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J. Differential Equations 261 (2016) 5035–5070

Journal of Differential Equations

www.elsevier.com/locate/jde

Boundary layer problem on a hyperbolic system arising from chemotaxis

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Received 9 February 2016; revised 6 July 2016

Available online 1 August 2016

Abstract

This paper is concerned with the boundary layer problem for a hyperbolic system transformed via a Cole–Hopf type transformation from a repulsive chemotaxis model with logarithmic sensitivity proposed in [23,34] modeling the biological movement of reinforced random walkers which deposit a non-diffusible (or slowly moving) signal that modifies the local environment for succeeding passages. By prescribing the Dirichlet boundary conditions to the transformed hyperbolic system in an interval $(0, 1)$, we show that the system has the boundary layer solutions as the chemical diffusion coefficient $\varepsilon \to 0$, and further use the formal asymptotic analysis to show that the boundary layer thickness is $\varepsilon^{1/2}$. Our work justifies the boundary layer phenomenon that was numerically found in the recent work [25]. However we find that the original chemotaxis system does not possess boundary layer solutions when the results are reverted to the pre-transformed system.

MSC: 35A01; 35B40; 35B44; 35K57; 35Q92; 92C17

Keywords: Chemotaxis; Weighted L^2-estimates; Boundary layers; BL-thickness

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http://dx.doi.org/10.1016/j.jde.2016.07.018
0022-0396/

1. Introduction

Chemotaxis is a common phenomenon in biology describing the change of motion of species in response to a chemical stimulus spread in the environment. The consequence of chemotaxis is that the species changes its movement toward (or away from) a higher concentration of the chemical stimulus. The first chemotaxis model was derived by Keller and Segel in a series of works [20–22] to describe abundant biological processes including the aggregation phase of cellular slime mold and traveling waves formed by bacterial chemotaxis. Since then, numerous variants of the Keller–Segel model have been developed to interpret the various biological phenomena/processes, such as aggregative patterns of bacteria [33,44], slime mould formation [14], fish pigmentation patterning [35], angiogenesis in tumor progression [4], primitive streak formation [36], blood vessel formation [9], wound healing [38], and so on. The Keller–Segel model, in its general form, reads

$$\begin{cases} n_t = [Dn_x - \chi n\phi(c)_x]_x, \\ c_t = \varepsilon c_{xx} + g(n,c), \end{cases} \tag{1.1}$$

where $n(x,t)$ and $c(x,t)$ denote cell density and chemical concentration, respectively. The parameter $D > 0$ is the diffusivity of endothelial cells, χ is the chemotactic coefficient and $\varepsilon \geq 0$ denotes the chemical diffusion rate. The chemotaxis is said to be attractive if $\chi > 0$ and repulsive if $\chi < 0$ with $|\chi|$ measuring the intensity of chemotaxis. The function $\phi(c)$ is commonly called the chemotactic sensitivity function accounting for the chemical signal detection mechanism and $g(n,c)$ denotes the chemical kinetics.

With $\chi > 0$, $\phi(c) = \ln c$ and $g(n,c) = -knc^m$ $(k > 0, m \geq 0)$, the model (1.1) was well-known as the Keller–Segel model proposed in [20] to describe the traveling band propagation of bacterial chemotaxis observed in the famous experiment of Adler [1,2]. The analytical studies of this model have been continuously undertaken in a series of works (see survey papers [17,46] and references therein). When $\chi > 0$, $\phi(c) = c$ and $g(n,c) = n - c$, the model (1.1) was well-known as classical Keller–Segel model first proposed in [21] to describe the aggregation phase of slime mold amoebae *Dictyostelium discoideum*, which has attracted extensive attentions in the past few decades (see survey articles [3,13,16]). In contrast to the attractive chemotaxis models, the studies of repulsive chemotaxis (i.e. $\chi < 0$) are much less and not many results have been developed. It is generally believed that repulsive chemotaxis is a stabilizing factor for the dynamics, but its mathematical mechanism has not been completely understood (see [52]). In this paper, we shall consider the following Keller–Segel type repulsive chemotaxis model

$$\begin{cases} n_t = [Dn_x - \chi n(\ln c)_x]_x, \\ c_t = \varepsilon c_{xx} + nc - \mu c, \end{cases} \tag{1.2}$$

where $\chi < 0$. This model was developed in [23,34] to model the biological movement of reinforced random walkers that deposit a non-diffusible (or slowly moving) substance that modifies the local environment for succeeding passages with little or no transport of the modifying substance.

The characteristic of model (1.2) lies in the logarithmic sensitivity function $\ln c$ which is singular at $c = 0$. This singularity brings great difficulties in analytical studies such as the stability of traveling waves and well-posedness problem. Among other things, the foremost mathematical question is therefore how to resolve this singularity such that analysis can be undertaken. The way used in the literature (see [23,30]) was to apply a Cole–Hopf type transformation

$$q = (\ln c)_x = \frac{c_x}{c}$$

and transform (1.2) into a hyperbolic system augmented with initial data as follows:

$$\begin{cases} p_t - (pq)_x = p_{xx}, \\ q_t - (\varepsilon q^2 + p)_x = \varepsilon q_{xx}, \\ (p,q)(x,0) = (p_0,q_0)(x), \end{cases} \tag{1.3}$$

where we have set $p = n$ and assumed $-\chi = D = 1$ without loss of generality since specific values of χ and D are not important in our analysis. Clearly the transformed system (1.3) removes the singularity and becomes both analytically and numerically tractable. Numerous results of (1.3) have been obtained in recent years from different perspectives. We briefly recall these results according to the value of ε that is particularly relevant to the present paper.

- **Case of** $\varepsilon = 0$. The global well-posedness of (1.3) was studied in [11] for $x \in \mathbb{R}$ and in [5,12, 24] for $x \in \mathbb{R}^N (N \geq 2)$. Furthermore the existence and stability of traveling wave solutions in $\mathbb{R}$ was established in [19,27,29,30,47]. The global existence of solutions of (1.3) in the interval $(0, 1)$ subject to the Neumann-Dirichlet (ND) boundary condition, namely $p_x = q = 0$ at $x = 0, 1$, was obtained in [53] for small data and later in [43] for large data. In the multidimensional bounded domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$, the global existence and exponential decay rates of solutions under Neumann boundary conditions were obtained in [28] for small initial data.
- **Case of** $\varepsilon > 0$. First the existence and stability of traveling wave solutions were established by the second author with his collaborators in a series of works [26,31,32]. For $x \in \mathbb{R}^N (N = 2, 3)$, the global existence and asymptotic behavior of classical solution of (1.3) was recently established in [37,48] for small data, whereas for $x \in (0, 1)$ the global existence and asymptotic behavior of large-data solution was established in [43] with ND boundary conditions and in [25] with Dirichlet boundary conditions.

Except the above-mentioned results, for the model (1.3), it is particularly relevant to consider the solution behavior for small $\varepsilon > 0$ since it conforms to the original idea of [23,30] modeling the movement of reinforced random walks towards a non-diffusible (or slowly moving) substance. This paper will be focused on the asymptotic behavior of solutions of (1.3) as $\varepsilon \to 0$ to understand how the solution behavior could be different with respect to ε. In the works [37,46,48], it has been shown that if the spatial domain is unbounded (i.e. $x \in \mathbb{R}^N$, $N \geq 1$), both traveling wave solutions (see [46]) and global solutions of well-posedness problem (see [37,48]) are uniformly convergent in ε, namely the solution with $\varepsilon > 0$ converges to that with $\varepsilon = 0$ as $\varepsilon \to 0$. If the domain is an interval say $(0, 1)$, and ND boundary condition is prescribed for $\varepsilon \geq 0$:

$$p_x|_{x=0,x=1} = \bar{p} \geq 0, \quad q|_{x=0,x=1} = \bar{q}, \text{ if } \varepsilon \geq 0,$$

then the solution is still uniformly convergent in ε (see [49]). However if the Dirichlet boundary conditions are imposed in $(0, 1)$ as follows:

$$\begin{cases} p|_{x=0,x=1} = \bar{p} \geq 0, \quad q|_{x=0,x=1} = \bar{q}, & \text{if } \varepsilon > 0 \\ p|_{x=0,x=1} = \bar{p} \geq 0, & \text{if } \varepsilon = 0 \end{cases} \tag{1.4}$$

it was numerically illustrated in [25] recently that ε is a singular parameter and a boundary layer for q would arise as $\varepsilon \to 0$. This is a new phenomenon discovered for the chemotaxis model showing that boundary condition might be a significant factor to affect the solution behavior as $\varepsilon > 0$. Mathematical justification of boundary layers was left open in [25]. It is the purpose of this paper to rigorously prove the occurrence of boundary layer process for the solution component q of (1.3)–(1.4) as $\varepsilon \to 0$ and complement the numerical discovery of [25] with analytical justifications. As stressed in [25], the boundary condition of q can not be imposed for $\varepsilon = 0$ since otherwise the problem is over-determined. Due to this special structure, a boundary layer for q may arise as $\varepsilon \to 0$ if the value of q does not match at the boundary between $\varepsilon > 0$ and $\varepsilon = 0$. This is the key observation and starting point of our present work.

The theory of boundary layers has been one of the fundamental and important issues in physics and fluid mechanics [41] since the pioneering work by Prandtl [39] in 1904. The boundary layer phenomenon usually occurs when the inviscid limit of the Navier–Stokes equations near a boundary is considered (cf. [6,7,18,45,50,51]). Moreover, the boundary layer problem also arises in the theory of hyperbolic systems when parabolic equations with small viscosity are applied as perturbations (cf. [8,10,42]). With these empirical results, it is natural to expect that (1.3) may possess boundary layer solutions as ε tends to zero by regarding it as a viscosity coefficient. To the best of our knowledge, this is a new phenomenon found for mathematical models related to chemotaxis and has never been studied before in spite of a large body of works on chemotaxis models. We hope our work can arouse a new interest in chemotaxis researches.

2. Main results

For convenience, we first state some notations.

Notation. Throughout this paper, unless specified, we denote by C a generic positive constant which is independent of ε and may change from one line to another. Without loss of generality, we assume $0 \leq \varepsilon < 1$ for we consider the diffusion limit problem as $\varepsilon \to 0$. For simplicity we denote $\|f\|_{L^p} = \|f\|_{L^p(0,1)}$ for $1 \leq p \leq \infty$, and $H^k = W^{k,2}(0,1)$ denotes the k-th Sobolev space with norm $\|f\|_{H^k}$. The Banach space $W^{1,p}(0,T;X)$ consists of all functions $f \in L^p(0,T;X)$ such that $\partial_t f$ exists in the weak sense and belongs to $L^p(0,T;X)$, with norm denoted by

$$\|f\|_{W^{1,p}(0,T;X)} = \left(\int_0^T \|f(t)\|_X^p + \|\partial_t f(t)\|_X^p \, dt \right)^{1/p}, \quad 1 \leq p < \infty$$

In [25], the authors show that the system (1.3)–(1.4) admits a unique global classical solution. We cite the results below for later use.

Lemma 2.1 *([25]). Suppose that* $(p_0, q_0) \in H^2$ *and satisfies the compatible condition* $(p_0, q_0)(0) = (\bar{p}, \bar{q})$. *Then for any* $\varepsilon \geq 0$, *there exists a unique global classical solution* $(p^\varepsilon, q^\varepsilon)$ *to the initial-boundary value problem (1.3)–(1.4) such that the following hold true.*

(i) *If* $\varepsilon > 0$, *then* $(p^\varepsilon - \bar{p}, q^\varepsilon - \bar{q}) \in C([0,\infty); H^2) \cap L^2([0,\infty); H^3)$, *and for all* $t > 0$, *there is a constant* $C > 0$ *independent of* t *such that*

$$\|(p^\varepsilon - \bar{p})(t)\|_{H^2}^2 + \|(q^\varepsilon - \bar{q})(t)\|_{H^2}^2 + \int_0^t (\|(p^\varepsilon - \bar{p})(\tau)\|_{H^3}^2 + \|(q^\varepsilon - \bar{q})(\tau)\|_{H^3}^2) d\tau \le C. \tag{2.1}$$

(ii) *If $\varepsilon = 0$, then it holds that*

$$(p^0 - \bar{p}) \in C([0,\infty); H^2) \cap L^2([0,\infty); H^3),$$
$$(q^0 - \hat{q}) \in C([0,\infty); H^2) \cap L^2([0,\infty); H^2),$$

and when $\bar{p} > 0$, for all $t > 0$, it holds that

$$\|(p^0 - \bar{p})(t)\|_{H^2}^2 + \|(q^0 - \hat{q})(t)\|_{H^2}^2 + \int_0^t (\|(p^0 - \bar{p})(\tau)\|_{H^3}^2 + \|(q^0 - \hat{q})(\tau)\|_{H^2}^2) d\tau \le C,$$

for some constant $C > 0$ independent of t, where $\hat{q} = \int_0^1 q_0(x)dx$.

Remark 2.2. We remark that the constant C in estimate (2.1), which is obtained by the standard energy methods in [25], depends reciprocally on ε. In [25], the decay rates of solution as $t \to \infty$ were derived also and omitted here for they will not be used in this paper.

Next we recall the conventional definition of boundary layer (BL)-thickness (cf. [7]).

Definition 2.3. Let $(p^\varepsilon, q^\varepsilon)$ and (p^0, q^0) be the solution of (1.3)–(1.4) with $\varepsilon > 0$ and $\varepsilon = 0$, respectively. If there is a non-negative function $\delta = \delta(\varepsilon)$ satisfying $\delta(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ such that

$$\lim_{\varepsilon\to 0} \|p^\varepsilon - p^0\|_{L^\infty(0,T;C[0,1])} = 0,$$
$$\lim_{\varepsilon\to 0} \|q^\varepsilon - q^0\|_{L^\infty(0,T;C[\delta,1-\delta])} = 0,$$
$$\liminf_{\varepsilon\to 0} \|q^\varepsilon - q^0\|_{L^\infty(0,T;C[0,1])} > 0$$

we say that the initial-boundary value problem (1.3)–(1.4) has a boundary layer solution as $\varepsilon \to 0$ and $\delta(\varepsilon)$ is called a BL-thickness.

