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Asymptotic stability of traveling waves of a chemotaxis model with singular sensitivity

Hai-Yang Jin [a], Jingyu Li [b,c], Zhi-An Wang [a,*]

[a] Department of Applied Mathematics, Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong
[b] Center for Partial Differential Equations, East China Normal University, Minhang, Shanghai, 200241, PR China
[c] School of Mathematics and Statistics, Northeast Normal University, Changchun, 130024, PR China

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ABSTRACT

This paper establishes the nonlinear stability of traveling wave solutions to a chemotaxis model with singular (or logarithmic) sensitivity and its transformed parabolic–hyperbolic system. Depending on the parameter signs, we discuss the linear instability of traveling wave solutions using the spectral analysis and nonlinear asymptotic stability of traveling wave solutions with zero end state by the weighted energy estimates, where the latter result solves the open question left in a previous work (Li and Wang, 2009 [7]).

1. Introduction

In many biological systems, the phenomenon that cells move in response to a diffusible or otherwise transported signal can be modeled by diffusion equations with advection terms (see Patlak [16]). However, other systems are more accurately modeled by random walkers that deposit a non-diffusible signal which modifies the local environment for succeeding passages and there is little or no transport of the modifying substance. Examples include myxobacteria which produce slime over which their cohorts can move more readily, and ants, which follow trails left by predecessors. Hence in [3,15],

* Corresponding author.
E-mail addresses: haiyang.jin@connect.polyu.hk (H.Y. Jin), lijy645@nenu.edu.cn (J.Y. Li), mawza@polyu.edu.hk (Z.A. Wang).

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a number of mathematical models have been derived based on the reinforced random walk framework to explore the question as to whether aggregation is possible with such strictly local modification or whether some form of longer-range communication is necessary. One of these models studied extensively in the past is a PDE–ODE hybrid model as follows

$$\begin{cases} u_t = \left[Du_x - \xi u(\log c)_x\right]_x, \\ c_t = \mu uc, \end{cases} \tag{1.1}$$

where $u(x,t)$ denotes the particle density and $c(x,t)$ the concentration of a non-diffusible chemical signal. The parameter $\xi \in \mathbb{R}$ is called the chemotactic coefficient, and the chemotaxis is said to be attractive if $\xi > 0$ and repulsive if $\xi < 0$.

When $\mu > 0$, a detailed qualitative and numerical analysis was presented in [3] where explicit solutions about aggregation, blow-up and collapse were constructed in one-dimensional space. The local and global existence of solutions was rigorously studied in [24,25] and elaborated subsequently in [12]. The existence and stability of spike solutions of (1.1) was studied in [17].

In [3], a Hopf–Cole transformation

$$v = \frac{(\log c)_x}{\mu} = \frac{1}{\mu}\frac{c_x}{c} \tag{1.2}$$

was employed to transform the system (1.1) into a conservational hyperbolic–parabolic system

$$\begin{cases} u_t - \chi(uv)_x = Du_{xx}, \\ v_t - u_x = 0 \end{cases} \tag{1.3}$$

with

$$\chi = -\mu\xi. \tag{1.4}$$

As a system of conservation laws derived from chemotaxis model, the transformed system (1.3) itself is of interest to study mathematically. For $(x,t) \in \mathbb{R} \times [0,\infty)$, Wang and Hillen [22] first established the existence of traveling wave solutions of (1.3) for both $\chi > 0$ and $\chi < 0$, subject to the initial data

$$(u,v)(x,0) = (u_0, v_0)(x) \to (u_\pm, v_\pm) \quad \text{as } x \to \pm\infty, \tag{1.5}$$

where $u_0(x) \geqslant 0$ and $u_\pm \geqslant 0$. When the chemical diffusion is included into the second equation of (1.1), the existence and stability of traveling wave solutions were given in [9,10]. If $u_+ > 0$, the nonlinear stability of traveling wave solutions of model (1.3) with $\chi > 0$ was recently studied in [7,8] and followed in [5] for composite waves, whereas the stability of traveling wave solutions for the case $u_+ = 0$ still remains an open question to date (see a review paper [21]). The main challenge is that there is a singularity in the energy estimates demonstrated in [7,8] if $u_+ = 0$. The primary goal of this paper is to solve this open problem and establish the nonlinear stability of traveling wave solutions of (1.3) with $\chi > 0$ for $u_+ = 0$ using the method of weighted energy estimates by carefully selecting an appropriate asymmetrical weight function. Furthermore we perform the spectral analysis and prove that the traveling wave solutions of (1.3) with $\chi < 0$ are linearly unstable by deriving that the essential spectrum of the linearized system of (1.3) at the traveling wave solution has the positive real part. Finally we shall transfer the results back to the original chemotaxis system (1.1) with the initial condition as follows:

$$(u,c)(x,0) = (u_0, c_0)(x) \to (u_\pm, c_\pm) \quad \text{as } x \to \pm\infty, \tag{1.6}$$

subject to a compatible condition to (1.2)

$$v_0(x) = \frac{(\log c_0)_x}{\mu} = \frac{1}{\mu}\frac{c_{0x}}{c_0}, \tag{1.7}$$

where $c_0(x) \geqslant 0$ for all $x \in \mathbb{R}$ and hence $c_\pm \geqslant 0$.

Before concluding this section, we shall recall some works on the transformed system (1.3). When $\chi > 0$, the initial–boundary value problem (IBVP) and Cauchy problem of (1.3) in one dimension were studied in [26] and in [1], respectively. Furthermore the Cauchy problem of (1.3) in multi-dimensional spaces for small initial data was investigated in [4]. The large-time behavior of classical solutions for the IBVP of (1.3) in one dimension with large initial data and in multi-dimensional spaces for small initial data were subsequently established in [6].

The rest of paper is organized as follows. In Section 2, we shall state our main results. In Section 3, the existence of traveling wave solutions will be briefly discussed. The linear instability analysis of traveling wave solutions for the case $\chi < 0$ will be given in Section 4, and then the proof of nonlinear stability of traveling wave solutions for $\chi > 0$ with zero right end state $u_+ = 0$ will be presented in Section 5. In Section 6, we transfer the results from the transformed system to the original chemotaxis model.

2. Statement of main results

Before proceeding, we present some notations for convenience. Throughout the paper, C denotes a generic positive constant which can change from one line to another. The integrals $\int_{\mathbb{R}} f(x)\,dx$ and $\int_0^t \int_{\mathbb{R}} f(x,\tau)\,dx\,d\tau$ will be abbreviated as $\int f(x)$ and $\int_0^t \int f(x,\tau)$, respectively, for convenience if no confusion is caused. $H^k(\mathbb{R})$ denotes the usual k-th order Sobolev space on $\mathbb{R}$ with norm $\|f\|_{H^k(\mathbb{R})} := (\sum_{j=0}^k \|\partial_x^j f\|_{L^2(\mathbb{R})}^2)^{1/2}$. $H_w^k(\mathbb{R})$ denotes the weighted space of measurable functions f so that $\sqrt{w}\partial_x^j f \in L^2$ for $0 \leqslant j \leqslant k$ with norm $\|f\|_{H_w^k(\mathbb{R})} := (\sum_{j=0}^k \int w(x)|\partial_x^j f|^2\,dx)^{1/2}$. For simplicity, we denote $\|\cdot\| := \|\cdot\|_{L^2(\mathbb{R})}$, $\|\cdot\|_w := \|\cdot\|_{L_w^2(\mathbb{R})}$, $\|\cdot\|_k := \|\cdot\|_{H^k(\mathbb{R})}$ and $\|\cdot\|_{k,w} := \|\cdot\|_{H_w^k(\mathbb{R})}$.

2.1. Main results of the transformed system (1.3)

The traveling wave solution of (1.3) with (1.5) is a non-constant special solution $(U,V) \in C^\infty(\mathbb{R})$ in the form of

$$(u,v)(x,t) = (U,V)(z), \quad z = x - st,$$

which satisfies equations

$$\begin{cases} -sU' - \chi(UV)' = DU'', \\ -sV' - U' = 0, \end{cases} \tag{2.1}$$

with boundary condition

$$U(\pm\infty) = u_\pm, \qquad V(\pm\infty) = v_\pm, \qquad U'(\pm\infty) = V'(\pm\infty) = 0, \tag{2.2}$$

where $' = \frac{d}{dz}$. Clearly we require $s \neq 0$ to ensure the existence of traveling wave solution, since otherwise it follows from the second equation of (2.1) that $U =$ constant. Integrating (2.1) in z over $\mathbb{R}$ yields the Rankine–Hugoniot condition as follows

$$\begin{cases} -s(u_+ - u_-) - \chi(u_+v_+ - u_-v_-) = 0, \\ -s(v_+ - v_-) - (u_+ - u_-) = 0 \end{cases} \tag{2.3}$$

which gives rise to

$$s^2 + \chi v_+ s - \chi u_- = 0. \tag{2.4}$$

In the paper, we only consider the case $s > 0$, which is obtained from (2.4)

$$s = \frac{-\chi v_+ + \sqrt{(\chi v_+)^2 + 4\chi u_-}}{2}, \tag{2.5}$$

and the analysis extends directly to the case $s < 0$.

Then the existence of traveling wave solutions to the system (1.3) is stated as follows.

Proposition 2.1. *Assume that $u_\pm$ and $v_\pm$ satisfy (2.3). Then there exists a monotone traveling wave solution (i.e. viscous shock wave) $(U, V)(x - st)$ to the system (2.1), which is unique up to a translation and satisfies:*

(i) *If $\chi > 0$, then $U_z < 0$, $V_z > 0$;*
(ii) *If $\chi < 0$, then $U_z > 0$, $V_z < 0$;*

where the wave speed s is given by (2.5).

When $\chi > 0$, $u_+ > 0$, the nonlinear asymptotic stability of traveling wave solution to (1.3) was proved in [7] by the L^2-energy estimates which, however, does not apply to the case $\chi < 0$ or $\chi > 0$, $u_+ = 0$. Next we use the spectral method to discuss the instability of traveling wave solutions of (1.3) for $\chi < 0$.

In the moving coordinate $z = x - st$, the solution $(u, v)(x, t) = (u, v)(z, t)$ of (1.3) satisfies

$$\begin{cases} u_t - \chi(uv)_z = su_z + Du_{zz}, \\ v_t - u_z = sv_z. \end{cases} \tag{2.6}$$

To derive the linearized stability/instability of traveling wave solutions, we consider a small perturbation of the traveling wave solution $(U, V)(z)$ in the form

$$\begin{cases} u(z, t) = U(z) + \varepsilon\tilde{u}(z, t), \\ v(z, t) = V(z) + \varepsilon\tilde{v}(z, t). \end{cases} \tag{2.7}$$

Substituting (2.7) into (2.6), keeping the first order terms in ε and dropping the tildes, we obtain the linearized system of (2.6) at (U, V)

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \mathcal{L}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{2.8}$$

where

$$\mathcal{L} = \begin{pmatrix} D\frac{\partial^2}{\partial z^2} + (s + \chi V)\frac{\partial}{\partial z} + \chi V_z & \chi U_z + \chi U\frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & s\frac{\partial}{\partial z} \end{pmatrix} \tag{2.9}$$

which is a closed operator in the space

$$X = \{(u, v) \mid (u, v) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})\}.$$

We then have the following theorem which asserts that traveling wave solutions $(U(z), V(z))$ are linearly unstable in X.

