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Journal of Differential Equations

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Asymptotic nonlinear stability of traveling waves to conservation laws arising from chemotaxis

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ARTICLE INFO

Article history:

Received 18 April 2010

Revised 23 August 2010

MSC:

35L45

35L60

92B05

92B99

92C15

92C17

Keywords:

Keller–Segel model

Chemotaxis

Nonlinear stability

Traveling waves

Conservation law

Large amplitude

Energy estimates

ABSTRACT

In this paper, we establish the existence and the nonlinear stability of traveling wave solutions to a system of conservation laws which is transformed, by a change of variable, from the well-known Keller–Segel model describing cell (bacteria) movement toward the concentration gradient of the chemical that is consumed by the cells. We prove the existence of traveling fronts by the phase plane analysis and show the asymptotic nonlinear stability of traveling wave solutions without the smallness assumption on the wave strengths by the method of energy estimates.

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1. Introduction

This paper is concerned with the existence and the asymptotic nonlinear stability of traveling wave solutions to a system of conservation laws

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$$\begin{cases} u_t - (uv)_x = Du_{xx}, \\ v_t + (\varepsilon v^2 - u)_x = \varepsilon v_{xx} \end{cases} \quad (1.1)$$

for $x \in \mathbb{R}$ and $t \geq 0$ with the initial data

$$(u, v)(x, 0) = (u_0, v_0)(x) \rightarrow \begin{cases} (u_-, v_-) & \text{as } x \rightarrow -\infty, \\ (u_+, v_+) & \text{as } x \rightarrow +\infty. \end{cases} \quad (1.2)$$

The conservation laws (1.1) are derived from the original well-known Keller–Segel model

$$\begin{cases} u_t = (Du_x - \chi uc^{-1}c_x)_x, \\ c_t = \varepsilon c_{xx} - uf(c) \end{cases} \quad (1.3)$$

which was proposed by Keller and Segel [7] to describe the traveling band behavior of bacteria due to the chemotactic response (i.e. the oriented movement of cells to the chemical concentration gradient) observed in experiments [1,2]. In model (1.3), $u(x, t)$ denotes the cell (i.e. bacteria) density and $c(x, t)$ denotes the chemical (i.e. oxygen) concentration, $D > 0$ and $\varepsilon > 0$ denote the diffusion coefficients of cells (bacteria) and the chemical, respectively. χ is a positive constant often referred to as chemosensitivity. $f(c)$ is a kinetic function describing the chemical reaction between cells and the chemical.

When $f(c)$ is a positive constant, namely, $f(c) = \alpha > 0$, the existence of traveling wave solutions of (1.3) with $\varepsilon = 0$ was established by Keller and Segel themselves [7]. When $\varepsilon \neq 0$, the existence and linear instability of traveling wave solutions of (1.3) were shown by Nagai and Ikeda [18] where the authors also obtained the diffusion limits of traveling wave solutions of (1.3) as ε approaches zero.

In this paper, we consider the case where $f(c) = \alpha c (\alpha > 0)$, which means that oxygen is consumed only when cells (bacteria) encounter the chemical (oxygen). We derive the system of conservation laws (1.1) from (1.3). The crucial step in the derivation is to make a change of variable through

$$v = -c^{-1}c_x = -(\ln c)_x \quad (1.4)$$

which was first introduced in [21] for a chemotaxis model proposed in [10] describing the chemotactic movement for non-diffusible chemicals (i.e. $\varepsilon = 0$), and was later applied in [12,13] to study the nonlinear stability of traveling wave solutions. It turns out this transformation also extends its capacity to the full Keller–Segel model (1.3) for $\varepsilon \neq 0$. Indeed, with the magic transformation (1.4), we derive the following viscous conservation laws from (1.3)

$$\begin{cases} u_t - \chi (uv)_x = Du_{xx}, \\ v_t + (\varepsilon v^2 - \alpha u)_x = \varepsilon v_{xx}. \end{cases} \quad (1.5)$$

Substituting the scalings

$$\tilde{t} = \alpha t, \quad \tilde{x} = \sqrt{\frac{\alpha}{\chi}}x, \quad \tilde{v} = \sqrt{\frac{\chi}{\alpha}}v, \quad \tilde{D} = \frac{D}{\chi}, \quad \tilde{\varepsilon} = \frac{\varepsilon}{\chi}$$

into (1.5) and dropping the tildes for convenience, we obtain the system of conservation laws (1.1).

In the present paper, we restrict our attention to the transformed system (1.1) instead of the original Keller–Segel model (1.3). It turns out that translating the results of the transformed system (1.1) back to the original Keller–Segel model (1.3) produces many interesting consequences, which will be presented in a separate paper.

Since $u(x, t)$ represents cell density, we assume that $u \geq 0$ and hence $u_{\pm} \geq 0$. By the same reason we suppose $c(x, t) \geq 0$. Because the Keller–Segel model (1.3) describes the directed movement of cells toward the chemical which is consumed by cells when they encounter, the wave is an “invasion”

pattern. That is, the wave profile of u decreases from its tail to front and that of c increases from its tail to the front, which requires $c_x > 0$. Due to the transformation (1.4) and $c(x, t) \geq 0$, we have $v(x, t) \leq 0$. Therefore we consider the solution (u, v) of system (1.1), (1.2) in the following physical region

$$\mathcal{X} = \{(u, v) \mid u \geq 0, v \leq 0, u_{\pm} \geq 0, v_{\pm} \leq 0\}. \tag{1.6}$$

In the region \mathcal{X} , we shall prove the existence of traveling wave solutions to the system of conservation laws (1.1) and that the solution to the Cauchy problem (1.1), (1.2), where the initial data is certain small perturbation of a traveling wave, approaches a shifted traveling wave as $t \rightarrow \infty$ without the smallness constraints on the wave strength. It is worthwhile to remark that the small wave strength is generally an assumption imposed in most of the studies for the stability of traveling waves for nonlinear conservation laws (e.g. see [11,14]).

The rest of this paper is organized as follows. We state the main results of our paper in Section 2, which consist in two parts: the existence and the stability of traveling wave solutions of system (1.1). In Section 3, we show the existence of traveling wave solutions for (1.1). We prove the nonlinear stability of the traveling wave solutions based on *a priori estimates* in Section 4. Finally, we give a brief summary in Section 5.

2. Statement of main results

In this section, we shall provide some preliminary analysis and then state the main results of this paper.

2.1. Existence of traveling waves

We first show that the system (1.1) without viscosity is a genuinely nonlinear hyperbolic system under some conditions for ε . In the absence of the viscous terms, (1.1) becomes

$$\begin{cases} u_t - (uv)_x = 0, \\ v_t + (\varepsilon v^2 - u)_x = 0. \end{cases} \tag{2.1}$$

The Jacobian matrix of (2.1) is

$$J(u, v) = \begin{bmatrix} -v & -u \\ -1 & 2\varepsilon v \end{bmatrix}$$

and its eigenvalues satisfy

$$\lambda^2 + (v - 2\varepsilon v)\lambda - 2\varepsilon v^2 - u = 0 \tag{2.2}$$

which has two real roots

$$\begin{aligned} \lambda_1(u, v) &= \frac{(2\varepsilon - 1)v}{2} - \frac{\sqrt{[(2\varepsilon + 1)v]^2 + 4u}}{2}, \\ \lambda_2(u, v) &= \frac{(2\varepsilon - 1)v}{2} + \frac{\sqrt{[(2\varepsilon + 1)v]^2 + 4u}}{2} \end{aligned} \tag{2.3}$$

with respective eigenvectors

$$\mathbf{r}_1(u, v) = \begin{bmatrix} -\lambda_1 + 2\varepsilon v \\ 1 \end{bmatrix}, \quad \mathbf{r}_2(u, v) = \begin{bmatrix} \lambda_2 - 2\varepsilon v \\ -1 \end{bmatrix}.$$

Since $u \geq 0$, hence $\lambda_1 < 0 < \lambda_2$ and system (2.1) is hyperbolic. Furthermore, direct calculations yields

$$\begin{aligned} \nabla\lambda_1(u, v) \cdot \mathbf{r}_1(u, v) &= -1 - \frac{v}{\sqrt{[(2\varepsilon + 1)v]^2 + 4u}} + \varepsilon - \frac{\varepsilon(2\varepsilon + 3)v}{\sqrt{[(2\varepsilon + 1)v]^2 + 4u}}, \\ \nabla\lambda_2(u, v) \cdot \mathbf{r}_2(u, v) &= 1 - \frac{v}{\sqrt{[(2\varepsilon + 1)v]^2 + 4u}} - \varepsilon - \frac{\varepsilon(2\varepsilon + 3)v}{\sqrt{[(2\varepsilon + 1)v]^2 + 4u}}. \end{aligned}$$

Noticing that $-1 - \frac{v}{\sqrt{[(2\varepsilon + 1)v]^2 + 4u}} < 0$ and $1 - \frac{v}{\sqrt{[(2\varepsilon + 1)v]^2 + 4u}} > 0$ due to $u \geq 0, v \leq 0$ and $\varepsilon > 0$, one shows that $\nabla\lambda_1(u, v) \cdot \mathbf{r}_1(u, v) < 0$ if

$$0 < \varepsilon < \frac{1 + \frac{v}{\sqrt{[(2\varepsilon + 1)v]^2 + 4u}}}{1 - \frac{(2\varepsilon + 3)v}{\sqrt{[(2\varepsilon + 1)v]^2 + 4u}}} \leq 1$$

and $\nabla\lambda_2(u, v) \cdot \mathbf{r}_2(u, v) > 0$ if $\varepsilon < 1$. We require

$$0 < \varepsilon < 1 \tag{2.4}$$

to ensure that the hyperbolic system (2.1) is genuinely nonlinear.

