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Boundedness and asymptotics of a reaction-diffusion system with density-dependent motility

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Abstract

We consider the initial-boundary value problem of a system of reaction-diffusion equations with density-dependent motility

$$\begin{cases} u_t = \Delta(\gamma(v)u) + \alpha u F(w) - \theta u, & x \in \Omega, \quad t > 0, \\ v_t = D\Delta v + u - v, & x \in \Omega, \quad t > 0, \\ w_t = \Delta w - uF(w), & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ (u, v, w)(x, 0) = (u_0, v_0, w_0)(x), & x \in \Omega, \end{cases} \tag{$*$}$$

in a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary, α and θ are non-negative constants and ν denotes the outward normal vector of $\partial\Omega$. The random motility function $\gamma(v)$ and functional response function $F(w)$ satisfy the following assumptions:

- $\gamma(v) \in C^3([0, \infty)), \; 0 < \gamma_1 \le \gamma(v) \le \gamma_2, \; |\gamma'(v)| \le \eta$ for all $v \ge 0$;
- $F(w) \in C^1([0, \infty))$, $F(0) = 0$, $F(w) > 0$ in $(0, \infty)$ and $F'(w) > 0$ on $[0, \infty)$

for some positive constants γ_1, γ_2 and η. Based on the method of energy estimates and Moser iteration, we prove that the problem ($*$) has a unique classical global solution uniformly bounded in time. Furthermore

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we show that if $\theta > 0$, the solution (u, v, w) will converge to $(0, 0, w_*)$ in L^∞ with some $w_* > 0$ as time tends to infinity, while if $\theta = 0$, the solution (u, v, w) will asymptotically converge to $(u_*, u_*, 0)$ in L^∞ with $u_* = \frac{1}{|\Omega|}(\|u_0\|_{L^1} + \alpha\|w_0\|_{L^1})$ if $D > 0$ is suitably large.

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Keywords: Density-dependent Motility; Global existence; Asymptotic stability

1. Introduction and main results

The reaction-diffusion models can generate a wide variety of exquisite spatio-temporal patterns arising in embryogenesis and development due to the diffusion-driven (Turing) instability [16,21]. In addition, colonies of bacteria and eukaryotes can also generate rich and complex patterns driven by chemotaxis, which typically result from coordinated cell movement, growth and differentiation that often involve the detection and processing of extracellular signals [5,6]. Many of these models invoke nonlinear diffusion which is enhanced by the local environment condition because of population pressure (cf. [20]), volume exclusion (cf. [8,22]) or avoidance of danger (cf. [21]) and so on. By employing a synthetic biology approach, the authors of [17] introduced the so-called "self-trapping" mechanism into programmed bacterial *Eeshcrichia coli* cells which excrete signalling molecules acyl-homoserine lactone (AHL) such that at low AHL level, the bacteria undergo run-and-tumble random motion and are motile, while at high AHL levels, the bacteria tumble incessantly and become immotile due to the vanishing macroscopic motility. As a result, *Eeshcrichia coli* cells formed the outward expanding stripe (wave) patterns in the petri dish. To gain a quantitative understanding of the patterning process in the experiment, the following three-component reaction-diffusion system has been proposed in [17]:

$$\begin{cases} u_t = \Delta(\gamma(v)u) + \frac{\alpha w^2 u}{w^2+\lambda}, & x \in \Omega, \quad t > 0, \\ v_t = D\Delta v + u - v, & x \in \Omega, \quad t > 0, \\ w_t = \Delta w - \frac{w^2 u}{w^2+\lambda}, & x \in \Omega, \quad t > 0, \end{cases} \tag{1.1}$$

where $u(x,t)$, $v(x,t)$, $w(x,t)$ denote the bacterial cell density, concentration of acyl-homoserine lactone (AHL) and nutrient density, respectively; $\alpha, \lambda, D > 0$ are constants and Ω is bounded domain in $\mathbb{R}^n (n \geq 2)$. The first equation of (1.1) describes the random motion of bacterial cells with an AHL-dependent motility coefficient $\gamma(v)$, and a cell growth due to the nutrient intake. The second equation of (1.1) describes the diffusion, production and turnover of AHL, while the third equation provides the dynamics of diffusion and consumption for the nutrient. The prominent feature of the system (1.1) is that the cell diffusion rate depends on a motility function $\gamma(v)$ satisfying $\gamma'(v) < 0$, which takes into account the repressive effect of AHL concentration on the cell motility (cf. [17]).

Though the system (1.1) may numerically reproduce some key features of experimental observations as illustrated in [17], the mathematical analysis remains open. Later an alternative simplified two-component so-called "density-suppressed motility" model was proposed in [9]:

$$\begin{cases} u_t = \Delta(\gamma(v)u) + \mu u(1-u), & x \in \Omega, \quad t > 0, \\ v_t = D\Delta v + u - v, & x \in \Omega, \quad t > 0, \end{cases} \tag{1.2}$$

where the reduced growth rate of cells at high density was used to approximate the nutrient depletion effect in the system (1.1). One can expand the Laplacian term in the first equation of (1.2) to obtain a chemotaxis model with signal-dependent motility. Hence the system (1.2) shares some features similar to the Keller-Segel type chemotaxis model. However due to the cross-diffusion and the density-suppressed motility (i.e., $\gamma'(v) < 0$), even for the simplified system (1.2), there are only few results obtained recently when the Neumann boundary conditions are imposed, as summarized below.

(1) $\mu > 0$: In this case, the first result on the global existence and large time behavior of solutions was established in [12]. More precisely, it is shown in [12] that the system (1.2) has a unique global classical solution in two dimensional spaces for the motility function $\gamma(v)$ satisfying the assumptions: $\gamma(v) \in C^3([0,\infty)), \gamma(v) > 0$ and $\gamma'(v) < 0$ on $[0,\infty)$, $\lim_{v\to\infty} \gamma(v) = 0$ and $\lim_{v\to\infty} \frac{\gamma'(v)}{\gamma(v)}$ exists. Moreover, the constant steady state $(1,1)$ of (1.2) is proved to be globally asymptotically stable if $\mu > \frac{K_0}{16}$ where $K_0 = \max_{0\le v\le\infty} \frac{|\gamma'(v)|^2}{\gamma(v)}$. Recently, the global existence result has been extended to the higher dimensions ($n \ge 3$) for large $\mu > 0$ in [31]. On the other hand, for small $\mu > 0$, the existence/nonexistence of nonconstant steady states of (1.2) was rigorously established under some constraints on the parameters in [19] and the periodic pulsating wave is analytically obtained by the multi-scale analysis. When $\gamma(v)$ is a constant step-wise function, the dynamics of discontinuity interface was studied in [25].

(2) $\mu = 0$: The existence of global classical solutions of (1.2) in any dimensions has been established in [37] in the case of $\gamma(v) = c_0/v^k (k > 0)$ for small $c_0 > 0$. The smallness assumption on c_0 is removed lately for the parabolic-elliptic case with $0 < k < \frac{2}{n-2}$ in [1]. Moreover, the global classical solution in two dimensions and global weak solution in three dimensions of (1.2) with $\mu = 0$ are obtained in [30] under the following assumptions:
    (H1) $\gamma(v) \in C^3([0,\infty))$, and there exist $\gamma_1, \gamma_2, \eta > 0$ such that $0 < \gamma_1 \le \gamma(v) \le \gamma_2$, $|\gamma'(v)| \le \eta$ for all $v \ge 0$.

    Without the lower-upper bound hypotheses for $\gamma(v)$ as assumed in (H1), if $\gamma(v)$ decays algebraically and $1 \le n \le 3$, the global existence of weak solutions with large initial data was established in [7]. Moreover, if $\gamma(v)$ decays to zero fastly like exponential decay, the solution of (1.2) with $\mu = 0$ may blow up. For example, if $\gamma(v) = e^{-\chi v}$, by constructing a Lyapunov functional, it is proved in [15] that there exists a critical mass $m_* = \frac{4\pi}{\chi}$ such that the solution of (1.2) with $\mu = 0$ exists globally with uniform-in-time bound if $\int_\Omega u_0 dx < m_*$ while blows up if $\int_\Omega u_0 dx > m_*$ in two dimensions, where u_0 denotes the initial value of u.

Except the above mentioned results on the simplified model (1.2), to our knowledge, there are not any results available for the original three-component system (1.1) proposed in [17]. The purpose of this paper is to develop some analytical results on the system (1.1). More generally we shall consider the following initial-boundary value problem

$$\begin{cases} u_t = \Delta(\gamma(v)u) + \alpha u F(w) - \theta u, & x \in \Omega, \quad t > 0, \\ v_t = D\Delta v + u - v, & x \in \Omega, \quad t > 0, \\ w_t = \Delta w - uF(w), & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ (u, v, w)(x, 0) = (u_0, v_0, w_0)(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where $\theta \geq 0$ accounts for the natural death rate. We assume that the motility function $\gamma(v)$ satisfies the assumption (H1) as used in [30] and the intake rate function $F(w)$ satisfies the following conditions

(H2) $F(w) \in C^1([0, \infty))$, $F(0) = 0$, $F(w) > 0$ in $(0, \infty)$ and $F'(w) > 0$ on $[0, \infty)$.

The conditions in (H2) can be satisfied by a wide class of functions such as

$$F(w) = w, \quad F(w) = \frac{w}{\lambda + w}, \quad F(w) = \frac{w^m}{\lambda + w^m},$$

with constants $\lambda > 0$ and $m > 1$, which are called the Holling type functional response functions in the predator-prey system (cf. [13,14,35,36]). Therefore the system (1.1) is a special case of the equations in (1.3) with $\theta = 0$ and $F(w) = \frac{w^2}{\lambda + w^2}$. In the sequel, for brevity we shall drop the differential element in the integrals without confusion, namely abbreviating $\int_\Omega f dx$ as $\int_\Omega f$ and $\int_0^t \int_\Omega f dx d\tau$ as $\int_0^t \int_\Omega f$. With the assumptions (H1)-(H2), we first prove the existence of globally bounded solutions to the system (1.3) in two dimensions as follows.

Theorem 1.1 *(Global boundedness). Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and the assumptions (H1)-(H2) hold. Assume $(u_0, v_0, w_0) \in [W^{1,\infty}(\Omega)]^3$ with $u_0, v_0, w_0 \gneqq 0$. Then for any $\theta \geq 0$, the problem (1.3) has a unique global classical solution $(u, v, w) \in [C(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))]^3$ satisfying $u, v, w \gneqq 0$ for all $t > 0$ and*

$$\|u(\cdot, t)\|_{L^\infty} \leq M,$$

where $M > 0$ is s constant such that

$$M := C_1(1 + \alpha)^{13}(1 + \frac{1}{D})^{12} e^{C_2(1+\alpha)^6(1+\frac{1}{D})^4}, \tag{1.4}$$

with some constants $C_1, C_2 > 0$ independent of D, α and t.

We remark that we precise the dependence of constant M on α in (1.4) so that the results of Theorem 1.1 can be applied to the case $\alpha = 0$. The explicit dependence of M on D will be used later to derive the asymptotic stability of solutions when imposing some conditions on D as shown in the next theorem.

Theorem 1.2 *(Asymptotic stability of solutions). Let the assumptions in Theorem 1.1 hold and (u, v, w) be the classical solution of (1.3) obtained in Theorem 1.1. Then the following asymptotic stability results hold.*

(1) *If $\theta > 0$, then it holds that*

$$\lim_{t\to\infty}(\|u(\cdot,t)\|_{L^\infty}+\|v(\cdot,t)\|_{L^\infty}+\|w(\cdot,t)-w_*\|_{L^\infty})=0,$$

where $w_*>0$ is a constant determined by $w_*=\frac{1}{|\Omega|}\|w_0\|_{L^1}-\frac{1}{|\Omega|}\int_0^\infty\int_\Omega uF(w)$.

(2) *If $\theta = 0$, there exists a constant $D_0>0$ such that if $D\ge D_0$, then*

$$\lim_{t\to\infty}(\|u(\cdot,t)-u_*\|_{L^\infty}+\|v(\cdot,t)-u_*\|_{L^\infty}+\|w(\cdot,t)\|_{L^\infty})=0,$$

where $u_*=\frac{1}{|\Omega|}(\|u_0\|_{L^1}+\alpha\|w_0\|_{L^1})$.

Remark 1.1. The results of Theorem 1.2 hold for any $\alpha\ge 0$. In the case $\theta=0$ and $\alpha=0$, the system (1.3) reduces to

$$\begin{cases} u_t=\Delta(\gamma(v)u), & x\in\Omega,\quad t>0,\\ v_t=D\Delta v+u-v, & x\in\Omega,\quad t>0,\\ \frac{\partial u}{\partial\nu}=\frac{\partial v}{\partial\nu}=0, & x\in\partial\Omega,\quad t>0,\\ (u,v)(x,0)=(u_0,v_0)(x), & x\in\Omega. \end{cases} \tag{1.5}$$

The global existence of classical solutions of (1.5) in two dimensions has been established in [30], whereas the large time behavior of solution is left open. The result of Theorem 1.2(2) solves this open question for large $D>0$.

