemma gives the first part of Theorem 1.2.

Lemma 5.2. *Assume that* $0 \leq (u_0, v_0) \in [W^{1,\infty}(\Omega)]^2$ *and* $\theta > 0$. *If* $\int_\Omega u_0(x) dx < \frac{4\pi}{\theta}$, *then there exists a unique triple* (u, v, w) *of nonnegative bounded functions in* $C(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))$ *which solves* (3.1) *classically. Furthermore, there exists a constant* C *independent of* t *such that* $\|u(\cdot, t)\|_{L^\infty} \leq C$.

Proof. If $\theta > 0$ and $\int_\Omega u_0(x)dx < \frac{4\pi}{\theta}$, from Lemma 5.1, one has (3.6). Then Lemma 5.2 is an immediate consequence of Lemma 3.6. $\square$

5.2. Blowup for supercritical mass

In this subsection, we are devoted to proving the second part of Theorem 1.2 concerning the blowup of solutions for supercritical mass. For the convenience of constructing the initial date of blowup solutions, we introduce the following change of variables:

$$\tilde{v} = v - \bar{v}, \qquad \tilde{w} = w - \bar{w}, \tag{5.15}$$

where $\bar{f} = \frac{1}{|\Omega|}\int_\Omega f dx$. From the second and third equation of (3.1), we have $\bar{v}_t = \alpha\bar{u} - \beta\bar{v}$ and $\gamma\bar{u} = \delta\bar{w}$, respectively. Substituting these results and (5.15) into (3.1) and dropping the tildes for convenience, we obtain

$$\begin{cases} u_t = \Delta u - \nabla\cdot(\chi u \nabla v) + \nabla\cdot(\xi u\nabla w), & x\in\Omega,\ t>0,\\ v_t = \Delta v + \alpha(u-\bar{u}) - \beta v, & x\in\Omega,\ t>0,\\ 0 = \Delta w + \gamma(u-\bar{u}) - \delta w, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x), & x\in\Omega. \end{cases} \tag{5.16}$$

Then the stationary problem of (5.16) reads

$$\begin{cases} 0 = \Delta u - \nabla\cdot(\chi u \nabla v) + \nabla\cdot(\xi u\nabla w), & x\in\Omega,\ t>0,\\ 0 = \Delta v + \alpha(u-\bar{u}) - \beta v, & x\in\Omega,\ t>0,\\ 0 = \Delta w + \gamma(u-\bar{u}) - \delta w, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0,\\ \int_\Omega u dx = M,\ \int_\Omega v dx = \int_\Omega w dx = 0. \end{cases} \tag{5.17}$$

To proceed, we denote

$$\phi = \frac{v}{\alpha} - \frac{w}{\gamma},\ \tilde{\theta} = \frac{\theta}{\alpha} = \frac{\chi\alpha - \xi\gamma}{\alpha}.$$

Solving the first equation of (5.17) subject to the Neumann boundary conditions gives

$$u = \lambda e^{\chi v - \xi w} = \lambda e^{\xi\gamma\phi} e^{\frac{(\chi\alpha-\xi\gamma)v}{\alpha}} = \lambda e^{\xi\gamma\phi} e^{\tilde{\theta} v} \tag{5.18}$$

where $\lambda > 0$ is a constant satisfying

$$\lambda = \frac{\int\limits_\Omega u dx}{\int\limits_\Omega e^{\chi v - \xi w} dx} = \frac{M}{\int\limits_\Omega e^{\xi\gamma\phi} e^{\tilde{\theta} v} dx}. \tag{5.19}$$

Then substituting (5.18) into the second equation of (5.17), and using the second and third equations of (5.17), we can reduce the stationary problem (5.17) to the following one:

$$\begin{cases} -\Delta v + \beta v = \alpha\lambda e^{\xi\gamma\phi}e^{\tilde{\theta}v} - \frac{\alpha M}{|\Omega|}, & x \in \Omega, \\ -\Delta\phi + \delta\phi = \frac{(\delta-\beta)v}{\alpha}, & x \in \Omega \\ \frac{\partial v}{\partial \nu} = \frac{\partial \phi}{\partial \nu} = 0, & x \in \partial\Omega, \\ \int_\Omega v dx = \int_\Omega \phi dx = 0 \end{cases} \tag{5.20}$$

where u is determined by (5.18) under the constraint (5.19). The existence of nontrivial solutions of the problem (5.20) still remains open. This is, however, not needed to achieve our goal. We only need the following result.

Lemma 5.3. *Let (v, ϕ) satisfy (5.20). Then there exists a constant $C > 0$ such that*

$$\|\phi\|_{W^{1,\infty}} \leq C. \tag{5.21}$$

Proof. Note that $\|\alpha\lambda e^{\xi\gamma\phi}e^{\tilde{\theta}v} - \frac{\alpha M}{|\Omega|}\|_{L^1} = \alpha\|u - \bar{u}\|_{L^1} = \alpha\int_\Omega |u - \bar{u}|dx \leq 2\alpha M$. Then by the L^1-regularity theory (see [35]), it follows that $v \in W^{1,q}(\Omega)$ with $q < \frac{n}{n-1}$ with space dimension n. With the Sobolev embedding: $W^{1,\frac{6}{5}}(\Omega) \hookrightarrow L^3(\Omega)$ with $n = 2$, one has $\|v\|_{L^3} \leq c_1$. Now applying the Agmon–Douglis–Nirenberg L^p-estimate to ϕ satisfying the second equation of (5.20), we have

$$\|\phi\|_{W^{2,3}} \leq c_2\|v\|_{L^3} \leq c_1c_2,$$

which implies (5.21) by the Sobolev embedding theorem with space dimension $n = 2$. □

Noting that $F(u, v, w)$ defined by (4.1) is also a Lyapunov functional of the transformed system (5.16), we obtain the following result.