Now we define the traveling wave ansatz

$$(u, v)(x, t) = (U, V)(z), \quad z = x - st$$

where s denotes the wave speed and z is the traveling wave variable. Substituting the above ansatz into (1.1), one obtains the traveling wave equations

$$\begin{cases} -sU_z - (UV)_z = DU_{zz}, \\ -sV_z + (\varepsilon V^2 - U)_z = \varepsilon V_{zz} \end{cases} \tag{2.5}$$

with boundary conditions

$$(U, V)(z) \rightarrow (u_{\pm}, v_{\pm}) \quad \text{as } z \rightarrow \pm\infty \tag{2.6}$$

where $u_{\pm} \geq 0$ and $v_{\pm} \leq 0$.

Integrating (2.5) once yields that

$$\begin{cases} DU_z = -sU - UV + \varrho_1 =: F(U, V), \\ \varepsilon V_z = -sV + \varepsilon V^2 - U + \varrho_2 =: G(U, V) \end{cases} \tag{2.7}$$

where ϱ_1 and ϱ_2 are constants satisfying

$$\begin{aligned} \varrho_1 &= su_- + u_-v_- = su_+ + u_+v_+, \\ \varrho_2 &= sv_- - \varepsilon(v_-)^2 + u_- = sv_+ - \varepsilon(v_+)^2 + u_+. \end{aligned} \tag{2.8}$$

Rearranging (2.8), we deduce that the wave speed s is determined by

$$\begin{cases} -s(u_+ - u_-) - (u_+v_+ - u_-v_-) = 0, \\ -s(v_+ - v_-) + [\varepsilon(v_+)^2 - u_+ - \varepsilon(v_-)^2 + u_-] = 0 \end{cases} \tag{2.9}$$

which can be reduced to a quadratic equation of s ,

$$s^2 + v_-s + u_+ \left(\frac{\varepsilon(v_+^2 - v_-^2)}{u_+ - u_-} - 1 \right) = 0. \tag{2.10}$$

Note that when ε is small such that

$$\frac{\varepsilon(v_+^2 - v_-^2)}{u_+ - u_-} < 1, \tag{2.11}$$

the discriminant of the quadratic (2.10) is positive and hence (2.10) gives two solutions with opposite signs, where the positive s corresponds to the waves speed of second characteristic family of system (1.1) and the negative s corresponds to that of the first characteristic family. Hereafter we only consider the case of $s > 0$ and the analysis extends to the case $s < 0$. The positive wave speed s is given by

$$s = -\frac{v_-}{2} + \frac{1}{2} \sqrt{v_-^2 + 4u_+ \left(1 - \varepsilon \frac{v_+^2 - v_-^2}{u_+ - u_-} \right)}. \tag{2.12}$$

Since $v_- < 0$ and $\frac{\varepsilon(v_+^2 - v_-^2)}{u_+ - u_-} - 1 < 0$, then $s > |v_-| = -v_-$, which is equivalent to

$$s + v_- > 0. \tag{2.13}$$

The entropy condition for the shock of second characteristic family (e.g. see [9]) is

$$\lambda_2(u_+, v_+) < s < \lambda_2(u_-, v_-) \tag{2.14}$$

where $\lambda_2(u, v)$ is defined in (2.3). When ε is small, we derive from the entropy inequality (2.14) that

$$0 \leq u_+ < u_-, \quad v_- < v_+ \leq 0 \tag{2.15}$$

which will be used later to prove the existence of traveling waves.

The first result of the paper concerning the existence of traveling wave solutions of (1.1), namely, the existence of solutions to (2.5), (2.6), is as follows.

Theorem 2.1. *Let (2.14) hold. If ε is small, then there exists a monotone shock profile $(U, V)(x - st)$ to system (2.5), (2.6), which is unique up to a translation and satisfies $U_z < 0$ and $V_z > 0$, where the wave speed s is given by (2.12).*

Remark 2.1. Since $U_z < 0$ and $V_z > 0$, $0 \leq u_+ < U < u_-$ and $v_- < V < v_+ \leq 0$. By (2.5) and (2.7), one easily shows that U_z, V_z, U_{zz} and V_{zz} are all bounded. Furthermore, from (2.3) we see that system (2.1) is strictly hyperbolic, i.e., $\lambda_1 \neq \lambda_2$, when $0 \leq u_+ < U < u_-$ and $v_- < V < v_+ \leq 0$. The same is true if (u, v) is a small perturbation of the traveling wave.

2.2. Nonlinear stability of traveling waves

The second result of this paper is the asymptotic stability of traveling wave solutions obtained in Theorem 2.1 under the small initial perturbation of the form

$$\int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x) - U(x) \\ v_0(x) - V(x) \end{pmatrix} dx = x_0 \begin{pmatrix} u_+ - u_- \\ v_+ - v_- \end{pmatrix} + \eta \mathbf{r}_1(u_-, v_-) \tag{2.16}$$

where $\mathbf{r}_1(u_-, v_-)$ is the first eigenvector evaluated at (u_-, v_-) . The coefficients x_0 and η are uniquely determined by the initial data $(u_0(x), v_0(x))$. For the stability of small-amplitude waves of conservation laws corresponding to the case $\eta \neq 0$, the reader is referred to [15,20] and references therein. In this paper, we assume that $\eta = 0$ as in [4,6]. However, we consider the large-amplitude waves in contrast to the small-amplitude waves in [4,6]. Now by conservation laws (1.1) and system (2.5), we derive that

$$\begin{aligned} \int_{-\infty}^{+\infty} \begin{pmatrix} u(x, t) - U(x + x_0 - st) \\ v(x, t) - V(x + x_0 - st) \end{pmatrix} dx &= \int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x) - U(x + x_0) \\ v_0(x) - V(x + x_0) \end{pmatrix} dx \\ &= \int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x) - U(x) \\ v_0(x) - V(x) \end{pmatrix} dx + \int_{-\infty}^{+\infty} \begin{pmatrix} U(x) - U(x + x_0) \\ V(x) - V(x + x_0) \end{pmatrix} dx \\ &= \int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x) - U(x) \\ v_0(x) - V(x) \end{pmatrix} dx - x_0 \begin{pmatrix} u_+ - u_- \\ v_+ - v_- \end{pmatrix} = \vec{0}. \end{aligned}$$

We decompose the solution (u, v) of systems (1.1), (1.2) by

$$(u, v)(x, t) = (U, V)(x - st + x_0) + (\phi_x, \psi_x)(x, t) \tag{2.17}$$

where

$$(\phi(x, t), \psi(x, t)) = \int_{-\infty}^x (u(y, t) - U(y + x_0 - st), v(y, t) - V(y + x_0 - st)) dy$$

for all $x \in \mathbb{R}$ and $t \geq 0$.

It then holds that

$$\phi(\pm\infty, t) = 0, \quad \psi(\pm\infty, t) = 0 \quad \text{for all } t > 0.$$

We further assume, without loss of generality, that the translation $x_0 = 0$. Then (2.16) becomes

$$\int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x) - U(x) \\ v_0(x) - V(x) \end{pmatrix} dx = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.18}$$

The initial conditions of the perturbation (ϕ, ψ) are thus given by

$$(\phi_0, \psi_0)(x) = \int_{-\infty}^x (u_0 - U, v_0 - V)(y) dy. \tag{2.19}$$

The asymptotic stability of traveling wave solutions of (1.1) means that $(u - U, v - V)(x, t) = (\phi_x, \psi_x)(x, t) \rightarrow 0$ as $t \rightarrow \infty$.

Next, we introduce some notations. Let $\|f\|$ denote the L^2 norm of any function $f \in L^2(\mathbb{R})$,

$$\|f\| = \left(\int |f(x)|^2 dx \right)^{1/2}$$

where the integral lacking limits of integration means the integral over the whole real line \mathbb{R} . Let $H^p(\mathbb{R})$ denote the usual Sobolev space $W^{p,2}(\mathbb{R})$, and we use $\|f\|_p$ to denote the H^p norm for any $f \in H^p(\mathbb{R})$ where $p \geq 1$,

$$\|f\|_p = \left(\int \sum_{i=0}^p \left| \frac{d^i}{dx^i} f(x) \right|^2 dx \right)^{1/2}.$$

The main theorem on the asymptotic stability is as follows.