Theorem 2.2. *If $\chi < 0$, then $\sigma(\mathcal{L}) \cap \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\} \neq \emptyset$, where $\sigma(\mathcal{L})$ is the spectrum of $\mathcal{L}$ in X.*

Since the spectrum of $\mathcal{L}$ contains the positive real part, it is natural to introduce a weight function and investigate the spectrum of $\mathcal{L}$ in a weighted space. In this paper, we shall show that if $\chi < 0$, the traveling wave solution (1.3) is still linearly unstable in a general weighted space

$$X_w = H_w^1(\mathbb{R}) \times L_w^2(\mathbb{R}) \triangleq \big\{(u, v) \bigm| (wu, wv) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})\big\},$$

with norm $\|(u, v)\|_{X_w} = \|(wu, wv)\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})}$, where the weight function $w(z)$ satisfies

$$\frac{w_z}{w}(-\infty) = \alpha, \qquad \frac{w_z}{w}(+\infty) = \gamma, \qquad \left(\frac{w_z}{w}\right)_z(\pm\infty) = 0, \quad \alpha, \gamma \in \mathbb{R}. \tag{2.10}$$

Define the operator $\mathcal{L}_w : H_w^2(\mathbb{R}) \times H_w^1(\mathbb{R}) \to H_w^1(\mathbb{R}) \times L_w^2(\mathbb{R})$ by

$$\mathcal{L}_w\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{L}\begin{pmatrix} u \\ v \end{pmatrix}, \quad \text{if } \begin{pmatrix} u \\ v \end{pmatrix} \in H_w^2(\mathbb{R}) \times H_w^1(\mathbb{R}).$$

Then we have the following theorem.

Theorem 2.3. *Let $u_\pm$ and $v_\pm$ satisfy (2.3) with $\chi < 0$, and $\sigma(\mathcal{L}_w)$ denote the spectrum of $\mathcal{L}$ in X_w. Then the following results hold.*

(i) *If $u_- > 0$, then $\sigma(\mathcal{L}_w) \cap \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\} \neq \emptyset$ for all $\alpha, \gamma \in \mathbb{R}$;*
(ii) *If $u_- = 0$, then $\sigma(\mathcal{L}_w) \cap \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\} \neq \emptyset$ for all $(\alpha, \gamma) \neq (0, \frac{u_+}{Dv_+})$ where $\alpha, \gamma \in \mathbb{R}$.*

Remark 2.1. The instability/stability of traveling wave solutions of (1.3) with $\chi < 0$, $u_- = 0$ in the weighted space for the case $\alpha = 0$, $\gamma = \frac{u_+}{Dv_+}$ remains unknown in the present paper, and we will leave it as an open question. Theorems 2.2 and 2.3 mean that the traveling waves are linearly unstable in the spaces X and X_w. However these theorems do not necessarily mean the linearized instability for perturbations in other weighted spaces. Further spectral analysis in more appropriate weighted spaces is worthwhile to be pursued.

Next result for the transformed model (1.3)–(1.5) is the nonlinear asymptotic stability of traveling wave solutions with $\chi > 0$, $u_+ = 0$, which was an open problem in [7].

Theorem 2.4. *Let $\chi > 0$, $u_+ = 0$ and $(U, V)(x - st)$ be a traveling wave solution obtained in Proposition 2.1. Then there exists a constant $\varepsilon_0 > 0$ such that if $\|u_0 - U\|_{1,w} + \|v_0 - V\|_{1,w} + \|(\phi_0, \psi_0)\|_w \leqslant \varepsilon_0$, where*

$$(\phi_0, \psi_0)(x) = \int_{-\infty}^{x} \big(u_0(y) - U(y), v_0(y) - V(y)\big)\, dy, \tag{2.11}$$

the Cauchy problem (1.3)–(1.5) has a unique global solution $(u, v)(x, t)$ satisfying

$$(u - U, v - V) \in C\big([0, \infty); H_w^1\big) \cap L^2\big((0, \infty); H_w^1\big),$$

where the weight function w is defined by

$$w(z) := 1 + e^{\eta z}, \quad z \in \mathbb{R} \tag{2.12}$$

with $\eta = \frac{\chi u_-}{Ds} > 0$*. Furthermore, the solution has the following asymptotic stability*:

$$\sup_{x\in\mathbb{R}}\big|(u,v)(x,t)-(U,V)(x-st)\big| \to 0, \quad \text{as } t\to+\infty.$$

Remark 2.2. Analogous essential spectral analysis for $\chi < 0$ applied to the case $\chi > 0$, $u_+ = 0$ (the details are omitted for brevity) can show that the essential spectrum in X contains the imaginary axis, and the essential spectrum in X_w with weight function (2.12) contains the zero. This cannot conclude the linear or nonlinear stability of traveling waves, and however indicates the traveling waves do not have exponential stability in both spaces X and X_w.

2.2. Main results of the original chemotaxis model (1.1)

Now we transfer the results back to the original chemotaxis model (1.1) from the results for the transformed system (1.3). Noting that the solution component u remains the same in both (1.1) and (1.3), we only need to consider the solution component c in (1.1) whose traveling wave ansatz is

$$c(x,t) = \mathcal{C}(z), \qquad \mathcal{C}(\pm\infty) = c_\pm \geqslant 0, \quad z = x - st$$

which satisfies

$$-s\mathcal{C}_z = \mu U(z)\mathcal{C}(z). \tag{2.13}$$

The existence theorem of traveling wave solutions to (1.1) is as follows.

Theorem 2.5. *Let* $\xi, \mu \in \mathbb{R}\setminus\{0\}$*. Then the chemotaxis model* (1.1) *does not have a traveling wave solution if* $\xi < 0$*. If* $\xi > 0$*, we have the following*:

- (i) *If* $\mu < 0$*, the chemotaxis model* (1.1) *has a unique monotone traveling wave solution* $(U,\mathcal{C})(z)$ *such that* $U_z < 0$, $\mathcal{C}_z > 0$ *for any given* $0 = u_+ < u_-$, $0 = c_- < c_+$.
- (ii) *If* $\mu > 0$*, the chemotaxis model* (1.1) *does not have a traveling wave solution.*

From Theorem 2.5, we see that the physical parameter regime for the existence of traveling wave solutions to the original chemotaxis system (1.1) is

$$\mathcal{X} = \big\{(\xi,\mu,u_-,u_+,c_-,c_+)\ \big|\ \xi > 0,\ \mu < 0,\ u_+ = c_- = 0,\ u_-, c_+ > 0\big\}.$$

The nonlinear stability of traveling wave solutions of (1.1) is stated as follows.

Theorem 2.6. *Let* $(U,\mathcal{C})(x-st)$ *be a traveling wave profile of* (1.1) *with initial data* (1.6) *obtained in Theorem* 2.5*. If* $(\xi,\mu,u_-,u_+,c_-,c_+)\in\mathcal{X}$*, then there exists a constant* $\varepsilon_0 > 0$ *such that if* $\|u_0 - U\|_{1,w} + \|(\ln c_0)_x - (\ln\mathcal{C})_x\|_{1,w} + \|(\phi_0,\psi_0)\|_w \leqslant \varepsilon_0$*, where*

$$\phi_0(x) = \int_{-\infty}^{x} (u_0 - U)(y)\,dy, \qquad \psi_0(x) = -\ln c_0(x) + \ln\mathcal{C}(x),$$

then the Cauchy problem (1.1), (1.6) *has a unique global solution* $(u,c)(x,t)$ *with*

$$(u-U, c_x/c - \mathcal{C}_x/\mathcal{C}) \in C\big([0,\infty); H_w^1\big) \cap L^2\big((0,\infty); H_w^1\big)$$

and the following asymptotic behavior

$$\sup_{x\in\mathbb{R}}\left|(u,c)(x,t)-(U,\mathcal{C})(x-st)\right|\to 0 \quad \text{as } t\to\infty.$$

3. Existence of traveling wave solutions of (1.3)

In this section, we are devoted to proving Proposition 2.1 and hence establish the existence of traveling wave solutions to the system (1.3) with (1.5).

Proof of Proposition 2.1. Integrating the second equation of (2.1), one has that

$$sV+U=\varrho_1=sv_++u_+=sv_-+u_-. \tag{3.1}$$

We substitute (3.1) into the first equation of (2.1) to get

$$U''=\sigma U'(U-\beta), \tag{3.2}$$

where $\sigma=\frac{2\chi}{Ds}$ and $\beta=\frac{u_++u_-}{2}$. Now we let $H=U'$ and from (3.2) we have

$$\begin{cases} U'=H,\\ H'=\sigma H(U-\beta). \end{cases} \tag{3.3}$$

It is clear that system (3.3) has a continuum of equilibrium $(\theta,0)$, where $\theta\geqslant 0$ due to the particle density $U\geqslant 0$. The corresponding Jacobian matrix at the equilibrium $(\theta,0)$ is

$$J=\begin{bmatrix} 0 & 1\\ 0 & \sigma(\theta-\beta)\end{bmatrix}, \tag{3.4}$$

whose eigenvalues are $\lambda_1=0$, $\lambda_2=\sigma(\theta-\beta)$. So, $\theta=\beta=\frac{u_++u_-}{2}>0$ is a critical point which separates the steady state into stable and unstable parts. From (3.3), one has $\frac{dH}{dU}=\sigma(U-\beta)$, which implies $H(U)=\frac{1}{2}\sigma U^2-\sigma\beta U+\varrho_2$, where $\varrho_2=-\frac{1}{2}\sigma u_\pm^2+\sigma\beta u_\pm$. Then substituting it into the first equation of (3.3) leads to

$$U'=\frac{1}{2}\sigma U^2-\sigma\beta U+\varrho_2, \tag{3.5}$$

which yields explicit solution $U(z)$ and hence $V(z)$ as

$$U(z)=u_++\frac{u_+-u_-}{\kappa e^{\frac{\sigma(u_--u_+)}{2}z}-1}, \qquad V(z)=\frac{\varrho_1-U}{s} \tag{3.6}$$

where $\kappa<0$ is an arbitrary constant. It is helpful to note that κ represents a constant of translation of the traveling wave solution (3.6). Indeed, letting $\tau=\frac{-2\ln(-\kappa)}{\sigma(u_--u_+)}$, then $U(z+\tau)$ corresponds to (3.6) with $\kappa=-1$. Hence in the sense of translation, the solution (U,V) given in (3.6) is unique. Further calculations give rise to

$$U'=\frac{\sigma\kappa(u_--u_+)^2e^{\frac{\sigma(u_--u_+)}{2}z}}{2(\kappa e^{\frac{\sigma(u_--u_+)}{2}z}-1)^2}, \qquad V'=-\frac{U'}{s}.$$

Noticing that $\sigma = \frac{2\chi}{Ds}$ and $s > 0$, we can easily find that

$$U' \begin{cases} > 0, & \text{if } \chi < 0, \\ < 0, & \text{if } \chi > 0, \end{cases} \qquad V' \begin{cases} < 0, & \text{if } \chi < 0, \\ > 0, & \text{if } \chi > 0 \end{cases}$$

which completes the proof of Proposition 2.1. □

Remark 3.1. When $\chi > 0$, $u_+ = 0$, $U(z) = -\frac{u_-}{\kappa e^{\eta z} - 1}$ and hence $\frac{1}{U(z)} = \frac{1}{u_-} - \frac{\kappa}{u_-} e^{\eta z}$. By the definition (2.12) of weight function w, one can easily find two constants $C_2 > C_1 > 0$ such that

$$C_1 w(z) \leqslant \frac{1}{U(z)} \leqslant C_2 w(z) \quad \text{for all } z \in \mathbb{R}. \tag{3.7}$$

4. Linearized instability of traveling wave solutions

In this section, we perform the spectral analysis for the linearized system of (1.3) at the traveling wave solution and prove Theorem 2.2 and Theorem 2.3.