Lemma 5.4. *Suppose that (u, v, w) is a global and bounded solution of (5.16). Then there exist a sequence of times $t_k \to \infty$ and nonnegative function $(u_\infty, v_\infty, w_\infty) \in [C^2(\bar{\Omega})]^3$ such that $(u(\cdot, t_k), v(\cdot, t_k), w(\cdot, t_k)) \to (u_\infty, v_\infty, w_\infty)$ in $[C^2(\bar{\Omega})]^3$. Furthermore, $(u_\infty, v_\infty, w_\infty)$ is a solution of (5.17), such that*

$$F(u_\infty, v_\infty, w_\infty) \leq F(u_0, v_0, w_0). \tag{5.22}$$

Proof. From the boundedness of (u, v, w) and Schauder regularity theory (e.g. see [12]), it follows that $(u(\cdot, t), v(\cdot, t), w(\cdot, t))_{t>1}$ is relatively compact in $[C^2(\bar{\Omega})]^3$. Hence we can find a suitable sequence of times $(t_k)_{k\geq 1}$ such that $(u(\cdot, t_k), v(\cdot, t_k), w(\cdot, t_k)) \to (u_\infty, v_\infty, w_\infty)$ in $[C^2(\bar{\Omega})]^3$ as $t_k \to \infty$. Note that $F(u, v, w)$ is bounded from below (see (5.10)). Then Lemma 4.1 implies that

$$\frac{\chi}{\alpha}\int_0^\infty\int_\Omega v_t^2 dxd\tau + \int_0^\infty\int_\Omega u|\nabla(\ln u - \chi v + \xi w)|^2 dxd\tau < \infty. \tag{5.23}$$

Then by the Arzelà–Ascoli theorem, a sequence of times, still denoted by $(t_k)_{k\geq 1}$, can be extracted such that

$$v_t(\cdot, t_k) \to 0 \quad \text{in } L^2(\Omega) \tag{5.24}$$

and

$$u(\cdot, t_k)|\nabla(\ln u(\cdot, t_k) - \chi v(\cdot, t_k) + \xi w(\cdot, t_k))|^2 \to 0 \quad \text{a.e. in } \bar{\Omega} \tag{5.25}$$

as $t_k \to \infty$. Evaluating the second equation of (5.16) at $t = t_k$ and letting $k \to \infty$, we have

$$-\Delta v_\infty + \beta v_\infty = \alpha(u_\infty - \bar{u}). \tag{5.26}$$

Using (5.25) and taking $k \to \infty$, we obtain

$$u_\infty|\nabla(\ln u_\infty - \chi v_\infty + \xi w_\infty)|^2 = 0 \quad \text{in } \bar{\Omega}.$$

By the same argument as in [40] (details are omitted here for brevity), one can show that $u_\infty > 0$ for all $x \in \bar{\Omega}$. Hence

$$\nabla(\ln u_\infty - \chi v_\infty + \xi w_\infty) = 0 \quad \text{in } \bar{\Omega}.$$

which indicates

$$u_\infty = \lambda e^{\chi v_\infty - \xi w_\infty}, \ \lambda = \frac{M}{\int_\Omega e^{\chi v_\infty - \xi w_\infty} dx}. \tag{5.27}$$

Furthermore, from the third equation of (5.16), we have

$$-\Delta w_\infty + \delta w_\infty = \gamma(u_\infty - \bar{u}). \tag{5.28}$$

Thus the combination of (5.26), (5.27) and (5.28) shows that $(u_\infty, v_\infty, w_\infty)$ satisfy (5.17) by noting (5.18). Since $(u(\cdot, t_k), v(\cdot, t_k), w(\cdot, t_k)) \to (u_\infty, v_\infty, w_\infty)$ in $[C^2(\bar{\Omega})]^3$ and thus

$$F(u(\cdot, t_k), v(\cdot, t_k), w(\cdot, t_k)) \to F(u_\infty, v_\infty, w_\infty), \text{ as } t_k \to \infty$$

then (5.22) follows from the property $F[t] \leq F[0]$. The proof of Lemma 5.4 is completed. □

5.2.1. Lower bound for steady-state energy

Next, we use an idea in [16,17,42] to show that if $\int_\Omega u_0(x)dx \neq \frac{4\pi m}{\theta}$ for any $m \in \mathbb{N}^+$, then there exists a constant $K > 0$ such that $F(u, v, w) \geq -K$ for all solutions of system (5.17). In summary, we can obtain the following results.

Lemma 5.5. *Suppose $M \neq \frac{4\pi m}{\theta}$ for all $m \in \mathbb{N}^+$. Then there exists a constant $K > 0$ such that*

$$F(u, v, w) \geq -K \tag{5.29}$$

holds for any solution v of system (5.17).

Proof. We will prove the lemma by the argument of contradiction. Suppose that there is no constant K such that (5.29) holds true for all solutions of (5.17). Then we claim there exists a sequence $(v_k)_{k\in\mathbb{N}}$ of solutions of (5.20) such that

$$\|\nabla v_k\|_{L^2} \to \infty, \tag{5.30}$$

$$\int_\Omega e^{\tilde{\theta} v_k} dx \to \infty, \tag{5.31}$$

and

$$\max_{x\in\bar{\Omega}} v_k(x) \to \infty \tag{5.32}$$

as $k \to \infty$. Indeed, if (5.30) does not hold, which means that there exists a constant $c_1 > 0$ such that $\|\nabla v_k\|_{L^2} \le c_1$ as $k \to \infty$. Then, using the Poincaré inequality and the fact $-\Delta v_k + \beta v_k = \alpha(u_k - \bar{u})$ and $\int_\Omega v_k dx = 0$, we can find a constant $c_2 > 0$ depending on Ω such that

$$\int_\Omega u_k v_k dx = \frac{1}{\alpha}\int_\Omega |\nabla v_k|^2 dx + \frac{\beta}{\alpha}\int_\Omega v_k^2 dx \le \left(\frac{1}{\alpha} + c_2\right)\int_\Omega |\nabla v_k|^2 dx \le c_1^2\left(\frac{1}{\alpha} + c_2\right),$$

which implies that $F(u_k, v_k, w_k)$ $(\ge -c_1^2(\frac{1}{\alpha} + c_2))$ is bounded from below, which contradicts our assumption, where $u_k = \lambda_k e^{\xi\gamma\phi_k} e^{\tilde{\theta} v_k}$ with $\lambda_k = \frac{\int_\Omega u_k dx}{\int_\Omega e^{\xi\gamma\phi_k} e^{\tilde{\theta} v_k} dx}$ and $w_k = \gamma(\frac{v_k}{\alpha} - \phi_k)$. Next substituting (5.2) into (5.1), using the Jensen's inequality (see (5.4)) by the facts $-\ln u$ is a convex function for all $u \ge 0$ and $\int_\Omega \frac{u}{M} dx = 1$, we can derive from (5.3) that

$$\begin{aligned} F(u, v, w) &\ge \int_\Omega u \ln u dx - \tilde{\theta}\int_\Omega uv dx + \frac{\tilde{\theta}}{2\alpha}\int_\Omega |\nabla v|^2 dx + \frac{\chi\alpha\beta - \xi\gamma\delta}{2\alpha^2}\int_\Omega v^2 dx \\ &\ge -\int_\Omega u \ln \frac{e^{\tilde{\theta} v}}{u} dx - c_3 \ge -M \ln\left(\frac{1}{M}\int_\Omega e^{\tilde{\theta} v} dx\right) - c_3. \end{aligned} \tag{5.33}$$