Remark 2.4. As mentioned in [7], the above definition does not determine the BL-thickness uniquely since any function $\delta_*(\varepsilon)$ satisfying inequality $\delta_*(\varepsilon) > \delta(\varepsilon)$ is also a BL-thickness. In Appendix A, we shall perform a formal asymptotic analysis based on WKB method to show that BL-thickness of the problem (1.3)–(1.4) is $\varepsilon^{1/2}$. But this can not be rigorously proved by the analysis developed in the present paper. We shall study the boundary layer stability and hence justify that BL-thickness of the problem (1.3)–(1.4) is exactly $\varepsilon^{1/2}$ in a future work.

We are now in a position to state the main results of this paper on the boundary layer problem. To this end, we first need a uniform-in-ε bound of solutions to the system (1.3)–(1.4), which is the key to show the existence of boundary layer solutions.

Theorem 2.5 *(Uniform-in-ε estimates). Assume that $(p_0, q_0) \in H^2$ and satisfies the compatible condition $(p_0, q_0)(0) = (\bar{p}, \bar{q})$. Let $(p^\varepsilon, q^\varepsilon)$ be the unique global solution of the system (1.3)–(1.4) with $\varepsilon > 0$ obtained in Lemma 2.1. Then for any $0 < T < \infty$, the following estimates hold*

$$\sup_{0 \le t \le T} \left(\|p_x^\varepsilon\|_{L^2}^2 + \varepsilon^{1/2} \|q_x^\varepsilon\|_{L^2}^2 \right)(t) + \int_0^T \left(\varepsilon^{1/2} \|p_{xx}^\varepsilon\|_{L^2}^2 + \varepsilon^{3/2} \|q_{xx}^\varepsilon\|_{L^2}^2 \right) dt \le C, \tag{2.2}$$

where C is a positive constant independent of ε.

Then the results on the existence of boundary layers for the transformed problem (1.3)–(1.4) are given in the following theorem.

Theorem 2.6. *Assume the conditions of Theorem 2.5 hold. Let $(p^\varepsilon, q^\varepsilon)$ and (p^0, q^0) be the solution of system (1.3)–(1.4) corresponding to $\varepsilon > 0$ and $\varepsilon = 0$, respectively. Then any non-negative function $\delta(\varepsilon)$ satisfying*

$$\delta(\varepsilon) \to 0 \text{ and } \varepsilon^{1/2}/\delta(\varepsilon) \to 0, \text{ as } \varepsilon \to 0$$

is a BL-thickness of (1.3)–(1.4), such that for any $0 < T < \infty$

$$\|p^\varepsilon - p^0\|_{L^\infty(0,T;C[0,1])}^2 < C\varepsilon^{1/2} \tag{2.3}$$

and

$$\|q^\varepsilon - q^0\|_{L^\infty(0,T;C[\delta,1-\delta])}^2 < C\delta^{-1}\varepsilon^{1/2}, \tag{2.4}$$

$$\liminf_{\varepsilon \to 0} \|q^\varepsilon - q^0\|_{L^\infty(0,T;C[0,1])} > 0, \tag{2.5}$$

if and only if

$$\int_0^t p_x^0(0,\tau) d\tau \cdot \int_0^t p_x^0(1,\tau) d\tau \neq 0, \quad \textit{for some } t \in [0,T], \tag{2.6}$$

where the constant C is independent of ε. That is the problem (1.3)–(1.4) has a boundary layer solution as $\varepsilon \to 0$ iff (2.6) holds.

Remark 2.7. If $p_{0x}(0) \cdot p_{0x}(1) \neq 0$, then the condition (2.6) in Theorem 2.6 is satisfied.

Before proceeding, we outline the main ideas employed in this paper to prove Theorem 2.6. The uniform-in-ε estimate (2.2) is the key for the proof of Theorem 2.6. The standard energy method as employed in [25] only can give the estimates depending on ε due to appearance of the boundary term $\varepsilon(q_x^\varepsilon p_x^\varepsilon)|_{x=0}^{x=1}$. For example, the following estimate was obtained in [25, Lemma 2.3]):

$$\|p_x^\varepsilon\|^2_{L^\infty(0,T;L^2)} + \|q_x^\varepsilon\|^2_{L^\infty(0,T;L^2)} + \|p_{xx}^\varepsilon\|^2_{L^2(0,T;L^2)} + \varepsilon\|q_{xx}^\varepsilon\|^2_{L^2(0,T;L^2)} \le C\varepsilon^{-1}$$

where C is a constant independent of ε. Thus to derive the solution convergence as $\varepsilon \to 0$, one needs a new approach to get the estimates of the boundary term $\varepsilon(q_x^\varepsilon p_x^\varepsilon)|_{x=0}^{x=1}$. Observing that by integrating $(1.3)_1$ with respect to x, $p_x^\varepsilon|_{x=0,x=1}$ can be expressed in terms of p_t^ε, and hence bounded by $\|p_t^\varepsilon\|_{L^2}$ and other controllable terms, where $\|p_t^\varepsilon\|_{L^2}$ can be estimated by the routine L^2-energy estimate thanks to the condition $p_t^\varepsilon|_{x=0,x=1} = q_t^\varepsilon|_{x=0,x=1} = 0$ (see Lemma 3.2). Based on this crucial observation, we undertake a refined estimate for $\varepsilon(q_x^\varepsilon p_x^\varepsilon)|_{x=0}^{x=1}$, which readily gives rise to (2.2) by employing various inequalities (see the proof of Lemma 3.3). With the key estimate (2.2), we prove Theorem 2.6 by exploiting the weighted L^2-method, inspired from a work [18]. By a delicate computation, we succeed in deriving the weighted L^2-estimate (see Lemma 4.3):

$$\int_0^1 \xi(x)|(q^\varepsilon - q^0)_x|^2(x,t)\,dx + \varepsilon\int_0^t\int_0^1 \xi(x)|(q^\varepsilon - q^0)_{xx}|^2(x,\tau)\,dxd\tau \le C\varepsilon^{1/2},$$

where $\xi(x) := x^2(1-x)^2,\ x \in [0,1]$. Then we can readily derive (2.4) based on the above estimates.

Theorem 2.6 asserts that the boundary layer for q will arise for the transformed system (1.3)–(1.4) as $\varepsilon \to 0$. Naturally we shall ask whether the original chemotaxis model (1.2) with corresponding boundary conditions will have boundary layer or not. To this end, we need to use the Cole–Hopf type transformation $q = \frac{c_x}{c}$ to pass the results of transformed system (1.3)–(1.4) to the pre-transformed chemotaxis system (1.2) with the corresponding boundary conditions as follows:

$$\begin{cases} n_t = [Dn_x - \chi n(\ln c)_x]_x, \\ c_t = \varepsilon c_{xx} + nc - \mu c, \\ (n,c)(x,0) = (n_0,c_0)(x), \\ n|_{x=0,x=1} = \bar{n}, \quad \dfrac{c_x}{c}|_{x=0,x=1} = \bar{c}, & \text{if } \varepsilon > 0 \\ n|_{x=0,x=1} = \bar{n}, & \text{if } \varepsilon = 0 \end{cases} \tag{2.7}$$

where $D > 0,\ \chi < 0,\ \mu \ge 0$ are constant parameters, and $\bar{n} \ge 0,\ \bar{c} \in \mathbb{R}$ are constants. The global existence of solutions to (2.7) has been obtained in [25] and here we only address the diffusion limit of solutions as $\varepsilon \to 0$.

Proposition 2.8. *Let the initial data satisfy $n_0(x) \ge 0$, $c_0(x) > 0$ and the compatibility conditions: $n_0|_{x=0,x=1} = \bar{n}$, $\frac{c_{0x}}{c_0}|_{x=0,x=1} = \bar{c}$. Assume that $n_0 \in H^2$, $\ln c_0 \in H^3$ and let $(n^\varepsilon, c^\varepsilon)$ and (n^0, c^0) be the unique global solution of system (2.7) with $\varepsilon > 0$ and $\varepsilon = 0$, respectively. Then for any $0 < T < \infty$, it holds that*

$$\|n^\varepsilon - n^0\|^2_{L^\infty(0,T;C[0,1])} < C\varepsilon^{1/2} \tag{2.8}$$

and

$$\|c^\varepsilon - c^0\|^2_{L^\infty(0,T;C[0,1])} < C\varepsilon^{1/2},$$

where the constant C is independent of ε but depends on T.

Remark 2.9. The results in Proposition 2.8 says that the original chemotaxis model (1.2) does not have boundary layer phenomenon as $\varepsilon \to 0$ although the transformed system does. This indicates that non-diffusive problem (2.7) can be approximated by the diffusive problem with small diffusion. Moreover the boundary condition $\frac{c_x}{c}|_{x=0,x=1} = \bar{c}$ may lead to standard boundary conditions. For example, if $\bar{c} = 0$, then it indicates the Neumann boundary condition $c_x|_{x=0,x=1} = 0$, whilst if $\bar{c} \neq 0$ it implies the Robin boundary condition $(c - \frac{1}{\bar{c}} c_x)|_{x=0,x=1} = 0$. We further note that the results in Proposition 2.8 do not include the case where c also has the Dirichlet boundary condition. In other words, it is unknown whether there is a boundary layer if the Dirichlet boundary condition is prescribed to c directly.

3. Proof of Theorem 2.5

Suppose that $(p^\varepsilon, q^\varepsilon)$ is the unique global solution to the system (1.3)–(1.4) with $\varepsilon > 0$ given in Lemma 2.1. In this section we are devoted to deriving the uniform estimate (2.2) for p^ε and q^ε, and thus prove Theorem 2.5. Let $\tilde{p} = p^\varepsilon - \bar{p}$, $\tilde{q} = q^\varepsilon - \bar{q}$. Substituting $\tilde{p}$ and $\tilde{q}$ into (1.3) and (1.4), we can reformulate the problem (1.3)–(1.4) as

$$\begin{cases} \tilde{p}_t = \tilde{p}_{xx} + (\tilde{p}\tilde{q})_x + \bar{p}\tilde{q}_x + \bar{q}\tilde{p}_x, \\ \tilde{q}_t = \varepsilon\tilde{q}_{xx} + \varepsilon[(\tilde{q})^2]_x + 2\varepsilon\bar{q}\tilde{q}_x + \tilde{p}_x, \\ (\tilde{p}, \tilde{q})(x,0) = (p_0 - \bar{p}, q_0 - \bar{q})(x), \\ \tilde{p}|_{x=0,x=1} = 0, \quad \tilde{q}|_{x=0,x=1} = 0. \end{cases} \tag{3.1}$$

The proof of (2.2) consists of a series of lemmas. The first one is the uniform L^2 estimates proved in [25]. We cite it here for later use.

Lemma 3.1 *([25, Lemma 2.2]). Suppose that the assumptions in Theorem 2.5 hold. Then there exists a positive constant C, independent of ε and t, such that*

$$\|(p^\varepsilon - \bar{p})(t)\|^2_{L^2} + \|(q^\varepsilon - \bar{q})(t)\|^2_{L^2} + \int_0^t (\|p^\varepsilon_x\|^2_{L^2} + \varepsilon\|q^\varepsilon_x\|^2_{L^2})\, d\tau \le C.$$

Next we proceed to derive the higher order estimates.

Lemma 3.2. *Suppose that the assumptions in Theorem 2.5 hold. Then for any $0 < T < \infty$, there exists a positive constant C, independent of ε but dependent on T, such that*

$$\sup_{0\le t\le T} \left(\|p^\varepsilon_t(t)\|^2_{L^2} + \|q^\varepsilon_t(t)\|^2_{L^2}\right) + \int_0^T \left(\|p^\varepsilon_{xt}\|^2_{L^2} + \varepsilon\|q^\varepsilon_{xt}\|^2_{L^2}\right) dt \le C.$$

Proof. Differentiating (3.1)$_1$ with respect to t, we have

$$\tilde{p}_{tt} = \tilde{p}_{xxt} + (\tilde{p}\tilde{q})_{xt} + \bar{p}\tilde{q}_{xt} + \bar{q}\,\tilde{p}_{xt}.$$

Taking the L^2 inner product of this equation with $\tilde{p}_t$, integrating the result by parts over $(0, 1)$, and using the boundary conditions, we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\tilde{p}_t\|_{L^2}^2 + \|\tilde{p}_{xt}\|_{L^2}^2 \\
&= -\int_0^1 (\tilde{p}\tilde{q})_t\,\tilde{p}_{xt}\,dx - \bar{p}\int_0^1 \tilde{q}_t\,\tilde{p}_{xt}\,dx \\
&= -\int_0^1 \tilde{p}_t\tilde{q}\,\tilde{p}_{xt}\,dx - \int_0^1 \tilde{p}\tilde{q}_t\,\tilde{p}_{xt}\,dx - \bar{p}\int_0^1 \tilde{q}_t\,\tilde{p}_{xt}\,dx \\
&:= I_1 + I_2 + I_3.
\end{aligned} \tag{3.2}$$

Observing that $\tilde{p}_t|_{x=0,x=1} = 0$, then by Hölder and Gagliardo–Nirenberg interpolation inequalities, we have

$$\begin{aligned}
I_1 &\le \|\tilde{p}_t\|_{L^\infty}\|\tilde{p}_{xt}\|_{L^2}\|\tilde{q}\|_{L^2} \\
&\le C\|\tilde{p}_t\|_{L^2}^{1/2}\|\tilde{p}_{xt}\|_{L^2}^{3/2}\|\tilde{q}\|_{L^2} \\
&\le \frac{1}{8}\|\tilde{p}_{xt}\|_{L^2}^2 + C\|\tilde{q}\|_{L^2}^4\|\tilde{p}_t\|_{L^2}^2.
\end{aligned}$$

Due to the boundary conditions and Sobolev embedding inequality, I_2 is estimated as follows:

$$\begin{aligned}
I_2 &\le \|\tilde{p}\|_{L^\infty}\|\tilde{q}_t\|_{L^2}\|\tilde{p}_{xt}\|_{L^2} \\
&\le C\|\tilde{p}\|_{H^1}\|\tilde{q}_t\|_{L^2}\|\tilde{p}_{xt}\|_{L^2} \\
&\le C\|\tilde{p}_x\|_{L^2}\|\tilde{q}_t\|_{L^2}\|\tilde{p}_{xt}\|_{L^2} \\
&\le \frac{1}{8}\|\tilde{p}_{xt}\|_{L^2}^2 + C\|\tilde{p}_x\|_{L^2}^2\|\tilde{q}_t\|_{L^2}^2.
\end{aligned}$$