4.1. Proof of Theorem 2.2

From (2.9), we can deduce that the asymptotic operators of $\mathcal{L}$ at $z = \pm\infty$ are

$$\mathcal{L}^{\pm} = \begin{pmatrix} D\frac{\partial^2}{\partial z^2} + (s + \chi v_{\pm})\frac{\partial}{\partial z} & \chi u_{\pm}\frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & s\frac{\partial}{\partial z} \end{pmatrix}. \tag{4.1}$$

Denote

$$A^{\pm}(\mu) = \begin{pmatrix} -D\mu^2 + (s + \chi v_{\pm})\mu i & \chi u_{\pm}\mu i \\ \mu i & s\mu i \end{pmatrix}, \quad \text{for } \mu \in \mathbb{R}$$

and the curves $S^{\pm}$ by

$$\begin{aligned} S^{\pm} &= \{\lambda \in \mathbb{C} \mid \det(\lambda I - A^{\pm}(\mu)) = 0, \text{ for some } \mu \in \mathbb{R}\} \\ &= \{\lambda \in \mathbb{C} \mid (\lambda - s\mu i)(\lambda + D\mu^2 - (s + \chi v_{\pm})\mu i) + \mu^2 \chi u_{\pm} = 0, \text{ for some } \mu \in \mathbb{R}\}. \end{aligned} \tag{4.2}$$

By applying essential spectral theory in [2], the boundary of the essential spectrum of $\mathcal{L}$ is described by curves $S^{\pm}$. If $\lambda \in S^+$, then λ satisfies

$$\lambda^2 + a(\mu)\lambda + b(\mu) = 0, \tag{4.3}$$

where

$$a(\mu) = D\mu^2 - (2s + \chi v_+)\mu i, \qquad b(\mu) = -s\mu^2(s + \chi v_+) + \chi u_+ \mu^2 - Ds\mu^3 i.$$

Let

$$d(\mu) = a^2(\mu) - 4b(\mu) = \operatorname{Re} d(\mu) + i \operatorname{Im} d(\mu), \tag{4.4}$$

and

$$\sqrt{d(\mu)} = c_1 + ic_2, \quad c_1, c_2 \in \mathbb{R}. \tag{4.5}$$

The combination (4.4) and (4.5) implies that

$$c_1^2 - c_2^2 + 2c_1c_2 i = d(\mu) = \operatorname{Re} d(\mu) + i \operatorname{Im} d(\mu). \tag{4.6}$$

Solving (4.6), one has

$$c_1^2 = \frac{\operatorname{Re} d(\mu) + \sqrt{(\operatorname{Re} d(\mu))^2 + (\operatorname{Im} d(\mu))^2}}{2}. \tag{4.7}$$

Since $\operatorname{Re} a(\mu) > 0$ and $\operatorname{Re}\lambda = \frac{1}{2}(-\operatorname{Re} a(\mu) \pm c_1)$, we have that

$$\operatorname{Re}\lambda < 0 \quad \Longleftrightarrow \quad \bigl(\operatorname{Re} a(\mu)\bigr)^2 > c_1^2. \tag{4.8}$$

Let

$$I(\mu) \triangleq \bigl(\operatorname{Im} d(\mu)\bigr)^2 + 4\bigl(\operatorname{Re} a(\mu)\bigr)^2 \operatorname{Re} d(\mu) - 4\bigl(\operatorname{Re} a(\mu)\bigr)^4. \tag{4.9}$$

Then from (4.7), (4.8) and (4.9), we can derive that

$$\operatorname{Re}\lambda < 0 \quad \Longleftrightarrow \quad I(\mu) < 0. \tag{4.10}$$

Substituting

$$\operatorname{Re} a^2(\mu) = \bigl(\operatorname{Re} a(\mu)\bigr)^2 - \bigl(\operatorname{Im} a(\mu)\bigr)^2, \qquad \operatorname{Re} d(\mu) = \operatorname{Re} a^2(\mu) - 4\operatorname{Re} b(\mu)$$

into (4.9), one has

$$\begin{aligned} I(\mu) &= \bigl(\operatorname{Im} d(\mu)\bigr)^2 - 4\bigl(\operatorname{Re} a(\mu)\bigr)^2\bigl(\operatorname{Im} a(\mu)\bigr)^2 - 16\bigl(\operatorname{Re} a(\mu)\bigr)^2 \operatorname{Re} b(\mu) \\ &= \bigl[4Ds\mu^3 - 2D\mu^3(2s + \chi v_+)\bigr]^2 - 4\bigl(D\mu^2\bigr)^2\bigl[(\chi v_+ + 2s)\mu\bigr]^2 \\ &\quad + 16\bigl(D\mu^2\bigr)^2\bigl[-\chi u_+\mu^2 + s\mu^2(s + \chi v_+)\bigr] \\ &= -16D^2\mu^6\chi u_+. \end{aligned} \tag{4.11}$$

When $\chi < 0$, one has $0 \leqslant u_- < u_+$ by Proposition 2.1. From (4.11), it follows that there exists some $\mu \neq 0$ such that $I(\mu) > 0$, which implies $\operatorname{Re}\lambda > 0$ for $\mu \neq 0$. That is, $\sigma_{ess}(\mathcal{L}) \cap \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\} \neq \emptyset$. Since $\sigma_{ess}(\mathcal{L}) \subset \sigma(\mathcal{L})$, the proof of Theorem 2.2 is finished. □

The following remark will be used later.

Remark 4.1. When $\lambda \in S^-$, we have

$$\lambda^2 + a(\mu)\lambda + b(\mu) = 0, \tag{4.12}$$

where

$$a(\mu) = D\mu^2 - (2s + \chi v_-)\mu i, \qquad b(\mu) = -Ds\mu^3 i - s\mu^2(s + \chi v_-) + \chi u_-\mu^2. \tag{4.13}$$

Applying the same procedure as for the case $\lambda \in S^+$, we have

$$\begin{aligned} I(\mu) &= \left[4Ds\mu^3 - 2D\mu^3(2s+\chi v_-)\right]^2 - 4\left(D\mu^2\right)^2\left[(\chi v_- + 2s)\mu\right]^2 \\ &\quad + 16\left(D\mu^2\right)^2\left[-\chi u_-\mu^2 + s\mu^2(s+\chi v_-)\right] \\ &= -16D^2\mu^6\chi u_-. \end{aligned} \tag{4.14}$$

4.2. *Proof of Theorem 2.3*

In this section, we shall calculate the essential spectrum of the operator $\mathcal{L}_w$. To this end, we define the operator $\widetilde{\mathcal{L}}_w : H^2(\mathbb{R}) \times H^1(\mathbb{R}) \to H^1(\mathbb{R}) \times L^2(\mathbb{R})$ by

$$\widetilde{\mathcal{L}}_w \begin{pmatrix} u \\ v \end{pmatrix} = w\mathcal{L}\begin{pmatrix} w^{-1}u \\ w^{-1}v \end{pmatrix}, \quad \text{if } \begin{pmatrix} u \\ v \end{pmatrix} \in H^2(\mathbb{R}) \times H^1(\mathbb{R}). \tag{4.15}$$

Then it follows that $\sigma(\mathcal{L}_w) = \sigma(\widetilde{\mathcal{L}}_w)$ and $\sigma_{ess}(\mathcal{L}_w) = \sigma_{ess}(\widetilde{\mathcal{L}}_w)$, see [13,20].

Using (4.15), we derive that the asymptotic operators $\widetilde{\mathcal{L}}_w$ at $z = -\infty$ is

$$\widetilde{\mathcal{L}}_w^- = \begin{pmatrix} D\frac{\partial^2}{\partial z^2} - 2D\alpha\frac{\partial}{\partial z} + D\alpha^2 + (s+\chi v_-)(\frac{\partial}{\partial z} - \alpha) & \chi u_-(\frac{\partial}{\partial z} - \alpha) \\ \frac{\partial}{\partial z} - \alpha & s(\frac{\partial}{\partial z} - \alpha) \end{pmatrix}. \tag{4.16}$$

Let

$$A_\alpha^-(\mu) = \begin{pmatrix} D(i\mu-\alpha)^2 + (s+\chi v_-)(i\mu-\alpha) & \chi u_-(i\mu-\alpha) \\ i\mu-\alpha & s(i\mu-\alpha) \end{pmatrix}. \tag{4.17}$$

Define the curves S_α^- by

$$S_\alpha^- = \left\{\lambda \in \mathbb{C} \,\middle|\, \det\left(\lambda I - A_\alpha^-(\mu)\right) = 0, \text{ for some } \mu \in \mathbb{R}\right\}.$$

If $\lambda \in S_\alpha^-$, one has

$$\left[\lambda - D(i\mu-\alpha)^2 - (s+\chi v_-)(i\mu-\alpha)\right]\left[\lambda - s(i\mu-\alpha)\right] - \chi u_-(i\mu-\alpha)^2 = 0, \tag{4.18}$$

which implies that λ satisfies

$$\lambda^2 + a(\mu)\lambda + b(\mu) = 0, \tag{4.19}$$

where

$$\begin{cases} a(\mu) = -D(i\mu-\alpha)^2 - (2s+\chi v_-)(i\mu-\alpha), \\ b(\mu) = Ds(i\mu-\alpha)^3 + \left[s(s+\chi v_-) - \chi u_-\right](i\mu-\alpha)^2. \end{cases} \tag{4.20}$$

Using

$$\begin{cases} (i\mu-\alpha)^2 = \alpha^2 - \mu^2 - 2\alpha\mu i, \\ (i\mu-\alpha)^3 = 3\alpha\mu^2 - \alpha^3 + \left(3\alpha^2\mu - \mu^3\right)i, \\ (i\mu-\alpha)^4 = \alpha^4 + \mu^4 - 6\alpha^2\mu^2 + \left(4\alpha\mu^3 - 4\alpha^3\mu\right)i, \end{cases}$$

one can derive that

$$\begin{aligned} d(\mu) &= a(\mu)^2 - 4b(\mu) \\ &= D^2(i\mu-\alpha)^4 + 2D\chi v_-(i\mu-\alpha)^3 + \left(\chi^2 v_-^2 + 4\chi u_-\right)(i\mu-\alpha)^2 \\ &= \operatorname{Re} d(\mu) + i \operatorname{Im} d(\mu), \end{aligned}$$

where

$$\begin{cases} \operatorname{Re} d(\mu) = D^2\left(\alpha^4+\mu^4-6\alpha^2\mu^2\right) + 2D\chi v_-\left(3\alpha\mu^2-\alpha^3\right) + \left(\chi^2 v_-^2 + 4\chi u_-\right)\left(\alpha^2-\mu^2\right), \\ \operatorname{Im} d(\mu) = \left(6D\chi\alpha^2 v_- - 4D^2\alpha^3 - 8\alpha\chi u_- - 2\alpha\chi^2 v_-^2\right)\mu + \left(4D^2\alpha - 2D\chi v_-\right)\mu^3. \end{cases}$$

We can readily get that

$$\begin{cases} \operatorname{Re} a(\mu) = D\left(\mu^2-\alpha^2\right) + \alpha(2s+\chi v_-), \\ \operatorname{Im} a(\mu) = 2D\alpha\mu - (2s+\chi v_-)\mu, \\ \operatorname{Re} b(\mu) = \left(s^2+s\chi v_- - \chi u_-\right)\left(\alpha^2-\mu^2\right) + 3D\alpha s\mu^2 - D\alpha^3 s, \\ \operatorname{Im} b(\mu) = Ds\left(3\alpha^2\mu-\mu^3\right) - 2\left(s^2+s\chi v_- - \chi u_-\right)\alpha\mu. \end{cases}$$