Theorem 2.2. *Let $(U, V)(x - st)$ be a viscous shock profile of (1.1) obtained in Theorem 2.1. If ε is small and $u_+ > 0$, then there exists a constant $\varepsilon_0 > 0$ such that if $\|u_0 - U\|_1 + \|v_0 - V\|_1 + \|(\phi_0, \psi_0)\| \leq \varepsilon_0$ and $\eta = 0$ in (2.16), the Cauchy problem (1.1), (1.2) has a unique global solution $(u, v)(x, t)$ satisfying $u(x, t) \geq \delta_0 > 0$ for some $\delta_0 > 0$ for all $x \in \mathbb{R}, t \geq 0$, and*

$$(u - U, v - V) \in (C([0, \infty); H^1) \cap L^2([0, \infty); H^1))^2.$$

Furthermore, the solution (u, v) has the following asymptotic nonlinear stability

$$\sup_{x \in \mathbb{R}} |(u, v)(x, t) - (U, V)(x - st)| \rightarrow 0 \quad \text{as } t \rightarrow +\infty. \tag{2.20}$$

Remark 2.2. It is worthwhile to remark that the above nonlinear stability results hold true regardless of the strengths of the waves. That is, the wave amplitude $|u_+ - u_-| + |v_+ - v_-|$ can be large, in contrast to the previous results related to the nonlinear stability of traveling waves to conservation laws, where various smallness conditions on wave strengths were imposed (e.g. see Ref. [4,5,11,15,17, 20]).

3. Proof of existence Theorem 2.1

In this section, we shall perform the phase plane analysis to prove the existence of traveling wave solutions of (1.1) as stated in Theorem 2.1. By the definition of constants ϱ_1 and ϱ_2 in (2.8), we see that the ODE system (2.7) has two and only two equilibria (u_-, v_-) and (u_+, v_+) . The Jacobian matrix of the linearized system of (2.7) about equilibrium (u_{\pm}, v_{\pm}) is

$$\hat{J}(u_{\pm}, v_{\pm}) = \begin{bmatrix} \frac{-s-v_{\pm}}{D} & -\frac{u_{\pm}}{D} \\ -\frac{1}{\varepsilon} & \frac{-s+2\varepsilon v_{\pm}}{\varepsilon} \end{bmatrix}$$

whose eigenvalues σ satisfy

$$\sigma^2 + \left(\frac{s + v_{\pm}}{D} + \frac{s - 2\varepsilon v_{\pm}}{\varepsilon} \right) \sigma + \frac{1}{\varepsilon D} ((s + v_{\pm})(s - 2\varepsilon v_{\pm}) - u_{\pm}) = 0. \quad (3.1)$$

One easily verifies that the discriminant of quadratic (3.1) is non-negative in the region \mathcal{X} defined in (1.6). Hence all roots of (3.1) are real. Furthermore, the two roots σ_1 and σ_2 satisfy

$$\sigma_1 \sigma_2 = \frac{1}{\varepsilon D} (s^2 + (v_{\pm} - 2\varepsilon v_{\pm})s - 2\varepsilon v_{\pm}^2 - u_{\pm}) =: \mathcal{H}(s, u_{\pm}, v_{\pm}).$$

From (2.2), we see that $\mathcal{H}(\lambda_2(u_-, v_-), u_-, v_-) = \mathcal{H}(\lambda_2(u_+, v_+), u_+, v_+) = 0$. Using the entropy condition (2.14), we show that

$$\begin{aligned} \sigma_1 \sigma_2|_{(u_-, v_-)} &= \mathcal{H}(s, u_-, v_-) < \mathcal{H}(\lambda_2(u_-, v_-), u_-, v_-) = 0, \\ \sigma_1 \sigma_2|_{(u_+, v_+)} &= \mathcal{H}(s, u_+, v_+) > \mathcal{H}(\lambda_2(u_+, v_+), u_+, v_+) = 0. \end{aligned}$$

On the other hand, $s + v_+ > s + v_- > 0$ from (2.13). If ε is small such that

$$2\varepsilon v_{\pm} < s, \quad (3.2)$$

then

$$\sigma_1 + \sigma_2 = - \left(\frac{s + v_{\pm}}{D} + \frac{s - 2\varepsilon v_{\pm}}{\varepsilon} \right) < 0.$$

Therefore the equilibrium (u_-, v_-) is a saddle and (u_+, v_+) is a stable node. Next we shall prove that there is a heteroclinic connection between (u_-, v_-) and (u_+, v_+) .

To this end, we look at the nullclines of system (2.5) which are given by

$$\begin{cases} U(V + s) = \varrho_1, \\ U = \varepsilon V^2 - sV + \varrho_2. \end{cases} \quad (3.3)$$

The first equation of (3.3) gives a hyperbola (see the solid line in Fig. 1) and second equation gives a parabola (see the dashed line in Fig. 1). Due to the definition of constants ϱ_1 and ϱ_2 in (2.8), these two curves must intersect at the equilibrium points (u_-, v_-) and (u_+, v_+) in the fourth quadrant of (U, V) plane (possibly including axes) due to $u_{\pm} \geq 0$ and $v_{\pm} \leq 0$ (see (2.15)). We shall show that the region (see an illustration in Fig. 1) enclosed by these two curves in the fourth quadrant comprises an invariant region of system (2.7) (2.6), which is defined by

$$\mathcal{G} = \left\{ (U, V) \mid \frac{\varrho_1}{V + s} \leq U \leq \varepsilon V^2 - sV + \varrho_2 \right\}. \quad (3.4)$$

The edges of the region \mathcal{G} are denoted by

$$\begin{aligned} \Gamma_1 &= \{(U, V) \mid F(U, V) = -U(V + s) + \varrho_1 = 0, u_+ < U < u_-, v_- < V < v_+\}, \\ \Gamma_2 &= \{(U, V) \mid G(U, V) = \varepsilon V^2 - sV - U + \varrho_2 = 0, u_+ < U < u_-, v_- < V < v_+\}. \end{aligned}$$

Note that Γ_1 is a portion of the hyperbola and Γ_2 is a portion of the parabola (see Fig. 1). Along the edge $\Gamma_1 : F(U, V) = 0, U_z = 0$ and the direction field of (2.7) is vertical. We shall show that indeed

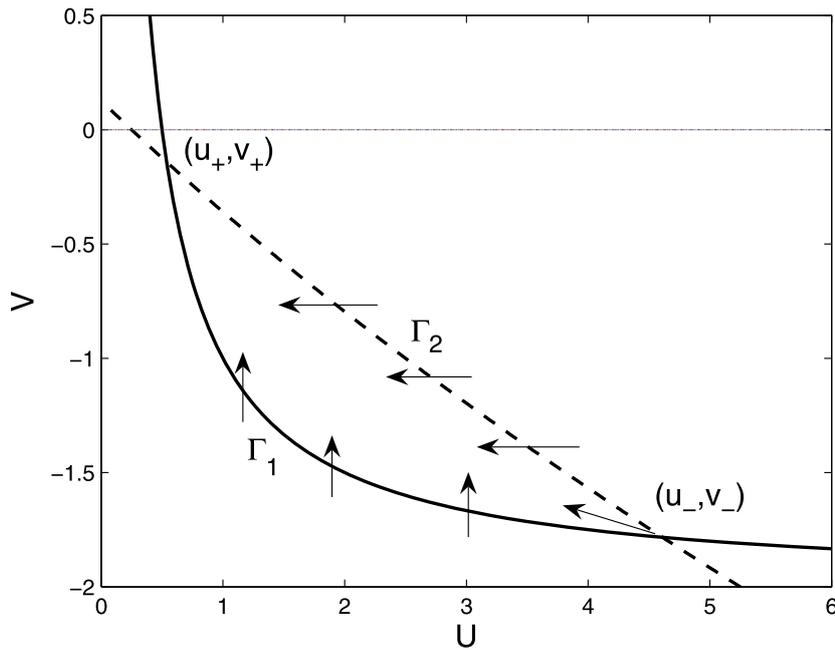


Fig. 1. A numerical plot of the phase plane of system (2.7), where the solid line (hyperbola) represent the nullcline $U(V+s) = \varrho_1$ and dashed line (parabola) represents the nullcline $U = \varepsilon V^2 - sV + \varrho_2$. The parameter values are $\varrho_1 = 1$, $\varrho_2 = 1/4$, $s = 2$, $\varepsilon = 1/2$.

the direction field of (2.7) along the edge Γ_1 points upward, which requires that $V_z|_{(U,V) \in \Gamma_1} > 0$. Note that along the edge $\Gamma_2 : G(U, V) = 0, V_z = 0$. It is straightforward to see that if a point (\tilde{U}, \tilde{V}) is below the edge Γ_2 , then $V_z = G(\tilde{U}, \tilde{V}) > 0$. Since the edge Γ_1 is below Γ_2 , $V_z|_{(U,V) \in \Gamma_1} > 0$, which immediately shows that the trajectory of (2.7) along the edge Γ_1 points upward. Similar idea leads to $U_z|_{(U,V) \in \Gamma_2} < 0$ and hence the direction field of (2.7) along the edge Γ_2 points to the left horizontally. Therefore \mathcal{G} is an invariant region of system (2.7). The phase portrait is illustrated in Fig. 1.

To show that there is a heteroclinic connection between (u_-, v_-) and (u_+, v_+) , it suffices to show that the unstable manifold of system (2.7) emanating from the saddle (u_-, v_-) points inside the invariant region \mathcal{G} .