Moreover, the Cauchy–Schwarz inequality, yields

$$I_3 \le \frac{1}{4}\|\tilde{p}_{xt}\|_{L^2}^2 + \bar{p}^2\|\tilde{q}_t\|_{L^2}^2.$$

Substituting above estimates for I_1-I_3 into (3.2), we obtain

$$\frac{d}{dt}\|\tilde{p}_t\|_{L^2}^2 + \|\tilde{p}_{xt}\|_{L^2}^2 \le C\|\tilde{q}\|_{L^2}^4\|\tilde{p}_t\|_{L^2}^2 + C(\|\tilde{p}_x\|_{L^2}^2 + \bar{p}^2)\|\tilde{q}_t\|_{L^2}^2. \tag{3.3}$$

We next estimate $\|\tilde{q}_t\|_{L^2}$. Differentiating (3.1)$_2$ with respect to t, gives

$$\tilde{q}_{tt} = \varepsilon \tilde{q}_{xxt} + \varepsilon[(\tilde{q})^2]_{xt} + 2\varepsilon\bar{q}\tilde{q}_{xt} + \tilde{p}_{xt},$$

which, multiplied by $\tilde{q}_t$ and integrated by parts with respect to x over (0, 1), results in

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\tilde{q}_t\|_{L^2}^2 + \varepsilon\|\tilde{q}_{xt}\|_{L^2}^2 &= -\varepsilon\int_0^1 [(\tilde{q})^2]_t \tilde{q}_{xt}\,dx + \int_0^1 \tilde{p}_{xt}\tilde{q}_t\,dx \\ &= -2\varepsilon\int_0^1 \tilde{q}\tilde{q}_t\tilde{q}_{xt}\,dx + \int_0^1 \tilde{p}_{xt}\tilde{q}_t\,dx \\ &:= I_4 + I_5. \end{aligned} \tag{3.4}$$

Upon using Hölder, Poincaré and Sobolev embedding inequalities, we estimate I_4 as

$$\begin{aligned} I_4 &\le 2\varepsilon\|\tilde{q}\|_{L^\infty}\|\tilde{q}_t\|_{L^2}\|\tilde{q}_{xt}\|_{L^2} \\ &\le C\varepsilon\|\tilde{q}\|_{H^1}\|\tilde{q}_t\|_{L^2}\|\tilde{q}_{xt}\|_{L^2} \\ &\le C(\varepsilon^{1/2}\|\tilde{q}_x\|_{L^2})\|\tilde{q}_t\|_{L^2}(\varepsilon^{1/2}\|\tilde{q}_{xt}\|_{L^2}) \\ &\le \frac{1}{2}\varepsilon\|\tilde{q}_{xt}\|_{L^2}^2 + C(\varepsilon\|\tilde{q}_x\|_{L^2}^2)\|\tilde{q}_t\|_{L^2}^2. \end{aligned}$$

With Cauchy–Schwarz inequality, I_5 can be easily estimated as

$$I_5 \le \frac{1}{4}\|\tilde{p}_{xt}\|_{L^2}^2 + \|\tilde{q}_t\|_{L^2}^2.$$

Inserting above estimates for I_4 and I_5 into (3.4), we obtain

$$\frac{d}{dt}\|\tilde{q}_t\|_{L^2}^2 + \varepsilon\|\tilde{q}_{xt}\|_{L^2}^2 \le \frac{1}{2}\|\tilde{p}_{xt}\|_{L^2}^2 + C(\varepsilon\|\tilde{q}_x\|_{L^2}^2 + 1)\|\tilde{q}_t\|_{L^2}^2,$$

which, combined with (3.3), yields

$$\begin{aligned} &\frac{d}{dt}(\|\tilde{p}_t\|_{L^2}^2 + \|\tilde{q}_t\|_{L^2}^2) + (\|\tilde{p}_{xt}\|_{L^2}^2 + \varepsilon\|\tilde{q}_{xt}\|_{L^2}^2) \\ &\quad \le C(\|\tilde{q}\|_{L^2}^4 + \|\tilde{p}_x\|_{L^2}^2 + \varepsilon\|\tilde{q}_x\|_{L^2}^2 + \bar{p}^2 + 1)(\|\tilde{p}_t\|_{L^2}^2 + \|\tilde{q}_t\|_{L^2}^2). \end{aligned}$$

This, along with Gronwall's inequality and Lemma 3.1, gives

$$\|\tilde{p}_t(t)\|_{L^2}^2 + \|\tilde{q}_t(t)\|_{L^2}^2 + \int_0^t \left(\|\tilde{p}_{xt}\|_{L^2}^2 + \varepsilon\|\tilde{q}_{xt}\|_{L^2}^2\right)d\tau \le C,$$

where the constant C is independent of ε but depends on t. The proof of Lemma 3.2 is thus finished. □

Lemma 3.3. *Suppose that the assumptions in Theorem 2.5 hold. Then for any $0 < T < \infty$, there exists a positive constant C, independent of ε but dependent on T, such that*

$$\sup_{0\le t\le T}\left(\|p_x^\varepsilon(t)\|_{L^2}^2+\varepsilon^{1/2}\|q_x^\varepsilon(t)\|_{L^2}^2\right)+\int_0^T\left(\varepsilon^{1/2}\|p_{xx}^\varepsilon\|_{L^2}^2+\varepsilon^{3/2}\|q_{xx}^\varepsilon\|_{L^2}^2\right)dt\le C.$$

Proof. Taking the L^2 inner product of $(3.1)_1$ with $(-\varepsilon\tilde{p}_{xx})$, integrating the result by parts over $(0, 1)$, and using the boundary conditions, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\varepsilon\|\tilde{p}_x\|_{L^2}^2\right)+\varepsilon\|\tilde{p}_{xx}\|_{L^2}^2\\
&\quad=-\varepsilon\int_0^1\tilde{p}_x\tilde{q}\tilde{p}_{xx}\,dx-\varepsilon\int_0^1\tilde{p}\tilde{q}_x\tilde{p}_{xx}\,dx\\
&\qquad-\varepsilon\bar{p}\int_0^1\tilde{q}_x\tilde{p}_{xx}\,dx-\varepsilon\bar{q}\int_0^1\tilde{p}_x\tilde{p}_{xx}\,dx\\
&\quad:=J_1+J_2+J_3+J_4.
\end{aligned}\tag{3.5}$$

We next estimate J_1–J_4. First by the boundary conditions and Hölder and Sobolev embedding inequalities, we infer that

$$\begin{aligned}
J_1&\le\varepsilon\|\tilde{p}_x\|_{L^2}\|\tilde{q}\|_{L^\infty}\|\tilde{p}_{xx}\|_{L^2}\\
&\le C\varepsilon\|\tilde{p}_x\|_{L^2}\|\tilde{q}\|_{H^1}\|\tilde{p}_{xx}\|_{L^2}\\
&\le C\|\tilde{p}_x\|_{L^2}(\varepsilon^{1/2}\|\tilde{q}_x\|_{L^2})(\varepsilon^{1/2}\|\tilde{p}_{xx}\|_{L^2})\\
&\le\frac{1}{8}\varepsilon\|\tilde{p}_{xx}\|_{L^2}^2+C\|\tilde{p}_x\|_{L^2}^2(\varepsilon\|\tilde{q}_x\|_{L^2}^2)
\end{aligned}$$

and

$$\begin{aligned}
J_2&\le\varepsilon\|\tilde{p}\|_{L^\infty}\|\tilde{q}_x\|_{L^2}\|\tilde{p}_{xx}\|_{L^2}\\
&\le C\varepsilon\|\tilde{p}\|_{H^1}\|\tilde{q}_x\|_{L^2}\|\tilde{p}_{xx}\|_{L^2}\\
&\le C\|\tilde{p}_x\|_{L^2}(\varepsilon^{1/2}\|\tilde{q}_x\|_{L^2})(\varepsilon^{1/2}\|\tilde{p}_{xx}\|_{L^2})\\
&\le\frac{1}{8}\varepsilon\|\tilde{p}_{xx}\|_{L^2}^2+C\|\tilde{p}_x\|_{L^2}^2(\varepsilon\|\tilde{q}_x\|_{L^2}^2).
\end{aligned}$$

Furthermore, the Cauchy–Schwarz inequality gives

$$\begin{aligned}
J_3+J_4&\le\left(\bar{p}\varepsilon^{1/2}\|\tilde{q}_x\|_{L^2}+|\bar{q}|\varepsilon^{1/2}\|\tilde{p}_x\|_{L^2}\right)(\varepsilon^{1/2}\|\tilde{p}_{xx}\|_{L^2})\\
&\le\frac{1}{4}\varepsilon\|\tilde{p}_{xx}\|_{L^2}^2+2\bar{p}^2(\varepsilon\|\tilde{q}_x\|_{L^2}^2)+2\bar{q}^2(\varepsilon\|\tilde{p}_x\|_{L^2}^2).
\end{aligned}$$

Then it follows from (3.5) that

$$\frac{d}{dt}(\varepsilon\|\tilde{p}_x\|_{L^2}^2)+\varepsilon\|\tilde{p}_{xx}\|_{L^2}^2 \le C(\|\tilde{p}_x\|_{L^2}^2+\bar{p}^2+\bar{q}^2)(\varepsilon\|\tilde{p}_x\|_{L^2}^2+\varepsilon\|\tilde{q}_x\|_{L^2}^2). \tag{3.6}$$

We are now in a position to estimate $\|\tilde{q}_x\|_{L^2}$. Taking the L^2 inner product of $(3.1)_2$ with $(-\varepsilon\tilde{q}_{xx})$, and integrating the result by parts, we derive

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\varepsilon\|\tilde{q}_x\|_{L^2}^2\right)+\varepsilon^2\|\tilde{q}_{xx}\|_{L^2}^2 &= -2\varepsilon^2\int_0^1 \tilde{q}\tilde{q}_x\tilde{q}_{xx}\,dx - 2\varepsilon^2\bar{q}\int_0^1 \tilde{q}_x\tilde{q}_{xx}\,dx \\ &\quad + \varepsilon\int_0^1 \tilde{p}_{xx}\tilde{q}_x\,dx - \varepsilon(\tilde{p}_x\tilde{q}_x)|_{x=0}^{x=1} \\ &:= J_5+J_6+J_7+J_8.\end{aligned} \tag{3.7}$$

We proceed to estimate J_5 - J_8. Using Hölder and Sobolev embedding inequalities and the boundary conditions, we deduce

$$\begin{aligned} J_5 &\le 2\varepsilon^2\|\tilde{q}\|_{L^\infty}\|\tilde{q}_x\|_{L^2}\|\tilde{q}_{xx}\|_{L^2} \\ &\le C\varepsilon^2\|\tilde{q}\|_{H^1}\|\tilde{q}_x\|_{L^2}\|\tilde{q}_{xx}\|_{L^2} \\ &\le C(\varepsilon\|\tilde{q}_x\|_{L^2}^2)(\varepsilon\|\tilde{q}_{xx}\|_{L^2}) \\ &\le \frac{1}{8}\varepsilon^2\|\tilde{q}_{xx}\|_{L^2}^2 + C(\varepsilon\|\tilde{q}_x\|_{L^2}^2)^2. \end{aligned}$$

By Cauchy–Schwarz inequality and the assumption that $0<\varepsilon<1$, we obtain

$$\begin{aligned} J_6 &\le 2\varepsilon^2\bar{q}\|\tilde{q}_x\|_{L^2}\|\tilde{q}_{xx}\|_{L^2} \\ &\le \frac{1}{8}\varepsilon^2\|\tilde{q}_{xx}\|_{L^2}^2 + C\varepsilon\bar{q}^2(\varepsilon\|\tilde{q}_x\|_{L^2}^2) \\ &\le \frac{1}{8}\varepsilon^2\|\tilde{q}_{xx}\|_{L^2}^2 + C\bar{q}^2(\varepsilon\|\tilde{q}_x\|_{L^2}^2) \end{aligned}$$

and

$$J_7 \le \frac{1}{4}\varepsilon\|\tilde{p}_{xx}\|_{L^2}^2 + \varepsilon\|\tilde{q}_x\|_{L^2}^2.$$

To estimate J_8, we rewrite $\tilde{p}_x|_{x=0,x=1}$ as follows. First, integrating $(3.1)_1$ over $(x,1)$, we have

$$\begin{aligned}\tilde{p}_x(1,t) &= \tilde{p}_x(x,t) + \int_x^1 \tilde{p}_{yy}\,dy \\ &= \tilde{p}_x(x,t) + \int_x^1 \tilde{p}_t\,dy - \int_x^1 (\tilde{p}\tilde{q})_y\,dy - \bar{p}\int_x^1 \tilde{q}_y\,dy - \bar{q}\int_x^1 \tilde{p}_y\,dy \\ &= \tilde{p}_x(x,t) + \int_x^1 \tilde{p}_t\,dy - [(\tilde{p}\tilde{q})(1,t) - (\tilde{p}\tilde{q})(x,t)] \\ &\quad - \bar{p}[\tilde{q}(1,t) - \tilde{q}(x,t)] - \bar{q}[\tilde{p}(1,t) - \tilde{p}(x,t)] \\ &= \tilde{p}_x(x,t) + \int_x^1 \tilde{p}_t\,dy + (\tilde{p}\tilde{q})(x,t) + \bar{p}\tilde{q}(x,t) + \bar{q}\tilde{p}(x,t),\end{aligned} \tag{3.8}$$

where we have used the boundary conditions $\tilde{p}(1,t) = \tilde{q}(1,t) = 0$. Then integrating (3.8) over $(0, 1)$ with respect to x, and using the boundary conditions again, we end up with

$$\begin{aligned}\tilde{p}_x(1,t) &= \int_0^1 \tilde{p}_x(x,t)\,dx + \int_0^1\int_x^1 \tilde{p}_t\,dydx \\ &\quad + \int_0^1 (\tilde{p}\tilde{q})(x,t)\,dx + \bar{p}\int_0^1 \tilde{q}(x,t)\,dx + \bar{q}\int_0^1 \tilde{p}(x,t)\,dx \\ &= \int_0^1\int_x^1 \tilde{p}_t\,dydx + \int_0^1 (\tilde{p}\tilde{q})(x,t)\,dx \\ &\quad + \bar{p}\int_0^1 \tilde{q}(x,t)\,dx + \bar{q}\int_0^1 \tilde{p}(x,t)\,dx,\end{aligned}$$

which, upon the application of Hölder inequality, gives

$$|\tilde{p}_x(1,t)| \le \|\tilde{p}_t\|_{L^2} + \|\tilde{p}\|_{L^2}\|\tilde{q}\|_{L^2} + \bar{p}\|\tilde{q}\|_{L^2} + |\bar{q}|\|\tilde{p}\|_{L^2}. \tag{3.9}$$

