STABILITY OF BOUNDARY LAYERS FOR A VISCOUS HYPERBOLIC SYSTEM ARISING FROM CHEMOTAXIS: ONE-DIMENSIONAL CASE*

QIANQIAN HOU†, CHENG-JIE LIU‡, YA-GUANG WANG§, AND ZHIAN WANG†

Abstract. This paper is concerned with the stability of boundary layer solutions for a viscous hyperbolic system transformed via a Cole–Hopf transformation from a singular chemotactic system modeling the initiation of tumor angiogenesis proposed in [H. A. Levine, B. Sleeman, and M. Nilsen-Hamilton, *Math. Biosci.*, 168 (2000), pp. 71–115]. It was previously shown in [Q. Hou, Z. Wang, and K. Zhao, *J. Differential Equations*, 261 (2016), pp. 5035–5070] that when prescribed with Dirichlet boundary conditions, the system possesses boundary layers at the boundaries in an bounded interval $(0,1)$ as the chemical diffusion rate (denoted by $\varepsilon > 0$) is small. This paper proceeds to prove the stability of boundary layer solutions and identify the precise structure of boundary layer solutions. Roughly speaking, we justify that the solution with $\varepsilon > 0$ converges to the solution with $\varepsilon = 0$ (outer layer solution) plus the inner layer solution with the optimal rate at order of $\mathcal{O}(\varepsilon^{1/2})$ as $\varepsilon \to 0$, where the outer and inner layer solutions are well determined and the relation between outer and inner layer solutions can be explicitly identified. Finally we transfer the results to the original pretransformed chemotaxis system and discuss the implications of our results.

Key words. boundary layers, chemotaxis, logarithmic singularity, asymptotic analysis, energy estimates

AMS subject classifications. 35A01, 35B40, 35B44, 35K57, 35Q92, 92C17

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1. Introduction. Chemotaxis, the movement of an organism in response to a chemical stimulus, has been an important mechanism of various biological phenomena/processes, such as aggregation of bacteria [49,63], slime mold formation [23], fish pigmentation [53], tumor angiogenesis [7,8,9], primitive streak formation [54], blood vessel formation [17], and wound healing [56]. The prototypical chemotaxis model, known as the Keller–Segel (KS) model due to their pioneering works of [30, 31, 32], reads in its general form as

$$(1.1) \qquad \begin{cases} u_t = [Du_x - \chi u(\phi(c))_x]_x, \\ c_t = \varepsilon c_{xx} + g(u,c), \end{cases}$$

where $u(x,t)$ and $c(x,t)$ denote the cell density and chemical (signal) concentration at position x and time t, respectively. The function $\phi(c)$ is called the chemotactic sensitivity function accounting for the signal response mechanism and $g(u,c)$ is the chemical kinetics (birth and death). $D > 0$ and $\varepsilon \geq 0$ are cell and chemical diffusion

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coefficients, respectively. $\chi \neq 0$ is referred to as the chemotactic coefficient with $|\chi|$ measuring the strength of the chemotactic sensitivity, where the chemotaxis is said to be attractive if $\chi > 0$ and repulsive if $\chi < 0$. The application of (1.1) generically depends on the specific forms of $\phi(c)$ and $g(u,c)$. There are two major classes of chemotactic sensitivity functions: linear law $\phi(c) = c$ and logarithmic law $\phi(c) = \ln c$. The former was originally derived in [30, 31] by Keller and Segel to model the self-aggregation of *Dictyostelium discoideum* in response to cyclic adenosine monophosphate (cAMP), while the latter was first employed in [32] to model the wave propagation of bacterial chemotaxis, though it has many other prominent applications in biology (cf. [2, 3, 10, 29, 34, 52]). Compared with massive well-known results on the KS system with linear chemotactic sensitivity (cf. [4, 5, 22, 25]), few results are available for the logarithmic sensitivity due to its singularity nature (at $c = 0$). This paper is concerned with the following KS system with logarithmic sensitivity:

$$(1.2) \qquad \begin{cases} u_t = [Du_x - \chi u(\ln c)_x]_x, \\ c_t = \varepsilon c_{xx} - \mu uc; \end{cases}$$

this was a specialized KS model with linear nutrient consumption proposed in [30], and it later found applications in [35] to describe the dynamical interactions between vascular endothelial cells (VECs), denoted by u, and signaling molecules vascular endothelial growth factor (VEGF), denoted by c, in the initiation of tumor angiogenesis. Except this, the model (1.2) has also been used in [50] to model the boundary movement of chemotactic bacterial populations. Though bearing specific applications, the logarithmic sensitivity brings considerable challenges to mathematical analysis due to its singularity nature. The common approach currently used to overcome this singularity is the Cole–Hopf type transformation (cf. [34, 43])

$$(1.3) \qquad v = -\frac{\sqrt{\chi\mu}}{\mu}(\ln c)_x = -\frac{\sqrt{\chi\mu}}{\mu}\frac{c_x}{c},$$

which transforms the model (1.2) into a nonsingular system of conservation laws

$$(1.4) \qquad \begin{cases} u_t - (uv)_x = u_{xx}, \\ v_t - (u - \frac{\varepsilon}{\chi}v^2)_x = \frac{\varepsilon}{D}v_{xx}, \\ (u,v)(x,0) = (u_0, v_0)(x), \end{cases}$$

where the temporal-spatial rescalings $\tilde{t} = \frac{\chi\mu}{D}t$ and $\tilde{x} = \frac{\sqrt{\chi\mu}}{D}x$ have been used and tildes have been suppressed in (1.4) for convenience.