This indicates that if (5.31) is false then $F(u_k, v_k, w_k)$ is bounded from below, which again contradicts our assumption. Lastly if (5.32) does not hold, then $e^{\tilde{\theta} v_k}$ is bounded and hence $F(u_k, v_k, w_k)$ is bounded from below from (5.33). This verifies our claim that (5.30)–(5.32) will hold if (5.29) is false. Let $\tilde{v}_k = v_k + \frac{\alpha M}{\beta|\Omega|}$. Then from (5.20), we know that each $\tilde{v}_k$ solves the problem

$$\begin{cases} -\Delta \tilde{v}_k + \beta \tilde{v}_k = \mu_k e^{\xi\gamma\phi_k} e^{\tilde{\theta}\tilde{v}_k}, & x \in \Omega \\ \frac{\partial \tilde{v}_k}{\partial \nu} = 0, & x \in \partial\Omega, \\ \int_\Omega \tilde{v}_k dx = \frac{\alpha M}{\beta}, \end{cases} \tag{5.34}$$

where $\|\phi_k\|_{W^{1,\infty}} \le c_4$ (see Lemma 5.3) and

$$\mu_k = \frac{\alpha M}{\int_\Omega e^{\xi\gamma\phi_k} e^{\tilde{\theta}\tilde{v}_k} dx} \to 0 \quad \text{as } k \to \infty. \tag{5.35}$$

Now we claim that (5.30)–(5.32) imply that there exists a subsequence of $(\tilde{v}_k)_{k\in\mathbb{N}}$ (denoted by $(\tilde{v}_k)_{k\in\mathbb{N}}$ again for simplicity) such that for some $m \in \mathbb{N}^+$

$$\mu_k \int_\Omega e^{\xi\gamma\phi_k} e^{\tilde{\theta}\tilde{v}_k} dx \to \frac{4\pi m}{\tilde{\theta}}, \quad \text{as } k \to \infty, \tag{5.36}$$

which contradicts the assumption $M \neq \frac{4\pi m}{\theta} = \frac{4\pi m}{\alpha\tilde{\theta}}$ since $\mu_k \int_\Omega e^{\xi\gamma\phi_k} e^{\tilde{\theta}\tilde{v}_k} dx = \alpha M$ from (5.35). Then the proof of the lemma is completed under the claim (5.36). □

Note that the proof of Lemma 5.5 replies on the claim (5.36). The rest of this subsection will be devoted to proving (5.36). Under the assumption that (5.29) does not hold for any constant $K > 0$, by the proof of Lemma 5.5, a sequence $(\tilde{v}_k)_{k\in\mathbb{N}}$ of solutions of (5.34) satisfying (5.30)–(5.32) is obtained. First, we establish the following Pohozaev's identity for the system (5.34)–(5.35).

Lemma 5.6. *Let $\tilde{v}_k$ be a solution of (5.34). Then the following Pohozaev's identity holds:*

$$\begin{aligned}
&2\int_\Omega \mu_k e^{\xi\gamma\phi_k} F(\tilde{v}_k) dx + \xi\gamma \int_\Omega (x\cdot\nabla\phi_k)\mu_k e^{\xi\gamma\phi_k} F(\tilde{v}_k) dx - \beta \int_\Omega \tilde{v}_k^2 dx \\
&= -\frac{1}{2}\int_{\partial\Omega} (x\cdot\nu)|\nabla\tilde{v}_k|^2 dS + \int_{\partial\Omega} (x\cdot\nabla\tilde{v}_k)\frac{\partial\tilde{v}_k}{\partial\nu} dS \\
&\quad + \int_{\partial\Omega} (x\cdot\nu)\mu_k e^{\xi\gamma\phi_k} F(\tilde{v}_k) dS - \frac{\beta}{2}\int_{\partial\Omega} (x\cdot\nu)\tilde{v}_k^2 dS
\end{aligned} \tag{5.37}$$

where $F(\tilde{v}_k) = \frac{1}{\tilde{\theta}}\left(e^{\tilde{\theta}\tilde{v}_k} - 1\right)$.

Proof. We multiply the first equation of system (5.34) by $x \cdot \nabla\tilde{v}_k = \sum_{j=1}^{2} x_j \frac{\partial\tilde{v}_k}{\partial x_j}$ and integrate the resulting equation by parts in Ω to obtain

$$\begin{aligned}
&-\int_\Omega \Delta\tilde{v}_k (x\cdot\nabla\tilde{v}_k) dx \\
&= -\int_\Omega \nabla\cdot(\nabla\tilde{v}_k)(x\cdot\nabla\tilde{v}_k) \\
&= \int_\Omega \left(|\nabla\tilde{v}_k|^2 + \sum_{i,j=1}^{2} x_j \frac{\partial\tilde{v}_k}{\partial x_i}\frac{\partial^2\tilde{v}_k}{\partial x_i\partial x_j} \right) dx - \sum_{i,j=1}^{2}\int_{\partial\Omega} \frac{\partial\tilde{v}_k}{\partial x_i}\frac{\partial\tilde{v}_k}{\partial x_j} x_j \nu_i dS
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega |\nabla \tilde{v}_k|^2 dx + \frac{1}{2}\int_\Omega \sum_{j=1}^{2} x_j \frac{\partial}{\partial x_j}(|\nabla \tilde{v}_k|^2) dx - \sum_{i,j=1}^{2}\int_{\partial\Omega} (x\cdot\nabla\tilde{v}_k)\frac{\partial \tilde{v}_k}{\partial \nu} dS \\
&= \int_\Omega |\nabla \tilde{v}_k|^2 dx - \int_\Omega |\nabla \tilde{v}_k|^2 dx + \frac{1}{2}\int_{\partial\Omega} (x\cdot\nu)|\nabla\tilde{v}_k|^2 dS - \sum_{i,j=1}^{2}\int_{\partial\Omega} (x\cdot\nabla\tilde{v}_k)\frac{\partial \tilde{v}_k}{\partial \nu} dS \\
&= \frac{1}{2}\int_{\partial\Omega} (x\cdot\nu)|\nabla\tilde{v}_k|^2 dS - \int_{\partial\Omega} (x\cdot\nabla\tilde{v}_k)\frac{\partial \tilde{v}_k}{\partial \nu} dS. \qquad (5.38)
\end{aligned}
$$