In a similar fashion as to obtain (3.9), we derive

$$|\tilde{p}_x(0,t)| \le \|\tilde{p}_t\|_{L^2} + \|\tilde{p}\|_{L^2}\|\tilde{q}\|_{L^2} + \bar{p}\|\tilde{q}\|_{L^2} + |\bar{q}|\|\tilde{p}\|_{L^2}. \tag{3.10}$$

Combination of (3.9), (3.10) and Gagliardo–Nirenberg interpolation inequality, gives

$$\begin{aligned}
J_8 &\le \varepsilon\|\tilde{q}_x\|_{L^\infty}(|\tilde{p}_x(0,t)| + |\tilde{p}_x(1,t)|)\\
&\le 2\varepsilon\|\tilde{q}_x\|_{L^\infty}(\|\tilde{p}_t\|_{L^2} + \|\tilde{p}\|_{L^2}\|\tilde{q}\|_{L^2} + \bar{p}\|\tilde{q}\|_{L^2} + |\bar{q}|\|\tilde{p}\|_{L^2})\\
&\le C\varepsilon(\|\tilde{q}_x\|_{L^2} + \|\tilde{q}_x\|_{L^2}^{1/2}\|\tilde{q}_{xx}\|_{L^2}^{1/2})\\
&\quad\times(\|\tilde{p}_t\|_{L^2} + \|\tilde{p}\|_{L^2}\|\tilde{q}\|_{L^2} + \bar{p}\|\tilde{q}\|_{L^2} + |\bar{q}|\|\tilde{p}\|_{L^2})\\
&\le \frac{1}{4}\varepsilon^2\|\tilde{q}_{xx}\|_{L^2}^2 + \varepsilon\|\tilde{q}_x\|_{L^2}^2\\
&\quad + C\varepsilon^{1/2}(\|\tilde{p}_t\|_{L^2}^2 + \|\tilde{p}\|_{L^2}^2\|\tilde{q}\|_{L^2}^2 + \bar{p}^2\|\tilde{q}\|_{L^2}^2 + \bar{q}^2\|\tilde{p}\|_{L^2}^2),
\end{aligned}$$

where the assumption that $0 < \varepsilon < 1$ has been used. Substituting above estimates for J_5–J_8 into (3.7), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\varepsilon\|\tilde{q}_x\|_{L^2}^2\right) + \varepsilon^2\|\tilde{q}_{xx}\|_{L^2}^2\\
&\quad\le \frac{1}{2}\varepsilon\|\tilde{p}_{xx}\|_{L^2}^2 + C(\varepsilon\|\tilde{q}_x\|_{L^2}^2 + \bar{q}^2 + 1)(\varepsilon\|\tilde{q}_x\|_{L^2}^2)\\
&\qquad + C\varepsilon^{1/2}(\|\tilde{p}_t\|_{L^2}^2 + \|\tilde{p}\|_{L^2}^2\|\tilde{q}\|_{L^2}^2 + \bar{p}^2\|\tilde{q}\|_{L^2}^2 + \bar{q}^2\|\tilde{p}\|_{L^2}^2),
\end{aligned}$$

which, added to (3.6), yields

$$\begin{aligned}
&\frac{d}{dt}(\varepsilon\|\tilde{p}_x\|_{L^2}^2 + \varepsilon\|\tilde{q}_x\|_{L^2}^2) + (\varepsilon\|\tilde{p}_{xx}\|_{L^2}^2 + \varepsilon^2\|\tilde{q}_{xx}\|_{L^2}^2)\\
&\quad\le C(\|\tilde{p}_x\|_{L^2}^2 + \varepsilon\|\tilde{q}_x\|_{L^2}^2 + \bar{p}^2 + \bar{q}^2 + 1)(\varepsilon\|\tilde{p}_x\|_{L^2}^2 + \varepsilon\|\tilde{q}_x\|_{L^2}^2)\\
&\qquad + C\varepsilon^{1/2}(\|\tilde{p}_t\|_{L^2}^2 + \|\tilde{p}\|_{L^2}^2\|\tilde{q}\|_{L^2}^2 + \bar{p}^2\|\tilde{q}\|_{L^2}^2 + \bar{q}^2\|\tilde{p}\|_{L^2}^2).
\end{aligned}$$

This, combined with Lemma 3.1, Lemma 3.2 and Gronwall's inequality, gives

$$\varepsilon\|\tilde{p}_x(t)\|_{L^2}^2 + \varepsilon\|\tilde{q}_x(t)\|_{L^2}^2 + \int_0^t \left(\varepsilon\|\tilde{p}_{xx}\|_{L^2}^2 + \varepsilon^2\|\tilde{q}_{xx}\|_{L^2}^2\right) d\tau \le C\varepsilon^{1/2},$$

where the constant C is independent of ε but depends on t, which implies

$$\varepsilon^{1/2}\|\tilde{q}_x(t)\|_{L^2}^2 + \int_0^t \left(\varepsilon^{1/2}\|\tilde{p}_{xx}\|_{L^2}^2 + \varepsilon^{3/2}\|\tilde{q}_{xx}\|_{L^2}^2\right) d\tau \le C. \tag{3.11}$$

As a consequence of Lemma 3.1 and Lemma 3.2, we get

$$\|\tilde{p}_x\|_{W^{1,2}(0,T;L^2)}^2 \le C,$$

which, along with the Sobolev embedding inequality, yields

$$\|\tilde{p}_x\|_{L^\infty(0,T;L^2)}^2 \le C\|\tilde{p}_x\|_{W^{1,2}(0,T;L^2)}^2 \le C.$$

This, combined with (3.11) completes the proof. □

Proof of Theorem 2.5. By Lemma 3.3, we derive estimate (2.2), which finishes the proof of Theorem 2.5.

4. Boundary layer problem (proof of Theorem 2.6)

Recall that $(p^\varepsilon, q^\varepsilon)$ denote the global solution of (1.3)–(1.4) with $\varepsilon \geq 0$. For convenience, we set

$$u = p^\varepsilon - p^0,\ v = q^\varepsilon - q^0 \tag{4.1}$$

Then from system (1.3)-(1.4), we deduce that (u, v) satisfies the following initial-boundary value problem:

$$\begin{cases} u_t = u_{xx} + (p^\varepsilon v + uq^0)_x, \\ v_t = \varepsilon v_{xx} + \varepsilon q^0_{xx} + \varepsilon[(q^\varepsilon)^2]_x + u_x, \\ u|_{x=0,x=1} = 0, \quad v|_{x=0,x=1} = (\bar{q} - q^0)|_{x=0,x=1}, \\ u(x,0) = 0, \quad v(x,0) = 0. \end{cases} \tag{4.2}$$

Based on the reformulated problem (4.2), we shall derive a series of results below.

Lemma 4.1. *Suppose that the assumptions in Theorem 2.6 hold. Then for any $0 < T < \infty$, there exists a positive constant C, independent of ε but dependent on T, such that*

$$\sup_{0\leq t\leq T} \left(\|(p^\varepsilon - p^0)(t)\|^2_{L^2} + \|(q^\varepsilon - q^0)(t)\|^2_{L^2} \right) + \int_0^T \left(\|(p^\varepsilon - p^0)_x\|^2_{L^2} + \varepsilon \|(q^\varepsilon - q^0)_x\|^2_{L^2} \right) dt \leq C\varepsilon^{1/2}.$$

Proof. Testing $(4.2)_1$ with u, integrating the result by parts, with Hölder and Sobolev embedding inequalities we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2} + \|u_x\|^2_{L^2} &= -\int_0^1 p^\varepsilon v u_x\, dx - \int_0^1 uq^0 u_x\, dx \\ &\leq \|p^\varepsilon\|_{L^\infty}\|v\|_{L^2}\|u_x\|_{L^2} + \|u\|_{L^2}\|q^0\|_{L^\infty}\|u_x\|_{L^2} \\ &\leq \frac{1}{4}\|u_x\|^2_{L^2} + C\|p^\varepsilon\|^2_{H^1}\|v\|^2_{L^2} + C\|q^0\|^2_{H^1}\|u\|^2_{L^2}. \end{aligned} \tag{4.3}$$

Taking the L^2 inner product of $(4.2)_2$ with v, and using the integration by parts again, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2+\varepsilon\|v_x\|_{L^2}^2&=\varepsilon\int_0^1 q_{xx}^0 v\,dx+2\varepsilon\int_0^1 q^\varepsilon q_x^\varepsilon v\,dx+\int_0^1 u_x v\,dx+\varepsilon(vv_x)|_{x=0}^{x=1}\\&:=K_1+K_2+K_3+K_4.\end{aligned}\tag{4.4}$$

By Hölder and Sobolev embedding inequalities, we estimate K_1–K_3 as follows:

$$\begin{aligned}K_1&\le\varepsilon^2\|q_{xx}^0\|_{L^2}^2+\|v\|_{L^2}^2,\\K_2&\le 2\varepsilon\|q^\varepsilon\|_{L^\infty}\|q_x^\varepsilon\|_{L^2}\|v\|_{L^2}\\&\le C\varepsilon(\|q^\varepsilon\|_{L^2}+\|q_x^\varepsilon\|_{L^2})\|q_x^\varepsilon\|_{L^2}\|v\|_{L^2}\\&\le C\varepsilon\|q_x^\varepsilon\|_{L^2}^2+C(\varepsilon\|q^\varepsilon\|_{L^2}^2+\varepsilon\|q_x^\varepsilon\|_{L^2}^2)\|v\|_{L^2}^2\end{aligned}$$

and

$$K_3\le\frac{1}{8}\|u_x\|_{L^2}^2+2\|v\|_{L^2}^2.$$

With boundary conditions in (4.2), we rewrite K_4 as follows:

$$\begin{aligned}K_4&=\varepsilon[(\bar q-q^0)(q_x^\varepsilon-q_x^0)]|_{x=0}^{x=1}\\&=\varepsilon[(\bar q-q^0)q_x^\varepsilon]|_{x=0}^{x=1}-\varepsilon[(\bar q-q^0)q_x^0]|_{x=0}^{x=1}\\&:=M_1+M_2.\end{aligned}$$

By Hölder and Gagliardo–Nirenberg interpolation inequalities, we deduce

$$\begin{aligned}M_1&\le 2\varepsilon(\bar q+\|q^0\|_{L^\infty})\|q_x^\varepsilon\|_{L^\infty}\\&\le C\varepsilon(\bar q+\|q^0\|_{H^1})\left(\|q_x^\varepsilon\|_{L^2}+\|q_x^\varepsilon\|_{L^2}^{1/2}\|q_{xx}^\varepsilon\|_{L^2}^{1/2}\right)\\&=C\varepsilon^{1/2}(\bar q+\|q^0\|_{H^1})\left(\varepsilon^{1/2}\|q_x^\varepsilon\|_{L^2}+(\varepsilon^{1/8}\|q_x^\varepsilon\|_{L^2}^{1/2})(\varepsilon^{3/8}\|q_{xx}^\varepsilon\|_{L^2}^{1/2})\right)\\&\le C\varepsilon^{1/2}\left((\bar q+\|q^0\|_{H^1})^2+\varepsilon\|q_x^\varepsilon\|_{L^2}^2+\varepsilon^{1/2}\|q_x^\varepsilon\|_{L^2}^2+\varepsilon^{3/2}\|q_{xx}^\varepsilon\|_{L^2}^2\right)\end{aligned}$$

and

$$\begin{aligned}M_2&\le 2\varepsilon(\bar q+\|q^0\|_{L^\infty})\|q_x^0\|_{L^\infty}\\&\le C\varepsilon(\bar q+\|q^0\|_{H^2})^2.\end{aligned}$$

With the above estimates for M_1 and M_2, and keeping in mind that $0<\varepsilon<1$, we get

$$K_4\le C\varepsilon^{1/2}\left((\bar q+\|q^0\|_{H^2})^2+\varepsilon^{1/2}\|q_x^\varepsilon\|_{L^2}^2+\varepsilon^{3/2}\|q_{xx}^\varepsilon\|_{L^2}^2\right),$$

which, combined with the above estimates for K_1–K_3 and (4.4), leads to

$$\begin{aligned}&\frac{d}{dt}\|v\|_{L^2}^2+\varepsilon\|v_x\|_{L^2}^2\\&\le\frac{1}{4}\|u_x\|_{L^2}^2+C(\|q^\varepsilon\|_{L^2}^2+\varepsilon\|q_x^\varepsilon\|_{L^2}^2+1)\|v\|_{L^2}^2\\&\quad+C\varepsilon^{1/2}\left((\bar{q}+\|q^0\|_{H^2})^2+\varepsilon^{1/2}\|q_x^\varepsilon\|_{L^2}^2+\varepsilon^{3/2}\|q_{xx}^\varepsilon\|_{L^2}^2\right).\end{aligned}$$

This, along with (4.3) gives

$$\begin{aligned}&\frac{d}{dt}(\|u\|_{L^2}^2+\|v\|_{L^2}^2)+(\|u_x\|_{L^2}^2+\varepsilon\|v_x\|_{L^2}^2)\\&\le C(\|p^\varepsilon\|_{H^1}^2+\|q^0\|_{H^1}^2+\|q^\varepsilon\|_{L^2}^2+\varepsilon\|q_x^\varepsilon\|_{L^2}^2+1)(\|u\|_{L^2}^2+\|v\|_{L^2}^2)\\&\quad+C\varepsilon^{1/2}\left((\bar{q}+\|q^0\|_{H^2})^2+\varepsilon^{1/2}\|q_x^\varepsilon\|_{L^2}^2+\varepsilon^{3/2}\|q_{xx}^\varepsilon\|_{L^2}^2\right).\end{aligned}$$