Though the transformed system (1.4) no longer has singularity, it has a quadratic nonlinear convection term and the parameter ε in (1.4) plays a dual role: coefficient of both diffusion and nonlinear convection, which is a prominent feature compared to existing viscous hyperbolic systems as far as we know (cf. [16, 19, 60]). How to make a balance between the diffusion and nonlinear convection becomes an art of analysis. The transformed model (1.4) in multidimensions still remains poorly understood so far and available results are limited to small-data solutions (cf. [11, 21, 36, 40, 55, 66]). In contrast the model (1.4) has been well understood in one dimension to a large extent such as the existence and stability of traveling wave solutions (cf. [6, 28, 39, 41, 42, 43, 44]) and large-data solutions (cf. [37, 48]) in $\mathbb{R}$ or in bounded intervals with various boundary conditions (cf. [40, 61, 68, 73]).

The present paper will investigate the zero-limit problem of (1.4) as $\varepsilon \to 0$, which is motivated by the fact pointed out in [35] that the magnitude of the diffusion rate

ε of the chemical VEGF can be negligible compared to the diffusion of VECs in the initiation of tumor angiogenesis. Moreover the diffusion rate ε was assumed to be zero in the analysis of [32] and many subsequent works (cf. [65]) for simplicity. Hence whether the nondiffusive model (i.e., $\varepsilon = 0$) is a good approximation of the diffusive model when $\varepsilon > 0$ is small is of importance. This promises a relevance to explore the zero-limit problem of (1.4) in elucidating this question. From a mathematical point of view, the zero-limit problem of (1.4) as $\varepsilon \to 0$ is of independent interest due to the dual role of ε which causes challenges in deriving uniform-in-ε estimates. This topic has been investigated in several circumstances. First in unbounded domains, it has been shown that both traveling wave solutions (cf. [65]) in $\mathbb{R}$ and the global small-data solution of the Cauchy problem (cf. [55, 66]) in $\mathbb{R}^N (N = 2, 3)$ are uniformly convergent in ε, namely, the solutions of (1.4) with $\varepsilon > 0$ converge to those with $\varepsilon = 0$ in the L^∞-norm as $\varepsilon \to 0$. However, the ε-limit problem in bounded domains appears to be more involved. This is closely related to the boundary layer theory, which has been an important topic in fluid mechanics stimulating a large body of outstanding works (cf. [13, 14, 16, 19, 20, 27, 60, 64, 70, 72]). A fundamental question in fluid mechanics is whether solutions of the incompressible Navier–Stokes equations (NSE) converge to those of the Euler equations as the viscosity vanishes. The positive answer of this question has been given to the incompressible NSE under Lions- or Navier-type boundary conditions (cf. [45, 69]). However, the convergence under no-slip (zero Dirichlet) boundary condition is elusive due to the appearance of degenerate Prandtl-type boundary layers (cf. [1, 18, 47, 51, 71]). This imposes an interesting question: under what type of boundary conditions do the solutions of (1.4) converge as $\varepsilon \to 0$? This topic has been recently studied in [38, 67] in a bounded interval. Hereafter we assume $D = \chi = \mu = 1$ without loss of generality for simplicity since the specific values of them are not of importance to our analysis. For illustration, let's first consider the initial-boundary value problem of system (1.4) in an interval $(0, 1)$:

$$(1.5) \qquad \begin{cases} u_t - (uv)_x = u_{xx}, & (x, t) \in (0, 1) \times (0, \infty), \\ v_t - (u - \varepsilon|v|^2)_x = \varepsilon v_{xx}, \\ (u, v)(x, 0) = (u_0, v_0)(x). \end{cases}$$