If $\alpha < 0$, then $\operatorname{Re} a(\mu) < 0$ for small μ which indicates that either $\operatorname{Re}\lambda = \frac{1}{2}(-\operatorname{Re} a(\mu) + c_1) > 0$ or $\operatorname{Re}\lambda = \frac{1}{2}(-\operatorname{Re} a(\mu) - c_1) > 0$ for all $c_1 \in \mathbb{R}$. We then proceed to consider the case $\alpha \geqslant 0$. Using the same argument as in previous subsection, one derives that

$$\begin{aligned} I_\alpha^-(\mu) &= \left(\operatorname{Im} d(\mu)\right)^2 - 4\left(\operatorname{Re} a(\mu)\right)^2\left(\operatorname{Im} a(\mu)\right)^2 - 16\left(\operatorname{Re} a(\mu)\right)^2 \operatorname{Re} b(\mu) \\ &= I_1^-(D,\alpha,s,\chi,u_-,v_-)\mu^2 + I_2^-(D,\alpha,s,\chi,u_-,v_-)\mu^4 + I_3^-(D,\alpha,s,\chi,u_-,v_-)\mu^6 \\ &\quad - 16\left[\alpha(2s+\chi v_-) - D\alpha^2\right]^2\left[\left(s^2+s\chi v_- - \chi u_-\right)\alpha^2 - Ds\alpha^3\right] \\ &= I_1^-(D,\alpha,s,\chi,u_-,v_-)\mu^2 + I_2^-(D,\alpha,s,\chi,u_-,v_-)\mu^4 + I_3^-(D,\alpha,s,\chi,u_-,v_-)\mu^6 \\ &\quad + 16\left[\alpha(2s+\chi v_-) - D\alpha^2\right]^2\left[\chi(u_- - u_+)\alpha^2 + Ds\alpha^3\right], \end{aligned} \tag{4.21}$$

where

$$\begin{aligned} I_1^-(D,\alpha,s,\chi,u_-,v_-) &= 16\alpha^5 D^3 s - 16\alpha^2\chi\left(s^2 v_-^2\chi + 4su_-v_-\chi\right. \\ &\quad \left. + sv_-^3\chi^2 - 4u_-^2\chi - u_-v_-^2\chi^2 + 4u_-s^2\right) \\ &\quad + 16D^2\alpha^4\left(6s^2 - 2sv_-\chi + u_-\chi\right) \\ &\quad - 32D\alpha^3\left(4s^3 + 3s^2v_-\chi - 4su_-\chi - sv_-^2\chi^2 + u_-v_-\chi^2\right), \\ I_2^-(D,\alpha,s,\chi,u_-,v_-) &= -16\alpha^3 D^3 s - 16\alpha^2 D^2\left(5s^2 - sv_-\chi + u_-\chi\right) \\ &\quad - 8\alpha D\left(2sv_-^2\chi^2 + 8u_-s\chi\right), \\ I_3^-(D,\alpha,s,\chi,u_-,v_-) &= -16D^2(\chi u_- + Ds\alpha). \end{aligned}$$

Concerning the values of α and u_-, we have four different cases to consider:

$$\begin{aligned} &\text{(i)} \quad u_- > 0, \qquad \alpha > 0; \\ &\text{(ii)} \quad u_- > 0, \qquad \alpha = 0; \\ &\text{(iii)} \quad u_- = 0, \qquad \alpha > 0; \\ &\text{(iv)} \quad u_- = 0, \qquad \alpha = 0. \end{aligned} \tag{4.22}$$

For case (i) of (4.22), one has $\chi(u_- - u_+)\alpha^2 + Ds\alpha^3 > 0$. When $\alpha(2s + \chi v_-) - D\alpha^2 \neq 0$, we can observe from (4.21) that if μ is small, then $I_\alpha^-(\mu) > 0$, which implies $\operatorname{Re}\lambda > 0$. When $\alpha(2s + \chi v_-) - D\alpha^2 = 0$, one can derive that

$$\begin{cases} \operatorname{Re} a(\mu) = D\mu^2, \qquad \operatorname{Im} a(\mu) = D\alpha\mu, \\ \operatorname{Re} b(\mu) = -\left[\chi(u_- - u_+)\alpha^2 + Ds\alpha^3\right] + \left[\chi(u_- - u_+)\alpha^2 + 3Ds\alpha\right]\mu^2, \\ \operatorname{Im} d(\mu) = -\left(16s^2 + 8\chi u_- + 4s\chi v_-\right)\alpha\mu + \left(4D^2\alpha - 2D\chi v_-\right)\mu^3. \end{cases} \tag{4.23}$$

Hence we write $I_\alpha^-(\mu)$ as

$$\begin{aligned} I_\alpha^-(\mu) &= \left(\operatorname{Im} d(\mu)\right)^2 - 4\left(\operatorname{Re} a(\mu)\right)^2\left(\operatorname{Im} a(\mu)\right)^2 - 16\left(\operatorname{Re} a(\mu)\right)^2 \operatorname{Re} b(\mu) \\ &= \left(16s^2 + 8\chi u_- + 4s\chi v_-\right)^2\alpha^2\mu^2 + 16D^2\left[\chi(u_- - u_+)\alpha^2 + Ds\alpha^3\right]\mu^4 \\ &\quad - 2\alpha\left(16s^2 + 8\chi u_- + 4s\chi v_-\right)\left(4D^2\alpha - 2D\chi v_-\right)\mu^4 \\ &\quad + \left(12D^4\alpha^2 - 16D^3\chi v_-\alpha + 4D^2\chi^2 v_-^2 + 16D^2\chi\alpha^2(u_+ - u_-) - 48D^3 s\alpha\right)\mu^6. \end{aligned} \tag{4.24}$$

Since $\chi(u_- - u_+)\alpha^2 + Ds\alpha^3 > 0$, we find that $I_\alpha^-(\mu) > 0$ for sufficiently small $\mu \neq 0$, which implies $\operatorname{Re}\lambda > 0$. Hence in case (i) of (4.22), $\operatorname{Re}\lambda > 0$ for small $\mu \neq 0$. In case (ii) of (4.22), $A_\alpha^-(\mu) = A^-(\mu)$ and hence from Remark 4.1 it follows that $I_\alpha^-(\mu) = -16D^2\mu^6\chi u_- > 0$ for all $\mu \neq 0$ due to $\chi < 0$. This indicates that $\operatorname{Re}\lambda > 0$. Hence above analysis shows that $\sigma_{ess}(\mathcal{L}_w) \cap \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\} \neq \emptyset$ for all $\alpha, \gamma \in \mathbb{R}$. Then the fact $\sigma_{ess}(\mathcal{L}_w) \subset \sigma(\mathcal{L}_w)$ completes the proof of Theorem 2.3(i). Now it remains to consider the case (iii) and (iv) of (4.22) to complete the proof of Theorem 2.3(ii). For case (iii) of (4.22), it holds that $\chi(u_- - u_+)\alpha^2 + Ds\alpha^3 > 0$ for $u_- = 0$ and we hence may let μ sufficiently small in (4.21) such that $I_\alpha^-(\mu) > 0$ which implies $\operatorname{Re}\lambda > 0$. To finish the proof of Theorem 2.3(ii), we now proceed to consider case (iv) where we have $I_\alpha^-(\mu) = -16D^2\mu^6\chi u_- = 0$, which is inclusive for the sign of $\operatorname{Re}\lambda$. In this scenario, we proceed to determine the spectrum of asymptotic operators $\widetilde{\mathcal{L}}_w$ at $z = +\infty$. Before proceeding, we want to underline in case (iv) where $u_- = 0$, one can obtain from (2.5) that

$$s = -\chi v_+ > 0. \tag{4.25}$$

Using (4.15) and (4.25), we derive that the asymptotic operators $\widetilde{\mathcal{L}}_w$ at $z = +\infty$ is

$$\widetilde{\mathcal{L}}_w^+ = \begin{pmatrix} D\frac{\partial^2}{\partial z^2} - 2D\gamma\frac{\partial}{\partial z} + D\gamma^2 & \chi u_+(\frac{\partial}{\partial z} - \gamma) \\ \frac{\partial}{\partial z} - \gamma & s(\frac{\partial}{\partial z} - \gamma) \end{pmatrix}. \tag{4.26}$$

Let

$$A_\gamma^+(\mu) = \begin{pmatrix} D(i\mu - \gamma)^2 & \chi u_+(i\mu - \gamma) \\ i\mu - \gamma & s(i\mu - \gamma) \end{pmatrix}. \tag{4.27}$$

Define the curves S_γ^+ by

$$S_\gamma^+ = \{\lambda \in \mathbb{C} \mid \det(\lambda I - A_\gamma^+(\mu)) = 0, \text{ for some } \mu \in \mathbb{R}\}.$$

If $\lambda \in S_\gamma^+$, one has

$$[\lambda - D(i\mu - \gamma)^2][\lambda - s(i\mu - \gamma)] - \chi u_+(i\mu - \gamma)^2 = 0, \tag{4.28}$$

which implies that λ satisfies

$$\lambda^2 + a(\mu)\lambda + b(\mu) = 0, \tag{4.29}$$

where

$$\begin{cases} a(\mu) = -D(i\mu - \gamma)^2 - s(i\mu - \gamma), \\ b(\mu) = Ds(i\mu - \gamma)^3 - \chi u_+(i\mu - \gamma)^2. \end{cases}$$

Hence

$$\begin{cases} \operatorname{Re} a(\mu) = \gamma s - D\gamma^2 + D\mu^2, \\ \operatorname{Im} a(\mu) = (2D\gamma - s)\mu, \\ \operatorname{Re} b(\mu) = -(\chi u_+\gamma^2 + D\gamma^3 s) + (3D\gamma s + \chi u_+)\mu^2, \\ \operatorname{Im} b(\mu) = (3Ds\gamma^2 + 2\chi u_+\gamma)\mu - Ds\mu^3. \end{cases} \tag{4.30}$$

If $\gamma s - D\gamma^2 < 0$, then $\operatorname{Re} a(\mu) < 0$ for small $\mu \neq 0$. This implies that either $\operatorname{Re}\lambda = \frac{1}{2}(-\operatorname{Re} a(\mu) + c_1) > 0$ or $\operatorname{Re}\lambda = \frac{1}{2}(-\operatorname{Re} a(\mu) - c_1) > 0$ for all $c_1 \in \mathbb{R}$. Hence we consider the case $\gamma s - D\gamma^2 \geqslant 0$ in what follows. One can derive that

$$\begin{aligned} \operatorname{Im} d(\mu) &= \operatorname{Im} a(\mu)^2 - 4\operatorname{Im} b(\mu) \\ &= 2\operatorname{Re} a(\mu)\operatorname{Im} a(\mu) - 4\operatorname{Im} b(\mu) \\ &= (4D^2\gamma + 2Ds)\mu^3 - (4D^2\gamma^3 + 2\gamma s^2 + 8\chi u_+\gamma + 6Ds\gamma^2)\mu. \end{aligned} \tag{4.31}$$

Using (4.30) and (4.31) yields that

$$\begin{aligned} I_\gamma^+(\mu) &= (\operatorname{Im} d(\mu))^2 - 4(\operatorname{Re} a(\mu))^2(\operatorname{Im} a(\mu))^2 - 16(\operatorname{Re} a(\mu))^2 \operatorname{Re} b(\mu) \\ &= J_1^+(D,\gamma,\chi,u_+,s)\mu^2 + J_2^+(D,\gamma,\chi,u_+,s)\mu^4 + J_3^+(D,\gamma,\chi,u_+,s)\mu^6 \\ &\quad + 16\gamma^2(s\gamma - D\gamma^2)^2(\chi u_+ + Ds\gamma), \end{aligned} \tag{4.32}$$

where

$$\begin{aligned} &J_1^+(D,\gamma,\chi,u_+,s) \\ &= (4D^2\gamma^3 + 2\gamma s^2 + 8\chi u_+\gamma + 6Ds\gamma^2)^2 - 4(\gamma s - D\gamma^2)^2(2D\gamma - s)^2 \\ &\quad - 16(\gamma s - D\gamma^2)^2(3D\gamma s + \chi u_+) + 32D(\gamma s - D\gamma^2)(\chi u_+\gamma^2 + D\gamma^3 s) \\ &= 16D\gamma^3 s^2(2D\gamma - s) + 16\gamma^2(D^2\gamma^2 + s^2 + 6Ds\gamma + 4\chi u_+)(\chi u_+ + Ds\gamma), \end{aligned} \tag{4.33}$$

$$\begin{aligned} &J_2^+(D,\gamma,\chi,u_+,s)\\ &\quad=-8D\big(\gamma s-D\gamma^2\big)(2D\gamma-s)^2-32D\big(\gamma s-D\gamma^2\big)(3D\gamma s+\chi u_+)\\ &\qquad-2\big(4D^2\gamma+2Ds\big)\big(4D^2\gamma^3+2\gamma s^2+8\chi u_+\gamma+6Ds\gamma^2\big)+16D^2\gamma^2(\chi u_++D\gamma s)\\ &\quad=-16D\gamma s\big(s^2+2D\gamma s\big)-16D\gamma(4s+D\gamma)(\chi u_++Ds\gamma), \end{aligned} \tag{4.34}$$