Applying Gronwall's inequality to this, and using Part (ii) of Lemma 2.1, Theorem 2.5 and Lemma 3.1, we arrive at

$$\|u(t)\|_{L^2}^2+\|v(t)\|_{L^2}^2+\int_0^t\left(\|u_x\|_{L^2}^2+\varepsilon\|v_x\|_{L^2}^2\right)d\tau\le C\varepsilon^{1/2},\tag{4.5}$$

where the constant C is independent of ε but depends on t. This, along with the convention (4.1), completes the proof. □

Lemma 4.2. *Suppose that the assumptions in Theorem 2.6 hold. Then for any $0<T<\infty$, there exists a positive constant C, independent of ε but dependent on T, such that*

$$\varepsilon\sup_{0\le t\le T}\|(q^\varepsilon-q^0)_x(t)\|_{L^2}^2+\int_0^T\|(q^\varepsilon-q^0)_t\|_{L^2}^2\,dt\le C\varepsilon^{1/2}\tag{4.6}$$

and

$$\sup_{0\le t\le T}\|(p^\varepsilon-p^0)_x(t)\|_{L^2}^2+\int_0^T\|(p^\varepsilon-p^0)_t\|_{L^2}^2\,dt\le C\varepsilon^{1/2}.\tag{4.7}$$

Proof. We first estimate (4.6). Taking the L^2 inner product of $(4.2)_2$ with v_t and integrating the result by parts, we derive

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\varepsilon\|v_x\|_{L^2}^2\right)+\|v_t\|_{L^2}^2&=\varepsilon\int_0^1 q_{xx}^0 v_t\,dx+2\varepsilon\int_0^1 q^\varepsilon q_x^\varepsilon v_t\,dx\\&\quad+\int_0^1 u_x v_t\,dx+\varepsilon(v_x v_t)|_{x=0}^{x=1}\\&:=K_5+K_6+K_7+K_8.\end{aligned}\tag{4.8}$$

First by Cauchy–Schwarz inequality and Part (ii) of Lemma 2.1, we obtain

$$K_5\le 2\varepsilon^2\|q_{xx}^0\|_{L^2}^2+\frac{1}{8}\|v_t\|_{L^2}^2\le C\varepsilon^2+\frac{1}{8}\|v_t\|_{L^2}^2.$$

Then using Hölder and Sobolev embedding inequalities, we estimate K_6 and K_7 as follows:

$$\begin{aligned}K_6&\le 2\varepsilon\|q^\varepsilon\|_{L^\infty}\|q_x^\varepsilon\|_{L^2}\|v_t\|_{L^2}\\&\le C\varepsilon(\|q^\varepsilon\|_{L^2}+\|q_x^\varepsilon\|_{L^2})\|q_x^\varepsilon\|_{L^2}\|v_t\|_{L^2}\\&\le\frac{1}{4}\|v_t\|_{L^2}^2+C\varepsilon(\varepsilon\|q_x^\varepsilon\|_{L^2}^2\|q^\varepsilon\|_{L^2}^2+\varepsilon\|q_x^\varepsilon\|_{L^2}^4)\end{aligned}$$

and

$$K_7\le\frac{1}{8}\|v_t\|_{L^2}^2+2\|u_x\|_{L^2}^2.$$

With the boundary conditions, Sobolev embedding $W^{1,2}(0,1)\hookrightarrow C([0,1])$, and Gagliardo–Nirenberg interpolation inequality, K_8 is estimated as follows:

$$\begin{aligned}K_8&=\varepsilon[(q_x^\varepsilon-q_x^0)(-q_t^0)]|_{x=0}^{x=1}\\&\le 2\varepsilon\|q_x^\varepsilon\|_{L^\infty}\|q_t^0\|_{L^\infty}+2\varepsilon\|q_x^0\|_{L^\infty}\|q_t^0\|_{L^\infty}\\&\le C\varepsilon(\|q_x^\varepsilon\|_{L^2}+\|q_x^\varepsilon\|_{L^2}^{1/2}\|q_{xx}^\varepsilon\|_{L^2}^{1/2})\|q_t^0\|_{H^1}+C\varepsilon\|q^0\|_{H^2}\|q_t^0\|_{H^1}\\&=C\varepsilon^{1/2}\left((\varepsilon^{1/2}\|q_x^\varepsilon\|_{L^2})+(\varepsilon^{1/8}\|q_x^\varepsilon\|_{L^2}^{1/2})(\varepsilon^{3/8}\|q_{xx}^\varepsilon\|_{L^2}^{1/2})\right)\|q_t^0\|_{H^1}\\&\quad+C\varepsilon\|q^0\|_{H^2}\|q_t^0\|_{H^1}\\&\le C\varepsilon^{1/2}\left(\varepsilon^{1/2}\|q_x^\varepsilon\|_{L^2}^2+\varepsilon^{3/2}\|q_{xx}^\varepsilon\|_{L^2}^2+\|q_t^0\|_{H^1}^2\right)+C\varepsilon\|q^0\|_{H^2}\|q_t^0\|_{H^1},\end{aligned}\tag{4.9}$$

where we have used the assumption that $0<\varepsilon<1$. We proceed to estimate $\|q_t^0\|_{H^1}$ in the right-hand side of (4.9). By equation $(1.3)_2$ with $\varepsilon=0$ and Part (ii) of Lemma 2.1, we derive

$$\|q_t^0\|_{H^1}=\|p_x^0\|_{H^1}\le\|p^0\|_{H^2}\le C.$$

Putting the above estimates into (4.9), and using Part (ii) of Lemma 2.1 again, we obtain that for $0<\varepsilon<1$

$$K_8 \le C\varepsilon^{1/2}(\varepsilon^{1/2}\|q_x^\varepsilon\|_{L^2}^2 + \varepsilon^{3/2}\|q_{xx}^\varepsilon\|_{L^2}^2 + 1).$$

Substituting the above estimates for K_5–K_8 into (4.8), using Theorem 2.5 and Lemma 3.1 we deduce

$$\begin{aligned}
&\frac{d}{dt}\left(\varepsilon\|v_x\|_{L^2}^2\right) + \|v_t\|_{L^2}^2 \\
&\quad \le 4\|u_x\|_{L^2}^2 + C\varepsilon\|q_x^\varepsilon\|_{L^2}^2\left(\|q^\varepsilon\|_{L^2}^2 + \varepsilon^{1/2}\|q_x^\varepsilon\|_{L^2}^2\right) \\
&\qquad + C\varepsilon^{1/2}\left(\varepsilon^{1/2}\|q_x^\varepsilon\|_{L^2}^2 + \varepsilon^{3/2}\|q_{xx}^\varepsilon\|_{L^2}^2 + 1\right) \\
&\quad \le 4\|u_x\|_{L^2}^2 + C\varepsilon^{1/2}\left(\varepsilon^{1/2}\|q_x^\varepsilon\|_{L^2}^2 + \varepsilon^{3/2}\|q_{xx}^\varepsilon\|_{L^2}^2 + 1\right),
\end{aligned}$$

where the assumption that $0 < \varepsilon < 1$ has been used. Integrating this inequality over $(0, t)$ and using Theorem 2.5 and Lemma 4.1, we obtain

$$\varepsilon\|v_x(t)\|_{L^2}^2 + \int_0^t \|v_t\|_{L^2}^2\, d\tau \le C\varepsilon^{1/2}, \tag{4.10}$$

where the constant C is independent of ε but depends on t. The above estimate completes the proof of (4.6).

We next prove (4.7). Testing $(4.2)_1$ with u_t, integrating the result by parts, and using the boundary conditions, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_x\|_{L^2}^2 + \|u_t\|_{L^2}^2 &= -\int_0^1 p^\varepsilon v u_{xt}\, dx + \int_0^1 (uq^0)_x u_t\, dx \\
&= -\frac{d}{dt}\int_0^1 p^\varepsilon v u_x\, dx + \int_0^1 (p^\varepsilon v)_t u_x\, dx + \int_0^1 (uq^0)_x u_t\, dx \\
&= -\frac{d}{dt}\int_0^1 \left(\left(p^\varepsilon v + \frac{u_x}{2}\right)^2 - (p^\varepsilon v)^2 - \frac{u_x^2}{4}\right) dx \\
&\quad + \int_0^1 (p^\varepsilon v)_t u_x\, dx + \int_0^1 (uq^0)_x u_t\, dx \\
&= -\frac{d}{dt}\int_0^1 \left(p^\varepsilon v + \frac{u_x}{2}\right)^2 dx + \frac{d}{dt}\int_0^1 (p^\varepsilon v)^2\, dx + \frac{1}{4}\frac{d}{dt}\int_0^1 u_x^2\, dx \\
&\quad + \int_0^1 (p^\varepsilon v)_t u_x\, dx + \int_0^1 (uq^0)_x u_t\, dx,
\end{aligned}$$

which, gives

$$\frac{1}{4}\frac{d}{dt}\|u_x\|_{L^2}^2+\frac{d}{dt}\left\|\left(p^\varepsilon v+\frac{u_x}{2}\right)\right\|_{L^2}^2+\|u_t\|_{L^2}^2$$
$$=\frac{d}{dt}\int_0^1(p^\varepsilon v)^2\,dx+\int_0^1(p^\varepsilon v)_t u_x\,dx+\int_0^1(uq^0)_x u_t\,dx.$$

For fixed $t\in(0,T]$, integrating this equation over $(0,t)$ and using the initial conditions, we deduce

$$\begin{aligned}&\frac{1}{4}\|u_x(t)\|_{L^2}^2+\left\|\left(p^\varepsilon v+\frac{u_x}{2}\right)(t)\right\|_{L^2}^2+\int_0^t\|u_t\|_{L^2}^2\,d\tau\\&=\|(p^\varepsilon v)(t)\|_{L^2}^2+\int_0^t\int_0^1(p^\varepsilon v)_t u_x\,dx\,d\tau+\int_0^t\int_0^1(uq^0)_x u_t\,dx\,d\tau\\&:=K_9+K_{10}+K_{11}.\end{aligned}\tag{4.11}$$

Let us estimate K_9–K_{11}. First by Theorem 2.1, Lemma 3.1, Lemma 4.1 and the Sobolev embedding inequality, we get

$$\begin{aligned}K_9&\le\|p^\varepsilon(t)\|_{L^\infty}^2\|v(t)\|_{L^2}^2\\&\le C\left(\|p^\varepsilon(t)\|_{L^2}^2+\|p_x^\varepsilon(t)\|_{L^2}^2\right)\|v(t)\|_{L^2}^2\\&\le C\varepsilon^{1/2}.\end{aligned}$$

With Hölder and Sobolev embedding inequalities, we have

$$\begin{aligned}K_{10}&=\int_0^t\int_0^1 p_t^\varepsilon v u_x\,dx\,d\tau+\int_0^t\int_0^1 p^\varepsilon v_t u_x\,dx\,d\tau\\&\le\|p_t^\varepsilon\|_{L^2(0,T;L^\infty)}\|v\|_{L^\infty(0,T;L^2)}\|u_x\|_{L^2(0,T;L^2)}\\&\quad+\|p^\varepsilon\|_{L^\infty(0,T;L^\infty)}\|v_t\|_{L^2(0,T;L^2)}\|u_x\|_{L^2(0,T;L^2)}\\&\le C(\|p_t^\varepsilon\|_{L^2(0,T;L^2)}+\|p_{xt}^\varepsilon\|_{L^2(0,T;L^2)})\|v\|_{L^\infty(0,T;L^2)}\|u_x\|_{L^2(0,T;L^2)}\\&\quad+C(\|p^\varepsilon\|_{L^\infty(0,T;L^2)}+\|p_x^\varepsilon\|_{L^\infty(0,T;L^2)})\|v_t\|_{L^2(0,T;L^2)}\|u_x\|_{L^2(0,T;L^2)}\\&\le C\varepsilon^{1/2},\end{aligned}$$

where we have used Theorem 2.5, Lemma 3.1, Lemma 3.2, Lemma 4.1 and (4.10). It follows from Poincaré and Sobolev embedding inequalities that

$$
\begin{aligned}
K_{11} &= \int_0^t \int_0^1 u_x q^0 u_t \, dx \, d\tau + \int_0^t \int_0^1 u q_x^0 u_t \, dx \, d\tau \\
&\le \|u_x\|_{L^2(0,T;L^2)} \|q^0\|_{L^\infty(0,T;L^\infty)} \|u_t\|_{L^2(0,t;L^2)} \\
&\quad + \|u\|_{L^2(0,T;L^\infty)} \|q_x^0\|_{L^\infty(0,T;L^2)} \|u_t\|_{L^2(0,t;L^2)} \\
&\le \frac{1}{4} \|u_t\|^2_{L^2(0,t;L^2)} + C\|q^0\|^2_{L^\infty(0,T;H^1)} \|u\|^2_{L^2(0,T;H^1)} \\
&\le \frac{1}{4} \|u_t\|^2_{L^2(0,t;L^2)} + C\|q^0\|^2_{L^\infty(0,T;H^1)} \|u_x\|^2_{L^2(0,T;L^2)} \\
&\le \frac{1}{4} \|u_t\|^2_{L^2(0,t;L^2)} + C\varepsilon^{1/2},
\end{aligned}
$$

where Lemma 4.1 and Part (ii) of Lemma 2.1 have been used. Substituting the above estimates for K_9–K_{11} into (4.11), we obtain

$$
\|u_x(t)\|^2_{L^2} + \int_0^t \|u_t\|^2_{L^2} \, d\tau \le C\varepsilon^{1/2},
$$

where the constant C is independent of ε but depends on t. Thus, the proof of (4.7) is completed. □

Lemma 4.3. *Suppose that the assumptions in Theorem 2.6 hold. Define $\xi(x) = x^2(1-x)^2$ for $0 \le x \le 1$. Then for any $0 < T < \infty$, there exists a positive constant C, independent of ε but dependent on T, such that*

$$
\sup_{0\le t\le T} \left(\int_0^1 \xi(x)|(q^\varepsilon - q^0)_x|^2(x,t)\, dx \right) + \varepsilon \int_0^T \int_0^1 \xi(x)|(q^\varepsilon - q^0)_{xx}|^2(x,t)\, dxdt \le C\varepsilon^{1/2}.
$$

Proof. Differentiating $(4.2)_2$ with respect to x, we have

$$
v_{xt} = \varepsilon v_{xxx} + \varepsilon q^0_{xxx} + \varepsilon[(q^\varepsilon)^2]_{xx} + u_{xx}.
$$