If (1.5) is endowed with the mixed homogeneous Neumann–Dirichlet boundary conditions

$$u_x|_{x=0,1} = v|_{x=0,1} = 0,$$

it was shown in [67] that the solutions of (1.5) are uniformly convergent in ε, namely, the solutions of the diffusive problem ($\varepsilon > 0$) uniformly converge to those of the nondiffusive problem ($\varepsilon = 0$). However, if the Dirichlet boundary conditions are imposed, one cannot impose Dirichlet boundary conditions for v at boundaries if $\varepsilon = 0$ since otherwise the nondiffusive problem ($\varepsilon = 0$) may be overdetermined (cf. [38]). This indicates that the Dirichlet boundary conditions of (1.5) ought to be prescribed as

$$(1.6) \qquad \begin{cases} u|_{x=0,1} = \bar{u} \geq 0, \quad v|_{x=0,1} = \bar{v}, & \text{if } \varepsilon > 0, \\ u|_{x=0,1} = \bar{u} \geq 0 & \text{if } \varepsilon = 0, \end{cases}$$

where the boundary values of v when $\varepsilon = 0$ are determined by the second equation of (1.5) via u_x and may not equal to $\bar{v}$. Due to this possible mismatch of boundary conditions between $\varepsilon > 0$ and $\varepsilon = 0$, the L^∞-norm of v may diverge as $\varepsilon \to 0$ near end points $x = 0, 1$, and if so boundary layers will arise. Such suspicion has been numerically verified recently by Li and Zhao in [38], followed with a rigorous proof in [26]. Precisely

speaking, if letting $(u^\varepsilon, v^\varepsilon)$ and (u^0, v^0) denote the solutions of the initial-boundary value problem (1.5)–(1.6) for $\varepsilon > 0$ and $\varepsilon = 0$, respectively, then the work [26] showed that if initial data satisfy some compatibility conditions on boundaries, then for any function $\delta(\varepsilon)$ depending on ε and satisfying $\delta(\varepsilon) \to 0$ and $\varepsilon^{1/2}/\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, it holds that for any $T > 0$,

$$(1.7) \qquad \begin{aligned} &\lim_{\varepsilon \to 0} \|u^\varepsilon - u^0\|_{L^\infty([0,1] \times [0,T])} = 0, \\ &\lim_{\varepsilon \to 0} \|v^\varepsilon - v^0\|_{L^\infty([\delta, 1-\delta] \times [0,T])} = 0, \ \liminf_{\varepsilon \to 0} \|v^\varepsilon - v^0\|_{L^\infty([0,1] \times [0,T])} > 0, \end{aligned}$$

where the function $\delta(\varepsilon)$ is called a boundary layer thickness following the nomenclature of [14,15]. But, it is easy to see that the boundary layer thickness $\delta(\varepsilon)$ satisfying the above constraints is not unique; for example, the relations given in (1.7) hold for any $\delta(\varepsilon) = \varepsilon^\alpha$ with $\alpha < \frac{1}{2}$. A formal asymptotic analysis was further performed in [26] to show that the exact boundary layer thickness magnitude is of order $\varepsilon^{1/2}$.

Compared to the boundary layer theory developed in fluid mechanics, the study of boundary layer theory of chemotaxis models is still in its infant stage. The rigorous work [26] only showed the existence of boundary layer solutions of (1.5)–(1.6) and proved the convergence of the solution component v^ε as $\varepsilon \to 0$ outside the boundary layers. However, the structure of v^ε as $\varepsilon \to 0$ inside the boundary layers remains open. In this paper we shall exploit the structure of v^ε inside the boundary layers and hence establish the stability of boundary layer solutions of (1.5)–(1.6) in the entire interval $(0, 1)$. With the general boundary layer theory [57, 59] applied to (1.5)–(1.6), the solution profile $(u^\varepsilon, v^\varepsilon)$ of (1.5) for small $\varepsilon > 0$ is composed of two parts: outer layer profile and inner (boundary) layer profile. Since u^ε converges uniformly in ε and hence the inner layer profile part will be absent, $(u^\varepsilon, v^\varepsilon)$ is anticipated to possess the form

$$(1.8) \qquad \begin{aligned} u^\varepsilon &= u^0 + \mathcal{O}(\varepsilon^\alpha); \\ v^\varepsilon &= v^0 + v^L\left(\tfrac{x}{\sqrt{\varepsilon}}, t\right) + v^R\left(\tfrac{x-1}{\sqrt{\varepsilon}}, t\right) + \mathcal{O}(\varepsilon^\alpha) \end{aligned}$$

for some $\alpha \leq 1/2$, where (u^0, v^0) is the outer layer profile which is the solution of the nondiffusive problem of (1.5)–(1.6) with $\varepsilon = 0$, and the inner (boundary) layer profile v^L/v^R adjusts rapidly from a value away from the boundary to a different value on the left/right end point. Outside the boundary layer, the nondiffusive problem dominates. Inside the boundary layer, diffusion becomes important. The main goal of this paper is to explicitly derive the outer/inner layer profiles and justify (1.8) holds as $\varepsilon \to 0$ for $\alpha = 1/2$, which is the optimal convergence rate since the magnitude of boundary layer thickness is of order $\varepsilon^{1/2}$. Finally we convert the results of (1.5)–(1.6) back to the original chemotaxis model (1.2) and find that the chemical concentration has no boundary layer but its gradient does. This essentially means that chemotactic flux near the boundary will change drastically if the chemical diffusion rate is small, which implies that the chemical diffusion plays an important role in the tumor angiogenesis (see a more detailed discussion for the biological implications of our results in the end of section 2).