and

$$J_3^+(D,\gamma,\chi,u_+,s)=-16D^2(\chi u_++Ds\gamma). \tag{4.35}$$

First if $\gamma=0$, then $I_\gamma^+(\mu)=-16D^2\mu^6\chi u_+>0$ for all $\mu\neq0$, and hence $\operatorname{Re}\lambda>0$. Next we consider the case when $\gamma\neq0$ and split the analysis into two steps. (1) If $\chi u_++Ds\gamma>0$, since $J_1^+(D,\gamma,\chi,u_+,s)$, $J_2^+(D,\gamma,\chi,u_+,s)$, $J_3^+(D,\gamma,\chi,u_+,s)$ are bounded constants, then $I_\gamma^+(\mu)>0$ for sufficiently small $\mu\neq0$ when $s\gamma-D\gamma^2>0$. When $s\gamma-D\gamma^2=0$, using (4.32), (4.33), (4.34) and (4.35), we have

$$\begin{aligned} I_\gamma^+(\mu)&=\big(4D^2\gamma^3+2\gamma s^2+8\chi u_+\gamma+6Ds\gamma^2\big)^2\mu^2\\ &\quad-12D^2\gamma\big(4D^2\gamma^3+2\gamma s^2+8\chi u_+\gamma+6Ds\gamma^2\big)\mu^4\\ &\quad+16D^2\gamma^2(\chi u_++D\gamma s)\mu^4-16D^2(\chi u_++Ds\gamma)\mu^6, \end{aligned}$$

which also implies $I_\gamma^+(\mu)>0$ for small $\mu\neq0$. (2) If $\chi u_++Ds\gamma<0$, then $J_3^+(D,\gamma,\chi,u_+,s)>0$ and hence $I_\gamma^+(\mu)>0$ for sufficiently large $\mu\neq0$. Then above analysis asserts that when $\chi u_++Ds\gamma\neq0$, namely $\gamma\neq-\frac{\chi u_+}{Ds}=\frac{u_+}{Dv_+}$, $\sigma_{ess}(\widetilde{\mathcal{L}}_w)\cap\{\lambda\in\mathbb{C}\mid\operatorname{Re}\lambda>0\}\neq\emptyset$. Noticing that $\sigma_{ess}(\widetilde{\mathcal{L}}_w)=\sigma_{ess}(\mathcal{L}_w)$, we complete the proof of Theorem 2.3(ii). □

Remark 4.2. If $\chi u_++Ds\gamma=0$, namely $\gamma=-\frac{\chi u_+}{Ds}=\frac{u_+}{Dv_+}$, then

$$I_\gamma^+(\mu)=16D\gamma^3s\big(2D\gamma s-s^2\big)\mu^2-16D\gamma s\big(s^2+2D\gamma s\big)\mu^4. \tag{4.36}$$

Clearly the sign of $I_\gamma^+(\mu)$ cannot be determined since the sign of $2D\gamma s-s^2$ is unknown. Indeed from (4.25), it has that

$$2D\gamma s-s^2=-\chi\big(2u_++\chi v_+^2\big).$$

Since $\chi<0$, if we assume that $2u_++\chi v_+^2>0$, then $2D\gamma s-s^2>0$ and hence $I_\gamma^+(\mu)>0$ for sufficiently small $\mu\neq0$. This indicates $\operatorname{Re}\lambda>0$ for sufficiently small $\mu\neq0$ and the linear instability of traveling wave solutions can be asserted. Since $2u_++\chi v_+^2>0$ is a stringent condition, we do not incorporate it in the statement of Theorem 2.3(ii).

Remark 4.3. When $u_-=0$, we can calculate from (3.6) that

$$U(z)=\frac{u_+\kappa e^{-\frac{\chi u_+}{Ds}z}}{\kappa e^{-\frac{\chi u_+}{Ds}z}-1}$$

which indicates that the number $\gamma=-\frac{\chi u_+}{Ds}$ is the decay rate of $U(z)$ as $z\to\infty$.

5. Nonlinear asymptotic stability

In this section, we prove the nonlinear stability of the traveling wave solution of (1.3)–(1.5) with $\chi > 0$, $u_+ = 0$. The main result is that the solution of (1.3) with data (1.5) approaches the traveling wave solution $(U, V)(x - st)$, properly translated by an amount x_0, i.e.,

$$\sup_{x\in\mathbb{R}}\left|(u, v)(x,t) - (U, V)(x + x_0 - st)\right| \to 0, \quad \text{as } t \to +\infty$$

where x_0 satisfies the following identity derived from the "conservation of mass" principle

$$\int_{-\infty}^{+\infty}\begin{pmatrix} u_0(x) - U(x) \\ v_0(x) - V(x) \end{pmatrix} dx = x_0\begin{pmatrix} u_+ - u_- \\ v_+ - v_- \end{pmatrix} + \beta r_1(u_-, v_-)$$

where $r_1(u_-, v_-)$ denotes the first right eigenvector of the Jacobian matrix of (1.3) with $D = 0$ evaluated at (u_-, v_-), see details in [18]. The coefficient β yields the diffusion wave in general [7]. Both β and x_0 will be uniquely determined by the initial data (u_0, v_0). For the stability of small-amplitude shock waves of conservation laws with diffusion wave (i.e. $\beta \neq 0$), we refer to [11,19] for details. In the present paper, we will neglect the diffusion wave by assuming $\beta = 0$ and consider the stability of large-amplitude waves. Then by the conservation laws (1.3), we derive that

$$\begin{aligned}
\int_{-\infty}^{+\infty}\begin{pmatrix} u(x,t) - U(x + x_0 - st) \\ v(x,t) - V(x + x_0 - st) \end{pmatrix} dx &= \int_{-\infty}^{+\infty}\begin{pmatrix} u_0(x) - U(x + x_0) \\ v_0(x) - V(x + x_0) \end{pmatrix} dx \\
&= \int_{-\infty}^{+\infty}\begin{pmatrix} u_0(x) - U(x) \\ v_0(x) - V(x) \end{pmatrix} dx + \int_{-\infty}^{+\infty}\begin{pmatrix} U(x) - U(x + x_0) \\ V(x) - V(x + x_0) \end{pmatrix} dx \\
&= \int_{-\infty}^{+\infty}\begin{pmatrix} u_0(x) - U(x) \\ v_0(x) - V(x) \end{pmatrix} dx - x_0\begin{pmatrix} u_+ - u_- \\ v_+ - v_- \end{pmatrix} = \vec{0}.
\end{aligned} \tag{5.1}$$

This allows us to employ the technique of taking anti-derivative to decompose the solution as

$$(u, v)(x,t) = (U, V)(x + x_0 - st) + (\phi_z, \psi_z)(z,t) \tag{5.2}$$

where $z = x - st$. That is

$$\big(\phi(z,t), \psi(z,t)\big) = \int_{-\infty}^{z}\big(u(y,t) - U(y + x_0 - st), v(y,t) - V(y + x_0 - st)\big)\,dy$$

for all $z \in \mathbb{R}$ and $t \geqslant 0$.

It then follows from (5.1) that

$$\phi(\pm\infty, t) = \psi(\pm\infty, t) = 0, \quad \text{for all } t > 0.$$

We assume, without loss of generality, that the translation $x_0 = 0$, which implies the zero integral of the initial perturbation

$$\int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x) - U(x) \\ v_0(x) - V(x) \end{pmatrix} dx = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.3}$$

The initial perturbation of (ϕ, ψ) is thus given by

$$(\phi_0, \psi_0)(z) = \int_{-\infty}^{z} (u_0 - U, v_0 - V)(y)\, dy, \tag{5.4}$$

with $(\phi_0, \psi_0)(\pm\infty) = 0$.

Substituting (5.2) into (1.3), using (2.1) and integrating the system with respect to z, we derive that the perturbation $(\phi, \psi)(z, t)$ satisfies

$$\begin{cases} \phi_t = D\phi_{zz} + (s + \chi V)\phi_z + \chi U \psi_z + \chi \phi_z \psi_z, & t > 0,\ z \in \mathbb{R}, \\ \psi_t = s\psi_z + \phi_z. \end{cases} \tag{5.5}$$

with initial perturbation $(\phi, \psi)(z, 0) = (\phi_0, \psi_0)(z)$ given in (5.4).

We look for solutions of the system (5.5) in the following solution space:

$$\begin{aligned} X(0, T) := \big\{ \big(\phi(z,t), \psi(z,t)\big) \,\big|\, & \phi \in C\big([0, T]; H_w^2\big),\ \phi_z \in L^2\big((0, T); H_w^2\big), \\ & \psi \in C\big([0, T]; H^2\big),\ \psi_z \in C\big([0, T]; H_w^1\big) \cap L^2\big((0, T); H_w^1\big) \big\}, \end{aligned}$$

where the weight function w is defined by (2.12).

Clearly, if $\phi \in H_w^2$, then $\phi \in H^2$ since $w \geqslant 1$. Define

$$N(t) := \sup_{\tau \in [0,t]} \big(\|\psi(\cdot, \tau)\|_2 + \|\phi(\cdot, \tau)\|_{2,w} + \|\psi_z(\cdot, \tau)\|_{1,w} \big).$$

By the Sobolev embedding theorem, it holds that

$$\sup_{\tau \in [0,t]} \big\{ \|\phi(\cdot, \tau)\|_{L^\infty}, \|\phi_z(\cdot, \tau)\|_{L^\infty}, \|\psi(\cdot, \tau)\|_{L^\infty}, \|\psi_z(\cdot, \tau)\|_{L^\infty} \big\} \leqslant N(t). \tag{5.6}$$

Then Theorem 2.4 is a consequence of the following theorem.

Theorem 5.1. *There exists a positive constant ε_1, such that if $N(0) \leqslant \varepsilon_1$, then the Cauchy problem (5.5) with (5.4) has a unique global solution $(\phi, \psi) \in X(0, \infty)$ such that*

$$\begin{aligned} & \|\phi\|_{2,w}^2 + \|\psi\|_2^2 + \|\psi_z\|_{1,w}^2 + \int_0^t \big(\|\phi_z(\tau)\|_{2,w}^2 + \|\psi_z(\tau)\|_{1,w}^2 \big) d\tau \\ & \quad \leqslant C \big(\|\phi_0\|_{2,w}^2 + \|\psi_0\|_2^2 + \|\psi_{0z}\|_{1,w}^2 \big) \leqslant C N^2(0) \end{aligned} \tag{5.7}$$

for any $t \in [0, \infty)$. Moreover, it follows that

$$\sup_{x \in \mathbb{R}} \big| (\phi_z, \psi_z)(z, t) \big| \to 0 \quad \text{as } t \to \infty. \tag{5.8}$$

The global existence of (ϕ, ψ) announced in Theorem 5.1 follows from the local existence theorem and the *a priori* estimates given below.

Proposition 5.2 *(Local existence). For any $\varepsilon_0 > 0$, there exists a positive constant T_0 depending on ε_0 such that if $(\phi_0, \psi_0) \in H_w^2$ with $N(0) \leqslant \varepsilon_0$, then the problem (5.5) with (5.4) has a unique solution $(\phi, \psi) \in X(0, T_0)$ satisfying $N(t) \leqslant 2N(0)$ for any $0 \leqslant t \leqslant T_0$.*

Proposition 5.3 *(A priori estimate). Assume that $(\phi, \psi) \in X(0, T)$ is a solution obtained in Proposition 5.2 for a positive constant T. Then there is a positive constant $\varepsilon_2 > 0$, independent of T, such that if*

$$N(t) \leqslant \varepsilon_2$$

for any $0 \leqslant t \leqslant T$, then the solution (ϕ, ψ) of (5.5) with (5.4) satisfies (5.7) for any $0 \leqslant t \leqslant T$.