Multiplying the above equation by $x^2(1-x)^2 v_x$, integrating the resulting equation with respect to x by parts, and using the fact that $\xi(x)|_{x=0,x=1} = 0$, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|x(1-x)v_x\|_{L^2}^2+\varepsilon\|x(1-x)v_{xx}\|_{L^2}^2 \\
&=-\varepsilon\int_0^1 2(1-2x)x(1-x)v_xv_{xx}\,dx+\varepsilon\int_0^1 x^2(1-x)^2v_xq^0_{xxx}\,dx \\
&\quad+2\varepsilon\int_0^1 x^2(1-x)^2(q^\varepsilon_x)^2v_x\,dx+2\varepsilon\int_0^1 x^2(1-x)^2q^\varepsilon v_xq^\varepsilon_{xx}\,dx \\
&\quad+\int_0^1 x^2(1-x)^2v_xu_{xx}\,dx \\
&:=K_{12}+K_{13}+K_{14}+K_{15}+K_{16}.
\end{aligned}
\tag{4.12}
$$

We proceed to estimate K_{12}–K_{16}. Starting with Cauchy–Schwarz inequality, we first have

$$
\begin{aligned}
K_{12}&\le 2\varepsilon\|x(1-x)v_{xx}\|_{L^2}\|(1-2x)v_x\|_{L^2} \\
&\le 2(\varepsilon^{1/2}\|x(1-x)v_{xx}\|_{L^2})(\varepsilon^{1/2}\|v_x\|_{L^2}) \\
&\le \frac{1}{8}\varepsilon\|x(1-x)v_{xx}\|_{L^2}^2+8\varepsilon\|v_x\|_{L^2}^2.
\end{aligned}
$$

The integration by parts with Hölder inequality yields

$$
\begin{aligned}
K_{13}&=-\varepsilon\int_0^1 x^2(1-x)^2v_{xx}q^0_{xx}\,dx-\varepsilon\int_0^1 2(1-2x)x(1-x)v_xq^0_{xx}\,dx \\
&\le \varepsilon\|q^0_{xx}\|_{L^2}\|x(1-x)v_{xx}\|_{L^2}+2\varepsilon\|v_x\|_{L^2}\|q^0_{xx}\|_{L^2} \\
&\le \frac{1}{8}\varepsilon\|x(1-x)v_{xx}\|_{L^2}^2+2(\varepsilon\|v_x\|_{L^2}^2+\varepsilon\|q^0_{xx}\|_{L^2}^2).
\end{aligned}
$$

By the assumption that $0<\varepsilon<1$ and Hölder and Gagliardo–Nirenberg interpolation inequalities, we derive

$$
\begin{aligned}
K_{14}&\le 2\varepsilon\|q^\varepsilon_x\|_{L^2}\|q^\varepsilon_x\|_{L^\infty}\|x(1-x)v_x\|_{L^2} \\
&\le C\varepsilon\|q^\varepsilon_x\|_{L^2}(\|q^\varepsilon_x\|_{L^2}+\|q^\varepsilon_x\|_{L^2}^{1/2}\|q^\varepsilon_{xx}\|_{L^2}^{1/2})\|x(1-x)v_x\|_{L^2} \\
&=C(\varepsilon\|q^\varepsilon_x\|_{L^2}^2)\|x(1-x)v_x\|_{L^2} \\
&\quad+C(\varepsilon^{1/2}\|q^\varepsilon_x\|_{L^2}^{3/2})(\varepsilon^{1/2}\|q^\varepsilon_{xx}\|_{L^2}^{1/2})\|x(1-x)v_x\|_{L^2} \\
&\le \|x(1-x)v_x\|_{L^2}^2+C\varepsilon^{1/2}\left((\varepsilon^{1/2}\|q^\varepsilon_x\|_{L^2}^2)^2+(\varepsilon^{1/2}\|q^\varepsilon_x\|_{L^2}^2)^3+\varepsilon^{3/2}\|q^\varepsilon_{xx}\|_{L^2}^2\right).
\end{aligned}
$$

Noting that $q^\varepsilon=v+q^0$, we have

$$
\begin{aligned}
K_{15} &= 2\varepsilon \int_0^1 x^2(1-x)^2 q^\varepsilon v_x v_{xx}\, dx + 2\varepsilon \int_0^1 x^2(1-x)^2 q^\varepsilon v_x q^0_{xx}\, dx \\
&\le 2\varepsilon \|q^\varepsilon\|_{L^\infty} \|x(1-x)v_x\|_{L^2} \|x(1-x)v_{xx}\|_{L^2} \\
&\quad + 2\varepsilon \|q^\varepsilon\|_{L^\infty} \|x(1-x)v_x\|_{L^2} \|q^0_{xx}\|_{L^2} \\
&\le C\varepsilon \|q^\varepsilon\|_{H^1} \|x(1-x)v_x\|_{L^2} \|x(1-x)v_{xx}\|_{L^2} \\
&\quad + C\varepsilon \|q^\varepsilon\|_{H^1} \|x(1-x)v_x\|_{L^2} \|q^0_{xx}\|_{L^2} \\
&\le \frac{1}{8}\varepsilon \|x(1-x)v_{xx}\|^2_{L^2} + C(\varepsilon \|q^\varepsilon\|^2_{H^1}) \|x(1-x)v_x\|^2_{L^2} + \varepsilon \|q^0_{xx}\|^2_{L^2},
\end{aligned}
$$

where we have used the Sobolev embedding $H^1 \hookrightarrow L^\infty$. For K_{16}, we use equation $(4.2)_1$ to rewrite it as

$$
\begin{aligned}
K_{16} &= \int_0^1 x^2(1-x)^2 v_x u_t\, dx \\
&\quad - \int_0^1 x^2(1-x)^2 v_x p^\varepsilon_x v\, dx - \int_0^1 x^2(1-x)^2 v_x p^\varepsilon v_x\, dx \\
&\quad - \int_0^1 x^2(1-x)^2 v_x u_x q^0\, dx - \int_0^1 x^2(1-x)^2 v_x u q^0_x\, dx \\
&:= R_1 + R_2 + R_3 + R_4 + R_5.
\end{aligned}
$$

To bound K_{16}, we estimate R_1–R_5 below. First Cauchy–Schwarz inequality leads to

$$
R_1 \le \|x(1-x)v_x\|^2_{L^2} + \|u_t\|^2_{L^2}.
$$

By Hölder and Sobolev embedding inequalities, we estimate R_2–R_5 as follows:

$$
\begin{aligned}
R_2 &\le \|x(1-x)v\|_{L^\infty} \|x(1-x)v_x\|_{L^2} \|p^\varepsilon_x\|_{L^2} \\
&\le C\|x(1-x)v\|_{H^1} \|x(1-x)v_x\|_{L^2} \|p^\varepsilon_x\|_{L^2} \\
&\le C(\|x(1-x)v\|_{L^2} + \|[x(1-x)v]_x\|_{L^2}) \|x(1-x)v_x\|_{L^2} \|p^\varepsilon_x\|_{L^2} \\
&\le C(\|v\|_{L^2} + \|x(1-x)v_x\|_{L^2}) \|x(1-x)v_x\|_{L^2} \|p^\varepsilon_x\|_{L^2} \\
&\le C(\|p^\varepsilon_x\|_{L^2} + \|p^\varepsilon_x\|^2_{L^2}) \|x(1-x)v_x\|^2_{L^2} + \|v\|^2_{L^2}, \\
R_3 &\le \|x(1-x)v_x\|^2_{L^2} \|p^\varepsilon\|_{L^\infty} \\
&\le C(\|p^\varepsilon\|_{L^2} + \|p^\varepsilon_x\|_{L^2}) \|x(1-x)v_x\|^2_{L^2}, \\
R_4 &\le \|x(1-x)v_x\|_{L^2} \|u_x\|_{L^2} \|q^0\|_{L^\infty} \\
&\le C\|q^0\|^2_{H^1} \|x(1-x)v_x\|^2_{L^2} + \|u_x\|^2_{L^2}
\end{aligned}
$$

and

$$R_5 \le \|x(1-x)v_x\|_{L^2}\|u\|_{L^2}\|q_x^0\|_{L^\infty}$$
$$\le C\|q^0\|_{H^2}^2\|x(1-x)v_x\|_{L^2}^2 + \|u\|_{L^2}^2.$$

With above estimates in hand, we obtain

$$K_{16} \le C(\|p^\varepsilon\|_{L^2} + \|p_x^\varepsilon\|_{L^2} + \|p_x^\varepsilon\|_{L^2}^2 + \|q^0\|_{H^2}^2 + 1)\|x(1-x)v_x\|_{L^2}^2$$
$$+ (\|v\|_{L^2}^2 + \|u\|_{L^2}^2 + \|u_x\|_{L^2}^2 + \|u_t\|_{L^2}^2).$$

Substituting the above estimates for K_{12}–K_{16} into (4.12), we derive that for $0 < \varepsilon < 1$

$$\frac{d}{dt}\|x(1-x)v_x\|_{L^2}^2 + \varepsilon\|x(1-x)v_{xx}\|_{L^2}^2$$
$$\le C(\|p^\varepsilon\|_{L^2} + \|p_x^\varepsilon\|_{L^2} + \|p_x^\varepsilon\|_{L^2}^2 + \varepsilon^{1/2}\|q^\varepsilon\|_{H^1}^2 + \|q^0\|_{H^2}^2 + 1)\|x(1-x)v_x\|_{L^2}^2$$
$$+ C(\|v\|_{L^2}^2 + \|u\|_{L^2}^2 + \|u_x\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \varepsilon\|v_x\|_{L^2}^2)$$
$$+ C\varepsilon^{1/2}\left((\varepsilon^{1/2}\|q_x^\varepsilon\|_{L^2}^2)^2 + (\varepsilon^{1/2}\|q_x^\varepsilon\|_{L^2}^2)^3 + \varepsilon^{3/2}\|q_{xx}^\varepsilon\|_{L^2}^2 + \|q^0\|_{H^2}^2\right).$$

Applying Gronwall's inequality to this, and using Part (ii) of Lemma 2.1, Theorem 2.5, Lemma 3.1, Lemma 4.1 and Lemma 4.2, we obtain

$$\int_0^1 x^2(1-x)^2|v_x|^2(x,t)\,dx + \varepsilon\int_0^t\int_0^1 x^2(1-x)^2|v_{xx}|^2(x,\tau)\,dxd\tau \le C\varepsilon^{1/2},$$

where the constant C is independent of ε but depends on t. Thus, the proof Lemma 4.3 is completed. □

With the help of Lemma 4.1, Lemma 4.2 and Lemma 4.3, we can estimate the thickness of boundary layers.

Proof of Theorem 2.6. By Lemma 4.1, Lemma 4.2 and Sobolev embedding inequality, we have that for any $t \in [0, T]$,

$$\|(p^\varepsilon - p^0)(t)\|_{C[0,1]}^2 \le C\left(\|(p^\varepsilon - p^0)(t)\|_{L^2(0,1)}^2 + \|(p^\varepsilon - p^0)_x(t)\|_{L^2(0,1)}^2\right) \le C\varepsilon^{1/2}.$$

Thus, we obtain (2.3) and

$$\|(p^\varepsilon - p^0)\|_{L^\infty(0,T;C[0,1])}^2 \to 0, \quad \text{as } \varepsilon \to 0.$$

Next, we prove (2.4). First one can prove that for any $\delta \in (0, 1/2)$,

$$\delta^2 \le 4x^2(1-x)^2, \quad \forall x \in (\delta, 1-\delta).$$

This, along with Lemma 4.3 gives

$$\delta^2 \int_\delta^{1-\delta} v_x^2(x,t)\,dx \le 4 \int_\delta^{1-\delta} x^2(1-x)^2 v_x^2(x,t)\,dx \le C\varepsilon^{1/2},$$

from which we derive that for any $\delta \in (0, 1/2)$,

$$\|(q^\varepsilon - q^0)_x(t)\|_{L^2(\delta,1-\delta)} \le C\delta^{-1}\varepsilon^{1/4}, \quad \forall t \in [0,T].$$

Combining this with Lemma 4.1, we have by Gagliardo–Nirenberg interpolation inequality that for any $t \in [0,T]$,

$$\begin{aligned}
&\|(q^\varepsilon - q^0)(t)\|^2_{C[\delta,1-\delta]} \\
&\quad \le C(\|(q^\varepsilon - q^0)(t)\|^2_{L^2(\delta,1-\delta)} + \|(q^\varepsilon - q^0)(t)\|_{L^2(\delta,1-\delta)}\|(q^\varepsilon - q^0)_x(t)\|_{L^2(\delta,1-\delta)}) \\
&\quad \le C(\varepsilon^{1/2} + \varepsilon^{1/2}\delta^{-1}) \\
&\quad \le C\varepsilon^{1/2}\delta^{-1},
\end{aligned}$$

where the constant C is independent of ε but depends on T. Hence, we obtain (2.4) and

$$\|(q^\varepsilon - q^0)\|^2_{L^\infty(0,T;C[\delta,1-\delta])} \to 0, \quad \text{as } \varepsilon \to 0,$$

provided that $\delta = \delta(\varepsilon)$ satisfies

$$\delta(\varepsilon) \to 0 \quad \text{and} \quad \varepsilon^{1/2}/\delta(\varepsilon) \to 0, \quad \text{as } \varepsilon \to 0.$$

Next we turn to show that (2.5) and (2.6) are equivalent. For this, we integrate $(1.3)_2$ with $\varepsilon = 0$ over $(0,t)$ and set $x = 0$ in the resulting integral equation to obtain

$$q^0(0,t) = q^0(0,0) + \int_0^t p_x^0(0,\tau)d\tau = q_0(0) + \int_0^t p_x^0(0,\tau)d\tau = \bar{q} + \int_0^t p_x^0(0,\tau)d\tau, \tag{4.13}$$

where we have used the compatible condition $q_0(0) = \bar{q}$. Then it follows from (4.13) that

$$q^0(0,t) - \bar{q} = \int_0^t p_x^0(0,\tau)d\tau,$$

which, along with the boundary condition $q^\varepsilon(0,t) = \bar{q}$ gives for any $\varepsilon > 0$ that

$$q^0(0,t) - q^\varepsilon(0,t) = \int_0^t p_x^0(0,\tau)d\tau. \tag{4.14}$$

If we assume that $\int_0^t p_x^0(0,\tau)d\tau \neq 0$ for some $t \in [0,T]$, then (2.5) holds and the boundary layer appears at $x = 0$. Similarly if we assume that $\int_0^t p_x^0(1,\tau)d\tau \neq 0$, then (2.5) holds and the boundary layer appears at $x = 1$. Thus, we have proved that (2.6) implies (2.5). It remains to show (2.5) implies (2.6) by argument of contradiction. Indeed if we assume (2.6) is false, that is

$$\int_0^t p_x^0(0,\tau)\,d\tau \cdot \int_0^t p_x^0(0,\tau)\,d\tau = 0, \quad \forall\, t \in [0,T],$$

then it follows from (4.14) that

$$q^0(0,t) - q^\varepsilon(0,t) = q^0(1,t) - q^\varepsilon(1,t) = 0, \quad \forall\, t \in [0,T]. \tag{4.15}$$