Sketch the proof. With the special structure of the first equation of (1.3), we shall use some ideas in [12,30] to show the boundedness of solutions. More precisely, let $\mathcal{A}$ be a self-adjoint realization of $-\Delta$ (see more details in [24]) defined on $D(\mathcal{A}):=\{\phi\in W^{2,2}(\Omega)\cap L^2(\Omega)|\int_\Omega\phi=0$ and $\frac{\partial\phi}{\partial\nu}=0$ on $\partial\Omega\}$. Let $\mathcal{B}$ denote the self-adjoint realization of $-\Delta+\delta$ under homogeneous Neumann boundary conditions in $L^2(\Omega)$ for some $\delta>0$. We can use the first and third equations of (1.3) to obtain

$$(u+\alpha w-\bar{u}-\alpha\bar{w})_t+\mathcal{A}(\gamma(v)u+\alpha w-\overline{\gamma(v)u}-\alpha\bar{w})=0,\ \text{if}\ \theta=0,$$

and

$$(u+\alpha w)_t+\mathcal{B}(\gamma(v)u+\alpha w)=(\delta\gamma(v)-\theta)u+\delta\alpha w,\ \text{if}\ \theta>0,$$

which enable us to find a constant $c_1>0$ independent of D and α such that $\int_t^{t+\tau}\int_\Omega u^2\le c_1(1+\alpha)^2$ for some appropriately small $\tau\in(0,1]$. Using the smoothing properties of the second equation of (1.3) we can obtain the boundedness of $\int_\Omega|\nabla v|^2$ and $\int_t^{t+\tau}\int_\Omega|\Delta v|^2$. Then we use the direct L^2 estimate of u as developed in [12] to find two positive constants c_2, c_3 independent of D and α such that

$$\|u(\cdot,t)\|_{L^2}\le c_2(1+\alpha)e^{c_3(1+\alpha)^6(1+\frac{1}{D})^4}\quad \text{for all}\ \ t\in(0,T_{max}),$$

see Lemma 3.3 for details. Then using the routine bootstrap argument and Moser-iteration method, we derive that $\|u(\cdot,t)\|_{L^\infty} \le M$ with M satisfying (1.4).

To study the asymptotic behavior, we divide our proofs into two cases: $\theta > 0$ and $\theta = 0$. When $\theta > 0$, we can obtain from the first equation of (1.3) that $\int_0^\infty \int_\Omega u < \infty$, which combined with the relative compactness of $(u(\cdot,t))_{t>1}$ in $C(\Omega)$ (see Lemma 4.1) gives $\|u(\cdot,t)\|_{L^\infty} \to 0$ and hence $\|v(\cdot,t)\|_{L^\infty} \to 0$ as $t \to \infty$ from the second equation of (1.3). Then using the semigroup estimates and the decay property of u, from the third equation we can show that $\|w(\cdot,t) - w_*\|_{L^\infty} \to 0$ as $t \to \infty$ for some $w_* > 0$, where $w_* > 0$ is proved by showing

$$\int_\Omega \ln w(x,t) \ge -c_4, \text{ for all } t \ge 1$$

for some constant $c_4 > 0$, see Lemma 4.3 for details.

When $\theta = 0$, from the third equation of (1.3) we have

$$\int_0^\infty \int_\Omega uF(w) + \int_0^\infty \int_\Omega |\nabla w|^2 < \infty,$$

which, combined with $\|u_0\|_{L^1} \le \|u(\cdot,t)\|_{L^1}$, entails us that

$$\|w(\cdot,t)\|_{L^\infty} \to 0 \text{ as } t \to \infty.$$

On the other hand, using the relations between M and D in (1.4), we can construct a L^2 energy functional of (u,v) which along with suitable regularity of (u,v) finally leads to the convergence of (u,v) as claimed. □

2. Local existence and Preliminaries

The existence and uniqueness of local solutions of (1.3) can be readily proved by the Amann's theorem [3,4] (cf. also [32, Lemma 2.6]) or the fixed point theorem along with the parabolic regularity theory [12,29]. We omit the details of the proof for brevity.

Lemma 2.1 *(Local existence). Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and the assumptions (H1) and (H2) hold. Assume $(u_0, v_0, w_0) \in [W^{1,\infty}(\Omega)]^3$ with $u_0, v_0, w_0 \gneqq 0$. Then there exists $T_{max} \in (0,\infty]$ such that the problem (1.3) has a unique classical solution $(u,v,w) \in [C(\bar\Omega \times [0,T_{max})) \cap C^{2,1}(\bar\Omega \times (0,T_{max}))]^3$ satisfying $u,v,w > 0$ for all $t > 0$. Moreover,*

$$if \quad T_{max} < \infty, \quad then \quad \|u(\cdot,t)\|_{L^\infty} \to \infty \quad as \quad t \nearrow T_{max}.$$

Lemma 2.2. *The solution (u,v,w) of (1.3) satisfies*

$$\|u(\cdot,t)\|_{L^1} + \alpha\|w(\cdot,t)\|_{L^1} + \theta\int_0^t \|u(\cdot,s)\|_{L^1} = \|u_0\|_{L^1} + \alpha\|w_0\|_{L^1}, \quad t \in (0,T_{max}), \tag{2.1}$$

and

$$\|w(\cdot,t)\|_{L^\infty} \text{ is decreasing in } t. \tag{2.2}$$

Moreover, for all $(x,t) \in \Omega \times (0, T_{\max})$*, it follows that*

$$F(w(x,t)) \le C_F = F(\|w_0\|_{L^\infty}). \tag{2.3}$$

Proof. We first multiply the third equation of (1.3) by α and add the resulting equation to the first equation of (1.3). Then integrating the result over $\Omega \times (0,t)$, we have (2.1) directly. The application of the maximum principle to the third equation of (1.3) gives (2.2). Furthermore, since $F'(w) > 0$ for all $w \ge 0$, one has (2.3) by using (2.2). □

Next, we list some well-known estimates for the Neumann heat semigroup for later use.

Lemma 2.3 *([33]). Let* $(e^{t\Delta})_{t\ge 0}$ *be the Neumann heat semigroup in* Ω*, and let* $\lambda_1 > 0$ *denote the first nonzero eigenvalue of* $-\Delta$ *in* Ω *under Neumann boundary conditions. Then for all* $t > 0$*, there exist some constants* $k_i (i = 1, 2, 3)$ *depending only on* Ω *such that*

(i) *If* $1 \le q \le p \le \infty$*, then*

$$\|e^{t\Delta} z\|_{L^p} \le k_1 \left(1 + t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right) e^{-\lambda_1 t} \|z\|_{L^q} \tag{2.4}$$

for all $z \in L^q(\Omega)$ *satisfying* $\int_\Omega z = 0$.

(ii) *If* $1 \le q \le p \le \infty$*, then*

$$\|\nabla e^{t\Delta} z\|_{L^p} \le k_2 \left(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right) e^{-\lambda_1 t} \|z\|_{L^q} \tag{2.5}$$

for all $z \in L^q(\Omega)$.

(iii) *If* $2 \le q \le p < \infty$*, then*

$$\|\nabla e^{t\Delta} z\|_{L^p} \le k_3 \left(1 + t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right) e^{-\lambda_1 t} \|\nabla z\|_{L^q} \tag{2.6}$$

for all $z \in W^{1,p}(\Omega)$.

The following lemma will be used to show the boundedness of solution, one can see [18, Lemma 3.3] or [26, Lemma 3.4] for details.

Lemma 2.4. *Let* $T > 0$*,* $\tau \in (0,T)$*,* $a > 0$ *and* $b > 0$*. Suppose that* $y : [0,T) \to [0,\infty)$ *is absolutely continuous and fulfils*

$$y'(t) + ay(t) \le h(t), \quad \text{for all } t \in (0,T),$$

with some nonnegative function $h \in L^1_{loc}([0,T))$ *satisfying*

$$\int_t^{t+\tau} h(s)ds \le b, \quad \text{for all } t \in [0, T-\tau).$$

Then

$$y(t) \le \max\left\{y(0)+b, \frac{b}{a\tau}+2b\right\}, \quad \textit{for all } t \in (0,T).$$

3. Boundedness of solutions (Proof of Theorem 1.1)

In this section, we shall establish the boundedness of solution in two dimensions.

Lemma 3.1. *Suppose the assumptions in Theorem 1.1 hold. For all $\theta \ge 0$, there exists a constant $K_1 > 0$ independent of D and α such that the solution to (1.3) satisfies*

$$\int_t^{t+\tau}\int_\Omega u^2 \le K_1(1+\alpha)^2, \quad \text{for all } \ t \in (0, \widetilde{T}_{max}), \tag{3.1}$$

where

$$\tau := \min\left\{1, \frac{1}{2}T_{max}\right\} \ \text{and} \ \widetilde{T}_{max} := T_{max} - \tau.$$

Proof. We divide the proof into two cases: $\theta = 0$ and $\theta > 0$.
Case 1: $\theta = 0$**.** In this case, multiplying the third equation of (1.3) by α and adding the result to the first equation of (1.3), one has

$$(u+\alpha w)_t = \Delta(\gamma(v)u + \alpha w). \tag{3.2}$$

Then integrating (3.2) with respect to x with the homogeneous Neumann boundary conditions, one has

$$\bar{u} + \alpha\bar{w} = \frac{1}{|\Omega|}\int_\Omega u_0 + \alpha\frac{1}{|\Omega|}\int_\Omega w_0 = \bar{u}_0 + \alpha\bar{w}_0, \tag{3.3}$$

where $\bar{f}$ denotes the mean of f, namely $\bar{f} = \frac{1}{|\Omega|}\int_\Omega f dx$. Let $\mathcal{A}$ be a self-adjoint realization of $-\Delta$ defined on $D(\mathcal{A}) := \{\phi \in W^{2,2}(\Omega) \cap L^2(\Omega) | \int_\Omega \phi = 0 \text{ and } \frac{\partial\phi}{\partial\nu} = 0 \text{ on } \partial\Omega\}$. Then using (3.3), we can rewrite (3.2) as

$$(u+\alpha w - \bar{u} - \alpha\bar{w})_t = -\mathcal{A}(\gamma(v)u + \alpha w - \overline{\gamma(v)u} - \alpha\bar{w}). \tag{3.4}$$

Multiplying (3.4) by $\mathcal{A}^{-1}(u+\alpha w - \bar{u} - \alpha\bar{w})$ and integrating the result by parts, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathcal{A}^{-\frac{1}{2}}(u+\alpha w-\bar{u}-\alpha\bar{w})|^2 \\
&=-\int_\Omega \mathcal{A}\Big(\gamma(v)u+\alpha w-\overline{\gamma(v)u}-\alpha\bar{w}\Big)\cdot\mathcal{A}^{-1}\left(u+\alpha w-\bar{u}-\alpha\bar{w}\right) \\
&=-\int_\Omega (u+\alpha w-\bar{u}-\alpha\bar{w})\cdot\Big(\gamma(v)u+\alpha w-\overline{\gamma(v)u}-\alpha\bar{w}\Big) \\
&=-\int_\Omega \gamma(v)(u-\bar{u})^2-\alpha^2\int_\Omega (w-\bar{w})^2-\alpha\int_\Omega \Big(1+\gamma(v)\Big)(u-\bar{u})(w-\bar{w}) \\
&\quad -\bar{u}\int_\Omega \gamma(v)(u-\bar{u})-\alpha\bar{u}\int_\Omega \gamma(v)(w-\bar{w}),
\end{aligned}
$$

which together with the facts $0<\gamma_1\le\gamma(v)\le\gamma_2$ and the nonnegativity of u, w, gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathcal{A}^{-\frac{1}{2}}(u+\alpha w-\bar{u}-\alpha\bar{w})|^2+\gamma_1\int_\Omega (u-\bar{u})^2+\alpha^2\int_\Omega (w-\bar{w})^2 \\
&=-\alpha\int_\Omega \Big(1+\gamma(v)\Big)(u-\bar{u})(w-\bar{w})-\bar{u}\int_\Omega \gamma(v)(u-\bar{u})-\alpha\bar{u}\int_\Omega \gamma(v)(w-\bar{w}) \\
&\le\alpha\bar{w}\int_\Omega \Big(1+\gamma(v)\Big)u+\alpha\bar{u}\int_\Omega \Big(1+\gamma(v)\Big)w+(\bar{u}^2+\alpha\bar{u}\bar{w})\int_\Omega \gamma(v) \\
&\le\frac{2\alpha+3\alpha\gamma_2}{|\Omega|}\|u\|_{L^1}\|w\|_{L^1}+\frac{\gamma_2}{|\Omega|}\|u\|_{L^1}^2,
\end{aligned}
\tag{3.5}
$$

in which we have used the fact $\int_\Omega(\varphi-\bar{\varphi})^2\le\int_\Omega\varphi^2$ for all $\varphi\in L^2(\Omega)$. We know from Lemma 2.2 that $\|u\|_{L^1}\le\|u_0\|_{L^1}+\alpha\|w_0\|_{L^1}\le|\Omega|(\|u_0\|_{L^\infty}+\alpha\|w_0\|_{L^\infty})$ and $\|w\|_{L^1}\le|\Omega|\|w\|_{L^\infty}\le|\Omega|\|w_0\|_{L^\infty}$. Therefore, (3.5) shows