To this end, we calculate the tangent directions of the nullclines at (u_-, v_-) . Solving V from the first equation of (3.3) and differentiating with respect to U , we derive

$$\left. \frac{dV}{dU} \right|_{(u_-, v_-)}^{\Gamma_1} = - \left. \frac{\varrho_1}{U^2} \right|_{(u_-, v_-)} = - \frac{s + v_-}{u_-}, \tag{3.5}$$

where $\left. \frac{dV}{dU} \right|_{(u_-, v_-)}^{\Gamma_1}$ denotes the tangent direction of Γ_1 at (u_-, v_-) . From the second equation of (3.3), we derive

$$\left. \frac{dV}{dU} \right|_{(u_-, v_-)}^{\Gamma_2} = \left. \frac{1}{dU/dV} \right|_{(u_-, v_-)} = \frac{1}{2\varepsilon v_- - s}, \tag{3.6}$$

where $\left. \frac{dV}{dU} \right|_{(u_-, v_-)}^{\Gamma_2}$ is the tangent direction of Γ_2 at (u_-, v_-) .

Next we compute the direction of the unstable manifold of system (2.7) at the saddle (u_-, v_-) and compare it with the directions of the nullclines calculated above. To this end, we consider the positive eigenvalue of $\hat{J}(u_-, v_-)$,

$$\sigma_2 = \frac{-\frac{s+v_-}{D} + \frac{-s+2\varepsilon v_-}{\varepsilon}}{2} + \frac{\sqrt{\left(-\frac{s+v_-}{D} - \frac{-s+2\varepsilon v_-}{\varepsilon}\right)^2 + \frac{4u_-}{\varepsilon D}}}{2}$$

which has the following eigenvector

$$\tilde{\mathbf{r}}_2 = \begin{bmatrix} -\frac{u_-}{D} \\ \sigma_2 + \frac{s+v_-}{D} \end{bmatrix}.$$

Tangent to the eigenvector $\tilde{\mathbf{r}}_2$, the direction of the unstable manifold of (2.7) at (u_-, v_-) is given by

$$\begin{aligned} \left. \frac{dV}{dU} \right|_{(u_-, v_-)} &= \frac{\sigma_2 + \frac{s+v_-}{D}}{-\frac{u_-}{D}} \\ &= \frac{\frac{1}{2}(\frac{s+v_-}{D} - \frac{s-2\varepsilon v_-}{\varepsilon}) + \frac{1}{2}\sqrt{(\frac{s+v_-}{D} - \frac{s-2\varepsilon v_-}{\varepsilon})^2 + \frac{4u_-}{\varepsilon D}}}{-\frac{u_-}{D}} \\ &= \frac{-2}{s - 2\varepsilon v_- - \frac{\varepsilon}{D}(s + v_-) + \sqrt{(s - 2\varepsilon v_- - \frac{\varepsilon}{D}(s + v_-))^2 + \frac{4\varepsilon u_-}{D}}} \\ &=: \frac{-2}{M}. \end{aligned} \tag{3.7}$$

Furthermore, for $\varepsilon > 0$ small, by Taylor expansion, the denominator M of (3.7) can be rewritten as

$$M = 2(s - 2\varepsilon v_-) + \frac{2\varepsilon}{D}M_1 + E_1$$

where

$$M_1 = \frac{u_-}{s - 2\varepsilon v_- - \frac{\varepsilon}{D}(s + v_-)} - (s + v_-)$$

and

$$E_1 = -\frac{1}{8} \left[\frac{4\varepsilon u_-}{D(s - 2\varepsilon v_- - \frac{\varepsilon}{D}(s + v_-))^2} \right]^2 + \mathcal{O}(\varepsilon^3)$$

which satisfies

$$\lim_{\varepsilon \rightarrow 0} \frac{E_1}{\varepsilon^2} = -\frac{2u_-^2}{D^2 s^2}. \tag{3.8}$$

Now we proceed with calculating

$$\begin{aligned} M_1 &= \frac{(u_- - u_+) + (u_+ - s^2 - sv_- - u_+ \frac{\varepsilon v_+^2 - \varepsilon v_-^2}{u_+ - u_-}) + E_2}{s - 2\varepsilon v_- - \frac{\varepsilon}{D}(s + v_-)} \\ &= \frac{(u_- - u_+) + E_2}{s - 2\varepsilon v_- - \frac{\varepsilon}{D}(s + v_-)} \end{aligned} \tag{3.9}$$

where Eq. (2.10) has been used and

$$E_2 = \varepsilon \left[\left(2v_- + \frac{s + v_-}{D} \right) (s + v_-) + u_+ \frac{v_+^2 - v_-^2}{u_+ - u_-} \right] = \mathcal{O}(\varepsilon). \tag{3.10}$$

Let $L_\varepsilon = (u_- - u_+) + E_2$. Taking into account that $u_+ \frac{v_+^2 - v_-^2}{u_+ - u_-} > 0$ and $s + v_- > 0$, if we choose ε such that

$$0 < \varepsilon < \frac{u_- - u_+}{|(2v_- + \frac{s+v_-}{D})(s + v_-) - u_+ \frac{v_+^2 - v_-^2}{u_- - u_+}|} \tag{3.11}$$

and

$$\varepsilon < \frac{s}{|2v_- + \frac{s+v_-}{D}|}, \tag{3.12}$$

then $L_\varepsilon > 0$ and $M_1 > 0$. Moreover it follows from (3.9) that $\lim_{\varepsilon \rightarrow 0} M_1 = \frac{u_- - u_+}{s} > 0$. Since $E_1 = \mathcal{O}(\varepsilon^2)$ (see (3.8)), then $\frac{2\varepsilon}{D}M_1 + E_1 > 0$ when ε is small. Thus if ε is small, it has that $M \geq 2(s - 2\varepsilon v_-)$. Therefore, we derive from (3.7) that

$$0 > \frac{\sigma_2 + \frac{s+v_-}{D}}{-\frac{u_-}{D}} > \frac{-2}{2(s - 2\varepsilon v_-)} = -\frac{1}{s - 2\varepsilon v_-}. \tag{3.13}$$

On the other hand, since $\sigma_2 > 0$, we have

$$\frac{\sigma_2 + \frac{s+v_-}{D}}{-\frac{u_-}{D}} < \frac{\frac{s+v_-}{D}}{-\frac{u_-}{D}} = -\frac{s + v_-}{u_-} < 0. \tag{3.14}$$

Therefore, combining (3.5), (3.6), (3.13) and (3.14), we end up with

$$\frac{dV}{dU} \Big|_{(u_-, v_-)}^{\Gamma_2} < \frac{dV}{dU} \Big|_{(u_-, v_-)} < \frac{dV}{dU} \Big|_{(u_-, v_-)}^{\Gamma_1} \tag{3.15}$$

where the left-hand side of (3.15) represents the tangential direction of the edge Γ_2 at (u_-, v_-) , the right-hand side of (3.15) is the tangential direction of the edge Γ_1 at (u_-, v_-) , and the middle term of (3.15) is the direction of unstable manifold of system (2.7) at (u_-, v_-) . Hence we conclude that the unstable manifold of (2.7) at (u_-, v_-) is between the tangent lines of Γ_2 and Γ_1 at (u_-, v_-) and points into the region \mathcal{G} . Since the manifold is trapped inside the invariant region \mathcal{G} , this unstable manifold has to go to the stable equilibrium (u_+, v_+) by the Poincaré–Bendixson theorem. This trajectory connecting (u_-, v_-) and (u_+, v_+) generates a solution for the system (2.7). From (2.7) and (3.4), we see that $U_z < 0$ and $V_z > 0$ and hence the proof of Theorem 2.1 is completed.

The following Lemma is established for later use.

Lemma 3.1. *There exists a constant $k > 0$ such that $V_z \leq k|U_z|$ for all $(U, V) \in \mathcal{G}$.*

Proof. Along the traveling wave trajectory, it has that from (2.7) and Theorem 2.1,

$$0 < \frac{\varepsilon V_z}{D|U_z|} = -\frac{G(U, V)}{F(U, V)} = -\frac{\varepsilon}{D} \frac{dV}{dU}.$$

From (3.15) we see $-\frac{dV}{dU} \Big|_{(u_-, v_-)} > 0$ is bounded. Therefore $\frac{\varepsilon V_z}{D|U_z|}$ is bounded at (u_-, v_-) . By continuity, $\frac{\varepsilon V_z}{D|U_z|}$ is bounded for (U, V) along the trajectory near (u_-, v_-) . The same argument shows that $\frac{\varepsilon V_z}{D|U_z|}$ is bounded for (U, V) along the trajectory near (u_+, v_+) .

Away from (u_{\pm}, v_{\pm}) , the trajectory is in the interior of the bounded region \mathcal{G} which does not contain equilibrium inside. Therefore along the trajectory, $-\frac{G(U,V)}{F(U,V)} > 0$ is bounded, and hence so is $\frac{\varepsilon V_z}{D|U_z|}$.

Thus $\frac{\varepsilon V_z}{D|U_z|}$ is bounded for any $(U, V) \in [u_+, u_-] \times [v_-, v_+]$ along the trajectory. The proof is finished. \square

4. Proof of stability Theorem 2.2

4.1. Reformulation of stability problem

The proof of Theorem 2.2 is based on iterative L^2 energy estimates. The method of energy estimates for the nonlinear stability of small-amplitude viscous shock profiles of conservation laws was first introduced independently by Matsumura and Nishihara in [17] and by Goodman in [4]. It has been further developed over the years, see [6,11,14,15,20]. We establish the stability estimates without the smallness assumption on the wave strengths. In what follows, we use C to denote a generic constant which changes from one line to another. An integral lacking limits of integration means an integral over the whole real line \mathbb{R} .