We shall show below that under (4.15) the boundary terms for v in the proof of Lemma 4.1 and Lemma 4.2 will vanish and hence lead to a estimates violating (2.5). In fact, with (4.15), we have $K_4 = 0$ in (4.4) and K_2 can be estimated in a more delicate way by

$$\begin{aligned}
K_2 &\le 2\varepsilon \|q^\varepsilon\|_{L^\infty} \|q_x^\varepsilon\|_{L^2} \|v\|_{L^2} \\
&\le C\varepsilon \left(\|q^\varepsilon\|_{L^2} + \|q^\varepsilon\|_{L^2}^{1/2} \|q_x^\varepsilon\|_{L^2}^{1/2} \right) \|q_x^\varepsilon\|_{L^2} \|v\|_{L^2} \\
&\le C\varepsilon^2 \|q^\varepsilon\|_{L^2}^2 \|q_x^\varepsilon\|_{L^2}^2 + C\varepsilon^2 \|q^\varepsilon\|_{L^2} \|q_x^\varepsilon\|_{L^2}^3 + \|v\|_{L^2}^2 \\
&\le C\varepsilon^{5/4} (\|q^\varepsilon\|_{L^2} + \|q^\varepsilon\|_{L^2}^2) \left(\varepsilon^{1/2} \|q_x^\varepsilon\|_{L^2}^2 + \varepsilon^{3/4} \|q_x^\varepsilon\|_{L^2}^3 \right) + \|v\|_{L^2}^2,
\end{aligned} \tag{4.16}$$

where the assumption $\varepsilon < 1$ has been used. Now we modify the proof of Lemma 4.1 directly by using $K_4 = 0$ and replacing K_2 in (4.4) with (4.16) and get by a similar argument as deriving (4.5) that

$$\sup_{0 \le t \le T} (\|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2) + \int_0^T (\|u_x\|_{L^2}^2 + \varepsilon \|v_x\|_{L^2}^2)\,dt \le C\varepsilon^{5/4}. \tag{4.17}$$

Similarly we can modify the proof of Lemma 4.2 directly to get a better estimates for v_x. First, differentiating (4.15) with respect to t gives $v_t|_{x=0,x=1} = 0$, which leads to $K_8 = 0$ in (4.8). Then using a similar argument as obtaining (4.16), we find

$$\begin{aligned}
K_6 &\le 2\varepsilon \|q^\varepsilon\|_{L^\infty} \|q_x^\varepsilon\|_{L^2} \|v_t\|_{L^2} \\
&\le \frac{1}{4} \|v_t\|_{L^2}^2 + C\varepsilon^{5/4} (\|q^\varepsilon\|_{L^2} + \|q^\varepsilon\|_{L^2}^2) \left(\varepsilon^{1/2} \|q_x^\varepsilon\|_{L^2}^2 + \varepsilon^{3/4} \|q_x^\varepsilon\|_{L^2}^3 \right).
\end{aligned}$$

Now substituting the above estimate for K_6 into (4.8), keeping the estimates of K_5, K_7 unchanged, and using the same arguments as deriving (4.10), one easily gets that

$$\varepsilon \sup_{0 \le t \le T} \|v_x(t)\|_{L^2}^2 + \int_0^T \|v_t\|_{L^2}^2\,dt \le C\varepsilon^{5/4},$$

which, entails that for $t \in [0, T]$

$$\|v_x(t)\|_{L^2}^2 \le C\varepsilon^{1/4}. \tag{4.18}$$

Then from (4.17) and (4.18), we deduce for $t \in [0, T]$ that

$$\begin{aligned} &\|(q^\varepsilon - q^0)(t)\|_{C[0,1]}^2 \\ &\quad \le C(\|(q^\varepsilon - q^0)(t)\|_{L^2(0,1)}^2 + \|(q^\varepsilon - q^0)(t)\|_{L^2(0,1)}\|(q^\varepsilon - q^0)_x(t)\|_{L^2(0,1)}) \\ &\quad \le C(\varepsilon^{5/4} + \varepsilon^{5/8} \cdot \varepsilon^{1/8}) \\ &\quad \le C\varepsilon^{3/4}, \end{aligned}$$

which, yields

$$\liminf_{\varepsilon \to 0} \|q^\varepsilon - q^0\|_{L^\infty(0,T;C[0,1])} = \lim_{\varepsilon \to 0} \|q^\varepsilon - q^0\|_{L^\infty(0,T;C[0,1])} = 0.$$

This contradicts (2.5) and hence (2.6) holds by argument of contradiction. The proof is completed. □

Note that in general the condition (2.6) in Theorem 2.6 is hardly checkable unless the term $p_x^0(0,\tau)$ or $p_x^0(1,\tau)$ is known. Below we shall show that the condition (2.6) can be ensured by assuming $p_{0x}(0) \cdot p_{0x}(1) \neq 0$. For example without loss of generality, we assume that $p_{0x}(0) \neq 0$ and furthermore $p_{0x}(0) > 0$. By part (ii) of Lemma 2.1, we know that $p^0 \in C([0,\infty); H^2)$, which along with Sobolev embedding theorem, entails for any $T \in (0,\infty)$ that

$$p_x^0(x,t) \in C([0,1] \times [0,T]),$$

which implies

$$p_x^0(0,t) \in C([0,T]). \tag{4.19}$$

We know from the initial conditions that $p^0(x,0) = p_0(x)$. Differentiating this equation with respect to x and then setting $x = 0$, we obtain

$$p_x^0(0,0) = p_{0x}(0) > 0. \tag{4.20}$$

Combing (4.19) and (4.20), we conclude that there exists a suitably small $T^* > 0$, such that

$$p_x^0(0,\tau) > 0, \quad \forall\, \tau \in [0, T^*]. \tag{4.21}$$

Then from (4.14) and (4.21), we have for any $\varepsilon > 0$ that

$$\|q^\varepsilon - q^0\|_{L^\infty(0,T;C[0,1])} \ge \int_0^T p_x^0(0,t)dt > 0, \quad \forall\, 0 < T < T^*$$

and

$$\|q^\varepsilon - q^0\|_{L^\infty(0,T;C[0,1])} \geq \int_0^{T^*} p_x^0(0,t)dt > 0, \quad \forall\, T \geq T^*.$$

From the above two inequalities, we conclude that there exists a positive constant $C(T, T^*)$ independent of ε but dependent on T and T^*, such that for any $\varepsilon > 0$

$$\|q^\varepsilon - q^0\|_{L^\infty(0,T;C[0,1])} \geq C(T, T^*) > 0.$$

Hence, for any $0 < T < \infty$

$$\liminf_{\varepsilon \to 0} \|q^\varepsilon - q^0\|_{L^\infty(0,T;C[0,1])} > 0.$$

Thus, we obtain (2.5) under the assumption that $p_{0x}(0) > 0$. The result can be extended to the case $p_{0x}(0) < 0$ similarly. In a similar fashion as above, one can derive (2.5) for the case $p_{0x}(1) \neq 0$. This yields the results of Remark 2.7.

5. Diffusion limit of the original model

In this section we transfer the diffusion limit results for the transformed system (1.3)–(1.4) back to the pre-transformed system (1.2) and prove Proposition 2.8. First passing the result of Lemma 2.1 to the pre-transformed chemotaxis system (1.2) gives the global existence of the problem (2.7), which has been done in [25, Proposition 1.1] and will be omitted here. To complete the proof of Proposition 2.8, it remains only to derive the solution convergence with respect to ε and the details are given below.

Proof of Proposition 2.8. We denote the solution of (2.7) with $\varepsilon \geq 0$ by $(n^\varepsilon, c^\varepsilon)$. Noticing that $n^\varepsilon = p^\varepsilon$ and $n^0 = p^0$, the convergence rate (2.8) is obtained directly from Theorem 2.6. We only need to prove the convergence for c^ε. Observing that

$$\begin{cases} (\ln c^\varepsilon)_t = \varepsilon (q^\varepsilon)^2 + \varepsilon q_x^\varepsilon + n^\varepsilon - \mu, \\ (\ln c^0)_t = n^0 - \mu, \end{cases}$$

where $q^\varepsilon = (\ln c^\varepsilon)_x$. We consider the difference of the two equations:

$$(\ln c^\varepsilon - \ln c^0)_t = \varepsilon (q^\varepsilon)^2 + \varepsilon q_x^\varepsilon + (n^\varepsilon - n^0),$$

which, integrated with respect to t, gives

$$\frac{c^\varepsilon(x,t)}{c^0(x,t)} = \frac{c^\varepsilon(x,0)}{c^0(x,0)} \exp\left\{ \int_0^t [(n^\varepsilon - n^0) + \varepsilon (q^\varepsilon)^2 + \varepsilon q_x^\varepsilon]\, d\tau \right\}.$$

It follows from the fact $c^\varepsilon(x,0) = c^0(x,0) = c_0(x)$ that

$$
\begin{aligned}
|c^\varepsilon(x,t)-c^0(x,t)| &= |c^0(x,t)|\cdot\left|\exp\left\{\int_0^t [(n^\varepsilon-n^0)+\varepsilon(q^\varepsilon)^2+\varepsilon q_x^\varepsilon]\,d\tau\right\}-1\right| \\
&= |c^0(x,t)|\cdot|e^{G^\varepsilon(x,t)}-1|
\end{aligned}
\tag{5.1}
$$

We denote $G^\varepsilon(x,t) := \int_0^t[(n^\varepsilon - n^0)+\varepsilon(q^\varepsilon)^2+\varepsilon q_x^\varepsilon]d\tau$ for convenience. By Hölder and Gagliardo–Nirenberg interpolation inequalities, we have for any $t\in(0,T]$

$$
\begin{aligned}
|G^\varepsilon(x,t)| &\le \int_0^T \left(\|n^\varepsilon-n^0\|_{L^\infty}+\varepsilon\|q^\varepsilon\|_{L^\infty}^2+\varepsilon\|q_x^\varepsilon\|_{L^\infty}\right)d\tau \\
&\le T\|n^\varepsilon-n^0\|_{L^\infty(0,T;L^\infty)} + C\varepsilon\int_0^T\left(\|q^\varepsilon\|_{L^2}^2+\|q^\varepsilon\|_{L^2}\|q_x^\varepsilon\|_{L^2}\right)d\tau \\
&\quad + C\varepsilon\int_0^T\left(\|q_x^\varepsilon\|_{L^2}+\|q_x^\varepsilon\|_{L^2}^{1/2}\|q_{xx}^\varepsilon\|_{L^2}^{1/2}\right)d\tau \\
&\le CT\varepsilon^{1/4}+CT\varepsilon\|q^\varepsilon\|_{L^\infty(0,T;L^2)}^2+CT\varepsilon\|q^\varepsilon\|_{L^\infty(0,T;L^2)}\|q_x^\varepsilon\|_{L^\infty(0,T;L^2)} \\
&\quad + CT\varepsilon\|q_x^\varepsilon\|_{L^\infty(0,T;L^2)}+CT^{3/4}\varepsilon\|q_x^\varepsilon\|_{L^\infty(0,T;L^2)}^{1/2}\|q_{xx}^\varepsilon\|_{L^2(0,T;L^2)}^{1/2} \\
&\le CT\varepsilon^{1/4}+CT\varepsilon+CT\varepsilon^{3/4}+CT^{3/4}\varepsilon^{1/2}
\end{aligned}
$$

where we have used (2.8), Theorem 2.5 and Lemma 3.1 and C is a constant independent of ε. Considering that $0<\varepsilon<1$, one has

$$
|G^\varepsilon(x,t)| \le C_1\varepsilon^{1/4} \tag{5.2}
$$

for some positive constant C_1 independent of ε (but depends on T). Since $c^0(x,t)$ satisfies the equation $c_t^0=(n^0-\mu)c^0$, then by using the results of Lemma 2.1, one can derive that

$$
|c^0(x,t)| \le C_2\exp\{(\bar{n}-\mu)t\},
$$

where the positive constant C_2 is independent of t if $\bar{n}>0$ and depends on t if $\bar{n}=0$ (see [25, Section 2.3]). We further apply the Taylor expansion and (5.2) with the assumption $0<\varepsilon<1$ to have

$$
|e^{G^\varepsilon(x,t)}-1| \le \sum_{k=1}^\infty \frac{1}{k!}|G^\varepsilon(x,t)|^k \le \sum_{k=1}^\infty |G^\varepsilon(x,t)|^k \le C_3\varepsilon^{1/4}
$$

where the constant $C_3>0$ is independent of ε. Combining the above two estimates with (5.1), we conclude the following result:

$$
\|c^\varepsilon-c^0\|_{L^\infty(0,T;L^\infty)} \le C_4\varepsilon^{1/4},
$$

where the constant $C_4 > 0$ is independent of ε and depends on t. This completes the proof of Proposition 2.8. □

Acknowledgments

The authors would like to thank the referee for the valuable comments and suggestions which lead to a significant improvement of the manuscript. The research of Z.A. Wang was supported by the Hong Kong RGC GRF grant No. PolyU 153032/15P. The research of K. Zhao was partially supported by the Louisiana Board of Regents Research Competitiveness Subprogram grant LEQSF(2015-18)-RD-A-24, the Louisiana EPSCoR Pilot fund LEQSF-EPS(2014)-PFUND-379, and a start up fund from the Mathematics Department of Tulane University.

Appendix A

Based on the Definition 2.3, the results in Theorem 2.6 can not give a unique boundary layer thickness. In this section, we shall show that the boundary layer thickness is $\varepsilon^{1/2}$ by the asymptotic analysis. We remark that our results are derived based on a formal instead of rigorous procedure by the WKB method (e.g. see [15, Chapter 4], [10,40]). However a rigorous proof is beyond the scope of this paper, see Remark 2.4. We split our analysis into four steps.