2. Statement of main results.
Notation. For clarity, we specify some notations below.
- Without loss of generality, we assume $0 \leq \varepsilon < 1$ throughout this paper for we consider the diffusion limit problem as $\varepsilon \to 0$.
- Unless specified, we use C to denote a generic positive constant which is independent of ε, depends on the time variable, and may vary in the context, while C_0 denotes a generic positive constant independent of ε and time t.

- L^p with $1 \le p \le \infty$ represents the Lebesgue space $L^p(0,1)$ with respect to $x \in (0,1)$. L_z^p and L_ξ^p denote $L^p(0,\infty)$ with respect to z and $L^p(-\infty,0)$ with respect to ξ, respectively. Similarly, H^k, H_z^k, and H_ξ^k denote the Sobolev spaces $W^{k,2}$ in $(0,1)$, $(0,\infty)$, and $(-\infty,0)$ with respect to x, z, and ξ, respectively.
- Denote $\langle z \rangle = \sqrt{1+z^2}$ for $z \in [0,\infty)$, and $\langle \xi \rangle = \sqrt{1+\xi^2}$ for $\xi \in (-\infty,0]$.
- $\mathbb{N}$ denotes the set of nonnegative integers, and $\mathbb{N}_+$ represents the set of positive integers.

2.1. Boundary layer profiles. In this subsection, we are devoted to using formal asymptotic analysis to find the equations of boundary layer profiles of (1.5) with small $\varepsilon > 0$. The boundary layer thickness has been formally justified as $\mathcal{O}(\varepsilon^{1/2})$ in the appendix of [26]. Thus based on the WKB method (cf. [19, 24, 58]), solutions of (1.5) with $\varepsilon > 0$ have the following expansions for $j \in \mathbb{N}$:

$$
(2.1)
\begin{aligned}
u^\varepsilon(x,t) &= \sum_{j=0}^\infty \varepsilon^{j/2} \left(u^{I,j}(x,t) + u^{B,j}(z,t) + u^{b,j}(\xi,t) \right), \\
v^\varepsilon(x,t) &= \sum_{j=0}^\infty \varepsilon^{j/2} \left(v^{I,j}(x,t) + v^{B,j}(z,t) + v^{b,j}(\xi,t) \right),
\end{aligned}
$$

with boundary layer coordinates (or stretching transformations) defined as

$$
(2.2) \qquad z = \frac{x}{\sqrt{\varepsilon}}, \qquad \xi = \frac{x-1}{\sqrt{\varepsilon}}, \quad x \in [0,1],
$$

where each term in (2.1) is assumed to be smooth, and the boundary layer profiles $(u^{B,j}, v^{B,j})$ and $(u^{b,j}, v^{b,j})$ enjoy the basic hypothesis (cf. [24, Chapter 4], [19], [58])

(H) $u^{B,j}$ and $v^{B,j}$ decay to zero exponentially as $z \to \infty$, while $u^{b,j}$ and $v^{b,j}$ decay to zero exponentially as $\xi \to -\infty$ for all $j \ge 0$.

To derive the equations of boundary layer profiles in (2.1), we split our analysis into three steps. We first insert expansions (2.1) into the initial data in (1.5) and into (1.6) to obtain the initial and boundary values of outer and inner layer profiles. Then in the second and third steps, equations for both outer and inner layer solutions will be derived by substituting (2.1) into the first and second equations of (1.5) successively. Proceeding with these procedures by the asymptotic matching method (details are given in the appendix), we derive that the leading-order outer layer solution pair $(u^{I,0}, v^{I,0})(x,t)$ satisfies the problem

$$
(2.3) \qquad
\begin{cases}
u_t^{I,0} = (u^{I,0}v^{I,0})_x + u_{xx}^{I,0}, & (x,t) \in (0,1) \times (0,\infty), \\
v_t^{I,0} = u_x^{I,0}, \\
(u^{I,0}, v^{I,0})(x,0) = (u_0, v_0)(x), \\
u^{I,0}(0,t) = u^{I,0}(1,t) = \bar{u},
\end{cases}
$$

which is exactly the nondiffusive form of (1.5)–(1.6) with $\varepsilon = 0$. Thus (u^0, v^0) solves (2.3) by uniqueness. The leading-order inner layer solution $v^{B,0}(z,t)$ near the left end point of $(0,1)$ satisfies

$$
(2.4) \qquad
\begin{cases}
v_t^{B,0} = -\bar{u}v^{B,0} + v_{zz}^{B,0}, & (z,t) \in (0,\infty) \times (0,\infty), \\
v^{B,0}(z,0) = 0, \\
v^{B,0}(0,t) = \bar{v} - v^{I,0}(0,t),
\end{cases}
$$

and $u^{B,0}(z,t) \equiv 0$, and the first-order inner layer solution $u^{B,1}(z,t)$ is determined by $v^{B,0}(z,t)$ through

$$(2.5) \qquad u^{B,1}(z,t) = \bar{u} \int_z^\infty v^{B,0}(y,t)\,dy, \quad z \in [0,\infty).$$