In view of (2.17), we decompose the solution of (1.1) in the form

$$(u, v)(x, t) = (U, V)(x - st) + (\phi_x, \psi_x)(x, t) = (U, V)(z) + (\bar{\phi}_z, \bar{\psi}_z)(z, t) \tag{4.1}$$

with $(\bar{\phi}, \bar{\psi})$ in some functional space which will be defined below. For simplicity of notation, we will omit the bars in $(\bar{\phi}, \bar{\psi})$ in the following text.

Substituting (4.1) into (1.1), using (2.5) and integrating the resulting equations with respect to z , we obtain the equations for the perturbation (ϕ, ψ)

$$\begin{cases} \phi_t = D\phi_{zz} + (s + V)\phi_z + U\psi_z + \phi_z\psi_z, \\ \psi_t = \varepsilon\psi_{zz} + s\psi_z + \phi_z - 2\varepsilon V\psi_z - \varepsilon\psi_z^2 \end{cases} \tag{4.2}$$

with initial data given by

$$(\phi, \psi)(z, 0) = (\phi_0, \psi_0)(z), \quad z \in \mathbb{R} \tag{4.3}$$

where (ϕ_0, ψ_0) is defined in (2.19).

We seek solutions of the reformulated problem (4.2)–(4.3) in the following solution space

$$X(0, T) = \{(\phi(z, t), \psi(z, t)): \phi, \psi \in C([0, T]; H^2) \cap C^1((0, T); H^1), \phi_z, \psi_z \in L^2((0, T); H^2)\}$$

with $0 \leq T \leq +\infty$.

Let

$$N(t) = \sup_{0 \leq \tau \leq t} \{\|\phi(\cdot, \tau)\|_2 + \|\psi(\cdot, \tau)\|_2\}. \tag{4.4}$$

By the Sobolev embedding theorem, we have

$$\sup_{z \in \mathbb{R}} \{|\phi|, |\phi_z|, |\psi|, |\psi_z|\} \leq N(t) \tag{4.5}$$

for $t \geq 0$.

Theorem 2.2 is proved based on the following theorem.

Theorem 4.1. *Let $(\phi_0, \psi_0) \in (H^2(\mathbb{R}))^2$ and $\varepsilon > 0$ small. If $u_+ > 0$, then there exists a constant $\delta_1 > 0$ such that if*

$$N(0) \leq \delta_1, \tag{4.6}$$

the Cauchy problem (4.2), (4.3) has a unique global solution $(\phi, \psi) \in X(0, +\infty)$ which satisfies

$$\begin{aligned} & \|(\phi(\cdot, t), \psi(\cdot, t))\|_2^2 + \int_0^t \int |U_z(z)| |(\phi(z, \tau), \psi(z, \tau))|^2 dz d\tau + \int_0^t \|(\phi_z(\cdot, \tau), \psi_z(\cdot, \tau))\|_2^2 d\tau \\ & \leq CN^2(0) \end{aligned} \tag{4.7}$$

for all $t \in [0, +\infty)$, where $C > 0$ is a constant. Moreover the following asymptotic behavior holds

$$\sup_{z \in \mathbb{R}} |(\phi_z, \psi_z)(z, t)| \rightarrow 0 \text{ as } t \rightarrow +\infty. \tag{4.8}$$

With Theorem 4.1, we can immediately show the positivity of u as claimed in Theorem 2.2. In fact, if the initial perturbation (4.3) satisfies (4.6), then by (4.7) there is a constant $C > 0$ such that

$$|\phi_z(z, t)| \leq \sqrt{2}N(t) \leq CN(0) \leq C\delta_1.$$

Thus for all $x \in \mathbb{R}$ and $t \geq 0$, it follows from (4.1) that

$$u(x, t) = (u(x, t) - U(z)) + U(z) = \phi_z(z, t) + U(z) \geq -CN(0) + u_+ \geq -C\delta_1 + u_+ = \delta_0 > 0$$

provided that δ_1 is suitably small and u_+ is positive.

The global existence of (ϕ, ψ) announced in Theorem 4.1 follows from the local existence theorem and the *a priori* estimates which are given below.

Proposition 4.2 (Local existence). *For any $\delta_2 > 0$, there exists a positive constant T depending on δ_2 such that if $(\phi_0, \psi_0) \in (H^2(\mathbb{R}))^2$ with $N(0) \leq \delta_2/2$, the problem (4.2), (4.3) has a unique solution $(\phi, \psi) \in X(0, T)$ satisfying*

$$N(t) < 2N(0) \tag{4.9}$$

for any $0 \leq t \leq T$.

Proposition 4.3 (A priori estimates). *Assume that $(\phi, \psi) \in X(0, T)$ is a solution obtained in Proposition 4.2 for a positive constant T . Then there is a positive constant $\delta_3 > 0$, independent of T , such that if*

$$N(t) < \delta_3$$

for any $0 \leq t \leq T$, the solution (ϕ, ψ) of (4.2), (4.3) satisfies (4.7) for any $0 \leq t \leq T$.

With the solution (ϕ, ψ) obtained in Theorem 4.1 and traveling wave solution (U, V) given in Lemma 2.1, we have the desired solution of the problem (1.1) (1.2) through the relation (4.1).

The local existence in Proposition 4.2 can be shown in a standard way (cf. [19]) and we omit the proof. Theorem 4.1 is a consequence of Proposition 4.2 and Proposition 4.3 by the continuation argument. It only remains to prove Proposition 4.3 which will be shown based on the following energy estimates.

Lemma 4.4. *Let $(\phi_0, \psi_0) \in (H^2(\mathbb{R}))^2$ and (ϕ, ψ) be a solution of (4.2), (4.3). If $u_+ > 0$ and $\varepsilon > 0$ is small, then there exists a constant $C > 0$, such that*

$$\begin{aligned} & \|\phi(\cdot, t)\|_2^2 + \|\psi(\cdot, t)\|_2^2 + \int_0^t \int |U_z(z)| |(\phi(z, \tau), \psi(z, \tau))|^2 dz d\tau \\ & + D \int_0^t \|\phi_z(\cdot, \tau)\|_2^2 d\tau + \varepsilon \int_0^t \|\psi_z(\cdot, \tau)\|_2^2 d\tau \\ & \leq C \left(\|\phi_0\|_2^2 + \|\psi_0\|_2^2 + \int_0^t \int (|\phi| + |\phi_z| + |\phi_{zz}|) |\phi_z \psi_z| dz d\tau \right) \\ & + C\varepsilon \left(\int_0^t \int |\psi_z^2 \psi_{zz}| dz d\tau + \int_0^t \int |\psi_z \psi_{zz} \psi_{zzz}| dz d\tau + \int_0^t \int |\psi \psi_z^2| dz d\tau \right) \\ & + C \int_0^t \int (|\phi_{zz}| + |\phi_{zzz}|) |(\phi_z \psi_z)_z| dz d\tau. \end{aligned} \tag{4.10}$$

Then Proposition 4.3 can be proved by Lemma 4.4.

Proof of Proposition 4.3. In fact it only remains to show that the *a priori estimate* (4.7) holds. To this end, we need to estimate the nonlinear terms on the right-hand side of (4.10). Indeed, by applying the Sobolev embedding theorem, all these nonlinear terms can be bounded by $CN(t)(\int_0^t \|(\phi_z(\cdot, \tau))\|_2^2 d\tau + \int_0^t \|\psi_z(\cdot, \tau)\|_2^2 d\tau)$ for some constant $C > 0$. Then from Lemma 4.4, we have

$$\begin{aligned} & N^2(t) + \int_0^t \int |U_z(z)| \cdot |(\phi(z, \tau), \psi(z, \tau))|^2 dz d\tau + \int_0^t \|\phi_z(\cdot, \tau)\|_2^2 d\tau + \int_0^t \|\psi_z(\cdot, \tau)\|_2^2 d\tau \\ & \leq CN^2(0) + CN(t) \left(\int_0^t \|\phi_z(\cdot, \tau)\|_2^2 d\tau + \int_0^t \|\psi_z(\cdot, \tau)\|_2^2 d\tau \right) \end{aligned}$$

for $t \in [0, T]$ and some constant $C > 0$.

Therefore, letting $N(t) \leq \frac{1}{2C}$, we obtain the following estimate for any $t \in [0, T]$,

$$N^2(t) + \int_0^t \int |U_z(z)| \cdot |(\phi(z, \tau), \psi(z, \tau))|^2 dz d\tau + \int_0^t \|\phi_z(\cdot, \tau)\|_2^2 d\tau + \int_0^t \|\psi_z(\cdot, \tau)\|_2^2 d\tau \leq CN^2(0)$$

which gives the desired estimate (4.7). \square

Proof of Theorem 4.1. By standard arguments (e.g. see [12,13,22]), we derive from the global estimate (4.7) that

$$\|(\phi_z(\cdot, t), \psi_z(\cdot, t))\|_1 \rightarrow 0 \text{ as } t \rightarrow +\infty. \tag{4.11}$$

Consequently, for all $z \in \mathbb{R}$, it follows that

$$\begin{aligned} \phi_z^2(z, t) &= 2 \int_{-\infty}^z \phi_z \phi_{zz}(y, t) dy \\ &\leq 2 \left(\int_{-\infty}^{+\infty} \phi_z^2 dy \right)^{1/2} \left(\int_{-\infty}^{+\infty} \phi_{zz}^2 dy \right)^{1/2} \rightarrow 0 \text{ as } t \rightarrow +\infty. \end{aligned} \tag{4.12}$$

Applying the same argument to ψ_z leads to, for all $z \in \mathbb{R}$,

$$\psi_z^2(z, t) \rightarrow 0 \text{ as } t \rightarrow +\infty.$$

Hence (4.8) is proved and the proof of Theorem 4.1 is completed. \square

4.2. Energy estimates

In this subsection, we are devoted to proving Lemma 4.4 which is a consequence of a series of energy estimates given below.