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega |\mathcal{A}^{-\frac{1}{2}}(u+\alpha w-\bar{u}-\alpha\bar{w})|^2+2\gamma_1\int_\Omega (u-\bar{u})^2+2\alpha^2\int_\Omega (w-\bar{w})^2 \\
&\quad\le c_1(1+\alpha)^2,
\end{aligned}
\tag{3.6}
$$

where $c_1=4(1+2\gamma_2)|\Omega|(\|u_0\|_{L^\infty}+\|w_0\|_{L^\infty})^2$. Because of $\int_\Omega\mathcal{A}^{-\frac{1}{2}}(u+\alpha w-\bar{u}-\alpha\bar{w})=0$, we can apply the Poincaré inequality with a positive constant c_2 and the fact $\|w\|_{L^\infty}\le\|w_0\|_{L^\infty}$ to obtain

$$
\begin{aligned}
&\int_\Omega |\mathcal{A}^{-\frac{1}{2}}(u+\alpha w-\bar{u}-\alpha\bar{w})|^2 \\
&\le c_2\int_\Omega |\nabla\mathcal{A}^{-\frac{1}{2}}(u+\alpha w-\bar{u}-\alpha\bar{w})|^2 \\
&= c_2\int_\Omega |u+\alpha w-\bar{u}-\alpha\bar{w}|^2 \\
&\le 2c_2\int_\Omega (u-\bar{u})^2+2c_2\alpha^2\int_\Omega (w-\bar{w})^2 \\
&\le 2c_2\int_\Omega (u-\bar{u})^2+2c_2\alpha^2|\Omega|\|w_0\|_{L^\infty}^2.
\end{aligned}
\tag{3.7}
$$

Substituting (3.7) into (3.6), and letting $X(t) := \int_\Omega |\mathcal{A}^{-\frac{1}{2}}(u+\alpha w-\bar{u}-\alpha\bar{w})|^2$, one yields

$$
X'(t)+\frac{\gamma_1}{2c_2}X(t)+\gamma_1\int_\Omega (u-\bar{u})^2 \le c_3(1+\alpha)^2, \tag{3.8}
$$

where $c_3 = c_1+\gamma_1|\Omega|\|w_0\|_{L^\infty}^2$. Then applying the Grönwall's inequality to (3.8), we first obtain

$$
X(t)=\int_\Omega |\mathcal{A}^{-\frac{1}{2}}(u+\alpha w-\bar{u}-\alpha\bar{w})|^2 \le c_4(1+\alpha)^2, \tag{3.9}
$$

where $c_4 = \frac{2c_2c_3}{\gamma_1}+2c_2|\Omega|(\|u_0\|_{L^\infty}+\|w_0\|_{L^\infty})^2$. Then integrating (3.8) over $(t, t+\tau)$ with $\tau := \min\left\{1, \frac{1}{2}T_{max}\right\}$ and using (3.9), one has

$$
\int_t^{t+\tau}\int_\Omega (u-\bar{u})^2 \le \frac{c_3\tau+c_4}{\gamma_1}(1+\alpha)^2 \le \frac{c_3+c_4}{\gamma_1}(1+\alpha)^2. \tag{3.10}
$$

By the fact $\int_\Omega (u-\bar{u})^2 = \int_\Omega u^2-\int_\Omega \bar{u}^2$, it follows from (3.10) that

$$
\int_t^{t+\tau}\int_\Omega u^2 = \int_t^{t+\tau}\int_\Omega (u-\bar{u})^2+\int_t^{t+\tau}\int_\Omega \bar{u}^2 \le \frac{c_3+c_4}{\gamma_1}(1+\alpha)^2+\bar{u}^2|\Omega|\tau,
$$

which yields (3.1) by using the fact $\bar{u} \le \|u_0\|_{L^\infty}+\alpha\|w_0\|_{L^\infty}$.

Case 2: $\theta > 0$**.** In this case, we let $\mathcal{B}$ denote the self-adjoint realization of $-\Delta+\delta$ under homogeneous Neumann boundary conditions in $L^2(\Omega)$, where $0<\delta<\frac{\theta}{\gamma_2}$. Then there exists a constant $c_5 > 0$ such that

$$
\|\mathcal{B}^{-1}\psi\|_{L^2} \le c_5\|\psi\|_{L^2} \quad \text{for all } \psi \in L^2(\Omega) \tag{3.11}
$$

and

$$\|\mathcal{B}^{-\frac{1}{2}}\psi\|_{L^2}^2 = \int_\Omega \psi \cdot \mathcal{B}^{-1}\psi \le c_5\|\psi\|_{L^2}^2 \quad \text{for all } \psi \in L^2(\Omega), \tag{3.12}$$

one can see the details in [18]. From the system (1.3), we have

$$(u+\alpha w)_t = \Delta(\gamma(v)u+\alpha w) - \theta u,$$

which can be rewritten as

$$(u+\alpha w)_t + \mathcal{B}(\gamma(v)u+\alpha w) = \delta(\gamma(v)u+\alpha w) - \theta u = (\delta\gamma(v)-\theta)u + \delta\alpha w. \tag{3.13}$$

With the fact $0<\delta<\frac{\theta}{\gamma_2}$ and the boundedness of w, we derive

$$(\delta\gamma(v)-\theta)u + \delta\alpha w \le (\delta\gamma_2-\theta)u + \delta\alpha\|w_0\|_{L^\infty} \le c_6\alpha, \tag{3.14}$$

where $c_6 = \frac{\theta\|w_0\|_{L^\infty}}{\gamma_2}$. Hence, multiplying (3.13) by $\mathcal{B}^{-1}(u+\alpha w)\ge 0$, and using (3.14), one has

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathcal{B}^{-\frac{1}{2}}(u+\alpha w)|^2 + \int_\Omega (\gamma(v)u+\alpha w)(u+\alpha w) \le c_6\alpha\int_\Omega \mathcal{B}^{-1}(u+\alpha w),$$

and hence

$$\frac{d}{dt}\int_\Omega |\mathcal{B}^{-\frac{1}{2}}(u+\alpha w)|^2 + 2c_7\int_\Omega (u+\alpha w)^2 \le 2c_6\alpha\int_\Omega \mathcal{B}^{-1}(u+\alpha w), \tag{3.15}$$

with $c_7 := \min\{\gamma_1, 1\}$. Using (3.11) and (3.12), we can derive that

$$\begin{aligned}
&\frac{c_7}{2c_5}\int_\Omega |\mathcal{B}^{-\frac{1}{2}}(u+\alpha w)|^2 + 2c_6\alpha\int_\Omega \mathcal{B}^{-1}(u+\alpha w)\\
&\le \frac{c_7}{2}\int_\Omega (u+\alpha w)^2 + 2c_5c_6\alpha|\Omega|^{\frac{1}{2}}\|u+\alpha w\|_{L^2}\\
&\le c_7\int_\Omega (u+\alpha w)^2 + \frac{2c_5^2c_6^2|\Omega|}{c_7}\alpha^2.
\end{aligned} \tag{3.16}$$

Substituting (3.16) into (3.15), and defining $Y(t) := \int_\Omega |\mathcal{B}^{-\frac{1}{2}}(u+\alpha w)|^2$, one has

$$Y'(t) + \frac{c_7}{2c_5}Y(t) + c_7\int_\Omega (u+\alpha w)^2 \le \frac{2c_5^2c_6^2|\Omega|}{c_7}\alpha^2,$$

which combined with the Grönwall's inequality gives

$$Y(t) \le c_5|\Omega| \left((\|u_0\|_{L^\infty} + \|w_0\|_{L^\infty})^2 + \frac{4c_5^2c_6^2}{c_7^2} \right) (1+\alpha)^2 := c_8(1+\alpha)^2$$

and thus

$$\int_t^{t+\tau}\int_\Omega u^2 \le \int_t^{t+\tau}\int_\Omega (u+\alpha w)^2 \le \frac{Y(t)}{c_7} + \frac{2c_5^2c_6^2|\Omega|\tau}{c_7^2}\alpha^2 \le c_9(1+\alpha)^2,$$

where $c_9 = \frac{c_8}{c_7} + \frac{2c_5^2c_6^2|\Omega|\tau}{c_7^2}$, which gives (3.1). Then we complete the proof of this lemma. □

Lemma 3.2. *Let the conditions in Theorem 1.1 hold. Then there exist two positive constants K_2, K_3 independent of D, α and t such that*

$$\int_\Omega |\nabla v|^2 \le K_2(1+\alpha)^2 \left(1 + \frac{1}{D}\right) \quad \text{for all } t \in (0, T_{max}), \tag{3.17}$$

and

$$\int_t^{t+\tau}\int_\Omega |\Delta v|^2 \le K_3(1+\alpha)^2 \left(1 + \frac{1}{D}\right)^2 \quad \text{for all } t \in (0, \widetilde{T}_{max}). \tag{3.18}$$

Proof. We multiply the second equation of (1.3) by $-\Delta v$ and integrate the result with Cauchy-Schwarz inequality to get for all $t \in (0, T_{max})$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v|^2 &= -D\int_\Omega |\Delta v|^2 - \int_\Omega u\Delta v + \int_\Omega v\Delta v \\ &\le -\frac{D}{2}\int_\Omega |\Delta v|^2 + \frac{1}{2D}\int_\Omega u^2 - \int_\Omega |\nabla v|^2,\end{aligned}$$

which leads to

$$\frac{d}{dt}\int_\Omega |\nabla v|^2 + D\int_\Omega |\Delta v|^2 + 2\int_\Omega |\nabla v|^2 \le \frac{1}{D}\int_\Omega u^2. \tag{3.19}$$

Letting $y(t) = \int_\Omega |\nabla v|^2$ and $h(t) = \frac{1}{D}\int_\Omega u^2$, we have from (3.19) that

$$y'(t) + 2y(t) \le h(t) \quad \text{for all } t \in (0, T_{max}). \tag{3.20}$$

Then applying Lemma 2.4 with the fact $\int_t^{t+\tau} h(s)ds \le \frac{K_1(1+\alpha)^2}{D}$ for $t \in (0, \widetilde{T}_{max})$ to (3.20) gives

$$\begin{aligned}\int_\Omega |\nabla v|^2 &\le \max\left\{\|\nabla v_0\|_{L^2}^2 + \frac{K_1(1+\alpha)^2}{D}, \frac{K_1(1+\alpha)^2}{2D\tau} + \frac{2K_1(1+\alpha)^2}{D}\right\}\\ &\le \|\nabla v_0\|_{L^2}^2 + \frac{2K_1(1+\alpha)^2}{D} + \frac{K_1(1+\alpha)^2}{2D\tau}\\ &\le \left(\|\nabla v_0\|_{L^2}^2 + 2K_1 + \frac{K_1}{2\tau}\right)\left(1+\frac{1}{D}\right)(1+\alpha)^2 \quad \text{for all } t\in(0,T_{max}),\end{aligned}$$

which yields (3.17) with $K_2 = \|\nabla v_0\|_{L^2}^2 + 2K_1 + \frac{K_1}{2\tau}$. On the other hand, integrating (3.19) over $(t,\ t+\tau)$ for $t \in (0, \widetilde{T}_{max})$ and using (3.17), we can derive that

$$\begin{aligned} D\int_t^{t+\tau}\int_\Omega |\Delta v|^2 &\le \frac{1}{D}\int_t^{t+\tau}\int_\Omega u^2 + \int_\Omega |\nabla v|^2\\ &\le \frac{K_1}{D}(1+\alpha)^2 + K_2\left(1+\frac{1}{D}\right)(1+\alpha)^2,\end{aligned}$$

which implies (3.18) with $K_3 = K_1 + K_2$. □

Lemma 3.3. *Let the assumptions in Theorem 1.1 hold. Then there exist two positive constants K_4 and K_5, which are independent of D and α, such that*

$$\|u(\cdot,t)\|_{L^2} \le K_4(1+\alpha)e^{K_5(1+\alpha)^6(1+\frac{1}{D})^4} \quad \text{for all } t\in(0,T_{max}). \tag{3.21}$$