The leading-order inner layer solution $v^{b,0}(\xi,t)$ near the right end point of $(0,1)$ satisfies

$$(2.6) \qquad \begin{cases} v_t^{b,0} = -\bar{u}v^{b,0} + v_{\xi\xi}^{b,0}, & (\xi,t) \in (-\infty,0) \times (0,\infty), \\ v^{b,0}(\xi,0) = 0, \\ v^{b,0}(0,t) = \bar{v} - v^{I,0}(1,t), \end{cases}$$

and $u^{b,0}(\xi,t) \equiv 0$, and the corresponding first-order inner layer solution $u^{b,1}(\xi,t)$ is given by

$$(2.7) \qquad u^{b,1}(\xi,t) = \bar{u} \int_\xi^{-\infty} v^{b,0}(y,t)\,dy, \quad \xi \in (-\infty,0].$$

To carry out our desired results, we need the estimates of the first-order outer layer solution pair $(u^{I,1}, v^{I,1})(x,t)$ which satisfies the following problem:

$$(2.8) \qquad \begin{cases} u_t^{I,1} = (u^{I,0}v^{I,1})_x + (u^{I,1}v^{I,0})_x + u_{xx}^{I,1}, & (x,t) \in (0,1) \times (0,\infty), \\ v_t^{I,1} = u_x^{I,1}, \\ (u^{I,1}, v^{I,1})(x,0) = (0,0), \\ u^{I,1}(0,t) = -\bar{u} \int_0^\infty v^{B,0}(z,t)\,dz, \\ u^{I,1}(1,t) = -\bar{u} \int_0^{-\infty} v^{b,0}(\xi,t)\,d\xi. \end{cases}$$

Moreover the inner layer profile $(u^{B,2}, v^{B,1})(z,t)$ satisfies

$$(2.9) \qquad \begin{cases} v_t^{B,1} = -\bar{u}v^{B,1} + v_{zz}^{B,1} - 2(v^{I,0}(0,t) + v^{B,0})v_z^{B,0} + \int_z^\infty \Phi(y,t)\,dy, \\ v^{B,1}(z,0) = 0, \\ v^{B,1}(0,t) = -v^{I,1}(0,t), \end{cases}$$

and

$$(2.10) \qquad u^{B,2}(z,t) = \bar{u} \int_z^\infty v^{B,1}(y,t)\,dy - \int_z^\infty \int_y^\infty \Phi(s,t)\,ds\,dy,$$

where $\Phi(z,t) := (u^{I,1}(0,t) + u^{B,1})v_z^{B,0} + u_x^{I,0}(0,t)v^{B,0} + u_z^{B,1}(v^{I,0}(0,t) + v^{B,0}) +$

$zu_x^{I,0}(0,t)v_z^{B,0}$. Correspondingly the inner layer profile $(u^{b,2}, v^{b,1})(\xi,t)$ satisfies

$$
(2.11) \quad
\begin{cases}
v_t^{b,1} = -\bar{u}v^{b,1} + v_{\xi\xi}^{b,1} - 2(v^{I,0}(1,t) + v^{b,0})v_\xi^{b,0} + \int_\xi^{-\infty} \Psi(y,t)\,dy, \\
v^{b,1}(\xi,0) = 0, \\
v^{b,1}(0,t) = -v^{I,1}(1,t),
\end{cases}
$$

and

$$
(2.12) \quad u^{b,2}(\xi,t) = \bar{u}\int_\xi^{-\infty} v^{b,1}(y,t)\,dy - \int_\xi^{-\infty}\int_y^{-\infty} \Psi(s,t)\,dsdy,
$$

where $\Psi(\xi,t) := (u^{I,1}(1,t) + u^{b,1})v_\xi^{b,0} + u_x^{I,0}(1,t)v^{b,0} + u_\xi^{b,1}(v^{I,0}(1,t) + v^{b,0}) + \xi u_x^{I,0}(1,t)v_\xi^{b,0}$. One can derive the initial-boundary value problems for higher-order layer profiles $(u^{I,j}, v^{I,j})$, $(u^{B,j+1}, v^{B,j})$, and $(u^{b,j+1}, v^{b,j})$ for $j \geq 2$. But (2.3)–(2.12) have been sufficient for our purpose. The detailed derivations of the above equations are postponed the appendix, since it is a little lengthy. The global solutions of (2.3) have been achieved in [38] (see Lemma 2.1 below) and their regularities will be shown in section 3, while the existence of global solutions of (2.4)–(2.12) with regularities will be detailed in section 3.

2.2. Main results (stability of boundary layer profiles). In [38], the authors proved the global well-posedness of classical solutions to system (1.5)–(1.6) with $\varepsilon \geq 0$. We cite the results below for later use.