The local existence in Proposition 5.2 can be proved using the standard argument (e.g. see [14]) and we omit the details for brevity. It is the key to establish the *a priori* estimates in Proposition 5.3. Next we are devoted to proving Proposition 5.3 with the L^2-energy estimates by modifying the idea of [7] where the singularity of $\frac{1}{U}$ was excluded by assuming $u_+ > 0$ since $u_+ < U(z) < u_-$. In the present paper, we deal with the case $u_+ = 0$ as $z \to +\infty$ by taking the singular term $\frac{1}{U}$ as the weight function in the energy estimates. Without loss of generality, we assume that $N(t) < 1$ in what follows.

Lemma 5.4. *Let the assumptions in Proposition 5.3 hold. Then there exists a constant $C > 0$ such that*

$$\|\psi\|^2 + \|\phi\|_w^2 + \int_0^t \|\phi_z\|_w^2 \leqslant C\|\psi_0\|^2 + C\|\phi_0\|_w^2 + CN(t)\int_0^t \int \frac{\psi_z^2}{U}. \tag{5.9}$$

Proof. Multiplying the first equation of (5.5) by ϕ/U and the second by $\chi\psi$, integrating the resultant equations with respect to z and adding them, we obtain

$$\frac{1}{2}\frac{d}{dt}\int\left(\frac{\phi^2}{U} + \chi\psi^2\right) = D\int\frac{\phi\phi_{zz}}{U} + s\int\frac{\phi_z\phi}{U} + \chi\int\frac{V\phi\phi_z}{U} + \chi\int\frac{\phi\phi_z\psi_z}{U}.$$

Noting that

$$\frac{\phi\phi_{zz}}{U} = \left(\frac{\phi\phi_z}{U}\right)_z - \frac{\phi_z^2}{U} - \phi\phi_z\left(\frac{1}{U}\right)_z = \left(\frac{\phi\phi_z}{U}\right)_z - \frac{\phi_z^2}{U} - \left(\frac{\phi^2}{2}\left(\frac{1}{U}\right)_z\right)_z + \frac{\phi^2}{2}\left(\frac{1}{U}\right)_{zz},$$

$$\frac{\phi_z\phi}{U} = \left(\frac{\phi^2}{2U}\right)_z - \frac{\phi^2}{2}\left(\frac{1}{U}\right)_z,$$

$$\frac{V\phi\phi_z}{U} = \frac{1}{2}\left(\frac{V\phi^2}{U}\right)_z - \frac{\phi^2}{2}\left(\frac{V}{U}\right)_z,$$

we get

$$\frac{1}{2}\frac{d}{dt}\int\left(\chi\psi^2 + \frac{\phi^2}{U}\right) + D\int\frac{\phi_z^2}{U} = \frac{1}{2}\int\phi^2\left[\left(\frac{D}{U}\right)_{zz} - \left(\frac{s+\chi V}{U}\right)_z\right] + \chi\int\frac{\phi\phi_z\psi_z}{U}. \tag{5.10}$$

By using (2.1) and the fact that $u_+ = 0$, it can be checked that

$$\left(\frac{D}{U}\right)_{zz} - \left(\frac{s+\chi V}{U}\right)_z = \frac{2u_+}{U^3}(s+\chi v_+)\cdot U_z = 0. \tag{5.11}$$

Substituting (5.11) into (5.10) and integrating the equation with respect to t, we derive

$$\begin{aligned}
&\frac{1}{2}\int\left(\chi\psi^2 + \frac{\phi^2}{U}\right) + D\int_0^t\int\frac{\phi_z^2}{U}\\
&= \frac{1}{2}\int\left(\chi\psi_0^2 + \frac{\phi_0^2}{U}\right) + \chi\int_0^t\int\frac{\phi_z\psi_z\phi}{U}\\
&\leqslant \frac{\chi}{2}\|\psi_0\|^2 + C\|\phi_0\|_w^2 + \frac{DN(t)}{2}\int_0^t\int\frac{\phi_z^2}{U} + \frac{N(t)\chi^2}{2D}\int_0^t\int\frac{\psi_z^2}{U},
\end{aligned}$$

where we have used the fact that $\|\phi(\cdot,t)\|_{L^\infty} \leqslant N(t)$ by (5.6). Using the fact that $\frac{1}{U} \leqslant Cw$ for $z \in \mathbb{R}$ (see Remark 3.1), we can derive (5.9). Then we complete the proof of Lemma 5.4. □

The next lemma gives the estimate of the first order derivatives of (ϕ, ψ).

Lemma 5.5. *Let the assumptions in Proposition 5.3 hold. Then it holds that*

$$\|\psi\|_1^2 + \|\phi\|_{1,w}^2 + \|\psi_z\|_w^2 + \int_0^t\left(\|\phi_z\|_{1,w}^2 + \|\psi_z\|_w^2\right) \leqslant C\left(\|\psi_{0z}\|_w^2 + \|\phi_0\|_{1,w}^2 + \|\psi_0\|_1^2\right) \tag{5.12}$$

where $C > 0$ is a constant.

Proof. We differentiate (5.5) with respect to z to get

$$\begin{cases}\phi_{zt} = D\phi_{zzz} + s\phi_{zz} + \chi U_z\psi_z + \chi U\psi_{zz} + \chi V_z\phi_z + \chi V\phi_{zz} + \chi(\phi_z\psi_z)_z,\\ \psi_{zt} = s\psi_{zz} + \phi_{zz}.\end{cases} \tag{5.13}$$

Multiplying the first equation of (5.13) by ϕ_z/U and the second by $\chi\psi_z$, integrating the resultant equations with respect to z and adding them, noticing that

$$\frac{\phi_{zzz}\phi_z}{U} = \left(\frac{\phi_z\phi_{zz}}{U}\right)_z - \frac{\phi_{zz}^2}{U} - \phi_z\phi_{zz}\left(\frac{1}{U}\right)_z = \left(\frac{\phi_z\phi_{zz}}{U}\right)_z - \frac{\phi_{zz}^2}{U} - \left[\frac{\phi_z^2}{2}\left(\frac{1}{U}\right)_z\right]_z + \frac{\phi_z^2}{2}\left(\frac{1}{U}\right)_{zz},$$

$$\frac{s\phi_{zz}\phi_z}{U} = \left(\frac{s\phi_z^2}{2U}\right)_z - \frac{\phi_z^2}{2}\left(\frac{s}{U}\right)_z,$$

$$\frac{V\phi_{zz}\phi_z}{U} = \left(\frac{V\phi_z^2}{2U}\right)_z - \frac{\phi_z^2}{2}\left(\frac{V}{U}\right)_z,$$

$$\frac{\phi_z(\phi_z\psi_z)_z}{U} = \left(\frac{\phi_z^2\psi_z}{U}\right)_z - \frac{\phi_{zz}\phi_z\psi_z}{U} + \frac{U_z\phi_z^2\psi_z}{U^2},$$

we have

$$\frac{1}{2}\frac{d}{dt}\int\left(\chi\psi_z^2+\frac{\phi_z^2}{U}\right)+D\int\frac{\phi_{zz}^2}{U}=\frac{1}{2}\int\phi_z^2\left[\left(\frac{D}{U}\right)_{zz}-\left(\frac{s+\chi V}{U}\right)_z\right]+\chi\int\frac{U_z\psi_z\phi_z}{U}$$
$$+\chi\int\frac{V_z\phi_z^2}{U}-\chi\int\frac{\phi_{zz}\phi_z\psi_z}{U}+\chi\int\frac{U_z\phi_z^2\psi_z}{U^2}. \tag{5.14}$$

Owing to (3.6), it is easy to see that

$$\frac{U_z}{U}=\frac{\kappa u_-\eta e^{\eta z}}{(\kappa e^{\eta z}-1)^2}\cdot\frac{\kappa e^{\eta z}-1}{-u_-}=-\frac{\kappa\eta e^{\eta z}}{\kappa e^{\eta z}-1},$$

which implies

$$\left|\frac{U_z}{U}\right|\leqslant\eta. \tag{5.15}$$

Integrating (5.14) over $[0,t]$ and using (5.11), $|V_z|\leqslant C$ and $\|\psi_z(\cdot,t)\|_{L^\infty}\leqslant N(t)<1$ for any $t\in[0,T]$ by (5.6), we get

$$\int\left(\chi\psi_z^2+\frac{\phi_z^2}{U}\right)+D\int_0^t\int\frac{\phi_{zz}^2}{U}\leqslant\chi\|\psi_{0z}\|^2+\|\phi_{0z}\|_w^2+C\int_0^t\int\frac{\phi_z^2}{U}+C\int_0^t\int U\psi_z^2$$
$$+\frac{DN(t)}{2}\int_0^t\int\frac{\phi_{zz}^2}{U}+CN(t)\int_0^t\int\frac{\phi_z^2}{U},$$

which in combination with (5.9) yields

$$\chi\int\psi_z^2+\int\frac{\phi_z^2}{U}+D\left(1-\frac{N(t)}{2}\right)\int_0^t\int\frac{\phi_{zz}^2}{U}$$
$$\leqslant C\left(\|\phi_0\|_{1,w}^2+\|\psi_0\|_1^2+N(t)\int_0^t\int\frac{\psi_z^2}{U}+\int_0^t\int U\psi_z^2\right). \tag{5.16}$$

Next, we claim

$$\int_0^t\int U\psi_z^2\leqslant C\left(\|\psi_{0z}\|^2+\|\psi_0\|^2+\|\phi_0\|_w^2+N(t)\int_0^t\int\frac{\psi_z^2}{U}\right),\quad\forall t\in[0,T]. \tag{5.17}$$

To prove (5.17), we multiply the first equation of (5.5) by ψ_z to get

$$\chi U\psi_z^2=\phi_t\psi_z-D\phi_{zz}\psi_z-s\phi_z\psi_z-\chi V\phi_z\psi_z-\chi\phi_z\psi_z^2. \tag{5.18}$$

Integrating (5.18) over $[0,t]\times\mathbb{R}$ and noting that by the second equation of (5.13)

$$\begin{aligned}\phi_t\psi_z &= (\phi\psi_z)_t-\phi\psi_{zt}=(\phi\psi_z)_t-\phi(s\psi_{zz}+\phi_{zz})\\ &= (\phi\psi_z)_t-s(\phi\psi_z)_z+s\phi_z\psi_z-(\phi\phi_z)_z+\phi_z^2,\\ \phi_{zz}\psi_z &= (\psi_{zt}-s\psi_{zz})\psi_z=\frac{1}{2}\big(\psi_z^2\big)_t-\frac{s}{2}\big(\psi_z^2\big)_z,\end{aligned}$$

we obtain

$$\begin{aligned}&\frac{D}{2}\int\psi_z^2+\chi\int_0^t\int U\psi_z^2\\ &\quad=\frac{D}{2}\int\psi_{0z}^2+\int\phi\psi_z-\int\phi_0\psi_{0z}+\int_0^t\int\phi_z^2-\chi\int_0^t\int V\phi_z\psi_z-\chi\int_0^t\int\phi_z\psi_z^2\\ &\quad\leqslant\frac{D+1}{2}\int\psi_{0z}^2+\frac{1}{2}\int\phi_0^2+\frac{1}{D}\int\phi^2+\frac{D}{4}\int\psi_z^2+\int_0^t\int\phi_z^2\\ &\qquad+C\big(1+N(t)\big)\int_0^t\int\frac{\phi_z^2}{U}+\frac{(1+N(t))\chi}{4}\int_0^t\int U\psi_z^2,\end{aligned}$$

where we have used the Young inequality and the fact $\|\psi_z(\cdot,t)\|_{L^\infty}\leqslant N(t)$, $|V|\leqslant C$. From this inequality and using $\phi^2\leqslant\frac{C\phi^2}{U}$, $\phi_z^2\leqslant\frac{C\phi_z^2}{U}$ due to $\frac{1}{U}\geqslant\frac{1}{u_-}$ and (5.9), it follows that

$$\begin{aligned}\int\psi_z^2+\int_0^t\int U\psi_z^2 &\leqslant C\left(\int\psi_{0z}^2+\int\phi_0^2+\int\frac{\phi^2}{U}+\int_0^t\int\frac{\phi_z^2}{U}\right)\\ &\leqslant C\left(\|\psi_{0z}\|^2+\|\psi_0\|^2+\|\phi_0\|_w^2+N(t)\int_0^t\int\frac{\psi_z^2}{U}\right).\end{aligned}\tag{5.19}$$