Lemma 4.5 (L^2 -estimates). *Let the assumptions in Lemma 4.4 hold. Then there exist constants $\nu_0 > 0$ and $C > 0$ such that the solution (ϕ, ψ) of (4.2), (4.3) satisfies*

$$\begin{aligned} &\|\phi(\cdot, t)\|^2 + \|\psi(\cdot, t)\|^2 + \nu_0 \int_0^t \int |U_z(z)| |(\phi(z, \tau), \psi(z, \tau))|^2 dz d\tau \\ &+ D \int_0^t \|\phi_z(\cdot, \tau)\|^2 d\tau + \varepsilon \int_0^t \|\psi_z(\cdot, \tau)\|^2 d\tau \\ &\leq C \left(\|\phi_0\|^2 + \|\psi_0\|^2 + \int_0^t \int |\phi \phi_z \psi_z| dz d\tau + \varepsilon \int_0^t \int |\psi \psi_z^2| dz d\tau \right). \end{aligned} \tag{4.13}$$

Proof. Multiplying the first equation of (4.2) by ϕ/U and the second by ψ , and adding them, we end up with the following equation after integrating the result w.r.t. z ,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \left(\frac{\phi^2}{U} + \psi^2 \right) dz + D \int \frac{\phi_z^2}{U} dz + \frac{1}{2} \int \delta(z) \phi^2 dz + \varepsilon \int \psi_z^2 dz \\ & = \int \frac{\phi \phi_z \psi_z}{U} dz + \varepsilon \int V_z \psi^2 dz - \varepsilon \int \psi \psi_z^2 dz \end{aligned} \tag{4.14}$$

where

$$\delta(z) = -\left(\frac{D}{U} \right)_{zz} + \left(\frac{s+V}{U} \right)_z. \tag{4.15}$$

From (2.5), (2.7), (2.8) and the fact $U_z < 0$, one infers that

$$\delta(z) = -\frac{2\varrho_1}{U^3} U_z = -\nu(z) U_z = \nu(z) |U_z| \tag{4.16}$$

where

$$\nu(z) = \frac{2\varrho_1}{U^3}. \tag{4.17}$$

Since $\varrho_1 = su_- + u_-v_- = u_-(s+v_-) > 0$ due to (2.13) and $0 < u_+ < U < u_-$, it follows that

$$\nu(z) = \frac{2\varrho_1}{U^3} > \frac{u_-(s+v_-)}{u_-^3} =: \nu_0 > 0. \tag{4.18}$$

Substituting (4.16) and (4.18) into (4.14), integrating the result with respect to t , and using the boundness of U , we have

$$\begin{aligned} & \|\phi(\cdot, t)\|^2 + \|\psi(\cdot, t)\|^2 + \nu_0 \int_0^t \int |U_z(z)| |\phi(z, \tau)|^2 dz d\tau + D \int_0^t \|\phi_z(\cdot, \tau)\|^2 d\tau \\ & + \varepsilon \int_0^t \|\psi_z(\cdot, \tau)\|^2 d\tau \\ & \leq C \left(\|\phi_0\|^2 + \|\psi_0\|^2 + \int_0^t \int |\phi \phi_z \psi_z| dz d\tau + \varepsilon \int_0^t \int |\psi \psi_z^2| dz d\tau + \varepsilon \int_0^t \int V_z \psi^2 dz d\tau \right) \end{aligned} \tag{4.19}$$

where $C > 0$ is a constant.

To finish the proof, it remains to estimate term $\int_0^t \int V_z \psi^2 dz d\tau$ in (4.19). Noticing that $0 < V_z \leq k|U_z|$ from Lemma 3.1, it suffices to estimate $\int_0^t \int |U_z| \psi^2 dz d\tau$. To this end, we multiply the first equation of (4.2) by ϕ , the second by $U\psi$, add them, and integrate the resulting equation w.r.t. z to have

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \int (\phi^2 + U\psi^2) dz + D \int \phi_z^2 dz + \frac{1}{2} \int V_z \phi^2 dz + \frac{s}{2} \int U_z \psi^2 dz \\
 &= \int U(\phi\psi_z + \psi\phi_z) dz + \varepsilon \int U\psi\psi_{zz} dz - 2\varepsilon \int UV\psi\psi_z dz \\
 & \quad + \int \phi\phi_z\psi_z dz - \varepsilon \int U\psi\psi_z^2 dz \\
 &= - \int U_z\phi\psi dz - \varepsilon \int U_z\psi\psi_z dz - \varepsilon \int U\psi_z^2 dz + \varepsilon \int (U_zV + UV_z)\psi^2 dz \\
 & \quad + \int \phi\phi_z\psi_z dz - \varepsilon \int U\psi\psi_z^2 dz. \tag{4.20}
 \end{aligned}$$

Integrating (4.20) with respect to t leads to

$$\begin{aligned}
 -\frac{s}{2} \int_0^t \int U_z \psi^2 dz d\tau &= \frac{s}{2} \int_0^t \int |U_z| \psi^2 dz d\tau \\
 &= \frac{1}{2} \int (\phi^2 + U\psi^2) dz - \frac{1}{2} \int (\phi_0^2 + U\psi_0^2) dz + D \int_0^t \int \phi_z^2 dz d\tau + \frac{1}{2} \int_0^t \int V_z \phi^2 dz d\tau \\
 & \quad + \int_0^t \int U_z \phi\psi dz d\tau + \varepsilon \int_0^t \int U_z \psi\psi_z dz d\tau + \varepsilon \int_0^t \int U\psi_z^2 dz d\tau \\
 & \quad - \varepsilon \int_0^t \int (U_zV + UV_z)\psi^2 dz d\tau - \int_0^t \int \phi\phi_z\psi_z dz d\tau + \varepsilon \int_0^t \int U\psi\psi_z^2 dz d\tau. \tag{4.21}
 \end{aligned}$$

By Cauchy–Schwarz inequality, we have the following inequalities

$$\begin{aligned}
 - \int_0^t \int U_z \phi\psi dz d\tau &\leq \frac{s}{4} \int_0^t \int |U_z| \psi^2 dz d\tau + \frac{1}{s} \int_0^t \int |U_z| \phi^2 dz d\tau, \\
 \varepsilon \int_0^t \int U_z \psi\psi_z dz d\tau &\leq \varepsilon \int_0^t \int |U_z| \psi^2 dz d\tau + \frac{\varepsilon}{4} \int_0^t \int |U_z| \psi_z^2 dz d\tau. \tag{4.22}
 \end{aligned}$$

Furthermore, by the boundedness of U, V and U_z and V_z as well as Lemma 3.1, there is a constant $m > 0$ which depends on u_{\pm} and v_{\pm} such that

$$-\varepsilon \int_0^t \int (U_zV + UV_z)\psi^2 dz d\tau \leq m\varepsilon \int_0^t \int |U_z| \psi^2 dz d\tau. \tag{4.23}$$

Substituting (4.22) and (4.23) into (4.21) yields

$$\begin{aligned}
 & \left(\frac{s}{4} - (m+1)\varepsilon\right) \int_0^t \int |U_z| \psi^2 dz d\tau \\
 & \leq \frac{1}{2} \int (\phi^2 + U\psi^2) dz - \frac{1}{2} \int (\phi_0^2 + U\psi_0^2) dz + D \int_0^t \int \phi_z^2 dz d\tau \\
 & \quad + \left(\frac{k}{2} + \frac{1}{s}\right) \int_0^t \int |U_z| \phi^2 dz d\tau + \varepsilon \int_0^t \int U \psi_z^2 dz d\tau + \frac{\varepsilon}{4} \int_0^t \int |U_z| \psi_z^2 dz d\tau \\
 & \quad - \int_0^t \int \phi \phi_z \psi_z dz d\tau + \varepsilon \int_0^t \int U \psi \psi_z^2 dz d\tau \\
 & \leq C \left(\|\phi_0\|^2 + \|\psi_0\|^2 + \int_0^t \int |\phi \phi_z \psi_z| dz d\tau + \varepsilon \int_0^t \int |\psi \psi_z^2| dz d\tau \right. \\
 & \quad \left. + \varepsilon \int_0^t \int V_z \psi^2 dz d\tau \right) \tag{4.24}
 \end{aligned}$$

where (4.19) and the boundedness of U and U_z have been used. Utilizing the condition $0 < V_z \leq k|U_z|$ from Lemma 3.1 again, one has

$$\begin{aligned}
 & \left(\frac{s}{4} - (m+1)\varepsilon - Ck\varepsilon\right) \int_0^t \int |U_z| \psi^2 dz d\tau \\
 & \leq C \left(\|\phi_0\|^2 + \|\psi_0\|^2 + \int_0^t \int |\phi \phi_z \psi_z| dz d\tau + \varepsilon \int_0^t \int |\psi \psi_z^2| dz d\tau \right). \tag{4.25}
 \end{aligned}$$