LEMMA 2.1 (see [38]). *Suppose that* $(u_0, v_0) \in H^2 \times H^2$ *with* $u_0 \geq 0$ *satisfy the compatibility conditions* $(u_0, v_0)(0) = (u_0, v_0)(1) = (\bar{u}, \bar{v})$. *Then for any* $\varepsilon \geq 0$, *the initial-boundary value problem* (1.5)–(1.6) *has a unique global classical solution* $(u^\varepsilon, v^\varepsilon)$ *satisfying the following properties:*
    (i) *If* $\varepsilon > 0$, *then* $(u^\varepsilon - \bar{u}, v^\varepsilon - \bar{v}) \in C([0,\infty); H^2 \times H^2) \cap L^2(0,\infty; H^3 \times H^3)$ *such that*

$$
\|(u^\varepsilon - \bar{u})(t)\|_{L^2}^2 + \|(v^\varepsilon - \bar{v})(t)\|_{L^2}^2 + \int_0^t (\|u_x^\varepsilon(\tau)\|_{L^2}^2 + \varepsilon\|v_x^\varepsilon(\tau)\|_{L^2}^2)\,d\tau \leq C,
$$

where C *is a positive constant independent of* ε.
    (ii) *If* $\varepsilon = 0$, *then* $(u^0 - \bar{u}, v^0) \in C([0,\infty); H^2 \times H^2) \cap L^2(0,\infty; H^3 \times H^2)$.

In order to prove the stability of boundary layer solutions of (1.5)–(1.6), we need some further compatibility conditions on boundaries and higher regularity on the initial data (u_0, v_0) to gain necessary estimates for solutions of (2.3)–(2.11). Precisely, we postulate that the initial data $(u_0, v_0) \in H^3 \times H^3$ satisfy

$$
(A) \quad
\begin{cases}
(u_0, v_0)|_{x=0,1} = (\bar{u}, \bar{v}), \\
u_{0x}|_{x=0,1} = 0, \\
[(u_0 v_0)_x + u_{0xx}]|_{x=0,1} = 0.
\end{cases}
$$

We underline that the condition (A) can be fulfilled by many functions, for instance, $u_0(x) = \bar{u} + ax^4(x-1)^4$, $v_0(x) = \bar{v} + bx^2(x-1)^2$ with $a \geq 0$ and $b \in \mathbb{R}$.
    Now we are in a position to state the main results of this paper as follows.

THEOREM 2.1. *Assume that* $(u_0, v_0) \in H^3 \times H^3$ *with* $u_0 \geq 0$ *satisfy the compatibility conditions* (A). *Denote by* $(u^\varepsilon, v^\varepsilon)$ *the unique global solution of* (1.5)–(1.6)
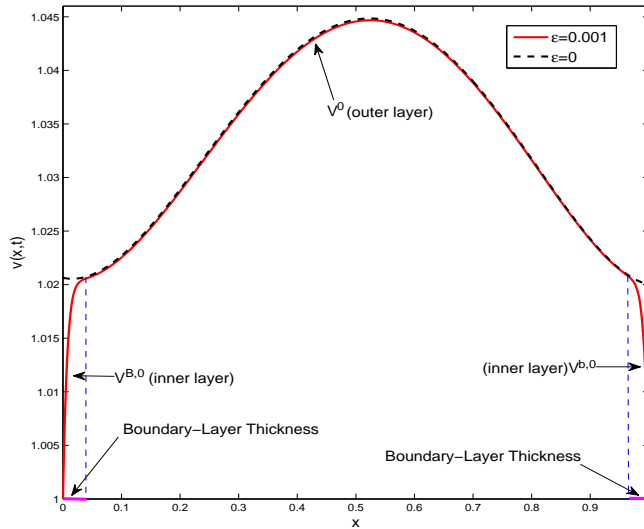
FIG. 2.1. *A numerical simulation of the boundary layer profile $v^\varepsilon(x,t)$ of the system* (1.5)– (1.6) *solved by the MATLAB PDE solver based on the finite difference scheme with time step size* $\Delta t = 0.01$ *and spatial step size* $\Delta x = 0.001$ *where initial data* $u_0(x) = 1 + x^4(x-1)^4$, $v_0(x) = 1 + x^2(x-1)^2$ *and boundary data* $\bar{u} = \bar{v} = 1$. *The profile consists of two parts: outer layer profile* v^0 *and inner layer profiles* $v^{B,0}$ *and* $v^{b,0}$ *near the left and right end points, respectively. Outside the boundary layer the profile* $v^\varepsilon(x,t)$ *matches well with the outer layer profile* $v^0(x,t)$, *whereas there is a rapid transition inside the boundary layer.*

with $\varepsilon \geq 0$. *Then as* $\varepsilon \to 0$, *the following asymptotic expansions hold in space* $L^\infty([0,1] \times [0,T])$ *for any fixed* $0 < T < \infty$:

$$
\begin{aligned}
(2.13) \qquad u^\varepsilon(x,t) &= u^0(x,t) + \mathcal{O}(\varepsilon^{1/2}), \\
v^\varepsilon(x,t) &= v^0(x,t) + v^{B,0}\Big(\frac{x}{\sqrt{\varepsilon}}, t\Big) + v^{b,0}\Big(\frac{x-1}{\sqrt{\varepsilon}}, t\Big) + \mathcal{O}(\varepsilon^{1/2}),
\end{aligned}
$$

where $(u^0, v^0) = (u^{I,0}, v^{I,0})$ *denotes the outer layer profile and the inner layer profile* $(v^{B,0}, v^{b,0})$ *is given by*

$$
\begin{aligned}
(2.14) \qquad v^{B,0}(z,t) &:= \int_0^t \int_{-\infty}^0 \frac{1}{\sqrt{\pi(t-s)}} e^{-\left(\frac{(z-y)^2}{4(t-s)} + \bar{u}(t-s)\right)} \left[\bar{u}(\bar{v} - v^0(0,s)) - v_s^0(0,s)\right] dy\, ds, \\
v^{b,0}(\xi,t) &:= \int_0^t \int_0^\infty \frac{1}{\sqrt{\pi(t-s)}} e^{-\left(\frac{(\xi-y)^2}{4(t-s)} + \bar{u}(t-s)\right)} \left[\bar{u}(\bar{v} - v^0(1,s)) - v_s^0(1,s)\right] dy\, ds.
\end{aligned}
$$

Remark 2.1. We remark that (2.13) displays an elaborate structure of v^ε in the entire interval $[0,1]$ including outer (approximated by v^0) and inner (approximated by $v^{B,0}$, $v^{b,0}$) boundary layer profiles, which significantly develops the result in previous work [26, Theorem 2.6], where only the outer layer profile for v^ε is obtained. The convergence rate $\mathcal{O}(\varepsilon^{1/2})$ in (2.13) is optimal, which improves the rate $\mathcal{O}(\varepsilon^{1/4})$ derived in [26]. Furthermore the inner layer profile is explicitly connected to the outer layer profile through (2.14).

A numerical simulation of the boundary layer solution component $v^\varepsilon(x,t)$ is plotted in Figure 2.1, where the structure of $v^\varepsilon(x,t)$ is graphically demonstrated.

The counterpart of the original system (1.2) in $[0, 1]$ corresponding to the initial-boundary value problem of the transformed system (1.5)–(1.6) reads as follows:

(2.15)
$$\begin{cases} u_t = [u_x - u(\ln c)_x]_x, \\ c_t = \varepsilon c_{xx} - uc, \\ (u, c)(x, 0) = (u_0, c_0)(x), \quad x \in [0, 1], \\ u|_{x=0,1} = \bar{u}, \quad \dfrac{c_x}{c}\Big|_{x=0,1} = -\bar{c} \qquad \text{if } \varepsilon > 0, \\ u|_{x=0,1} = \bar{u} \qquad\qquad\qquad\; \text{if } \varepsilon = 0. \end{cases}$$

Denote by $(u^0, c^0)(x, t)$ the solution of (2.15) with $\varepsilon = 0$. Then $c^0(x, t)$ can be solved from the second equation ($\varepsilon = 0$) of (2.15) as follows:

(2.16)
$$c^0(x, t) = c_0(x) e^{- \int_0^t u^0(x, \tau)\, d\tau}.$$

With (2.16) and the results obtained for the transformed system (1.5)–(1.6), we have the following assertions for the initial-boundary value problem (2.15).

THEOREM 2.2. *Suppose that the initial data* $(u_0, \ln c_0) \in H^3 \times H^4$ *satisfy* $u_0(x) \geq 0$, $c_0(x) > 0$ *and the compatibility conditions* (A) *with* $v_0 = -(\ln c_0)_x$ *and* $\bar{v} = \bar{c}$. *Let* $(u^\varepsilon, c^\varepsilon)$ *be the unique global solution of* (2.15) *with* $\varepsilon \geq 0$. *Then for any fixed* $0 < T < \infty$, *we have in space* $L^\infty([0, 1] \times [0, T])$ *that*

(2.17) $u^\varepsilon(x, t) = u^0(x, t) + \mathcal{O}(\varepsilon^{1/2}), \qquad c^\varepsilon(x, t) = c^0(x, t) + \mathcal{O}(\varepsilon^{1/2})$

and

(2.18) $c_x^\varepsilon(x, t) = c_x^0(x, t) - c^0(x, t) \Big[v^{B,0}\Big(\dfrac{x}{\sqrt{\varepsilon}}, t\Big) + v^{b,0}\Big(\dfrac{x-1}{\sqrt{\varepsilon}}, t\Big) \Big] + \mathcal{O}(\varepsilon^{1/2}),$

with the solution component $c^0(x, t)$ *explicitly given via* $u^0(x, t)$ *in* (2.16).