On the other hand, we can let $F(\tilde{v}_k) = \int_0^{\tilde{v}_k} e^{\tilde{\theta} s} ds = \frac{1}{\tilde{\theta}}\left(e^{\tilde{\theta}\tilde{v}_k} - 1\right)$ such that

$$
\begin{aligned}
&\int_\Omega \left(\mu_k e^{\xi\gamma\phi_k} e^{\tilde{\theta}\tilde{v}_k} - \beta\tilde{v}_k\right)(x\cdot\nabla\tilde{v}_k) dx \\
&\quad = \sum_{j=1}^{2}\int_\Omega \left(\mu_k e^{\xi\gamma\phi_k} x_j \frac{\partial F(\tilde{v}_k)}{\partial x_j} - \frac{\beta}{2} x_j \frac{\partial \tilde{v}_k^2}{\partial x_j}\right) dx \\
&\quad = -\int_\Omega 2\mu_k e^{\xi\gamma\phi_k} F(\tilde{v}_k) dx - \xi\gamma \int_\Omega (x\cdot\nabla\phi_k)\mu_k e^{\xi\gamma\phi_k} F(\tilde{v}_k) dx \\
&\qquad + \int_{\partial\Omega} (x\cdot\nu)\mu_k e^{\xi\gamma\phi_k} F(\tilde{v}_k) dS + \beta\int_\Omega \tilde{v}_k^2 dx - \frac{\beta}{2}\int_{\partial\Omega}(x\cdot\nu)\tilde{v}_k^2 dS. \qquad (5.39)
\end{aligned}
$$

The combination of (5.38) and (5.39) yields (5.37). □

Since we assume (5.29) does not hold, then we have (5.32) and define the following blowup set which is non-empty:

$$
\mathcal{S} := \left\{x \in \bar{\Omega} : \exists \mu_k \to 0 \text{ and } x_k \to x \text{ such that } \tilde{v}_k(x_k) \to \infty \text{ as } k \to \infty\right\}. \qquad (5.40)
$$

Since $(\mu_k e^{\xi\gamma\phi_k} e^{\tilde{\theta}\tilde{v}_k})_{k\in\mathbb{N}}$ is bounded in $L^1(\Omega)$, then using the Prokhorov's theorem we may extract a subsequence (still denoted $(\mu_k e^{\xi\gamma\phi_k} e^{\tilde{\theta}\tilde{v}_k})_{k\in\mathbb{N}}$ for simplicity) such that $\mu_k e^{\xi\gamma\phi_k} e^{\tilde{\theta}\tilde{v}_k}$ converges in the sense of measure on Ω to some nonnegative bounded measure η, i.e.

$$
\int_\Omega \mu_k e^{\xi\gamma\phi_k} e^{\tilde{\theta}\tilde{v}_k} \psi dx \to \int_\Omega \psi d\eta, \qquad (5.41)
$$

for every $\psi \in C_0^\infty(\Omega)$. Following the nomenclature in [16,42], we call $x_0 \in \bar{\Omega}$ a δ-regular point if there is a function $\psi \in C_0^\infty(\Omega)$, $0 \le \psi \le 1$, with $\psi = 1$ in a neighborhood of x_0 such that

$$
\int_\Omega \psi d\mu < \frac{4\pi}{\tilde{\theta}(1+3\delta)}. \qquad (5.42)
$$

We also denote by $\Sigma(\delta)$ the set of points which are not δ-regular points in $\bar{\Omega}$. Then using the same argument as in [16,42], we state the following proposition without proof.

Proposition 5.7. (i) *If x_0 is a δ-regular point, then the sequence $(\tilde{v}_k)_{k\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\bar{\Omega}\cap B_{R_0}(x_0))$ for some $R_0>0$.* (ii) *$\mathcal{S}=\Sigma(\delta)$ for any $\delta>0$.*

Furthermore we have the following result.

Proposition 5.8. *$1\le card\,\mathcal{S}<\infty$, where $card\,\mathcal{S}$ stands for the cardinality of set $\mathcal{S}$.*

Proof. Since $\max\limits_{x\in\bar{\Omega}} v_k(x)\to\infty$ as $k\to\infty$ (see (5.32)), we know that $card\,\mathcal{S}\ge 1$. Clearly $x_0\in\Sigma(\delta)$ iff $\eta(\{x_0\})\ge\frac{4\pi}{\tilde{\theta}(1+3\delta)}$. Since η is a bounded measure with $\int_\Omega d\eta=\alpha M$ form (5.41), it follows that $\Sigma(\delta)$ is finite and

$$card\,\Sigma(\delta)\le\frac{\theta(1+3\delta)M}{4\pi}<\infty. \tag{5.43}$$

Hence from (5.43) and Proposition 5.7 (ii), we have $1\le card\,\mathcal{S}=card\,\Sigma(\delta)<\infty$. The proof is completed. □

Due to $1\le card\,\mathcal{S}<\infty$, without loss of generality, we assume $\mathcal{S}=\{p_1,\cdots,p_N\}$. We decompose $\mathcal{S}$ into a boundary blowup set $\mathcal{S}_1=\mathcal{S}\cap\partial\Omega$ and an interior blowup set $\mathcal{S}_2=\mathcal{S}\cap\Omega$. For a small $r>0$, we set

$$\sigma_j^k(r)=\int\limits_{B_r(p_j)}\mu_k e^{\xi\gamma\phi_k}e^{\tilde{\theta}\tilde{v}_k}dx. \tag{5.44}$$

Then for all small $r>0$, we have the following equality:

$$\lim_{k\to\infty}\int\limits_\Omega\mu_k e^{\xi\gamma\phi_k}e^{\tilde{\theta}\tilde{v}_k}dx=\sum_{j=1}^N\lim_{k\to\infty}\sigma_j^k(r). \tag{5.45}$$

Then we can obtain the following equality by taking $r\to 0$ in (5.45)

$$\lim_{k\to\infty}\int\limits_\Omega\mu_k e^{\xi\gamma\phi_k}e^{\tilde{\theta}\tilde{v}_k}dx=\sum_{j=1}^N\lim_{r\to 0}\lim_{k\to\infty}\sigma_j^k(r), \tag{5.46}$$

which gives (5.36), provided that the following Lemma 5.9 holds.

Lemma 5.9. *Let $\sigma_j^k(r)$ be defined by (5.44). Then*

$$\lim_{r\to 0}\lim_{k\to\infty}\sigma_j^k(r)=\begin{cases}\dfrac{4\pi}{\tilde{\theta}}, & p_j\in\mathcal{S}_1,\\ \dfrac{8\pi}{\tilde{\theta}}, & p_j\in\mathcal{S}_2.\end{cases} \tag{5.47}$$

Proof. The proof of this lemma closely follows an argument in [16], Lemma 3.4. The differences lie in the modified Pohozaev's type inequality and an extra term ϕ_k whose regularity need to be proved. We first consider the case when $p_j \in \mathcal{S}_1$. Without loss of generality, we assume the blowup point $p_j = 0$. Let $U_r = B_r(0) \cap \bar{\Omega}$. Assume the function φ_k is a solution of the following problem

$$\begin{cases} \Delta\varphi - \beta\varphi = 0, & x \in U_r, \\ \frac{\partial\varphi}{\partial\nu} = \frac{\partial\tilde{v}_k}{\partial\nu}, & x \in \partial U_r. \end{cases} \tag{5.48}$$