Proof. Multiplying the first equation of (1.3) by u and integrating the result with assumptions (H1) and (2.3) gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 &= -\int_\Omega \nabla u\cdot\nabla(\gamma(v)u) + \alpha\int_\Omega F(w)u^2 - \theta\int_\Omega u^2\\ &\le -\int_\Omega \gamma(v)|\nabla u|^2 - \int_\Omega \gamma'(v)u\nabla u\cdot\nabla v + \alpha C_F\int_\Omega u^2\\ &\le -\gamma_1\int_\Omega |\nabla u|^2 + \eta\int_\Omega u|\nabla u||\nabla v| + \alpha C_F\int_\Omega u^2\\ &\le -\frac{\gamma_1}{2}\int_\Omega |\nabla u|^2 + \frac{\eta^2}{2\gamma_1}\int_\Omega u^2|\nabla v|^2 + \alpha C_F\int_\Omega u^2,\end{aligned}$$

which yields

$$\frac{d}{dt}\int_\Omega u^2 + \gamma_1\int_\Omega |\nabla u|^2 \le \frac{\eta^2}{\gamma_1}\int_\Omega u^2|\nabla v|^2 + 2\alpha C_F\int_\Omega u^2. \tag{3.22}$$

Moreover, applying Gagliardo-Nirenberg inequality and Young inequality to the first term on the right hand side of (3.22), we obtain a constant $c_1 > 0$ such that

$$\begin{aligned}\frac{\eta^2}{\gamma_1}\int_\Omega u^2|\nabla v|^2 \le& \frac{\eta^2}{\gamma_1}\|u\|_{L^4}^2\|\nabla v\|_{L^4}^2\\ \le& \frac{c_1\eta^2}{\gamma_1}\left(\|\nabla u\|_{L^2}\|u\|_{L^2}+\|u\|_{L^2}^2\right)\left(\|\Delta v\|_{L^2}\|\nabla v\|_{L^2}+\|\nabla v\|_{L^2}^2\right)\\ \le& \frac{c_1\eta^2}{\gamma_1}\|\nabla u\|_{L^2}\|u\|_{L^2}\|\Delta v\|_{L^2}\|\nabla v\|_{L^2}+\frac{c_1\eta^2}{\gamma_1}\|\nabla u\|_{L^2}\|u\|_{L^2}\|\nabla v\|_{L^2}^2\\ &+\frac{c_1\eta^2}{\gamma_1}\|u\|_{L^2}^2\|\Delta v\|_{L^2}\|\nabla v\|_{L^2}+\frac{c_1\eta^2}{\gamma_1}\|u\|_{L^2}^2\|\nabla v\|_{L^2}^2\\ \le& \gamma_1\|\nabla u\|_{L^2}^2+\frac{c_1\eta^2}{\gamma_1}\left(2+\frac{c_1\eta^2}{2\gamma_1^2}\|\nabla v\|_{L^2}^2\right)\|u\|_{L^2}^2\|\nabla v\|_{L^2}^2\\ &+\frac{c_1\eta^2}{\gamma_1}\left(\frac{1}{4}+\frac{c_1\eta^2}{2\gamma_1^2}\|\nabla v\|_{L^2}^2\right)\|u\|_{L^2}^2\|\Delta v\|_{L^2}^2.\end{aligned} \tag{3.23}$$

Substituting (3.23) into (3.22), and using (3.17), we conclude

$$\frac{d}{dt}\|u\|_{L^2}^2 \le c_2(1+\alpha)^4\left(1+\frac{1}{D}\right)^2(1+\|\Delta v\|_{L^2}^2)\|u\|_{L^2}^2, \tag{3.24}$$

where $c_2 = \frac{c_1\eta^2}{\gamma_1}\left(2K_2+\frac{1}{4}+\frac{c_1\eta^2K_2(1+K_2)}{2\gamma_1^2}\right)+2C_F$. On the other hand, using the facts (3.1) and (3.18), then for any $t\in(0,T_{max})$, we can find a $t_0\ge 0$ satisfying $t_0\in(0,\tilde{T}_{max})$ and $t_0\in((t-\tau)^+,t)$ such that

$$\|u(\cdot,t_0)\|_{L^2}^2 \le c_3(1+\alpha)^2, \tag{3.25}$$

and

$$\int_{t_0}^{t_0+\tau}\int_\Omega |\Delta v|^2 \le K_3(1+\alpha)^2\left(1+\frac{1}{D}\right)^2, \tag{3.26}$$

with $c_3 = \|u_0\|_{L^2}^2+\frac{K_1}{\tau}$. Then we integrate (3.24) over (t_0,t), and use the facts (3.25), (3.26) and $t\le t_0+\tau\le t_0+1$ to obtain

$$\begin{aligned}\|u(\cdot,t)\|_{L^2}^2 \le& \|u(\cdot,t_0)\|_{L^2}^2 e^{c_2(1+\alpha)^4\left(1+\frac{1}{D}\right)^2\int_{t_0}^t(1+\|\Delta v(\cdot,s)\|_{L^2}^2)ds}\\ \le& \|u(\cdot,t_0)\|_{L^2}^2 e^{c_2(1+\alpha)^4\left(1+\frac{1}{D}\right)^2\int_{t_0}^t ds+c_2(1+\alpha)^4\left(1+\frac{1}{D}\right)^2\int_{t_0}^t\|\Delta v(\cdot,s)\|_{L^2}^2ds}\\ \le& c_3(1+\alpha)^2e^{c_2(1+\alpha)^4\left(1+\frac{1}{D}\right)^2+c_2K_3(1+\alpha)^6\left(1+\frac{1}{D}\right)^4},\end{aligned}$$

which yields (3.21) with $K_4 = c_3$ and $K_5 = c_2(1 + K_3)$. Then we finish the proof of this lemma. □

Lemma 3.4. *Suppose the conditions in Theorem 1.1 hold. Let (u, v, w) be the solution of the system (1.3). Then it holds that*

$$\|u(\cdot,t)\|_{L^4} \le K_6(1+\alpha)^3\left(1+\frac{1}{D}\right)^2 e^{3K_5(1+\alpha)^6(1+\frac{1}{D})^4}, \quad \text{for all } t \in (0, T_{max}), \tag{3.27}$$

where $K_6 > 0$ is a constant independent of α, D and t.

Proof. With the fact that $0 \le F(w) \le C_F$ from (2.3) and the assumptions (H1), we multiply the first equation of (1.3) with u^3 and integrate the result to have

$$\begin{aligned}
\frac{1}{4}\frac{d}{dt}\int_\Omega u^4 &= -3\int_\Omega u^2\nabla u\cdot\nabla(\gamma(v)u) + \alpha\int_\Omega F(w)u^4 - \theta\int_\Omega u^4 \\
&\le -3\int_\Omega \gamma(v)u^2|\nabla u|^2 - 3\int_\Omega \gamma'(v)u^3\nabla u\cdot\nabla v + \alpha C_F\int_\Omega u^4 \\
&\le -3\gamma_1\int_\Omega u^2|\nabla u|^2 + 3\eta\int_\Omega u^3|\nabla u||\nabla v| + \alpha C_F\int_\Omega u^4 \\
&\le -\frac{3\gamma_1}{2}\int_\Omega u^2|\nabla u|^2 + \frac{3\eta^2}{2\gamma_1}\int_\Omega u^4|\nabla v|^2 + \alpha C_F\int_\Omega u^4,
\end{aligned}$$

which yields that

$$\frac{d}{dt}\int_\Omega u^4 + \frac{3\gamma_1}{2}\int_\Omega |\nabla u^2|^2 \le \frac{6\eta^2}{\gamma_1}\int_\Omega u^4|\nabla v|^2 + 4\alpha C_F\int_\Omega u^4. \tag{3.28}$$

Using Gagliardo-Nirenberg inequality and Young's inequality, along with the facts (3.33) and $\|u^2\|_{L^1} = \|u\|^2_{L^2}$, we can find a constant $c_1 > 0$ independent of α and D, such that

$$\begin{aligned}
\frac{6\eta^2}{\gamma_1}\int_\Omega u^4|\nabla v|^2 &\le \frac{6\eta^2}{\gamma_1}\left(\int_\Omega u^8\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla v|^4\right)^{\frac{1}{2}} \\
&= \frac{6\eta^2}{\gamma_1}\|u^2\|^2_{L^4}\|\nabla v\|^2_{L^4} \\
&\le \frac{6\eta^2 c_1}{\gamma_1}\left(\|\nabla u^2\|^{\frac{3}{2}}_{L^2}\|u^2\|^{\frac{1}{2}}_{L^1} + \|u^2\|^2_{L^1}\right)\|\nabla v\|^2_{L^4} \\
&\le \frac{6\eta^2 c_1}{\gamma_1}\|\nabla u^2\|^{\frac{3}{2}}_{L^2}\|u\|_{L^2}\|\nabla v\|^2_{L^4} + \frac{6\eta^2 c_1}{\gamma_1}\|u\|^4_{L^2}\|\nabla v\|^2_{L^4}
\end{aligned}$$

$$\leq \gamma_1 \|\nabla u^2\|_{L^2}^2 + c_2 \|\nabla v\|_{L^4}^2 \|u\|_{L^2}^4 (\|\nabla v\|_{L^4}^6 + 1), \tag{3.29}$$

where

$$c_2 := \frac{1}{4}\left(\frac{3}{4\gamma_1}\right)^3 \left(\frac{6\eta^2 c_1}{\gamma_1}\right)^4 + \frac{6\eta^2 c_1}{\gamma_1}.$$

Furthermore, using the Gagliardo-Nirenberg inequality and Young's inequality again, we can find a constant $c_3 > 0$ independent of D and α, such that

$$\begin{aligned}
(1+4\alpha C_F)\int_\Omega u^4 &= (1+4\alpha C_F)\|u^2\|_{L^2}^2 \\
&\leq c_3(1+4\alpha C_F)\left(\|\nabla u^2\|_{L^2}^{\frac{3}{2}} \|u^2\|_{L^{\frac{1}{2}}}^{\frac{1}{2}} + \|u^2\|_{L^{\frac{1}{2}}}^2\right) \\
&\leq c_3(1+4\alpha C_F)\left(\|\nabla u^2\|_{L^2}^{\frac{3}{2}} \|u\|_{L^1} + \|u\|_{L^1}^4\right) \\
&\leq \frac{\gamma_1}{2}\|\nabla u^2\|_{L^2}^2 + c_4(1+\alpha)^8,
\end{aligned} \tag{3.30}$$

where $c_4 = \left(\frac{c_3^4}{4}(4C_F+1)^4\left(\frac{3}{2\gamma_1}\right)^3 + c_3(4C_F+1)\right)(\|u_0\|_{L^1} + \|w_0\|_{L^1})^4$. Substituting (3.29) and (3.30) into (3.28), one has

$$\frac{d}{dt}\int_\Omega u^4 + \int_\Omega u^4 \leq c_2\|\nabla v\|_{L^4}^2 \|u\|_{L^2}^4 (\|\nabla v\|_{L^4}^6 + 1) + c_4(1+\alpha)^8 \tag{3.31}$$

By the scaling $\tilde{t} = Dt$, and applying the variation-of-constants formula to the second equation of (1.3), one has

$$v(\cdot,\tilde{t}) = e^{(\Delta - \frac{1}{D})\tilde{t}} v_0 + \frac{1}{D}\int_0^{\tilde{t}} e^{(\Delta-\frac{1}{D})(\tilde{t}-s)} u(\cdot,s)ds. \tag{3.32}$$

Then using the semigroup estimates (2.5) and (2.6), we derive from (3.32)

$$\begin{aligned}
\|\nabla v(\cdot,\tilde{t})\|_{L^4} &\leq \|\nabla e^{(\Delta-\frac{1}{D})\tilde{t}} v_0\|_{L^4} + \frac{1}{D}\int_0^{\tilde{t}} \|\nabla e^{(\Delta-\frac{1}{D})(\tilde{t}-s)} u(\cdot,s)\|_{L^4} ds \\
&\leq k_1 e^{-\lambda_1 \tilde{t}} \|\nabla v_0\|_{L^4} + \frac{k_2}{D}\int_0^{\tilde{t}} \left(1 + (\tilde{t}-s)^{-\frac{3}{4}}\right) e^{-\lambda_1(\tilde{t}-s)} \|u(\cdot,s)\|_{L^2} ds \\
&\leq k_1 \|\nabla v_0\|_{L^4} + \frac{k_2 K_4}{D\lambda_1}\left(1 + \Gamma(1/4)\lambda_1^{\frac{3}{4}}\right)(1+\alpha) e^{K_5(1+\alpha)^6 (1+\frac{1}{D})^4},
\end{aligned}$$

which gives

$$\|\nabla v(\cdot,t)\|_{L^4} \le c_5(1+\alpha)\left(1+\frac{1}{D}\right)e^{K_5(1+\alpha)^6(1+\frac{1}{D})^4}, \tag{3.33}$$

with $c_5 = k_1\|\nabla v_0\|_{L^4} + \frac{k_2K_4}{\lambda_1}\left(1+\Gamma(1/4)\lambda_1^{\frac{3}{4}}\right)$. Then substituting (3.33) into (3.31), one can find a constant $c_6 := c_2K_4^4c_5^2(c_5^6+1)+c_4$ to obtain

$$\frac{d}{dt}\int_\Omega u^4 + \int_\Omega u^4 \le c_6(1+\alpha)^{12}\left(1+\frac{1}{D}\right)^8 e^{12K_5(1+\alpha)^6(1+\frac{1}{D})^4}.$$

This along with the Grönwall's inequality yields a constant $c_7 = c_6 + \|u_0\|_{L^4}^4$ independent of D and α so that

$$\begin{aligned}\|u(\cdot,t)\|_{L^4}^4 &\le \|u_0\|_{L^4}^4 + c_6(1+\alpha)^{12}\left(1+\frac{1}{D}\right)^8 e^{12K_5(1+\alpha)^6(1+\frac{1}{D})^4}\\ &\le c_7(1+\alpha)^{12}\left(1+\frac{1}{D}\right)^8 e^{12K_5(1+\alpha)^6(1+\frac{1}{D})^4},\end{aligned}$$

which yields (3.27). □

Lemma 3.5. *Let the conditions in Lemma 3.4 hold. Suppose (u,v,w) is a solution of (1.3). Then it follows that*

$$\|u(\cdot,t)\|_{L^\infty} \le K_7(1+\alpha)^{13}\left(1+\frac{1}{D}\right)^{12} e^{12K_5(1+\alpha)^6(1+\frac{1}{D})^4} \quad \text{for all } t\in(0,T_{max}), \tag{3.34}$$

where the constant $K_7>0$ is independent of D and α.