Thus (5.17) holds. Substituting (5.17) into (5.16) yields

$$\int\psi_z^2+\int\frac{\phi_z^2}{U}+D\int_0^t\int\frac{\phi_{zz}^2}{U}\leqslant C\left(\|\psi_0\|_1^2+\|\phi_0\|_{1,w}^2+N(t)\int_0^t\int\frac{\psi_z^2}{U}\right).\tag{5.20}$$

Note that U is monotone decreasing in $(-\infty,\infty)$ and hence $0<\frac{u_-}{1-\kappa}=U(0)<U(z)<u_-$ for all $z\in(-\infty,0)$. Since $1<w(z)<2$ for all $z\in(-\infty,0)$, we have $U(z)>\frac{u_-}{2(1-\kappa)}w(z)$ for all $z\in(-\infty,0)$. From (5.19), it follows that

$$\int_{-\infty}^0\psi_z^2+\int_0^t\int_{-\infty}^0 w\psi_z^2\leqslant C\left(\|\psi_{0z}\|^2+\|\psi_0\|^2+\|\phi_0\|_w^2+N(t)\int_0^t\int\frac{\psi_z^2}{U}\right).\tag{5.21}$$

To complete the proof of (5.12), it remains only to estimate $\int_0^t \int \frac{\psi_z^2}{U}$. Due to (3.7), it suffices to estimate $\int_0^t \int w\psi_z^2$. For this purpose, we multiply the second equation of (5.13) by $w\psi_z$ and obtain that

$$s\psi_{zz}w\psi_z = \frac{sw(\psi_z^2)_z}{2} = \left(\frac{sw\psi_z^2}{2}\right)_z - \frac{sw'\psi_z^2}{2},$$

which leads to

$$\frac{sw'\psi_z^2}{2} + \left(\frac{w\psi_z^2}{2}\right)_t = \left(\frac{sw\psi_z^2}{2}\right)_z + w\psi_z\phi_{zz}. \tag{5.22}$$

Recalling that $w = 1 + e^{\eta z}$, we have $1 < w < 2$ and $w' = \eta e^{\eta z} \geqslant 0$ in $(-\infty, 0)$. Then integrating (5.22) with respect to z over $(-\infty, 0)$, we have

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{0} w\psi_z^2 + \frac{s\eta}{2}\int_{-\infty}^{0} e^{\eta z}\psi_z^2 = s\psi_z^2(0,t) + \int_{-\infty}^{0} w\psi_z\phi_{zz} \leqslant s\psi_z^2(0,t) + 2\int_{-\infty}^{0} |\psi_z\phi_{zz}|. \tag{5.23}$$

Integrating (5.22) over $(0, +\infty)$ and using the fact that $w' = \eta e^{\eta z} \geqslant \frac{\eta w}{2}$ in $(0, +\infty)$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^{+\infty} w\psi_z^2 + \frac{s\eta}{4}\int_0^{+\infty} w\psi_z^2 &\leqslant \frac{s\eta}{2}\int_0^{+\infty} e^{\eta z}\psi_z^2 + \frac{1}{2}\frac{d}{dt}\int_0^{+\infty} w\psi_z^2 \\ &= -s\psi_z^2(0,t) + \int_0^{+\infty} w\psi_z\phi_{zz}.\end{aligned} \tag{5.24}$$

Adding (5.24) and (5.23) together and integrating the resultant equation over $[0, t]$, it follows that

$$\begin{aligned}&\frac{1}{2}\int w\psi_z^2 + \frac{s\eta}{4}\int_0^t\int_0^{+\infty} w\psi_z^2 + \frac{s\eta}{2}\int_0^t\int_{-\infty}^{0} e^{\eta z}\psi_z^2 \\ &\leqslant \frac{1}{2}\int_{-\infty}^{+\infty} w\psi_{0z}^2 + \int_0^t\int_0^{+\infty} w\psi_z\phi_{zz} + 2\int_0^t\int_{-\infty}^{0} |\psi_z\phi_{zz}| \\ &\leqslant \frac{1}{2}\|\psi_{0z}\|_w^2 + \frac{s\eta}{8}\int_0^t\int_0^{+\infty} w\psi_z^2 + \frac{2}{s\eta}\int_0^t\int_0^{+\infty} w\phi_{zz}^2 + \int_0^t\int_{-\infty}^{0} \frac{\phi_{zz}^2}{U} + \int_0^t\int_{-\infty}^{0} U\psi_z^2,\end{aligned}$$

which in combination with (3.7), (5.17), (5.20) and (5.21) gives

$$\int w\psi_z^2 + \int_0^t\int w\psi_z^2 \leqslant C\left(\|\psi_{0z}\|_w^2 + \|\phi_0\|_{1,w}^2 + \|\psi_0\|_1^2 + N(t)\int_0^t\int w\psi_z^2\right).$$

Thus,

$$\int w\psi_z^2 + \big(1 - CN(t)\big)\int_0^t \int w\psi_z^2 \leqslant C\big(\|\psi_{0z}\|_w^2 + \|\phi_0\|_{1,w}^2 + \|\psi_0\|_1^2\big). \tag{5.25}$$

When $N(t)$ is small enough, by (5.9), (5.20) and (5.25), we derive (5.12). □

Next, we give the estimates of the second order derivative of (ϕ, ψ).

Lemma 5.6. *Let the assumptions in Proposition 5.3 hold. Then there exists a constant $C > 0$ such that*

$$\|\phi_{zz}\|_w^2 + \|\psi_{zz}\|^2 + \|\psi_{zz}\|_w^2 + \int_0^t \big(\|\phi_{zzz}\|_w^2 + \|\psi_{zz}\|_w^2\big) \leqslant C\big(\|\psi_{0z}\|_{1,w}^2 + \|\phi_0\|_{2,w}^2 + \|\psi_0\|_2^2\big). \tag{5.26}$$

Proof. We differentiate (5.13) with respect to z to get

$$\begin{cases} \phi_{zzt} = D\phi_{zzzz} + s\phi_{zzz} + \chi\big[(U_z\psi_z)_z + U_z\psi_{zz} + U\psi_{zzz} + (V_z\phi_z)_z + (V\phi_{zz})_z + (\phi_z\psi_z)_{zz}\big], \\ \psi_{zzt} = s\psi_{zzz} + \phi_{zzz}. \end{cases} \tag{5.27}$$

Multiplying the first equation of (5.27) by ϕ_{zz}/U and the second by $\chi\psi_{zz}$ and using

$$\frac{\phi_{zzzz}\phi_{zz}}{U} = \left(\frac{\phi_{zzz}\phi_{zz}}{U}\right)_z - \frac{\phi_{zzz}^2}{U} - \frac{1}{2}\left(\phi_{zz}^2\left(\frac{1}{U}\right)_z\right)_z + \frac{1}{2}\phi_{zz}^2\left(\frac{1}{U}\right)_{zz},$$
$$\frac{V\phi_{zzz}\phi_{zz}}{U} = \frac{1}{2}\left(\phi_{zz}^2\frac{V}{U}\right)_z - \frac{1}{2}\phi_{zz}^2\left(\frac{V}{U}\right)_z,$$

we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int\left(\chi\psi_{zz}^2 + \frac{\phi_{zz}^2}{U}\right) + D\int\frac{\phi_{zzz}^2}{U} \\ &\quad = \frac{1}{2}\int\phi_{zz}^2\left[\left(\frac{D}{U}\right)_{zz} - \left(\frac{s+\chi V}{U}\right)_z\right] + \chi\int\frac{(U_z\psi_z)_z\phi_{zz}}{U} \\ &\qquad + \chi\int\frac{U_z\psi_{zz}\phi_{zz}}{U} + \chi\int\frac{(V_z\phi_z)_z\phi_{zz}}{U} + \chi\int\frac{V_z\phi_{zz}^2}{U} + \chi\int\frac{(\phi_z\psi_z)_{zz}\phi_{zz}}{U}. \end{aligned} \tag{5.28}$$

Because $|\frac{U_z}{U}| \leqslant C$, $|U_{zz}| \leqslant C$, $|V_z| \leqslant C$, $|V_{zz}| \leqslant C$, $\|\psi_z(\cdot,t)\|_{L^\infty} \leqslant N(t)$ and $\|\phi_z(\cdot,t)\|_{L^\infty} \leqslant N(t)$ for any $t \in [0,T]$, we get by Cauchy–Schwarz inequality

$$\begin{aligned} \int\frac{(U_z\psi_z)_z\phi_{zz}}{U} &= \int\frac{U_{zz}\psi_z\phi_{zz}}{U} + \int\frac{U_z\psi_{zz}\phi_{zz}}{U} \\ &\leqslant C\int\frac{\psi_z^2}{U} + C\int\frac{\phi_{zz}^2}{U} + C\int U\psi_{zz}^2, \end{aligned}$$
$$\int\frac{(V_z\phi_z)_z\phi_{zz}}{U} \leqslant C\int\frac{\phi_z^2}{U} + C\int\frac{\phi_{zz}^2}{U},$$

$$\int \frac{(\phi_z\psi_z)_{zz}\phi_{zz}}{U} = -\int \frac{(\phi_z\psi_z)_z\phi_{zzz}}{U} + \int \frac{(\phi_z\psi_z)_z\phi_{zz}U_z}{U^2}$$
$$\leqslant \frac{N(t)}{4}\int \frac{\phi_{zzz}^2}{U} + CN(t)\int \frac{\phi_{zz}^2}{U} + CN(t)\int \frac{\psi_{zz}^2}{U}.$$

Substituting above three inequalities into (5.28) and using (5.12), one has

$$\int \psi_{zz}^2 + \int \frac{\phi_{zz}^2}{U} + D\int_0^t \int \frac{\phi_{zzz}^2}{U}$$
$$\leqslant C\left(\|\phi_0\|_{2,w}^2 + \|\psi_{0z}\|_w^2 + \|\psi_0\|_2^2 + \int_0^t \int U\psi_{zz}^2 + N(t)\int_0^t \int \frac{\psi_{zz}^2}{U}\right). \tag{5.29}$$

Next we estimate the term $\int_0^t \int U\psi_{zz}^2$.