Then clearly $\varphi_k = O(1)$ in $C^2(U_r)$ since $|\frac{\partial\tilde{v}_k}{\partial\nu}| \le C$ on ∂U_r. If we let $h_k = (\tilde{v}_k - \varphi_k)/\sigma_j^k(r)$, then $h_k \to G(\cdot, 0)$ in $C^2_{loc}(B_r(0) \cap \bar{\Omega} \setminus \{0\})$ as $k \to \infty$ (see the proof in [43], Lemma 2.6 or see [16]), where $G(\cdot, 0)$ satisfies

$$\begin{cases} -\Delta G + \beta G = \delta_0, & x \in U_r, \\ \frac{\partial G}{\partial\nu} = 0, & x \in \partial U_r, \end{cases}$$

with δ_0 denoting the Dirac measure on U_r giving unit mass to the point 0. By the potential theory, as $|x| = r$ is small, $G(\cdot, 0)$ has the following form (e.g., see [7])

$$G(\cdot, 0) = -\frac{1}{\pi}\ln|x| + H(r) \text{ in } \bar{U}_r$$

where $H(r)$ is of class C^1 in $\bar{U}_r$. Hence

$$\tilde{v}_k = \sigma_j^k(r)\Big(-\frac{1}{\pi}\ln|x| + H(r)\Big) \text{ in } C^1(\partial U_r). \tag{5.49}$$

From the second equation of (5.20), we have

$$-\Delta\phi_k + \delta\phi_k = \frac{\delta - \beta}{\alpha}\left(\tilde{v}_k - \frac{\alpha M}{\beta|\Omega|}\right) \text{ in } \bar{U}_r \setminus \mathcal{U}, \ \frac{\partial\phi_k}{\partial\nu} = 0 \text{ on } \mathcal{U}$$

where $\mathcal{U} = \partial U_r \cap \partial\Omega$. Then by the elliptic regularity theorem (e.g. Agmon–Douglis–Nirenberg theorem), we have

$$\phi_k \in C^2(\bar{U}_r). \tag{5.50}$$

Now using Lemma 5.6 in U_r, we have

$$\begin{aligned} &\frac{2}{\tilde{\theta}}\int_{U_r} \mu_k e^{\xi\gamma\phi_k}\left(e^{\tilde{\theta}\tilde{v}_k} - 1\right)dx + \frac{\xi\gamma}{\tilde{\theta}}\int_{U_r}(x\cdot\nabla\phi_k)\mu_k e^{\xi\gamma\phi_k}\left(e^{\tilde{\theta}\tilde{v}_k} - 1\right)dx - \beta\int_{U_r}\tilde{v}_k^2 dx \\ &= \int_{\partial U_r}(x\cdot\nabla\tilde{v}_k)\frac{\partial\tilde{v}_k}{\partial\nu}dS - \frac{1}{2}\int_{\partial U_r}(x\cdot\nu)|\nabla\tilde{v}_k|^2 dS - \frac{\beta}{2}\int_{\partial U_r}(x\cdot\nu)\tilde{v}_k^2 dS \\ &\quad + \frac{2}{\tilde{\theta}}\int_{\partial U_r}(x\cdot\nu)\mu_k e^{\xi\gamma\phi_k}\left(e^{\tilde{\theta}\tilde{v}_k} - 1\right)dS. \end{aligned} \tag{5.51}$$

Next, we will estimate the terms on both sides of (5.51). First with the fact that $\mu_k \int_\Omega e^{\xi\gamma\phi_k} e^{\tilde{\theta}\tilde{v}_k} dx = \alpha M$ from (5.35), one can derive that $\|\tilde{v}_k\|^2_{W^{1,4/3}} \le C$ for some constant $C > 0$ by the Agmon–Douglis–Nirenberg estimate [2] and Gagliardo–Nirenberg inequality. Then it follows that

$$\int_{U_r} \tilde{v}_k^2 dx = O(r\|\tilde{v}_k\|^2_{L^4}) = O(r\|\tilde{v}_k\|^2_{W^{1,4/3}}) = O(r). \tag{5.52}$$

Furthermore, we have the following estimates

$$\begin{aligned}
&\frac{2}{\tilde{\theta}}\int_{U_r} \mu_k e^{\xi\gamma\phi_k}\left(e^{\tilde{\theta}\tilde{v}_k}-1\right)dx + \frac{\xi\gamma}{\tilde{\theta}}\int_{U_r}(x\cdot\nabla\phi_k)\mu_k e^{\xi\gamma\phi_k}\left(e^{\tilde{\theta}\tilde{v}_k}-1\right)dx \\
&= \frac{2}{\tilde{\theta}}\int_{U_r} \mu_k e^{\xi\gamma\phi_k} e^{\tilde{\theta}\tilde{v}_k} dx - \frac{2}{\tilde{\theta}}\int_{U_r}\mu_k e^{\xi\gamma\phi_k}dx + \frac{\xi\gamma}{\tilde{\theta}}\int_{U_r}(x\cdot\nabla\phi_k)\mu_k e^{\xi\gamma\phi_k}\left(e^{\tilde{\theta}\tilde{v}_k}-1\right)dx \\
&= \frac{2}{\tilde{\theta}}\sigma_j^k(r) + O(\mu_k r^2) + O(r),
\end{aligned} \tag{5.53}$$

where we have used (5.21) which leads to

$$-\frac{2}{\tilde{\theta}}\int_{U_r}\mu_k e^{\xi\gamma\phi_k}dx + \frac{\xi\gamma}{\tilde{\theta}}\int_{U_r}(x\cdot\nabla\phi_k)\mu_k e^{\xi\gamma\phi_k}\left(e^{\tilde{\theta}\tilde{v}_k}-1\right)dx = O(\mu_k r^2) + O(r).$$

Using the equalities (5.49) and $\frac{\partial\tilde{v}_k}{\partial\nu} = \nu\cdot\nabla\tilde{v}_k$, we have

$$\begin{aligned}
\int_{\partial U_r}(x\cdot\nabla\tilde{v}_k)\frac{\partial\tilde{v}_k}{\partial\nu}dS &= \int_{\partial U_r}\left[\frac{x\cdot\nu}{r^2}\left(\frac{\sigma_j^k(r)}{\pi}\right)^2 + O(1)\right]dS \\
&= \left(\frac{\sigma_j^k(r)}{\pi}\right)^2\pi + O(r),
\end{aligned} \tag{5.54}$$

and

$$\frac{1}{2}\int_{\partial U_r}(x\cdot\nu)|\nabla\tilde{v}_k|^2 dS = \left(\frac{\sigma_j^k(r)}{\pi}\right)^2\frac{\pi}{2} + O(r). \tag{5.55}$$