Proof. Using (2.5), (3.27) and the estimate $\|\nabla e^{\tilde{t}\Delta}v_0\|_{L^\infty} \le c_1\|v_0\|_{W^{1,\infty}}$ for all $\tilde{t}>0$ (see [10]), from (3.32) we have

$$\begin{aligned}\|\nabla v(\cdot,\tilde{t})\|_{L^\infty} &\le \|\nabla e^{(\Delta-\frac{1}{D})\tilde{t}}v_0\|_{L^\infty} + \frac{1}{D}\int_0^{\tilde{t}}\|\nabla e^{(\Delta-\frac{1}{D})(\tilde{t}-s)}u(\cdot,s)\|_{L^\infty}ds\\ &\le c_1\|v_0\|_{W^{1,\infty}} + \frac{k_2}{D}\int_0^{\tilde{t}}\left(1+(\tilde{t}-s)^{-\frac{3}{4}}\right)e^{-\lambda_1(\tilde{t}-s)}\|u(\cdot,s)\|_{L^4}ds\\ &\le c_1\|v_0\|_{W^{1,\infty}} + \frac{k_2K_6}{D\lambda_1}\left(1+\Gamma(1/4)\lambda_1^{\frac{3}{4}}\right)(1+\alpha)^3\left(1+\frac{1}{D}\right)^2 e^{3K_5(1+\alpha)^6(1+\frac{1}{D})^4}\end{aligned}$$

which implies

$$\|\nabla v(\cdot,t)\|_{L^\infty} \le c_2(1+\alpha)^3\left(1+\frac{1}{D}\right)^3 e^{3K_5(1+\alpha)^6(1+\frac{1}{D})^4}, \tag{3.35}$$

where $c_2 := c_1\|v_0\|_{W^{1,\infty}} + \frac{k_2K_6}{\lambda_1}\left(1+\Gamma(1/4)\lambda_1^{\frac{3}{4}}\right)$. With (2.3) and (3.35), we multiply the first equation of (1.3) by $u^{p-1}(p\ge 2)$ and integrate the result to obtain

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p &= -(p-1)\int_\Omega u^{p-2}\nabla u\cdot\nabla(\gamma(v)u) + \alpha\int_\Omega F(w)u^p - \theta\int_\Omega u^p \\
&\le -(p-1)\int_\Omega \gamma(v)u^{p-2}|\nabla u|^2 - (p-1)\int_\Omega \gamma'(v)u^{p-1}\nabla u\cdot\nabla v + \alpha C_F\int_\Omega u^p \\
&\le -\gamma_1(p-1)\int_\Omega u^{p-2}|\nabla u|^2 + \eta(p-1)\int_\Omega u^{p-1}|\nabla u||\nabla v| + \alpha C_F\int_\Omega u^p \\
&\le -\frac{\gamma_1(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{\eta^2}{2\gamma_1}(p-1)\int_\Omega u^p|\nabla v|^2 + \alpha C_F(p-1)\int_\Omega u^p \\
&\le -\frac{\gamma_1(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \mathcal{K}_D(p-1)\int_\Omega u^p,
\end{aligned} \tag{3.36}$$

where $\mathcal{K}_D$ is independent of p and defined by

$$\mathcal{K}_D := \left(\frac{\eta^2c_2^2}{2\gamma_1} + C_F\right)(1+\alpha)^6\left(1+\frac{1}{D}\right)^6 e^{6K_5(1+\alpha)^6(1+\frac{1}{D})^4}.$$

Then using the identity $\int_\Omega u^{p-2}|\nabla u| = \frac{4}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|$, from (3.36) one has

$$\frac{d}{dt}\int_\Omega u^p + p(p-1)\int_\Omega u^p \le -\frac{2\gamma_1(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + (\mathcal{K}_D+1)p(p-1)\int_\Omega u^p. \tag{3.37}$$

Using the interpolation inequality and Young's inequality with ε, then for all $f\in W^{1,2}(\Omega)$, one has

$$\|f\|_{L^2}^2 \le \varepsilon\|\nabla f\|_{L^2}^2 + c_3(1+\varepsilon^{-1})\|f\|_{L^1}^2 \tag{3.38}$$

for any $\varepsilon>0$, where $c_3>0$ only depends on Ω. Then letting $f=u^{\frac{p}{2}}$ and $\varepsilon = \frac{2\gamma_1}{p^2(\mathcal{K}_D+1)}$ in (3.38), we can derive that

$$(\mathcal{K}_D+1)p(p-1)\int_\Omega u^p \le \frac{2\gamma_1(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \widetilde{\mathcal{K}}_D p(p-1)(1+p^2)\left(\int_\Omega u^{\frac{p}{2}}\right)^2, \tag{3.39}$$

where

$$\begin{aligned}\widetilde{\mathcal{K}}_D =& \frac{c_3(1+2\gamma_1)}{2\gamma_1}(\mathcal{K}_D+1)^2\\ =& \frac{c_3(1+2\gamma_1)}{2\gamma_1}\left(\frac{\eta^2 c_2^2}{2\gamma_1}+C_F+1\right)^2(1+\alpha)^{12}\left(1+\frac{1}{D}\right)^{12} e^{12K_5(1+\alpha)^6(1+\frac{1}{D})^4}.\end{aligned}$$

Substituting (3.39) into (3.37) and using the fact $1+p^2 \le (1+p)^2$, one has

$$\frac{d}{dt}\int_\Omega u^p + p(p-1)\int_\Omega u^p \le \widetilde{\mathcal{K}}_D p(p-1)(1+p)^2\left(\int_\Omega u^{\frac{p}{2}}\right)^2,$$

which gives

$$\int_\Omega u^p(x,t) \le \int_\Omega u_0^p(x) + \widetilde{\mathcal{K}}_D(1+p)^2 \sup_{0\le t\le T_{max}}\left(\int_\Omega u^{\frac{p}{2}}(x,t)\right)^2. \tag{3.40}$$

Then using the Moser iteration [2] (see also the similar argument as in [27,28]), from (3.40) one has

$$\|u(\cdot,t)\|_{L^\infty} \le 2^6\widetilde{\mathcal{K}}_D(1+|\Omega|)(1+\alpha)(\|u_0\|_{L^\infty}+\|w_0\|_{L^\infty}),$$

which gives (3.34). □

Proof of Theorem 1.1. For any fixed $D>0$ and $\alpha \ge 0$, from Lemma 3.5, we can find a constant $C>0$ independent of t such that

$$\|u(\cdot,t)\|_{L^\infty} \le C(1+\alpha)^{13}\left(1+\frac{1}{D}\right)^{12} e^{12K_5(1+\alpha)^6(1+\frac{1}{D})^4},$$

which combined with the local existence results in Lemma 2.1 proves Theorem 1.1. □

4. Asymptotic behavior (Proof of Theorem 1.2)

In this section, we will derive the asymptotic behavior of solutions as shown in Theorem 1.2. Before embarking on these details, we first use the standard parabolic property to improve the regularity of u, v and w as follows.

Lemma 4.1. *Let (u,v,w) be the nonnegative global classical solution of (1.3) obtained in Theorem 1.1. Then there exist $\sigma \in (0,1)$ and $C>0$ such that*

$$\|u(\cdot,t)\|_{C^{\sigma,\frac{\sigma}{2}}(\bar{\Omega}\times[t,t+1])} \le C \quad \text{for all} \quad t>1 \tag{4.1}$$

and

$$\|v(\cdot,t)\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\bar{\Omega}\times[t,t+1])}+\|w\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\bar{\Omega}\times[t,t+1])}\le C \quad \text{for all} \quad t>1. \tag{4.2}$$

Proof. Let $A(x,t,u,\nabla u)=\gamma(v)\nabla u+\gamma'(v)u\nabla v$ and $B(x,t,u)=\alpha F(w)u-\theta u$. Then we can rewrite the first equation of (1.3) as follows

$$u_t=\nabla\cdot A(x,t,u,\nabla u)+B(x,t,u).$$

Noting that Theorem 1.1 gives two positive constants c_1 and c_2 satisfying $\|u\|_{L^\infty}\le c_1$ and $\|v\|_{W^{1,\infty}}+\|w\|_{W^{1,\infty}}\le c_2$, we end up with

$$\begin{aligned}
A(x,t,u,\nabla u)\cdot\nabla u&=\big(\gamma(v)\nabla u+u\gamma'(v)\nabla v\big)\cdot\nabla u\\
&\le\gamma(v)|\nabla u|^2+\gamma'(v)u\nabla u\cdot\nabla v\\
&\le\frac{\gamma(v)}{2}|\nabla u|^2+\frac{(\gamma'(v))^2}{2\gamma(v)}u^2|\nabla v|^2\\
&\le\frac{\gamma_2}{2}|\nabla u|^2+\frac{c_1^2c_2^2\eta^2}{2\gamma_1}
\end{aligned}\tag{4.3}$$

and

$$\begin{aligned}
|A(x,t,u,\nabla u)|&=|\gamma(v)\nabla u+\gamma'(v)u\nabla v|\\
&\le|\gamma(v)||\nabla u|+|\gamma'(v)|\|u\|_{L^\infty}\|\nabla v\|_{L^\infty}\\
&\le\gamma_2|\nabla u|+c_1c_2\eta.
\end{aligned}\tag{4.4}$$

Moreover, since (2.3) guarantees $F(w)\le C_F$ and hence

$$\begin{aligned}
|B(x,t,u)|&=|\alpha F(w)u-\theta u|\\
&\le\alpha|F(w)|\|u\|_{L^\infty}+\theta\|u\|_{L^\infty}\\
&\le c_1(\alpha C_F+\theta).
\end{aligned}\tag{4.5}$$

With (4.3)–(4.5) in hand, we obtain (4.1) by applying [23, Theorem 1.3]. Furthermore, the standard parabolic regularity combined with (4.1) infers (4.2) directly. □

4.1. Case of $\theta>0$

In this subsection, we are devoted to studying the large time behavior of solutions for the case $\theta>0$. Notice that $\int_0^\infty\int_\Omega u<\infty$ and the relative compactness of $(u(\cdot,t))_{t>1}$ in $C(\Omega)$ (see Lemma 4.1) indicate some decay information for u and hence the decay properties of v from the second equation of (1.3). Precisely, we have the following results.

Lemma 4.2. *Let the conditions in Theorem 1.2 hold, and suppose* $\theta>0$ *and* (u,v,w) *is the solution of the system* (1.3). *Then it follows that*

$$\|u(\cdot,t)\|_{L^\infty}\to 0 \quad as \quad t\to\infty, \tag{4.6}$$

and

$$\|v(\cdot,t)\|_{L^\infty} \to 0 \quad as \quad t \to \infty. \tag{4.7}$$

Proof. First, we claim that

$$u(\cdot,t) \to 0 \ \text{ in } \ L^1(\Omega) \ \text{ as } \ t \to \infty. \tag{4.8}$$

Indeed, defining $A(t) := \int_\Omega u > 0$, we have $\int_0^\infty |A(t)| = \int_0^\infty \int_\Omega u < \infty$ from (2.1). Furthermore, from the first equation of (1.3) and the fact $w \in L^\infty(\Omega)$ (see Lemma 2.2), we can derive that

$$\int_0^\infty |A'(t)| = \int_0^\infty \left| \int_\Omega (\alpha F(w) - \theta)u \right| \le c_1 \int_0^\infty \int_\Omega u < \infty,$$

which together with the fact $\int_0^\infty |A(t)| = \int_0^\infty \int_\Omega u < \infty$ gives $A(t) \to 0$ as $t \to \infty$. This verifies the claim (4.8).