Multiplying the first equation of (5.13) by ψ_{zz}, we obtain

$$\chi U\psi_{zz}^2 = \phi_{zt}\psi_{zz} - \big(D\phi_{zzz} + s\phi_{zz} + \chi U_z\psi_z + \chi V_z\phi_z + \chi V\phi_{zz} + \chi(\phi_z\psi_z)_z\big)\psi_{zz}.$$

Noting that

$$\phi_{zt}\psi_{zz} = (\phi_z\psi_{zz})_t - \phi_z\psi_{zzt} = (\phi_z\psi_{zz})_t - \phi_z(s\psi_{zzz} + \phi_{zzz})$$
$$= (\phi_z\psi_{zz})_t - s(\phi_z\psi_{zz})_z + s\phi_{zz}\psi_{zz} - (\phi_z\phi_{zz})_z + \phi_{zz}^2,$$
$$\phi_{zzz}\psi_{zz} = (\psi_{zzt} - s\psi_{zzz})\psi_{zz} = \frac{1}{2}\big(\psi_{zz}^2\big)_t - \frac{s}{2}\big(\psi_{zz}^2\big)_z,$$

we have

$$\frac{D}{2}\frac{d}{dt}\int \psi_{zz}^2 + \chi\int U\psi_{zz}^2 = \frac{d}{dt}\int \phi_z\psi_{zz} + \int \phi_{zz}^2 - \chi\int \big[U_z\psi_z + V_z\phi_z + V\phi_{zz} + (\phi_z\psi_z)_z\big]\psi_{zz}.$$

Thus, integrating the above equation with respect to t over $[0,t]$ and using the Cauchy–Schwarz inequality, we have

$$\frac{D}{2}\int \psi_{zz}^2 + \chi\int_0^t \int U\psi_{zz}^2$$
$$\leqslant \frac{1}{D}\int \phi_z^2 + \frac{D}{4}\int \psi_{zz}^2 + \left(\frac{D+1}{2}\right)\int \psi_{0zz}^2 + \frac{1}{2}\int \phi_{0z}^2 + \int_0^t \int \phi_{zz}^2$$
$$+ \frac{(1+N(t))\chi}{4}\int_0^t \int U\psi_{zz}^2 + C\left(\int_0^t \int \frac{\psi_z^2}{U} + \int_0^t \int \frac{\phi_z^2}{U} + \int_0^t \int \frac{\phi_{zz}^2}{U}\right).$$

Then it follows from (5.12) that

$$\int \psi_{zz}^2 + \int_0^t \int U\psi_{zz}^2 \leqslant C\left(\|\psi_{0zz}\|^2 + \|\phi_0\|_{1,w}^2 + \|\psi_{0z}\|_w^2 + \|\psi_0\|_1^2 + N(t)\int_0^t \int \psi_{zz}^2\right), \tag{5.30}$$

which in conjunction with (5.29) leads to

$$\int \psi_{zz}^2 + \int \frac{\phi_{zz}^2}{U} + \int_0^t \int \frac{\phi_{zzz}^2}{U} \leqslant C\left(\|\phi_0\|_{2,w}^2 + \|\psi_{0z}\|_w^2 + \|\psi_0\|_2^2 + N(t)\int_0^t \int \frac{\psi_{zz}^2}{U}\right). \tag{5.31}$$

Using the same argument of deriving (5.21), we have from (5.30) that

$$\int_{-\infty}^0 \psi_{zz}^2 + \int_0^t \int_{-\infty}^0 w\psi_{zz}^2 \leqslant C\left(\|\psi_{0zz}\|^2 + \|\phi_0\|_{1,w}^2 + \|\psi_{0z}\|_w^2 + \|\psi_0\|_1^2 + N(t)\int_0^t \int \psi_{zz}^2\right). \tag{5.32}$$

To finish the proof of (5.26), we only need to estimate the term $\int_0^t \int \frac{\psi_{zz}^2}{U}$ or equivalently $\int_0^t \int w\psi_{zz}^2$ due to (3.7). Multiplying the second equation of (5.27) by $w\psi_{zz}$, as in (5.22), we get

$$\frac{sw'\psi_{zz}^2}{2} + \left(\frac{w\psi_{zz}^2}{2}\right)_t = \left(\frac{sw\psi_{zz}^2}{2}\right)_z + \phi_{zzz}w\psi_{zz}. \tag{5.33}$$

Integrating (5.33) with respect to z over $(-\infty, 0)$, using the facts that $1 < w < 2$ and $w' = \eta e^{\eta z} \geqslant 0$ in $(-\infty, 0)$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{-\infty}^0 w\psi_{zz}^2 &\leqslant \frac{1}{2}\frac{d}{dt}\int_{-\infty}^0 w\psi_{zz}^2 + \frac{s\eta}{2}\int_{-\infty}^0 e^{\eta z}\psi_{zz}^2 \\ &= s\psi_{zz}^2(0,t) + \int_{-\infty}^0 w\phi_{zzz}\psi_{zz} \leqslant s\psi_{zz}^2(0,t) + 2\int_{-\infty}^0 |\phi_{zzz}\psi_{zz}|. \end{aligned} \tag{5.34}$$

Integrating (5.33) over $[0, t] \times (0, +\infty)$ with $w' = \eta e^{\eta z} \geqslant \frac{\eta w}{2}$ for $z > 0$ and using (5.34), we obtain

$$\begin{aligned} \frac{1}{2}\int w\psi_{zz}^2 + \frac{s\eta}{4}\int_0^t \int_0^{+\infty} w\psi_{zz}^2 &\leqslant \frac{s\eta}{2}\int_0^{+\infty} e^{\eta z}\psi_{zz}^2 + \frac{1}{2}\frac{d}{dt}\int_0^{+\infty} w\psi_{zz}^2 + \frac{1}{2}\frac{d}{dt}\int_{-\infty}^0 w\psi_{zz}^2 \\ &\leqslant \frac{1}{2}\int_{-\infty}^{+\infty} w\psi_{0zz}^2 + \int_0^t \int_0^{+\infty} w\phi_{zzz}\psi_{zz} + 2\int_{-\infty}^0 |\phi_{zzz}\psi_{zz}| \\ &\leqslant \frac{1}{2}\|\psi_{0zz}\|_w^2 + \frac{s\eta}{8}\int_0^t \int_0^{+\infty} w\psi_{zz}^2 + \frac{2}{s\eta}\int_0^t \int_0^{+\infty} w\phi_{zzz}^2 \\ &\quad + \int_0^t \int_{-\infty}^0 \frac{\phi_{zzz}^2}{U} + \int_0^t \int_{-\infty}^0 U\psi_{zz}^2. \end{aligned}$$

By (3.7), it follows from (5.31), (5.32) and above inequality that

$$\int w\psi_{zz}^2 + \int_0^t \int w\psi_{zz}^2 \leqslant C\left(\|\phi_0\|_{2,w}^2 + \|\psi_{0z}\|_{1,w}^2 + \|\psi_0\|_2^2 + N(t)\int_0^t \int \frac{\psi_{zz}^2}{U}\right).$$

When $N(t)$ is small enough, the above inequality with (3.7) gives

$$\int w\psi_{zz}^2 + \int_0^t \int w\psi_{zz}^2 \leqslant C\big(\|\psi_{0z}\|_{1,w}^2 + \|\phi_0\|_{2,w}^2 + \|\psi_0\|_2^2\big),$$

which in combination with (5.31) and (3.7) gives (5.26). The proof of Lemma 5.6 is finished. □

Finally, the desired estimate (5.7) follows from (5.12) and (5.26), and the proof of Proposition 5.3 is completed. □

5.1. *Proof of Theorem 5.1*

Now we are in a position to prove Theorem 5.1. In fact we only need to prove (5.8) since the rest has been implied by Proposition 5.3. From global estimate (5.7), we have

$$\big\|\phi_z(\cdot,t), \psi_z(\cdot,t)\big\|_{1,w} \to 0 \quad \text{as } t \to +\infty.$$

Hence, for all $z \in \mathbb{R}$, it follows that

$$\begin{aligned}\phi_z^2(z,t) &= 2\int_{-\infty}^{z} \phi_z\phi_{zz}(y,t)\,dy \\ &\leqslant 2\left(\int_{-\infty}^{\infty} \phi_z^2\,dy\right)^{1/2}\left(\int_{-\infty}^{\infty} \phi_{zz}^2\,dy\right)^{1/2} \\ &\leqslant \big\|\phi_z(\cdot,t)\big\|_{1,w} \to 0 \quad \text{as } t \to +\infty.\end{aligned}$$

Applying the same procedure to ψ_z leads to

$$\psi_z(z,t) \to 0 \quad \text{as } t \to +\infty \text{ for all } z \in \mathbb{R}.$$

Hence (5.8) is proved. □

6. Passing the results to the chemotaxis model (1.1)

In this section, we shall prove Theorem 2.5 and Theorem 2.6.

Proof of Theorem 2.5. First notice that $s \neq 0$. Then the valuation of Eq. (2.13) at $z = \pm\infty$ yields that

$$u_\pm c_\pm = 0. \tag{6.1}$$

Furthermore we have from (2.13) that

$$\mathcal{C}_z = -\frac{\mu}{s} U(z)\mathcal{C}(z) \tag{6.2}$$

which yields

$$\mathcal{C}(z) = c_+ \exp\left(\frac{\mu}{s}\int_z^\infty U(y)\,dy\right). \tag{6.3}$$

Since $u(z)$ exponentially decays as $z \to \infty$, see (3.6), the integral $\int_z^\infty U(y)\,dy$ is bounded for any $z \in \mathbb{R}$. As a consequence, $\mathcal{C}(z) > 0$ for any $z \in \mathbb{R}$ since otherwise $c_+ \equiv 0$ and hence $\mathcal{C}(z) \equiv 0$ which is not desired. To proceed, we split our analysis into cases: $\mu > 0$ and $\mu < 0$.

Case 1: $\mu < 0$. This entails that $\mathcal{C}_z > 0$ for all $z \in \mathbb{R}$ and hence $0 \leqslant c_- < c_+$. The fact $c_+ > 0$ with (6.1) leads to $u_+ = 0$ which is only possible when $U_z < 0$ (see Proposition 2.1). Hence we require $\chi = -\xi\mu > 0$ which implies $\xi > 0$ since $\mu < 0$. Since $0 = u_+ < u_-$, it follows from (6.1) that $\mathcal{C}(-\infty) = c_- = 0$. This completes the proof of Theorem 2.5(i).

Case 2: $\mu > 0$. Then it follows from (6.2) that $\mathcal{C}_z < 0$ and hence $0 \leqslant c_+ < c_-$. Therefore (6.1) requires that $u_- = 0$. This is only possible when $U_z > 0$, or equivalently $\chi = -\xi\mu < 0$ (see Proposition 2.1) which requires $\xi > 0$ due to $\mu > 0$. In this scenario, we have from the transformation (1.2) that $v_- = \frac{1}{\mu}\frac{\mathcal{C}_z(-\infty)}{\mathcal{C}(-\infty)} = \frac{1}{\mu}\frac{\mathcal{C}_z(-\infty)}{c_-} = 0$. Therefore $v_+ < v_- = 0$ due to $V_z < 0$ from Proposition 2.1. However from (2.5), one obtains $s = \frac{1}{2}(-\chi v_+ + \chi v_+) = 0$ if $v_+ < 0$ and $u_- = 0$ which violates the condition $s > 0$. Therefore the traveling wave solution of (1.1) does not exist. This completes the proof of Theorem 2.5(ii).

From above analysis, we observe that the traveling wave solution of (1.1) does not exists for any $\mu \neq 0$ when $\xi < 0$ and the proof of Theorem 2.5 is completed. □

Proof of Theorem 2.6. The stability for u has been given in Theorem 2.4. It remains to pass the results from v to c. By the transformation (1.2) and (5.2), one deduces that

$$\frac{c(x,t)}{\mathcal{C}(x-st)} = e^{\int_{-\infty}^x (V(\xi-st)-v(\xi,t))\,d\xi} = e^{\psi(x,t)}.$$

Next we show that $\psi(x,t) \to 0$ as $t \to \infty$. By the results of Theorem 5.1, the standard argument (e.g., see [23]) entails that $\|\psi(\cdot,t)\|_{1,w} \to 0$ as $t \to \infty$. Then

$$\begin{aligned}
\psi^2(x,t) &= 2\int_{-\infty}^x \psi\psi_y(y,t)\,dy \\
&\leqslant 2\left(\int_{\mathbb{R}} \psi^2\,dy\right)^{1/2}\left(\int_{\mathbb{R}} \psi_y^2\,dy\right)^{1/2} \\
&\leqslant \big\|\psi(\cdot,t)\big\|_{1,w} \to 0 \quad \text{as } t \to \infty
\end{aligned}$$

which implies $\psi(x,t) \to 0$ as $t \to \infty$ for all $x \in \mathbb{R}$. Note that $\mathcal{C}(x-st)$ is a traveling wave solution which is bounded by $c_+ > 0$. Then

$$\begin{aligned}
\big|c(x,t) - \mathcal{C}(x-st)\big| &= \big|\mathcal{C}(x-st)e^{\psi(x,t)} - \mathcal{C}(x-st)\big| \\
&= \mathcal{C}(x-st)\big|1 - e^{\psi(x,t)}\big| \leqslant c_+\big|1 - e^{\psi(x,t)}\big| \to 0 \quad \text{as } t \to \infty
\end{aligned}$$

for all $x \in \mathbb{R}$.

The proof is completed. □

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