Using (5.49) and (5.50), we have

$$\int_{\partial U_r}(x\cdot\nu)\tilde{v}_k^2 dS = O(r), \tag{5.56}$$

and

$$\int_{\partial U_r} (x\cdot\nu)\mu_k e^{\xi\gamma\phi_k} e^{\bar{\theta}\tilde{v}_k} dS = O\Big(r\mu_k \max_{x\in\partial U_r}(e^{\xi\gamma\phi_k} e^{\bar{\theta}\tilde{v}_k})\Big) = O(\mu_k r). \tag{5.57}$$

Furthermore, we can derive that

$$\int_{\partial U_r} (x\cdot\nu)\mu_k e^{\xi\gamma\phi_k} dS = O(\mu_k r). \tag{5.58}$$

Substituting (5.52)–(5.58) into (5.51), and letting $k\to\infty$ first and then $r\to 0$, we can obtain that

$$\frac{2}{\tilde{\theta}}\lim_{r\to 0}\lim_{k\to\infty}\sigma_j^k(r) = \frac{\pi}{2}\cdot\frac{1}{\pi^2}\Big(\lim_{r\to 0}\lim_{k\to\infty}\sigma_j^k(r)\Big)^2,$$

which implies

$$\lim_{r\to 0}\lim_{k\to\infty}\sigma_j^k(r) = \frac{4\pi}{\tilde{\theta}}. \tag{5.59}$$

When the blowup point $0\in\mathcal{S}_2$, we consider φ_k satisfying

$$\begin{cases} \Delta\varphi - \beta\varphi = 0, & x\in U_r, \\ \varphi = \tilde{v}_k, & x\in\partial U_r. \end{cases} \tag{5.60}$$

Let $h_k = (\tilde{v}_k - \varphi)/\sigma_j^k(r)$. Then $h_k \to G(\cdot,0)$ in $C^2_{loc}(B_r(0)\cap\bar{\Omega}\setminus\{0\})$, where $G(\cdot,0)$ satisfies

$$\begin{cases} -\Delta G + \beta G = \delta_0, & x\in U_r, \\ G = 0, & x\in\partial U_r. \end{cases}$$

In this case, the Green's function has the following expansion near 0

$$G(\cdot,0) = -\frac{1}{2\pi}\ln|x| + H(r) \text{ in } \bar{U}_r$$

with $H(r)\in C^1(\bar{U}_r)$, which implies

$$\tilde{v}_k = \sigma_j^k(r)\Big(-\frac{1}{2\pi}\ln|x| + H(r)\Big). \tag{5.61}$$

Next we can follow the similar arguments and calculations for the case $0\in\mathcal{S}_1$ to obtain the same estimate for $0\in\mathcal{S}_2$ except that

$$\int_{\partial U_r} (x\cdot\nabla\tilde{v}_k)\frac{\partial\tilde{v}_k}{\partial\nu} dS = \left(\frac{\sigma_j^k(r)}{2\pi}\right)^2 2\pi + O(r), \tag{5.62}$$

and

$$\int_{\partial U_r} (x \cdot \nu) \frac{|\nabla \tilde{v}_k|^2}{2} dS = \left(\frac{\sigma_j^k(r)}{2\pi} \right)^2 \pi + O(r). \tag{5.63}$$

Then using the Pohozaev's identity in Lemma 5.6 again, we have

$$\frac{2}{\tilde{\theta}} \lim_{r \to 0} \lim_{k \to \infty} \sigma_j^k(r) = \frac{1}{4\pi} \left(\lim_{r \to 0} \lim_{k \to \infty} \sigma_j^k(r) \right)^2,$$

which yields

$$\lim_{r \to 0} \lim_{k \to \infty} \sigma_j^k(r) = \frac{8\pi}{\tilde{\theta}}. \tag{5.64}$$

Hence the proof of Lemma 5.9 is completed. □

Finally, we remark that the claim (5.36) is proved by (5.46), (5.59) and (5.64).

5.2.2. Initial data with large negative energy

In this subsection, we assert that there exist initial data with supercritical mass having energy below any prescribed bound. Using the third the equation of (5.16), we have

$$\frac{\xi}{2} \int_\Omega uw dx = \frac{\xi}{2\gamma} \int_\Omega (\delta w^2 + |\nabla w|^2) dx, \tag{5.65}$$

which implies the Lyapunov function $F(u, v, w)$ can be written as follows

$$\begin{aligned} F(u, v, w) = & \int_\Omega u \ln u dx - \chi \int_\Omega uv dx + \xi \int_\Omega uw dx \\ & + \frac{\chi}{2\alpha} \int_\Omega (\beta v^2 + |\nabla v|^2) dx - \frac{\xi}{2\gamma} \int_\Omega (\delta w^2 + |\nabla w|^2) dx. \end{aligned} \tag{5.66}$$

Next, we look for a sequence $(u_\varepsilon, v_\varepsilon, w_\varepsilon)_{\varepsilon \geq 0}$ satisfying $\int_\Omega v_\varepsilon(x) dx = \int_\Omega w_\varepsilon dx = 0$ and $\int_\Omega u_\varepsilon dx = M$ such that $\lim_{\varepsilon \to 0} F(u_\varepsilon, v_\varepsilon, w_\varepsilon) = -\infty$. From [44], p. 615, we know that the functions

$$\psi_\varepsilon(x) = \ln \left(\frac{8\varepsilon^2}{(\varepsilon^2 + \pi |x - x_0|^2)^2} \right), \ \varepsilon > 0, \ x_0 \in \mathbb{R}^2$$

are solutions of $-\Delta \psi(x) = e^{\psi(x)}$, $x \in \mathbb{R}^2$ satisfying $\int_{\mathbb{R}^2} e^{\psi(x)} dx < \infty$. We note that as $\varepsilon \to 0$, $\psi_\varepsilon(x) \to -\infty$ for all $x \neq x_0$ and $\psi_\varepsilon(x_0) \to \infty$. Using the same notation $\tilde{\theta} = \frac{\theta}{\alpha} = \frac{\chi\alpha - \xi\gamma}{\alpha}$ as in Section 5.2.1, we choose the sequence $(u_\varepsilon, v_\varepsilon, w_\varepsilon)_{\varepsilon \geq 0}$ with

$$\begin{aligned}
v_\varepsilon(x) &= \frac{\alpha}{\gamma} w_\varepsilon(x) \\
&= \frac{1}{\tilde{\theta}}\left(\psi_\varepsilon(x) - \frac{1}{|\Omega|}\int_\Omega \psi_\varepsilon(x)dx\right) \\
&= \frac{1}{\tilde{\theta}}\left[\ln\left(\frac{\varepsilon^2}{(\varepsilon^2+\pi|x-x_0|^2)^2}\right) - \frac{1}{|\Omega|}\int_\Omega \ln\left(\frac{\varepsilon^2}{(\varepsilon^2+\pi|x-x_0|^2)^2}\right)dx\right], \qquad (5.67)
\end{aligned}$$

and

$$u_\varepsilon(x) = \frac{Me^{\tilde{\theta}v_\varepsilon(x)}}{\int_\Omega e^{\tilde{\theta}v_\varepsilon(x)}dx}, \qquad (5.68)$$

as our candidate to obtain the property $\lim\limits_{\varepsilon\to 0} F(u_\varepsilon, v_\varepsilon, w_\varepsilon) = -\infty$ with supercritical mass.