With (4.8) in hand, we shall show (4.6) holds. In fact, if (4.6) is false, we can find a constant $c_2 > 0$ and a time sequence $(t_k)_{k\in\mathbb{N}} \subset (1,\infty)$ satisfying $t_k \to \infty$ as $k \to \infty$ such that

$$\|u(\cdot,t_k)\|_{L^\infty} \ge c_2 \ \text{ for all } k \in \mathbb{N}. \tag{4.9}$$

On the other hand, using (4.1) in Lemma 4.1 and the Arzelà-Ascoli theorem, we know that $(u(\cdot,t))_{t>1}$ is relatively compact in $C(\Omega)$. Hence, we can extract a subsequence, still denoted by $(t_k)_{k\in\mathbb{N}} \subset (1,\infty)$, such that

$$u(\cdot,t_k) \to u_\infty \ \text{ in } \ L^\infty(\Omega) \ \text{ as } \ k \to \infty,$$

which combined with (4.8) implies $u_\infty \equiv 0$. This however contradicts (4.9) and hence (4.6) is proved.

Next, we show (4.7) holds. To this end, we consider the following system

$$\begin{cases} v_t + v = D\Delta v + u, & x \in \Omega, t > 0, \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{4.10}$$

Let $v^*(t)$ be solutions of the ODE problem

$$\begin{cases} v_t^*(t) + v^*(t) = \|u(\cdot,t)\|_{L^\infty}, & t > 0, \\ v^*(0) = \|v_0\|_{L^\infty}. \end{cases} \tag{4.11}$$

By the comparison principle, we know that $v^*(t)$ is a super-solution of (4.10) satisfying $v(x,t) \le v^*(t)$ for all $x \in \Omega$, $t > 0$. Similarly, we can prove that $v(x,t) \ge -v^*(t)$ for all $x \in \Omega$, $t > 0$. Hence, one has

$$|v(x,t)| \le v^*(t) \ \text{ for all } \ x \in \Omega, \ t > 0. \tag{4.12}$$

On the other hand, from (4.11) and using the fact $\|u(\cdot,t)\|_{L^\infty} \to 0$ as $t \to \infty$ we have

$$v^*(t) \to 0 \ \text{ as } \ t \to \infty,$$

which combined with (4.12) gives

$$\|v(\cdot,t)\|_{L^\infty} \le v^*(t) \to 0 \ \text{ as } \ t \to \infty.$$

This yields (4.7) and completes the proof of Lemma 4.2. □

Lemma 4.3. *Suppose the conditions in Lemma 4.2 hold. Let (u, v, w) be the solution of the system (1.3). Then we have the following result*

$$\|w(\cdot,t) - w_*\|_{L^\infty} \to 0 \quad as \ \ t \to \infty, \tag{4.13}$$

where $w_* > 0$ is a constant determined by $w_* = \frac{1}{|\Omega|}\|w_0\|_{L^1} - \frac{1}{|\Omega|}\int_0^\infty \int_\Omega uF(w)$.

Proof. Let $\bar{w}(t) = \frac{1}{|\Omega|}\int_\Omega w = \frac{1}{|\Omega|}\|w\|_{L^1}$, then the third equation of (1.3) can be rewritten as

$$(w - \bar{w})_t = \Delta(w - \bar{w}) - uF(w) + \overline{uF(w)}. \tag{4.14}$$

Then applying the variation-of-constants formula to (4.14), we get

$$w(\cdot,t) - \bar{w}(t) = e^{\frac{t}{2}\Delta}\big(w(\cdot,t/2) - \bar{w}(t/2)\big) - \int_{\frac{t}{2}}^{t} e^{(t-s)\Delta}\left(uF(w) - \overline{uF(w)}\right) ds,$$

which, together with the fact $\|w(\cdot,t)\|_{L^\infty} \le c_1$ and (2.4), gives

$$\begin{aligned}
&\|w(\cdot,t) - \bar{w}(t)\|_{L^\infty} \\
&\le \|e^{\frac{t}{2}\Delta}\big(w(\cdot,t/2)) - \bar{w}(t/2)\big)\|_{L^\infty} + \int_{\frac{t}{2}}^{t} \|e^{(t-s)\Delta}(uF(w) - \overline{uF(w)})\|_{L^\infty} ds \\
&\le k_1 e^{-\frac{\lambda_1 t}{2}}\|w(\cdot,t/2) - \bar{w}(t/2)\|_{L^\infty} + k_1 C_F \int_{\frac{t}{2}}^{t} e^{-(t-s)\lambda_1}\|u(\cdot,s)\|_{L^\infty} ds \\
&\le 2k_1 c_1 e^{-\frac{\lambda_1 t}{2}} + \frac{k_1 C_F}{\lambda_1} \sup_{\frac{t}{2} \le s \le t} \|u(\cdot,s)\|_{L^\infty}.
\end{aligned} \tag{4.15}$$

Then using the decay property of u in (4.6), from (4.15) one has

$$\lim_{t\to\infty} \|w(\cdot,t) - \bar{w}(t)\|_{L^\infty} = 0. \tag{4.16}$$

Next we define a number w_* by

$$w_* = \frac{1}{|\Omega|}\|w_0\|_{L^1} - \frac{1}{|\Omega|}\int_0^\infty\int_\Omega uF(w). \tag{4.17}$$

Then integrating the third equation of (1.3) over $\Omega \times (0,t)$, we see that

$$\bar{w}(t) = w_* + \frac{1}{|\Omega|}\int_t^\infty\int_\Omega uF(w),$$

which implies

$$\|\bar{w}(t) - w_*\|_{L^\infty} \le \frac{C_F}{|\Omega|}\int_t^\infty \|u(\cdot,s)\|_{L^1} ds \to 0 \text{ as } t \to \infty. \tag{4.18}$$

Then combining (4.16) and (4.18), one has

$$\|w(\cdot,t) - w_*\|_{L^\infty} \le \|w(\cdot,t) - \bar{w}(t)\|_{L^\infty} + \|\bar{w}(t) - w_*\|_{L^\infty} \to 0, \text{ as } t \to \infty,$$

which yields (4.13).

Next, we shall show $w_* > 0$. Noting $F(w) \in C^1([0,\infty))$ and $F(0) = 0$ and using the boundedness of u and w, we can find $\xi \in (0,w)$ and $\mathcal{K} > 0$ such that

$$\frac{uF(w)}{w} = \frac{F(w) - F(0)}{w}\cdot u = F'(\xi)u \le \|F'(\xi)\|_{L^\infty}\|u\|_{L^\infty} := \mathcal{K}.$$

Let $\tilde{w}(x,t)$ be the solution of the following system

$$\begin{cases} \tilde{w}_t - \Delta\tilde{w} = -\mathcal{K}\tilde{w}, & x \in \Omega, t > 0, \\ \frac{\partial\tilde{w}}{\partial\nu} = 0, & x \in \partial\Omega, t > 0, \\ \tilde{w}(x,0) = w_0(x), & x \in \Omega. \end{cases}$$

Clearly, $\tilde{w}(x,t)$ is a sub-solution of $w(x,t)$ by the comparison principle, and hence

$$w(x,t) \ge \tilde{w}(x,t). \tag{4.19}$$

On the other hand, using [11, Lemma 3.1], we can find a constant $\Gamma_0 > 0$ such that for all $t \ge 1$

$$\tilde{w}(x,t) = e^{-\mathcal{K}t}e^{\Delta t}w_0 \ge e^{-\mathcal{K}t}\Gamma_0\int_\Omega w_0,$$

which combined with (4.19) gives

$$w(x,t) \geq e^{-\mathcal{K}t}\Gamma_0 \int_\Omega w_0, \text{ for all } t \geq 1. \tag{4.20}$$

Multiplying the third equation of (1.3) by $\frac{1}{w}$, and integrating by parts with respect to $x \in \Omega$, one has

$$\frac{d}{dt}\int_\Omega \ln w(x,t) = \int_\Omega \frac{|\nabla w|^2}{w^2} - \int_\Omega \frac{F(w)}{w}u \geq -\int_\Omega \frac{F(w)}{w}u,$$

which thus gives

$$\int_\Omega \ln w(x,t) \geq \int_\Omega \ln w(x,1) - \int_1^t \int_\Omega \frac{F(w)}{w}u. \tag{4.21}$$

Then using (4.20) and the fact $\int_0^t \int_\Omega u \leq c_7$, from (4.21) we can find a constant $c_8 > 0$ such that

$$\int_\Omega \ln w(x,t) \geq -c_8, \text{ for all } t \geq 1,$$

which combined with the fact (4.13) implies $w_* > 0$. □

In summary, we have the asymptotic behavior of solutions for the system (1.3) with $\theta > 0$.

Proposition 4.4. *Let the conditions of Theorem 1.2 hold and $\theta > 0$, the solution of system (1.3) satisfies*

$$\lim_{t\to\infty}(\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{L^\infty} + \|w(\cdot,t) - w_*\|_{L^\infty}) = 0,$$

where $w_* > 0$ defined by (4.17).

4.2. Case of $\theta = 0$

In this subsection, we shall study the large time behavior of the system (1.3) with $\theta = 0$. We first show the decay of w based on some ideas in [34].

Lemma 4.5. *Assume the conditions in Theorem 1.2 hold. Let (u, v, w) be the solution of the system (1.3) with $\theta = 0$. Then we have*

$$\int_0^\infty \int_\Omega uF(w) < \infty \tag{4.22}$$

and

$$\int_0^\infty \int_\Omega |\nabla w|^2 < \infty. \tag{4.23}$$

Proof. Integrating the third equation of (1.3) over Ω and using the homogeneous Neumann boundary condition, one has

$$\frac{d}{dt}\int_\Omega w + \int_\Omega uF(w) = 0,$$

which gives

$$\int_0^t \int_\Omega uF(w) \le \int_\Omega w_0, \quad \text{for all } t > 0, \tag{4.24}$$

and (4.22) is a direct result of (4.24). We multiply the third equation of (1.3) by w to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2 = -\int_\Omega |\nabla w|^2 - \int_\Omega uwF(w). \tag{4.25}$$

Integrating (4.25) with respect to t and using the nonnegativity of u and w, one can derive

$$\int_0^t \int_\Omega |\nabla w|^2 \le \frac{1}{2}\int_\Omega w_0^2,$$

which gives (4.23). □

Lemma 4.6. *Let the conditions in Lemma 4.5 hold. Then there exists a time sequence $(t_k)_{k\in\mathbb{N}} \subset (0,\infty)$ satisfying $t_k \to \infty$ as $k \to \infty$ such that*

$$\int_{t_k}^{t_k+1} \int_\Omega w \to 0 \quad \text{as} \quad k \to \infty. \tag{4.26}$$

Proof. From (4.22), we have

$$\int_j^{j+1} \int_\Omega uF(w) \to 0 \quad \text{as} \quad j \to \infty. \tag{4.27}$$

Defining $\bar{F}(w) := \frac{1}{|\Omega|}\int_\Omega F(w)$, we have

$$\int_j^{j+1}\int_\Omega uF(w) = \int_j^{j+1}\int_\Omega u(F(w)-\bar{F}(w)) + \int_j^{j+1}\int_\Omega u\bar{F}(w) = I_1(j)+I_2(j). \tag{4.28}$$

Using the Hölder inequality, Poincaŕe inequality and the boundedness of $\|u(\cdot,t)\|_{L^2}$ in (3.21), one has

$$\begin{aligned}
\left|I_1(j)\right| &\le \left(\int_j^{j+1}\int_\Omega u^2\right)^{\frac{1}{2}} \cdot \left(\int_j^{j+1}\int_\Omega |F(w)-\bar{F}(w)|^2\right)^{\frac{1}{2}} \\
&\le c_1\left(\int_j^{j+1}\int_\Omega |F(w)-\bar{F}(w)|^2\right)^{\frac{1}{2}} \\
&\le c_1\left(c_2\int_j^{j+1}\int_\Omega |\nabla F(w)|^2\right)^{\frac{1}{2}} \\
&\le c_3\left(\int_j^{j+1}\int_\Omega |\nabla w|^2\right)^{\frac{1}{2}} \to 0, \quad \text{as} \ \ j\to\infty,
\end{aligned} \tag{4.29}$$

where we have used (4.23) to derive the convergence. Then combining (4.27), (4.28) and (4.29), one has $I_2(j)\to 0$ as $j\to\infty$.