Step 1. Asymptotic expansions. Assume that the boundary layer thickness is ε^α for some $\alpha > 0$ to be determined later. By the asymptotic matching method (see [15, Chapter 2]), it is known that the solution consists of two parts: the outer solution and the boundary-layer solution near the endpoints $x = 0, 1$. We first introduce two boundary layer coordinates given as

$$z = \frac{x}{\varepsilon^\alpha}, \qquad \eta = \frac{x-1}{\varepsilon^\alpha}, \quad x \in [0,1] \tag{A.1}$$

where $0 \le z < \infty$ and $-\infty < \eta \le 0$. As $\varepsilon \to 0$, these coordinates have the effect of stretching the regions near $x = 0$ and $x = 1$. By the WKB method, we assume that the solutions of (1.3)–(1.4) with small $\varepsilon > 0$ have the following expansions:

$$\begin{aligned} p^\varepsilon(x,t) &= \sum_{j\ge 0} \varepsilon^{\alpha j}\left[p^{I,j}(x,t) + p^{B,j}(z,t) + p^{b,j}(\eta,t)\right], \ j = 0,1,2,\cdots, \\ q^\varepsilon(x,t) &= \sum_{j\ge 0} \varepsilon^{\alpha j}\left[q^{I,j}(x,t) + q^{B,j}(z,t) + q^{b,j}(\eta,t)\right], \ j = 0,1,2,\cdots, \end{aligned} \tag{A.2}$$

where each term in (A.2) is assumed to be smooth and the boundary-layer solutions enjoy the following basic hypothesis (see also [15, Chapter 4], [10,40]):

(H) $p^{B,j}$ and $q^{B,j}$ decay to zero exponentially as $z \to \infty$, while $p^{b,j}$ and $q^{b,j}$ decay to zero exponentially as $\eta \to -\infty$ all for $j \ge 0$.

Step 2. Initial and boundary conditions. Substituting the expansions (A.2) into the initial conditions in (1.3), we have

$$p_0(x) = \sum_{j\geq 0} \varepsilon^{\alpha j} \left[p^{I,j}(x,0) + p^{B,j}(z,0) + p^{b,j}(\eta,0) \right],$$
$$q_0(x) = \sum_{j\geq 0} \varepsilon^{\alpha j} \left[q^{I,j}(x,0) + q^{B,j}(z,0) + q^{b,j}(\eta,0) \right]. \tag{A.3}$$

Noticing that the initial data p_0, q_0 are independent of ε, it follows from (A.3) and the hypothesis (H) that

$$p^{I,0}(x,0) = p_0(x), \quad p^{B,0}(z,0) = p^{b,0}(\eta,0) = 0,$$
$$q^{I,0}(x,0) = q_0(x), \quad q^{B,0}(z,0) = q^{b,0}(\eta,0) = 0 \tag{A.4}$$

and for $j \geq 1$

$$p^{I,j}(x,0) = p^{B,j}(z,0) = p^{b,j}(\eta,0) = 0,$$
$$q^{I,j}(x,0) = q^{B,j}(z,0) = q^{b,j}(\eta,0) = 0.$$

To derive the boundary conditions, we substitute (A.2) into (1.4) and obtain from (A.1) that

$$\bar{p} = \sum_{j\geq 0} \varepsilon^{\alpha j} \left[p^{I,j}(0,t) + p^{B,j}(0,t) \right],$$
$$\bar{p} = \sum_{j\geq 0} \varepsilon^{\alpha j} \left[p^{I,j}(1,t) + p^{b,j}(0,t) \right],$$
$$\bar{q} = \sum_{j\geq 0} \varepsilon^{\alpha j} \left[q^{I,j}(0,t) + q^{B,j}(0,t) \right],$$
$$\bar{q} = \sum_{j\geq 0} \varepsilon^{\alpha j} \left[q^{I,j}(1,t) + q^{b,j}(0,t) \right],$$

where in the first expression given above, we have neglected the term $p^{b,j}(-\frac{1}{\varepsilon^\alpha}, t)$ due to the assumption (H) that $p^{b,j}(-\frac{1}{\varepsilon^\alpha}, t)$ decay to zero exponentially as $-\frac{1}{\varepsilon^\alpha} \to -\infty$ (i.e. as $\varepsilon \to 0$). For the same reason, the terms $p^{B,j}(\frac{1}{\varepsilon^\alpha}, t)$, $q^{b,j}(-\frac{1}{\varepsilon^\alpha}, t)$ and $q^{B,j}(\frac{1}{\varepsilon^\alpha}, t)$ have been neglected in the second, third and fourth expressions above, respectively. Since the above expressions hold for any $\varepsilon > 0$, we get

$$\bar{p} = p^{I,0}(0,t) + p^{B,0}(0,t),$$
$$\bar{p} = p^{I,0}(1,t) + p^{b,0}(0,t),$$
$$\bar{q} = q^{I,0}(0,t) + q^{B,0}(0,t),$$
$$\bar{q} = q^{I,0}(1,t) + q^{b,0}(0,t) \tag{A.5}$$

and for $j \geq 1$ we have

$$
\begin{aligned}
p^{I,j}(0,t)+p^{B,j}(0,t)&=0,\\
p^{I,j}(1,t)+p^{b,j}(0,t)&=0,\\
q^{I,j}(0,t)+q^{B,j}(0,t)&=0,\\
q^{I,j}(1,t)+q^{b,j}(0,t)&=0.
\end{aligned}
$$

Step 3. Profiles of $p^{I,j}$ and $p^{B,j}$. We first substitute (A.2) without the boundary layer solutions $p^{B,j}$, $p^{b,j}$, $q^{B,j}$ and $q^{b,j}$ into the first equation of (1.3) and immediately get the equation for the outer solution $p^{I,j}$:

$$
p_t^{I,j}-\sum_{k=0}^{j}(p^{I,k}q^{I,j-k})_x=p_{xx}^{I,j},\quad \text{for } j\ge 0. \tag{A.6}
$$

To find the profile for left boundary-layer solution $p^{B,j}$, we neglect the right boundary-layer solutions $p^{b,j}$ and $q^{b,j}$ in (A.2) and substitute the remaining terms of (A.2) into the first equation of (1.3). By using (A.6), after some calculations, we end up with

$$
\sum_{j\ge -2}\varepsilon^{\alpha j}G_j(x,z,t)=0, \tag{A.7}
$$

where

$$
\begin{cases}
G_{-2}=-p_{zz}^{B,0},\\
G_{-1}=-p^{I,0}q_z^{B,0}-q^{I,0}p_z^{B,0}-(p^{B,0}q^{B,0})_z-p_{zz}^{B,1},\\
G_j=p_t^{B,j}-\displaystyle\sum_{k=0}^{j}p^{B,k}q_x^{I,j-k}\\
\qquad -\displaystyle\sum_{k=0}^{j+1}(p^{I,k}+p^{B,k})q_z^{B,j+1-k}-\sum_{k=0}^{j}p_x^{I,k}q^{B,j-k}\\
\qquad -\displaystyle\sum_{k=0}^{j+1}p_z^{B,k}(q^{I,j+1-k}+q^{B,j+1-k})-p_{zz}^{B,j+2},\quad \text{for } j\ge 0.
\end{cases}
$$

Now using $x=\varepsilon^\alpha z$ and expanding $G_j(x,z,t)$ in ε by the Taylor expansion to get

$$
G_j(x,z,t)=G_j(\varepsilon^\alpha z,z,t)=G_j(0,z,t)+\sum_{k=1}^{\infty}\frac{1}{k!}(\varepsilon^\alpha z)^k\partial_x^kG_j(0,z,t),\ \ j\ge 0. \tag{A.8}
$$

Then feeding (A.8) into (A.7), we obtain

$$
\sum_{j\ge -2}\varepsilon^{\alpha j}\tilde{G}_j(z,t)=0, \tag{A.9}
$$

where

$$\begin{cases} \tilde{G}_{-2} = -p_{zz}^{B,0}, \\ \tilde{G}_{-1} = -p^{I,0}(0,t)q_z^{B,0} - q^{I,0}(0,t)p_z^{B,0} - (p^{B,0}q^{B,0})_z - p_{zz}^{B,1}, \\ \cdots\ \cdots \end{cases}$$

here we have omitted the terms $\tilde{G}_j$ for $j \geq 0$ for brevity. To make (A.9) true for any $\varepsilon > 0$, it is required that $\tilde{G}_j = 0$ for $j \geq -2$ where in particular $\tilde{G}_{-2} = 0$ implies $p_{zz}^{B,0} = 0$. This, upon the integration with respect to z over (z, ∞) along with the assumption (H), gives $p_z^{B,0} = 0$. Integrating this over (z, ∞) yields

$$p^{B,0}(z,t) = 0, \quad \text{for } (z,t) \in [0,\infty) \times [0,T] \tag{A.10}$$

which, applied to $\tilde{G}_{-1} = 0$, gives

$$p_{zz}^{B,1} = -p^{I,0}(0,t)q_z^{B,0}.$$

Integrating the above equation over (z, ∞) and using the assumption (H) again, we have that

$$p_z^{B,1} = -p^{I,0}(0,t)q^{B,0} = -\bar{p}q^{B,0}, \tag{A.11}$$

where (A.10) and the first identity in (A.5) have been used.

Step 4. Boundary layer thickness. For later use, we first derive the equation for $q^{I,0}$ for which we substitute (A.2) without the boundary-layer terms $p^{B,j}$, $p^{b,j}$, $q^{B,j}$ and $q^{b,j}$ into the second equation of (1.3) and immediately get

$$q_t^{I,0} - p_x^{I,0} = 0. \tag{A.12}$$

In what follows, we discuss the boundary layer at $x = 0$ only for brevity and similar arguments apply directly to the boundary layer at $x = 1$. Now we insert (A.2) by neglecting $p^{b,j}$ and $q^{b,j}$ into the second equation of (1.3). Using (A.12) and (A.10), we arrive at

$$\begin{aligned} &(q_t^{B,0} + \varepsilon^\alpha q_t^{I,1} + \varepsilon^\alpha q_t^{B,1} + \cdots) \\ &\quad - 2\varepsilon(q^{I,0} + q^{B,0} + \varepsilon^\alpha q^{I,1} + \varepsilon^\alpha q^{B,1} + \cdots) \\ &\quad \times (q_x^{I,0} + \varepsilon^{-\alpha} q_z^{B,0} + \varepsilon^\alpha q_x^{I,1} + q_z^{B,1} + \cdots) \\ &\quad - (\varepsilon^\alpha p_x^{I,1} + p_z^{B,1} + \cdots) \\ &\quad - \varepsilon(q_{xx}^{I,0} + \varepsilon^{-2\alpha} q_{zz}^{B,0} + \varepsilon^\alpha q_{xx}^{I,1} + \varepsilon^{-\alpha} q_{zz}^{B,1} + \varepsilon^{2\alpha} q_{xx}^{I,2} + q_{zz}^{B,2} + \cdots) = 0, \end{aligned} \tag{A.13}$$

where in each bracket the omitted terms are higher-order terms of ε^α. With $x = \varepsilon^\alpha z$, we substitute the Taylor expansions

$$p^{I,j}(x,t) = p^{I,j}(\varepsilon^\alpha z, t) = p^{I,j}(0,t) + \sum_{k=1}^{\infty} \frac{1}{k!}(\varepsilon^\alpha z)^k \partial_x^k p^{I,j}(0,t),$$

$$q^{I,j}(x,t) = q^{I,j}(\varepsilon^\alpha z, t) = q^{I,j}(0,t) + \sum_{k=1}^{\infty} \frac{1}{k!}(\varepsilon^\alpha z)^k \partial_x^k q^{I,j}(0,t),$$

into (A.13) and get

$$\underbrace{(q_t^{B,0}}_{S_1}+\cdots)-[\underbrace{2\varepsilon^{1-\alpha}(q^{I,0}(0,t)+q^{B,0})q_z^{B,0}}_{S_2}+\cdots]-(\underbrace{p_z^{B,1}}_{S_3}+\cdots)-(\underbrace{\varepsilon^{1-2\alpha}q_{zz}^{B,0}}_{S_4}+\cdots)=0, \tag{A.14}$$

where in each bracket we only write out the lowest order terms of ε, which suffices to find the value of α by finding the correct balancing so as to generate a boundary layer. Since $\alpha > 0$ gives rise to $1-\alpha \neq 1-2\alpha$, S_2 and S_4 can never be together to produce a balance. Hence there are three possible balancing as discussed below:

- $S_1 \sim S_3$ and S_2, S_4 are higher orders. Then the condition requires $1-\alpha > 0$ and $1-2\alpha > 0$, which leads to $\alpha < \frac{1}{2}$. But this will rule out the possibility of the boundary layer appearing for q^ε at $x = 0$. In fact, assuming $\alpha < \frac{1}{2}$ and letting $\varepsilon \to 0$ in (A.14), we have $q_t^{B,0} = p_z^{B,1}$. This along with (A.11) gives $q_t^{B,0} = -\bar{p}q^{B,0}$. Thus $q^{B,0}(z,t) = q^{B,0}(z,0)e^{-\bar{p}t} \equiv 0$ due to (A.4). This implies that q^ε does not have the boundary layer at $x = 0$ as $\varepsilon \to 0$. Hence this balancing is inappropriate.
- $S_1 \sim S_2 \sim S_3$ and S_4 is higher order. The condition $S_1 \sim S_2 \sim S_3$ requires that $1-\alpha = 0$ and so $\alpha = 1$ which indicates that $S_1, S_2, S_3 = O(1)$ and $S_4 = O(\varepsilon^{-1})$. This violates our assumption that S_4 is higher order.
- $S_1 \sim S_3 \sim S_4$ and S_2 is higher order. The condition $S_1 \sim S_2 \sim S_4$ requires that $1-2\alpha = 0$ and hence $\alpha = \frac{1}{2}$. With this, we have $S_1, S_3, S_4 = O(1)$ and $S_2 = O(\varepsilon^{1/2})$ which is consistent with our assumption that S_4 is higher order. This gives a (only) possible balancing to produce boundary layers.

Therefore from the above arguments, we may conclude that the boundary layer thickness must be exactly $\varepsilon^{1/2}$ if it exists.

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