If ε is small such that $\frac{s}{4} - (m+1)\varepsilon - Ck\varepsilon > 0$, namely,

$$0 < \varepsilon < \frac{s}{4(m+1+Ck)} \tag{4.26}$$

then (4.25) gives that

$$\begin{aligned}
 & \int_0^t \int |U_z| \psi^2 dz d\tau \\
 & \leq C \left(\|\phi_0\|^2 + \|\psi_0\|^2 + \int_0^t \int |\phi \phi_z \psi_z| dz d\tau + \varepsilon \int_0^t \int |\psi \psi_z^2| dz d\tau \right). \tag{4.27}
 \end{aligned}$$

Hence

$$\begin{aligned} \varepsilon \int_0^t \int V_z \psi^2 dz d\tau &\leq k\varepsilon \int_0^t \int |U_z| \psi^2 dz d\tau \\ &\leq C\varepsilon \left(\|\phi_0\|^2 + \|\psi_0\|^2 + \int_0^t \int |\phi \phi_z \psi_z| dz d\tau + \varepsilon \int_0^t \int |\psi \psi_z^2| dz d\tau \right). \end{aligned} \quad (4.28)$$

Finally, the substitution of (4.28) into (4.19) finishes the proof of (4.13). \square

Next we derive the estimates for the first order derivatives of (ϕ, ψ) .

Lemma 4.6 (H^1 -estimates). *Let the assumptions in Lemma 4.4 hold. Then the solution (ϕ, ψ) of (4.2), (4.3) satisfies*

$$\begin{aligned} &\|\phi_z(\cdot, t)\|^2 + \|\psi_z(\cdot, t)\|^2 + D \int_0^t \|\phi_{zz}(\cdot, \tau)\|^2 d\tau + \varepsilon \int_0^t \|\psi_{zz}(\cdot, \tau)\|^2 d\tau \\ &\leq C \left(\|\phi_0\|_1^2 + \|\psi_0\|_1^2 + \int_0^t \int (|\phi| + |\phi_z| + |\phi_{zz}|) |\phi_z \psi_z| dz d\tau \right) \\ &\quad + C\varepsilon \left(\int_0^t \int |\psi_z^2 \psi_{zz}| dz d\tau + \int_0^t \int |\psi \psi_z^2| dz d\tau \right). \end{aligned} \quad (4.29)$$

Proof. Multiplying the first equation of (4.2) by $-\phi_{zz}/U$, the second equation by $-\psi_{zz}$, adding them and integrating the resulting equation w.r.t. z , we obtain with some rearrangement

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int \left(\frac{\phi_z^2}{U} + \psi_z^2 \right) dz + D \int \frac{\phi_{zz}^2}{U} dz + \varepsilon \int V_z \psi_z^2 dz + \varepsilon \int \psi_{zz}^2 dz \\ &= \int \left[-\left(\frac{1}{U} \right)_z \phi_t \phi_z + \frac{1}{2} \left(\frac{s+V}{U} \right)_z \phi_z^2 \right] dz - \int \frac{1}{U} \phi_z \phi_{zz} \psi_z dz + \varepsilon \int \psi_z^2 \psi_{zz} dz. \end{aligned} \quad (4.30)$$

We now estimate the first term on the right-hand side of (4.30). For convenience, we denote

$$I = \int \left[-\left(\frac{1}{U} \right)_z \phi_t \phi_z + \frac{1}{2} \left(\frac{s+V}{U} \right)_z \phi_z^2 \right] dz.$$

Indeed, by using the first equation of (4.2), we have

$$I = \frac{1}{2} \int \left[\left(\frac{D}{U} \right)_{zz} - \left(\frac{s+V}{U} \right)_z \right] \phi_z^2 dz - \int U \left(\frac{1}{U} \right)_z \phi_z \psi_z' dz - \int \left(\frac{1}{U} \right)_z \phi_z^2 \psi_z dz. \quad (4.31)$$

In view of (4.15) and (4.16), one has

$$\left(\frac{D}{U} \right)_{zz} - \left(\frac{s+V}{U} \right)_z = -\delta(z) = \frac{2Q_1}{U^3} U_z. \quad (4.32)$$

By Cauchy–Schwarz inequality and the boundedness of U, U_z as well as U_{zz} , we deduce that

$$\begin{aligned}
 -U\left(\frac{1}{U}\right)_z \phi_z \psi_z &= -\left(U\left(\frac{1}{U}\right)_z \phi_z \psi\right)_z + U\left(\frac{1}{U}\right)_z \phi_{zz} \psi + \left(U_z\left(\frac{1}{U}\right)_z + U\left(\frac{1}{U}\right)_{zz}\right) \phi_z \psi \\
 &= -\left(U\left(\frac{1}{U}\right)_z \phi_z \psi\right)_z - \frac{U_z}{U} \phi_{zz} \psi + \left(U_z\left(\frac{1}{U}\right)_z + U\left(\frac{1}{U}\right)_{zz}\right) \phi_z \psi \\
 &\leq -\left(U\left(\frac{1}{U}\right)_z \phi_z \psi\right)_z + \frac{D}{2} \frac{\phi_{zz}^2}{U} + C(|U_z| \psi^2 + \phi_z^2).
 \end{aligned} \tag{4.33}$$

Substituting (4.32) and (4.33) into (4.31) yields

$$I \leq \frac{D}{2} \int \frac{\phi_{zz}^2}{U} dz + C \int (|U_z| \psi^2 + \phi_z^2) dz + C \int \phi_z^2 |\psi_z| dz \tag{4.34}$$

where the boundedness of U and U_z has been used again. Now we substitute (4.34) into (4.30) and integrate the result with respect to t . By the boundedness of U, V, U_z, V_z and inequality (4.13), we end up with

$$\begin{aligned}
 &\int (\phi_z^2 + \psi_z^2) dz + D \int_0^t \int \phi_{zz}^2 dz d\tau + \varepsilon \int_0^t \int \psi_{zz}^2 dz d\tau \\
 &\leq C(\|\phi_0\|_1^2 + \|\psi_0\|_1^2) + C \int_0^t \int (|\phi| + |\phi_z| + |\phi_{zz}|) |\phi_z \psi_z| dz d\tau \\
 &\quad + C\varepsilon \int_0^t \int |\psi \psi_z^2| dz d\tau + C\varepsilon \int_0^t \int |\psi_z^2 \psi_{zz}| dz d\tau,
 \end{aligned} \tag{4.35}$$

which completes the proof of (4.29). \square

Finally, we show the estimates for the second order derivatives of (ϕ, ψ) .

Lemma 4.7. *Let the assumptions in Theorem 4.4 hold. Then there is a constant $C > 0$ such that the solution (ϕ, ψ) of (4.2), (4.3) satisfies*

$$\begin{aligned}
 &\|\phi_{zz}(\cdot, t)\|^2 + \|\psi_{zz}(\cdot, t)\|^2 + D \int_0^t \|\phi_{zzz}(\cdot, \tau)\|^2 d\tau + \varepsilon \int_0^t \|\psi_{zzz}(\cdot, \tau)\|^2 d\tau \\
 &\leq C\left(\|\phi_0\|_2^2 + \|\psi_0\|_2^2 + \int_0^t \int (|\phi| + |\phi_z| + |\phi_{zz}|) |\phi_z \psi_z| dz d\tau\right) \\
 &\quad + C\varepsilon\left(\int_0^t \int |\psi_z^2 \psi_{zz}| dz d\tau + \int_0^t \int |\psi_z \psi_{zz} \psi_{zzz}| dz d\tau + \int_0^t \int |\psi \psi_z^2| dz d\tau\right) \\
 &\quad + C \int_0^t \int (|\phi_{zz}| + |\phi_{zzz}|) |(\phi_z \psi_z)_z| dz d\tau.
 \end{aligned} \tag{4.36}$$

Proof. We multiply the first equation of (4.2) by $1/U$, differentiate the resulting equation with respect to z twice and differentiate the second equation of (4.2) with respect to z twice to obtain

$$\begin{cases} \left(\frac{1}{U}\phi_t\right)_{zz} = \left(\frac{D}{U}\phi_{zz}\right)_{zz} + \left(\frac{s+V}{U}\phi_z\right)_{zz} + \psi_{zzz} + \left(\frac{1}{U}\phi_z\psi_z\right)_{zz}, \\ \psi_{tzz} = \varepsilon\psi_{zzzz} + s\psi_{zzz} + \phi_{zzz} - 2\varepsilon(V\psi_z)_{zz} - \varepsilon(\psi_z^2)_{zz}. \end{cases} \quad (4.37)$$