Lemma 5.10. *Assume $M > \frac{4\pi}{\tilde{\theta}}$. If $(u_\varepsilon, v_\varepsilon, w_\varepsilon)_{\varepsilon\geq 0}$ are defined by (5.67)–(5.68) and $x_0 \in \partial\Omega$, then as $\varepsilon \to 0$, we have*

$$F(u_\varepsilon, v_\varepsilon, w_\varepsilon) \to -\infty \quad \textit{and} \quad \int_\Omega |\nabla v_\varepsilon|^2 dx = \frac{\alpha}{\gamma}\int_\Omega |\nabla w_\varepsilon|^2 dx \to \infty. \qquad (5.69)$$

Proof. Without loss of generality, we assume $x_0 = 0$ for convenience. Using (5.66) and the fact $w_\varepsilon(x) = \frac{\gamma}{\alpha} v_\varepsilon(x)$, we obtain that

$$\begin{aligned}
&F(u_\varepsilon, v_\varepsilon, w_\varepsilon) \\
&= \int_\Omega u_\varepsilon \ln u_\varepsilon dx - \tilde{\theta}\int_\Omega u_\varepsilon v_\varepsilon dx + \frac{\tilde{\theta}}{2\alpha}\int_\Omega |\nabla v_\varepsilon|^2 dx + \frac{\chi\alpha\beta - \xi\gamma\delta}{2\alpha^2}\int_\Omega v_\varepsilon^2 dx. \qquad (5.70)
\end{aligned}$$

From (5.68), we can derive

$$\begin{aligned}
&\int_\Omega u_\varepsilon \ln u_\varepsilon dx \\
&= \frac{M}{\int_\Omega e^{\tilde{\theta}v_\varepsilon}dx}\int_\Omega e^{\tilde{\theta}v_\varepsilon}\left[\ln M + \tilde{\theta}v_\varepsilon - \ln\left(\int_\Omega e^{\tilde{\theta}v_\varepsilon}dx\right)\right]dx \\
&= M\ln M + \frac{\tilde{\theta}M}{\int_\Omega e^{\tilde{\theta}v_\varepsilon}dx}\int_\Omega v_\varepsilon e^{\tilde{\theta}v_\varepsilon}dx - M\ln\left(\int_\Omega e^{\tilde{\theta}v_\varepsilon}dx\right), \qquad (5.71)
\end{aligned}$$

and

$$\tilde{\theta}\int_\Omega u_\varepsilon v_\varepsilon dx = \frac{\tilde{\theta}M}{\int_\Omega e^{\tilde{\theta}v_\varepsilon}dx}\int_\Omega e^{\tilde{\theta}v_\varepsilon}v_\varepsilon dx. \tag{5.72}$$

Substituting (5.71) and (5.72) into (5.70), one has

$$\begin{aligned} F(u_\varepsilon, v_\varepsilon, w_\varepsilon) = \frac{\tilde{\theta}}{2\alpha}\int_\Omega |\nabla v_\varepsilon|^2\,dx + \frac{\chi\alpha\beta - \xi\gamma\delta}{2\alpha^2}\int_\Omega v_\varepsilon^2 dx \\ - M\ln\left(\int_\Omega e^{\tilde{\theta}v_\varepsilon}dx\right) + M\ln M. \end{aligned} \tag{5.73}$$

From (5.67), we have

$$\frac{\tilde{\theta}}{2\alpha}\int_\Omega |\nabla v_\varepsilon|^2 dx = \frac{8\pi^2}{\theta}\int_\Omega \frac{x^2}{(\varepsilon^2+\pi x^2)^2}dx. \tag{5.74}$$

Substituting $y = \frac{x}{\varepsilon}$, we obtain that

$$\frac{\tilde{\theta}}{2\alpha}\|\nabla v_\varepsilon\|_{L^2}^2 = \frac{8\pi^2}{\theta}\int_{\Omega_\varepsilon}\frac{|y|^2}{(1+\pi|y|^2)^2}dy, \tag{5.75}$$

where $\Omega_\varepsilon = \{y|\varepsilon y \in \Omega\}$. Applying the polar coordinates around origin $0 \in \partial\Omega$ to (5.75), and denoting the maximum distance between the pole and boundary of Ω by R, we obtain

$$\begin{aligned} \frac{\tilde{\theta}}{2\alpha}\|\nabla v_\varepsilon\|_{L^2}^2 &= \frac{8\pi^2}{\theta}\int_{\Omega_\varepsilon}\frac{|y|^2}{(1+\pi|y|^2)^2}dy \\ &\le \frac{8\pi^2}{\theta}\int_0^\pi\int_0^{\frac{R}{\varepsilon}}\frac{r^3}{(1+\pi r^2)^2}drd\theta \\ &\le \frac{4\pi}{\theta}\left(\ln\frac{1}{\varepsilon^2} + \ln(\varepsilon^2+\pi R^2) - 1 + \frac{\varepsilon^2}{\varepsilon^2+\pi R^2}\right) \\ &\le \frac{8\pi}{\theta}\ln\frac{1}{\varepsilon} + O_1(1), \end{aligned} \tag{5.76}$$

where $|O_1(1)| \le C$ as $\varepsilon \to 0$. Moreover, we can deduce that

$$v_\varepsilon^2 = \frac{1}{\tilde{\theta}^2}\left(\ln(\varepsilon^2+\pi|x|^2)^2 - \frac{1}{|\Omega|}\int_\Omega \ln(\varepsilon^2+\pi|x|^2)^2 dx\right)^2$$

$$=\frac{1}{\tilde{\theta}^2}\left[\left(\ln(\varepsilon^2+\pi|x|^2)^2\right)^2-\frac{2}{|\Omega|}\ln(\varepsilon^2+\pi|x|^2)^2\int_\Omega \ln(\varepsilon^2+\pi|x|^2)^2dx\right]$$

$$+\frac{1}{\tilde{\theta}^2|\Omega|^2}\left(\int_\Omega \ln(\varepsilon^2+\pi|x|^2)^2dx\right)^2, \tag{5.77}$$

which gives that

$$\frac{\chi\alpha\beta-\xi\gamma\delta}{2\alpha^2}\int_\Omega v_\varepsilon^2dx=\frac{\chi\alpha\beta-\xi\gamma\delta}{2\theta^2}\int_\Omega (\ln(\varepsilon^2+\pi|x|^2)^2)^2dx$$