On the other hand, using (2.1) and the fact $\|w(\cdot,t)\|_{L^1}\le\|w_0\|_{L^1}$, we have

$$\|u_0\|_{L^1}+\alpha\|w_0\|_{L^1} = \|u\|_{L^1}+\alpha\|w\|_{L^1} \le \|u\|_{L^1}+\alpha\|w_0\|_{L^1},$$

which implies

$$\|u_0\|_{L^1}\le\|u\|_{L^1}\le\|u_0\|_{L^1}+\alpha\|w_0\|_{L^1}. \tag{4.30}$$

Hence, using (4.30) and the fact $I_2(j)\to 0$ as $j\to\infty$, one has

$$\bar{u}_0\int_j^{j+1}\int_\Omega F(w) = \|u_0\|_{L^1}\int_j^{j+1}\bar{F}(w) \le I_2(j) = \int_j^{j+1}\int_\Omega u\bar{F}(w) \to 0, \ \ \text{as} \ \ j\to\infty,$$

which implies

$$\int_j^{j+1}\int_\Omega F(w)\to 0, \qquad \text{as} \ \ j\to\infty. \tag{4.31}$$

Next, we will show that (4.31) implies (4.26). In fact, if we define $w_j(x,s) := w(x, j + s)$, $(x,s) \in \Omega \times (0,1)$, $j \in \mathbb{N}$, then (4.31) implies

$$\int_0^1 \int_\Omega F(w_j(x,s)) \to 0, \qquad \text{as } \ j \to \infty.$$

Hence we can extract a subsequence $(j_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $j_k \to \infty$ and $F(w_{j_k}) \to 0$ almost everywhere in $\Omega \times (0,1)$ as $k \to \infty$. Because the function F is positive on $(0,\infty)$ and $F(0) = 0$, which requires that $w_{j_k} \to 0$ almost everywhere in $\Omega \times (0,1)$ as $k \to \infty$. Moreover, since $\|w(\cdot,t)\|_{L^\infty} \le c_4$ for all $t > 0$, then the sequence $(w_{j_k})_{k\in\mathbb{N}} \to 0$ in $L^1(\Omega \times (0,1))$ as $k \to \infty$. Choosing $t_k := j_k$, one has (4.26). Then the proof of this lemma is completed. □

Lemma 4.7. *Suppose the conditions in Lemma 4.5 hold. Let (u, v, w) be the solution of the system (1.3) with $\theta = 0$. Then it holds that*

$$\|w(\cdot,t)\|_{L^\infty} \to 0 \quad \text{as} \quad t \to \infty. \tag{4.32}$$

Proof. Letting $(t_k)_{k\in\mathbb{N}} \subset (0,\infty)$ be the sequence chosen in Lemma 4.6. Using the Gagliardo-Nirenberg inequality, one can find a constant $c_1 > 0$ such that

$$\begin{aligned}\|w(\cdot,t)\|_{L^\infty} &\le c_1 \|\nabla w(\cdot,t)\|_{L^4}^{\frac{4}{5}} \|w(\cdot,t)\|_{L^1}^{\frac{1}{5}} + c_1 \|w(\cdot,t)\|_{L^1} \\ &\le \mu \|\nabla w(\cdot,t)\|_{L^4} + c_2 \|w(\cdot,t)\|_{L^1},\end{aligned} \tag{4.33}$$

where $\mu > 0$ is an arbitrary constant, $c_2 > 0$ is a constant depending on μ. Noting the uniform boundedness of $\|\nabla w(\cdot,t)\|_{L^4}$ and the arbitrary of μ, and using (4.26), from (4.33) we get

$$\int_{t_k}^{t_k+1} \|w(\cdot,t)\|_{L^\infty} \to 0, \qquad \text{as } \ k \to \infty,$$

which implies

$$\liminf_{t\to\infty} \|w(\cdot,t)\|_{L^\infty} = 0. \tag{4.34}$$

The combination of (4.34) and the fact that $t \to \|w(\cdot,t)\|_{L^\infty}$ is monotone as shown in Lemma 2.2, one obtains (4.32) and completes the proof of Lemma 4.7. □

Lemma 4.8. *Let (u, v, w) be the solution of the system (1.3) with $\theta = 0$. Then it follows that*

$$\frac{d}{dt} \int_\Omega (u - u_*)^2 + \gamma_1 \int_\Omega |\nabla u|^2 \le \int_\Omega \frac{|\gamma'(v)|^2}{\gamma(v)} u^2 |\nabla v|^2 + \alpha M^2 \int_\Omega F(w), \tag{4.35}$$

where $M > 0$ is defined by (1.4) and $u_* = \bar{u}_0 + \alpha \bar{w}_0$.

Proof. We rewrite the first equation of the system (1.3) as

$$(u-u_*)_t = \Delta(\gamma(v)u) + \alpha u F(w). \tag{4.36}$$

Then multiplying (4.36) by $u-u_*$, and integrating it by parts, we end up with

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega (u-u_*)^2 + \int_\Omega \gamma(v)|\nabla u|^2 \\
&= -\int_\Omega \gamma'(v) u \nabla u \cdot \nabla v + \alpha \int_\Omega u F(w)(u-u_*) \\
&\le \frac{1}{2}\int_\Omega \gamma(v)|\nabla u|^2 + \frac{1}{2}\int_\Omega \frac{|\gamma'(v)|^2}{\gamma(v)} u^2 |\nabla v|^2 + \alpha \int_\Omega u^2 F(w),
\end{aligned}$$

which, together with the facts $\|u\|_{L^\infty} \le M$ and $\gamma(v) \ge \gamma_1$, gives

$$\frac{d}{dt}\int_\Omega (u-u_*)^2 + \gamma_1 \int_\Omega |\nabla u|^2 \le \int_\Omega \frac{|\gamma'(v)|^2}{\gamma(v)} u^2 |\nabla v|^2 + \alpha M^2 \int_\Omega F(w),$$

and hence (4.35) follows. □

Lemma 4.9. *The solution (u, v, w) of the system (1.3) with $\theta = 0$ satisfies*

$$\frac{d}{dt}\int_\Omega (v-u_*)^2 + 2D\int_\Omega |\nabla v|^2 + \int_\Omega (v-u_*)^2 \le \int_\Omega (u-u_*)^2. \tag{4.37}$$

Proof. Multiplying the second equation of the system (1.3) by $v-u_*$, and integrating by parts, we end up with

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega (v-u_*)^2 &= \int_\Omega (v-u_*)[D\Delta v + (u-u_*) - (v-u_*)] \\
&= -D\int_\Omega |\nabla v|^2 - \int_\Omega (v-u_*)^2 + \int_\Omega (v-u_*)(u-u_*) \\
&\le -D\int_\Omega |\nabla v|^2 - \frac{1}{2}\int_\Omega (v-u_*)^2 + \frac{1}{2}\int_\Omega (u-u_*)^2,
\end{aligned}$$

which yields (4.37). □

Lemma 4.10. *Let (u, v, w) be the solution of the system (1.3) with $\theta = 0$. Then there exists a positive constant D_1 such that if $D \ge D_1$, it holds that*

$$\lim_{t\to\infty} (\|u(\cdot,t) - u_*\|_{L^\infty} + \|v(\cdot,t) - u_*\|_{L^\infty}) = 0. \tag{4.38}$$

Proof. Applying the Poincaré inequality, we find a constant $C_p > 0$ such that

$$\int_\Omega (u-\bar{u})^2 \le C_p \int_\Omega |\nabla u|^2. \tag{4.39}$$

Then using the definition of u_*, we know from (2.1) that $u_* = \bar{u} + \alpha\bar{w}$. Then it follows from (4.39) that

$$\begin{aligned}\int_\Omega (u-u_*)^2 &\le 2\int_\Omega (u-\bar{u})^2 + 2\alpha^2 \int_\Omega \bar{w}^2 \\ &\le 2C_p \int_\Omega |\nabla u|^2 + \frac{2\alpha^2}{|\Omega|}\left(\int_\Omega w\right)^2,\end{aligned}$$

which implies

$$\frac{\gamma_1}{2C_p}\int_\Omega (u-u_*)^2 \le \gamma_1 \int_\Omega |\nabla u|^2 + \frac{\alpha^2\gamma_1}{C_p|\Omega|}\left(\int_\Omega w\right)^2. \tag{4.40}$$

Applying (4.40) into (4.35), and using the facts $\|u(\cdot,t)\|_{L^\infty} \le M$, $0<\gamma_1 \le \gamma(v)$ and $|\gamma'(v)| \le \eta$, we can derive that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (u-u_*)^2 + \frac{\gamma_1}{2C_p}\int_\Omega (u-u_*)^2 \\ &\le \int_\Omega \frac{|\gamma'(v)|^2}{\gamma(v)} u^2 |\nabla v|^2 + \alpha M^2 \int_\Omega F(w) + \frac{\alpha^2\gamma_1}{C_p|\Omega|}\left(\int_\Omega w\right)^2 \\ &\le \frac{\eta^2 M^2}{\gamma_1}\int_\Omega |\nabla v|^2 + \alpha M^2 \int_\Omega F(w) + \frac{\alpha^2\gamma_1}{C_p|\Omega|}\left(\int_\Omega w\right)^2.\end{aligned} \tag{4.41}$$

On the other hand, we multiply (4.37) by $\frac{\gamma_1}{4c_p}$, and use (4.41) to have

$$\begin{aligned}&\frac{d}{dt}\left(\int_\Omega (u-u_*)^2 + \frac{\gamma_1}{4C_p}\int_\Omega (v-u_*)^2\right) + \frac{\gamma_1}{4C_p}\int_\Omega (u-u_*)^2 + \frac{\gamma_1}{4C_p}\int_\Omega (v-u_*)^2 \\ &\le \left(\frac{\eta^2 M^2}{\gamma_1} - \frac{D\gamma_1}{2C_p}\right)\int_\Omega |\nabla v|^2 + \alpha M^2 \int_\Omega F(w) + \frac{\alpha^2\gamma_1}{C_p|\Omega|}\left(\int_\Omega w\right)^2.\end{aligned} \tag{4.42}$$

Using the definition of M in (1.4), one can find two constants $C_1, C_2 > 0$ independent of D such that

$$M := C_1(1+\alpha)^{13}\left(1+\frac{1}{D}\right)^{12} e^{C_2(1+\frac{1}{D})^4(1+\alpha)^6}.$$

Let D_* be the positive constant uniquely determined by the following identity

$$D_* = \frac{2\eta^2 C_p C_1^2}{\gamma_1^2}(1+\alpha)^{26}\left(1+\frac{1}{D_*}\right)^{24} e^{2C_2(1+\frac{1}{D_*})^4(1+\alpha)^6},$$

where C_p, γ_1 and η are independent of D_*. Then if $D \geq D_*$, one has $\frac{\eta^2 M^2}{\gamma_1} - \frac{D\gamma_1}{2C_p} \leq 0$, and hence the estimate (4.42) becomes

$$\begin{aligned} &\frac{d}{dt}\left(\int_\Omega (u-u_*)^2 + \frac{\gamma_1}{4C_p}\int_\Omega (v-u_*)^2\right) + \frac{\gamma_1}{4C_p}\int_\Omega (u-u_*)^2 + \frac{\gamma_1}{4C_p}\int_\Omega (v-u_*)^2 \\ &\leq \alpha M^2 \int_\Omega F(w) + \frac{\alpha^2\gamma_1}{C_p|\Omega|}\left(\int_\Omega w\right)^2. \end{aligned} \tag{4.43}$$

Define $Z(t) := \int_\Omega (u-u_*)^2 + \frac{\gamma_1}{4C_p}\int_\Omega (v-u_*)^2$ and $G(t) := \alpha M^2 \int_\Omega F(w) + \frac{\alpha^2\gamma_1}{C_p|\Omega|}\left(\int_\Omega w\right)^2$. Choosing $c_1 := \min\{1, \frac{\gamma_1}{4C_p}\}$, we have from (4.43)

$$Z'(t) + c_1 Z(t) \leq G(t). \tag{4.44}$$

Since $\|w(\cdot,t)\|_{L^\infty} \to 0$ as $t \to \infty$ (see Lemma 4.7), one has $G(t) \to 0$ as $t \to \infty$. Then from (4.44), we can derive that

$$Z(t) \to 0 \ \text{ as } \ t \to \infty.$$

This implies

$$\|u(\cdot,t) - u_*\|_{L^2} + \|v(\cdot,t) - u_*\|_{L^2} \to 0 \ \text{ as } \ t \to \infty. \tag{4.45}$$

Using the similar arguments as in Lemma 4.2 with (4.45), we obtain (4.38) directly. Then the proof of Lemma 4.10 is completed. □

In summary, we have the following asymptotic results for the case $\theta = 0$.