Now multiplying the first equation of (4.37) by ϕ_{zz} and the second equation by ψ_{zz} , integrating the results w.r.t. z and adding them, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \left(\frac{\phi_{zz}^2}{U} + \psi_{zz}^2\right) dz + \int \left[\left(\frac{1}{U}\right)_{zz} \phi_t + 2\left(\frac{1}{U}\right)_z \phi_{tz}\right] \phi_{zz} dz + \varepsilon \int \psi_{zzz}^2 dz \\ &= D \int \left(\frac{\phi_{zz}}{U}\right)_{zz} \phi_{zz} dz + \int \left(\frac{s+V}{U}\phi_z\right)_{zz} \phi_{zz} dz - 2\varepsilon \int (V\psi_z)_{zz} \psi_{zz} dz \\ & \quad - \varepsilon \int (\psi_z^2)_{zz} \psi_{zz} dz + \int \left(\frac{1}{U}\phi_z\psi_z\right)_{zz} \phi_{zz} dz \\ &=: I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \quad (4.38)$$

Using the integration by parts, we deduce that

$$\begin{cases} I_1 = \frac{D}{2} \int \left(\frac{1}{U}\right)_{zz} \phi_{zz}^2 dz - D \int \frac{1}{U} \phi_{zzz}^2 dz, \\ I_2 = -\frac{1}{2} \int \left(\frac{s+V}{U}\right)_{zzz} \phi_z^2 dz + \frac{3}{2} \int \left(\frac{s+V}{U}\right)_z \phi_{zz}^2 dz, \\ I_3 = -2\varepsilon \int V_{zz} \psi_z \psi_{zz} dz - 3\varepsilon \int V_z \psi_{zz}^2 dz. \end{cases} \quad (4.39)$$

Substituting (4.39) into (4.38) leads to

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \left(\frac{\phi_{zz}^2}{U} + \psi_{zz}^2\right) dz + D \int \frac{1}{U} \phi_{zzz}^2 dz \\ & \quad + \int \left[\left(\frac{1}{U}\right)_{zz} \phi_t + 2\left(\frac{1}{U}\right)_z \phi_{tz}\right] \phi_{zz} dz + \varepsilon \int \psi_{zzz}^2 dz + 3\varepsilon \int V_z \psi_{zz}^2 dz \\ &= -\frac{1}{2} \int \left(\frac{s+V}{U}\right)_{zzz} \phi_z^2 dz + \frac{1}{2} \int \left[3\left(\frac{s+V}{U}\right)_z + \left(\frac{D}{U}\right)_{zz}\right] \phi_{zz}^2 dz \\ & \quad - 2\varepsilon \int V_{zz} \psi_z \psi_{zz} dz - \varepsilon \int (\psi_z^2)_{zz} \psi_{zz} dz + \int \left(\frac{1}{U}\phi_z\psi_z\right)_{zz} \phi_{zz} dz. \end{aligned} \quad (4.40)$$

Now we estimate the third term on the left-hand side of (4.40). Substituting the first equation of (4.2) into it, one derives that

$$\begin{aligned} & \left[\left(\frac{1}{U}\right)_{zz} \phi_t + 2\left(\frac{1}{U}\right)_z \phi_{tz}\right] \phi_{zz} \\ &= \left(D\left(\frac{1}{U}\right)_z \phi_{zz}^2\right)_z + 2(s+V)\left(\frac{1}{U}\right)_z \phi_{zz}^2 + \left[(s+V)\left(\frac{1}{U}\right)_{zz} + 2V_z\left(\frac{1}{U}\right)_z\right] \phi_z \phi_{zz} \end{aligned}$$

$$\begin{aligned}
 & -U\left(\frac{1}{U}\right)_{zz} \psi_z \phi_{zz} + \left(\frac{1}{U}\right)_{zz} \phi_z \psi_z \phi_{zz} + 2\left(U\left(\frac{1}{U}\right)_z \psi_z \phi_{zz}\right)_z \\
 & -2U_z\left(\frac{1}{U}\right)_z \psi_z \phi_{zz} + 2\left(\frac{1}{U}\right)_z (\phi_z \psi_z)_z \phi_{zz}.
 \end{aligned} \tag{4.41}$$

Then the substitution of (4.41) into (4.40) yields

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \int \left(\frac{\phi_{zz}^2}{U} + \psi_{zz}^2\right) dz + D \int \frac{1}{U} \phi_{zzz}^2 dz + \varepsilon \int \psi_{zzz}^2 dz + 3\varepsilon \int V_z \psi_{zz}^2 dz \\
 & = -\frac{1}{2} \int \left(\frac{s+V}{U}\right)_{zzz} \phi_z^2 dz + \frac{1}{2} \int \left[3\left(\frac{s+V}{U}\right)_z + \left(\frac{D}{U}\right)_{zz} - 4(s+V)\left(\frac{1}{U}\right)_z\right] \phi_{zz}^2 dz \\
 & - \int \left[(s+V)\left(\frac{1}{U}\right)_{zz} + 2V_z\left(\frac{1}{U}\right)_z\right] \phi_z \phi_{zz} dz + \int \left[U\left(\frac{1}{U}\right)_{zz} + 2U_z\left(\frac{1}{U}\right)_z\right] \psi_z \phi_{zz} dz \\
 & - \int \left(\frac{1}{U}\right)_{zz} \phi_z \psi_z \phi_{zz} dz - 2 \int \left(\frac{1}{U}\right)_z (\phi_z \psi_z)_z \phi_{zz} dz \\
 & - 2\varepsilon \int V_{zz} \psi_z \psi_{zz} dz - \varepsilon \int (\psi_z^2)_{zz} \psi_{zz} dz + \int \left(\frac{1}{U} \phi_z \psi_z\right)_{zz} \phi_{zz} dz.
 \end{aligned} \tag{4.42}$$

If we write

$$\begin{aligned}
 & (\psi_z^2)_{zz} \psi_{zz} = ((\psi_z^2)_z \psi_{zz})_z - 2\psi_z \psi_{zz} \psi_{zzz}, \\
 & \left(\frac{1}{U} \phi_z \psi_z\right)_{zz} \phi_{zz} = \left(\frac{1}{U}\right)_{zz} \phi_z \psi_z \phi_{zz} + \left(\frac{1}{U}\right)_z (\phi_z \psi_z)_z \phi_{zz} + \left(\frac{1}{U} (\phi_z \psi_z)_z \phi_{zz}\right)_z - \frac{1}{U} (\phi_z \psi_z)_z \phi_{zzz}
 \end{aligned}$$

and use the boundedness of U, V as well as the boundedness of their derivatives (see Remark 2.1), and utilize the Cauchy–Schwarz inequality, then we have from (4.42) that

$$\begin{aligned}
 & \int (\phi_{zz}^2 + \psi_{zz}^2) dz + D \int_0^t \int \phi_{zzz}^2 dz d\tau + \varepsilon \int_0^t \int \psi_{zzz}^2 dz d\tau \\
 & \leq C \int (\phi_{0,zz}^2 + \psi_{0,zz}^2) dz + C \int_0^t \int (\phi_z^2 + \phi_{zz}^2 + \psi_z^2) dz d\tau \\
 & + C\varepsilon \int_0^t \int (\psi_z^2 + \psi_{zz}^2) dz d\tau + \varepsilon \int_0^t \int |\psi_z \psi_{zz} \psi_{zzz}| dz d\tau \\
 & + C \int_0^t \int |\phi_z \psi_z \phi_{zz}| dz d\tau + C \int_0^t \int |(\phi_z \psi_z)_z| (|\phi_{zz}| + |\phi_{zzz}|) dz d\tau.
 \end{aligned} \tag{4.43}$$

The combination of (4.43) with (4.13) and (4.29) finishes the proof of (4.36). \square

5. Summary

This paper established the existence and nonlinear stability of traveling wave solutions for a system of conservation laws which was derived from the well-known Keller–Segel model through an artful transformation. We proved the existence of traveling wave solutions by the phase plane analysis, and showed the asymptotic nonlinear stability of traveling wave solutions without the smallness assumption on the wave strength via the method of energy estimates. There are several interesting points rising from the current studies, as outlined below.

The first point is about the smallness assumption of ε . Usually the diffusion is a dissipative effect stabilizing the system. But in the problem considered here the diffusion coefficient ε needs to be small. Indeed, ε appears not only as a diffusion coefficient in the transformed system (1.1), but also as a coefficient of the first order term. As a result, we are able to show the existence and stability of traveling wave solutions if ε is small. The question for large ε is still open. We shall investigate such a problem in the future.

The second point is about the stability of traveling wave solutions when $u_+ = 0$. In the present paper, we showed the asymptotic nonlinear stability of traveling wave solutions for $u_+ > 0$. The methods exposed in this paper would fail if $u_+ = 0$. However $u_+ = 0$ is not a physical assumption in general, which indicates that cell density is zero in the region that has not been invaded by cells themselves, for instance the cancer cell density during the metastasis. It is interesting to note that when $u_+ = 0$, the traveling wave equation of the Keller–Segel model (1.3) can be converted to a traveling wave equation of the well-known Fisher–KPP type model [3,8], as shown in [16]. Therefore it is possible to investigate such a problem using the approaches for the study of Fisher–KPP equation.

This paper is focused on the transformed system (1.1) instead of the original Keller–Segel model itself. Our ultimate goal is to translate the results of system (1.1) back to the original Keller–Segel model (1.3) using the transformation (1.4). However it turns out this backward translation will produce some interesting consequences, and the exploration of these outcomes goes beyond the scope of this paper. Instead we shall explore the application of the results of system (1.1) to the original Keller–Segel model in a separate paper.

Acknowledgment

The authors would like to thank the referee for valuable comments which improve the exposition of the manuscript.

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