$$-\frac{\chi\alpha\beta-\xi\gamma\delta}{2\theta^2|\Omega|}\left(\int_\Omega \ln(\varepsilon^2+\pi|x|^2)^2dx\right)^2=O_2(1), \tag{5.78}$$

where $|O_2(1)|\le C$ as $\varepsilon\to 0$. Using (5.67), we have the estimates

$$\int_\Omega e^{\tilde{\theta}v_\varepsilon}dx=|\Omega|e^{-\frac{1}{|\Omega|}\int_\Omega \ln\left(\frac{\varepsilon^2}{(\varepsilon^2+\pi|x|^2)^2}\right)dx}\int_\Omega\left(\frac{\varepsilon^2}{(\varepsilon^2+\pi|x|^2)^2}\right)dx$$

and

$$\ln\left(\int_\Omega e^{\tilde{\theta}v_\varepsilon}dx\right)=\ln\left(|\Omega|\int_\Omega \frac{\varepsilon^2}{(\varepsilon^2+\pi|x|^2)^2}dx\right)-\frac{1}{|\Omega|}\int_\Omega \ln\left(\frac{\varepsilon^2}{(\varepsilon^2+\pi|x|^2)^2}\right)dx.$$

Then we have the following estimate

$$-M\ln\left(\int_\Omega e^{\tilde{\theta}v_\varepsilon}dx\right)$$

$$=-M\left[\ln\left(|\Omega|\int_\Omega \frac{\varepsilon^2}{(\varepsilon^2+\pi|x|^2)^2}dx\right)-\frac{1}{|\Omega|}\int_\Omega \ln\left(\frac{\varepsilon^2}{(\varepsilon^2+\pi|x|^2)^2}\right)dx\right]$$

$$=\frac{M}{|\Omega|}\int_\Omega \ln\varepsilon^2dx+\frac{M}{|\Omega|}\int_\Omega \ln(\varepsilon^2+\pi|x|^2)^2dx-M\ln\left(|\Omega|\int_\Omega \frac{\varepsilon^2}{(\varepsilon^2+\pi|x|^2)^2}dx\right).$$

By the polar coordinates, one can readily estimate that

$$1-\frac{\varepsilon^2}{\pi r_1^2+\varepsilon^2}\le\int_\Omega \frac{\varepsilon^2}{(\varepsilon^2+\pi|x|^2)^2}dx\le 1-\frac{\varepsilon^2}{\pi r_2^2+\varepsilon^2}$$

where r_1 and r_2 denote the maximum and minimum distance between the pole and the boundary of Ω. Hence we have

$$-M\ln\left(\int_\Omega e^{\tilde{\theta}v_\varepsilon}dx\right)=2M\ln\varepsilon+O_3(1), \tag{5.79}$$

where $|O_3(1)|\le C$ as $\varepsilon\to 0$. Then the combination of (5.76), (5.78) and (5.79) implies

$$F(u_\varepsilon,v_\varepsilon,w_\varepsilon)\le 2\left(\frac{4\pi}{\theta}-M\right)\ln\frac{1}{\varepsilon}+O(1), \tag{5.80}$$

where $O(1)=O_1(1)+O_2(1)+O_3(1)$ and $|O(1)|\le C$ as $\varepsilon\to 0$. Then (5.80) leads to the assertion of the lemma. $\square$

Remark 5.1. In this lemma, we only consider the case $x_0\in\partial\Omega$. If $x_0\in\Omega$, then we have the same estimates as above except changing the estimate in (5.76) to $\frac{\tilde{\theta}}{2\alpha}\|\nabla v_\varepsilon\|_{L^2}^2\le\frac{16\pi}{\theta}\ln\frac{1}{\varepsilon}+O_1(1)$. This leads to

$$F(u_\varepsilon,v_\varepsilon,w_\varepsilon)\le 2\left(\frac{8\pi}{\theta}-M\right)\ln\frac{1}{\varepsilon}+O(1), \tag{5.81}$$

which implies that $F(u_\varepsilon,v_\varepsilon,w_\varepsilon)\to-\infty$ as $\varepsilon\to 0$ if $M>\frac{8\pi}{\theta}$.

Lemma 5.11. *Assume $M>\frac{4\pi}{\theta}$ and $M\notin\{\frac{4\pi m}{\theta}:m\in\mathbb{N}^+\}$. Then there exists initial data (u_0,v_0) such that the corresponding solution of (3.1) blows up.*

Proof. Since $M\notin\{\frac{4\pi m}{\theta}:m\in\mathbb{N}^+\}$, then by Lemma 5.5, we can find a constant $K>0$ such that (5.29) holds. Furthermore, for this constant $K>0$, if $M>\frac{4\pi}{\theta}$, then by Lemma 5.10 we can choose a small $\varepsilon_0>0$ such that

$$v_{\varepsilon_0}(x)=\frac{\alpha}{\gamma}w_{\varepsilon_0}(x)=\frac{\alpha}{\theta}\left[\ln\left(\frac{\varepsilon_0^2}{(\varepsilon_0^2+\pi|x-x_0|^2)^2}\right)-\frac{1}{|\Omega|}\int_\Omega\ln\left(\frac{\varepsilon_0^2}{(\varepsilon_0^2+\pi|x-x_0|^2)^2}\right)dx\right],$$

and

$$u_{\varepsilon_0}(x)=\frac{Me^{\tilde{\theta}v_{\varepsilon_0}(x)}}{\int_\Omega e^{\tilde{\theta}v_{\varepsilon_0}(x)}dx}$$

such that

$$F(u_{\varepsilon_0},v_{\varepsilon_0},w_{\varepsilon_0})<-K.$$

It can be readily verified that $(u_{\varepsilon_0},v_{\varepsilon_0})\in[W^{1,\infty}(\Omega)]^2$ and $\int_\Omega u_{\varepsilon_0}(x)dx=M$. Hence, if we define $(u_0,v_0)=(u_{\varepsilon_0},v_{\varepsilon_0})$ as the initial data, then the corresponding solution of chemotaxis model (5.16) must blow up. Otherwise, if the corresponding solution (u,v,w) of (5.16) is global and bounded in $\Omega \times (0, \infty)$, then from Lemma 5.4, we have $F(u_\infty, v_\infty, w_\infty) \leq F(u_0, v_0, w_0) < -K$. But Lemma 5.5 says that $F(u_\infty, v_\infty, w_\infty) \geq -K$ since $(u_\infty, v_\infty, w_\infty)$ is a solution of (5.17) by Lemma 5.4, which is a contradiction. The lemma is proved. □

5.2.3. *Proof of Theorem 1.2*

Theorem 1.2 is a direct consequence of Lemma 5.2 and Lemma 5.11.

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