Proposition 4.11. *Suppose the conditions in Theorem 1.1 hold. Let (u, v, w) be the solution of the system (1.3) with $\theta = 0$. Then there exists constant $D_1 > 0$ such that if $D \geq D_1$, it has*

$$\lim_{t\to\infty}\left(\|u(\cdot,t) - u_*\|_{L^\infty} + \|v(\cdot,t) - u_*\|_{L^\infty} + \|w(\cdot,t)\|_{L^\infty}\right) = 0,$$

where $u_* = \frac{1}{|\Omega|}(\|u_0\|_{L^1} + \alpha\|w_0\|_{L^1})$.

Proof of Theorem 1.2. The proof of Theorem 1.2 is a direct consequence of Proposition 4.4 and Proposition 4.11. □

5. Simulations and discussions

5.1. Linear instability analysis

The results of Theorem 1.2 imply that the system (1.3) has no pattern formation if $\theta > 0$ or $\theta = 0$ and D is large. In this section, we will study the possible pattern arising from the system (1.3) with $\theta = 0$ and small D. To this end, we note that (1.3) with $\theta = 0$ has three constant equilibria $(0,0,0)$, $(0,0,\frac{u_*}{\alpha})$ and $(u_*, u_*, 0)$ for given initial value (u_0, v_0, w_0), where $u_* = \bar{u}_0 + \alpha\bar{w}_0$. We first consider the system (1.3) with $\theta = 0$ in the absence of spatial components, that is

$$\begin{cases} u_t = \alpha u F(w), \\ v_t = u - v, \\ w_t = -uF(w). \end{cases}$$

The linear stability/instability of each equilibrium is determined by the sign of the eigenvalues (ρ_1, ρ_2, ρ_3) defined by

$$(\rho_1,\rho_2,\rho_3) = \begin{cases} (0,-1,0), & \text{at } (0,0,0), \\ (0,-1,\alpha F(\frac{u_*}{\alpha})), & \text{at } (0,0,\frac{u_*}{\alpha}), \\ (0,-1,-u_*F'(0)), & \text{at } (u_*,u_*,0). \end{cases}$$

Since $F(\frac{u_*}{\alpha}) > 0$ and $F'(0) > 0$, we know the non-trivial steady state $(0,0,\frac{u_*}{\alpha})$ is linearly unstable, while $(0,0,0)$ and $(u_*,u_*,0)$ are linearly stable. Hence we study the possible patterns bifurcating from the constant equilibria $(u_c, u_c, 0)$ where $u_c = 0$ or $u_c = u_*$. To this end, we linearize the system (1.3) at the equilibrium $(u_c, u_c, 0)$ to obtain

$$\begin{cases} \Phi_t = A_1\Delta\Phi + B_1\Phi, & x \in \Omega,\ t > 0, \\ (\nu\cdot\nabla)\Phi = 0, & x \in \partial\Omega,\ t > 0, \\ \Phi(x,0) = (u_0 - u_c, v_0 - u_c, w_0)^{\mathcal{T}}, & x \in \Omega, \end{cases} \tag{5.1}$$

where $\mathcal{T}$ denotes the transpose and

$$\Phi = \begin{pmatrix} u - u_c \\ v - u_c \\ w \end{pmatrix}, \quad A_1 = \begin{pmatrix} \gamma(u_c) & \gamma'(u_c)u_c & 0 \\ 0 & D & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

as well as

$$B_1 = \begin{pmatrix} 0 & 0 & \alpha u_c F'(0) \\ 1 & -1 & 0 \\ 0 & 0 & -u_c F'(0) \end{pmatrix}$$

Noting that the linear system (5.1) has the solution of the form

$$\Phi(x,t) = \sum_{k\ge 0} c_k e^{\rho t} W_k(x), \tag{5.2}$$

where $W_k(x)$ denotes the eigenfunction of the following eigenvalue problem:

$$\Delta W_k(x) + k^2 W_k(x) = 0, \quad \frac{\partial W_k(x)}{\partial \nu} = 0,$$

and the constants c_k are determined by the Fourier expansion of the initial conditions in terms of $W_k(x)$ and ρ is the temporal eigenvalue. After some calculations, we know ρ is the eigenvalue of the following matrix

$$M_k = \begin{pmatrix} -\gamma(u_c)k^2 & -u_c\gamma'(u_c)k^2 & \alpha u_c F'(0) \\ 1 & -Dk^2 - 1 & 0 \\ 0 & 0 & -k^2 - u_c F'(0) \end{pmatrix}.$$

Obviously, $\rho(k^2) = -k^2 - u_u F'(0)$ is an eigenvalue, which is negative for all $k \neq 0$. Hence to get the possible pattern formation, we only need to consider the other two eigenvalues of the matrix M_k, which satisfy

$$\rho^2 + a_1(k^2)\rho + a_0(k^2) = 0,$$

where

$$\begin{cases} a_1(k^2) = 1 + (D + \gamma(u_c))k^2 > 0, \\ a_0(k^2) = D\gamma(u_c)k^4 + \left(\gamma(u_c) + u_c\gamma'(u_c)\right)k^2. \end{cases}$$

One can check that if $\gamma(u_c) + u_c\gamma'(u_c) \geq 0$ which is the case for $u_c = 0$, then $a_0(k^2) > 0$ for all $k \neq 0$, which implies the real part of the eigenvalues $\rho(k^2)$ are negative, and hence the steady state $(0, 0, 0)$ is linearly stable and no patterns will bifurcate from $(0, 0, 0)$. Next we consider the equilibrium $(u_*, u_*, 0)$. If $\gamma(u_*) + u_*\gamma'(u_*) < 0$, the real part of the eigenvalues $\rho(k^2)$ can be positive and hence the pattern formation may occur provided that the admissible wavenumber k satisfies

$$0 < k^2 < -\frac{\gamma(u_*) + u_*\gamma'(u_*)}{D\gamma(u_*)} =: \bar{k}. \tag{5.3}$$

Note the allowable wave numbers k are discrete in a bounded domain, for instance if $\Omega = (0, l)$ then $k = \frac{n\pi}{l}$ for $n = 1, 2, \cdots$. Hence the condition (5.3) is only necessary because the interval $(0, \bar{k})$ may not contain any desired discrete number k^2, for instance when $D > 0$ is sufficiently large. Hence we have the following conclusion.

Lemma 5.1. *Suppose $\gamma(v)$ satisfies the assumptions (H1). Then the homogeneous steady state $(u_*, u_*, 0)$ of the system (1.3) is linearly unstable if and only if $\gamma(u_*) + u_*\gamma'(u_*) < 0$ and there is at least an allowable wavenumber k satisfying condition (5.3).*

5.2. Simulations and questions

In section 5.1, we identify the instability parameter regimes for the possible pattern formation. But this linear instability result is not sufficient to conclude that there are non-constant stationary (pattern) solutions. Now we want to numerically test in one dimension whether non-constant stationary patterns exist for $\gamma(v)$ satisfying the conditions in Lemma 5.1. For definiteness in the simulation, we assume $\Omega = (0, l)$ and consider

$$\gamma(v) = \gamma_1 + \gamma_0 e^{-\lambda v}, \quad F(w) = \frac{w^2}{1+w^2}$$

where γ_0, γ_1 and λ are positive constants. Then the condition $\gamma(u_*) + u_*\gamma'(u_*) = \gamma_1 + \gamma_0 e^{-\lambda u_*}(1 - \lambda u_*) < 0$ in Lemma 5.1 amounts to

$$u_* > \frac{1}{\lambda} \text{ and } \frac{\gamma_1}{\gamma_0} e^{\lambda u_*} < \lambda u_* - 1. \tag{5.4}$$

Since $k = \frac{\pi}{l}$, then the condition (5.3) becomes

$$D < -\frac{\gamma(u_*) + u_*\gamma'(u_*)}{\gamma(u_*)} \cdot \frac{l^2}{(n\pi)^2} = \frac{\gamma_1 + \gamma_0 e^{-\lambda u_*}(1 - \lambda u_*)}{\gamma_1 + \gamma_0 e^{-\lambda u_*}} \cdot \frac{l^2}{(n\pi)^2}. \tag{5.5}$$

Therefore if we choose appropriate values of $\gamma_0, \gamma_1, \lambda, u_*$ and l so that the conditions (5.4)-(5.5) hold for some positive integer n, the pattern formation is expected from the results of Lemma 5.1. Note that $u_* = \bar{u}_0 + \alpha \bar{w}_0$. Hence for numerical simulations, we choose the initial value (u_0, v_0, w_0) as a small random perturbation of the equilibrium $(u_*, v_*, 0)$, and fix $\lambda = \alpha = 1$. The system (1.3) is numerically solved by the MATLAB PDEPE solver. We choose $l = 20$, $(u_*, v_*, 0) = (4, 4, 0)$, $\gamma_0 = 10$, $\gamma_1 = 0.1$ and show the numerical simulations for $D = 0.1$ and $D = 0.01$ in Fig. 1 where we do observe the aggregated stationary patterns. This indicates for suitably small $D > 0$, the system (1.3) with appropriate motility function $\gamma(v)$ admits the pattern formation, which complements the analytical results of Theorem 1.2. However the rigorous proof the existence of pattern (stationary) solutions leaves open in this paper and we shall investigate this question in the future. Note that the assumption (H1) rules out the possible degeneracy of motility function $\gamma(v)$, which plays a key role in proving the results of this paper. Therefore another interesting open question is the global dynamics of (1.3) without assuming that $\gamma(v)$ has a positive lower bound such as $\gamma(v) = (1 + v)^{-\lambda}$ or $\gamma(v) = e^{-\lambda v}$ with $\lambda > 0$. Such motility function $\gamma(v)$ without positive lower bound has been used to study the global boundedness/asymptotics of solutions and stationary solutions for the two-component density-suppressed motility model (1.2) in [12,19], where the quadratic decay $-\mu u^2$ plays an essential role. However the three-component system (1.3) does not have such nice decay term and hence novel ideas are anticipated to solve the above-mentioned open question.

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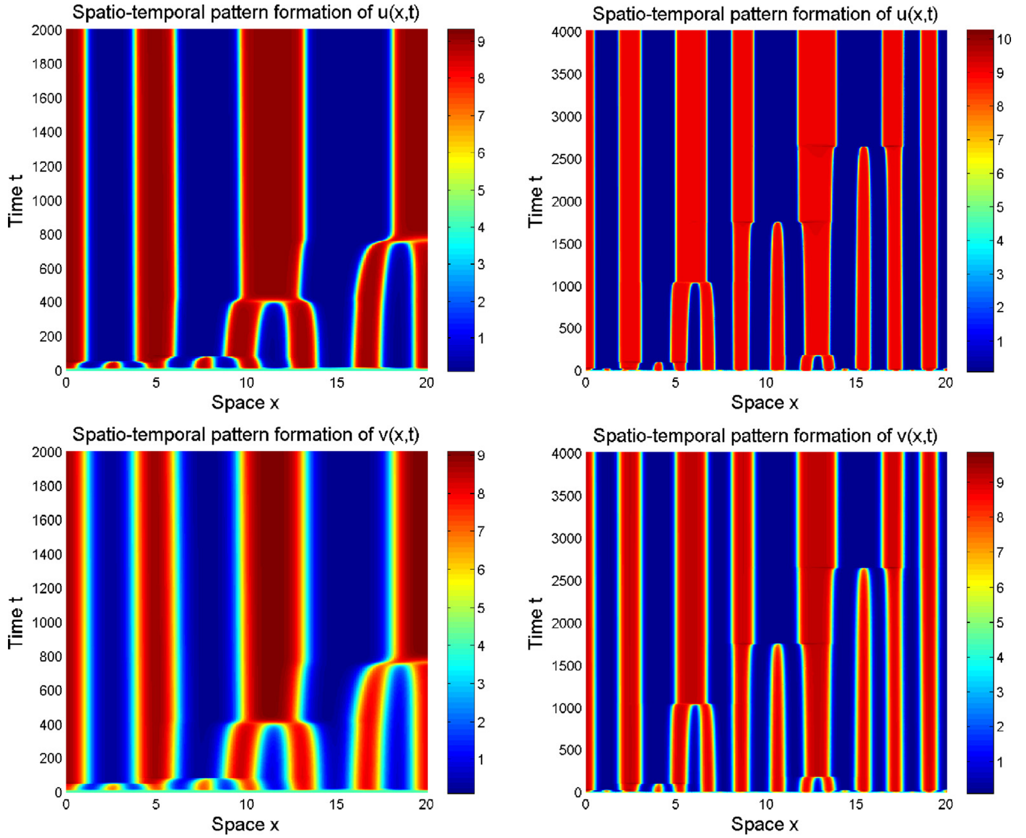

$D = 0.1$ $D = 0.01$

Fig. 1. Numerical simulations of the pattern formation to the system (1.3) in $[0, 20]$ with $\gamma(v) = 0.1 + 10e^{-v}$, $F(w) = \frac{w^2}{1+w^2}$, where $\lambda = \alpha = 1$ and (u_0, v_0, w_0) is set as a small random perturbation of the constant steady state $(4, 4, 0)$. There is no pattern formation for w and hence the numerical simulation of w is not shown here. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

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