Remark 2.2. Compared with the previous result in [26, Proposition 2.8], Theorem 2.2 enhances the convergence rate by $\varepsilon^{1/4}$ and gives the leading-order expansion for c_x^ε in (2.18), thanks to the elaborated approximation (2.13) for solutions $(u^\varepsilon, v^\varepsilon)$.

In view of model (1.2) and the transformation (1.3), we see that the quantity v represents the velocity of chemotactic flux crossing the boundary (in tumor angiogenesis the blood vessel wall can be understood as a boundary). Therefore the results in Theorem 2.2 assert that although both cell density and chemical concentration will have no boundary layer as chemical diffusion ε goes to zero, the chemotactic flux, namely, the term $u(\ln c)_x = -uv$, has a sharp transition near the boundary (i.e., the endothelial cells cross the blood vessel wall quickly). Hence our results indicate that the diffusion of the chemical signal (i.e., VEGF) plays an essential role in the transition of cell mass from boundaries to the field away from boundaries during the initiation of tumor angiogenesis. Our results further indicate that the nondiffusive model (2.15) with $\varepsilon = 0$ is not a good approximation of the diffusive model (2.15) for small $\varepsilon > 0$ near the boundary under the boundary conditions imposed in (2.15).

The rest of the paper is organized as follows. In section 3, we shall derive some regularity of global solutions to (2.3)–(2.12). In section 4, we reformulate our problem properly and then prove Theorem 2.1 by the refined energy estimates based on the regularity results derived in section 3. The proof of Theorem 2.2 will be given in section 5. In Appendix A we detail the asymptotical analysis of obtaining (2.3)–(2.12).

3. Regularity of outer/inner layer profiles. In this section, we shall devote ourselves to deriving some regularities for solutions of (2.3)–(2.12) for later use. We depart with a basic regularity result.

Let functions $f_1(x,t)$, $f_2(x,t)$, $f(x,t)$, and $g(x,t)$ defined on $[0,1] \times [0,\infty)$ satisfy the following regularity properties for any $m \in \mathbb{N}_+$ and $0 < T < \infty$:

$$\partial_t^k f_1 \in L^2(0,T;H^{2m-1-2k}), \quad \partial_t^k f_2 \in L^2(0,T;H^{2m-1-2k}),$$
$$\partial_t^k f \in L^2(0,T;H^{2m-2-2k}), \quad \partial_t^k g \in L^2(0,T;H^{2m-1-2k}),$$

where $k = 0, 1, \ldots, m-1$. To solve the outer layer solution pairs $(u^{I,j}, v^{I,j})(x,t), j = 0, 1$, from problems (2.3) and (2.8), we first consider the following auxiliary initial-boundary value problem:

$$(3.1) \quad \begin{cases} h_t = (f_1 h)_x + (f_2 w)_x + h_{xx} + f, & (x,t) \in (0,1) \times (0,T), \\ w_t = h_x + g, \\ (h,w)(x,0) = (h_0, w_0)(x), \\ h(0,t) = h(1,t) = 0. \end{cases}$$

To derive the desired regularity (3.2) for solutions (h,w) of (3.1) (see Proposition 3.1 below), some compatibility conditions on h_0, w_0, f_1, f_2, f, g are required. In what follows, by "h_0, w_0, f_1, f_2, f, and g satisfy the compatibility conditions up to order $(m-1)$ for the problem (3.1)," we mean that $\partial_t^k h|_{t=0}$, which is determined by h_0, w_0, f_1, f_2, f, g and their time derivatives through the equations in (3.1), is equal to zeros on boundaries for $0 \le k \le m-1$ (cf. [33, p. 319]).

Then the solution of (3.1) has the following regularity properties.

PROPOSITION 3.1. *Suppose that* $(h_0, w_0) \in H^{2m-1} \times H^{2m-1}$, f_1, f_2, f, *and* g *satisfy the compatibility conditions up to order* $(m-1)$ *for the problem* (3.1). *Then there exists a unique solution* (h,w) *to* (3.1) *for any* $0 < T < \infty$ *such that*

$$(3.2) \quad \begin{aligned} & \partial_t^k h \in L^2(0,T;H^{2m-2k}), \quad k = 0,1,\ldots,m; \\ & w \in L^\infty(0,T;H^{2m-1}); \quad \partial_t^k w \in L^2(0,T;H^{2m+1-2k}), \quad k = 1,\ldots,m. \end{aligned}$$

Proof. The proof of global existence and uniqueness of solutions to (3.1) is standard (see Lemma 2.1). The regularity given in (3.2) can be proved by mathematical induction. For $m = 1$, the conclusion follows from the standard energy method used in [38, Proof of Theorem 1.1] and we hence omit the details. The remaining procedure of mathematical induction is routine (e.g., see details in [12, pp. 387–388]) and will be skipped for brevity. □

To solve inner layer profiles $v^{B,0}(z,t)$ and $v^{B,1}(z,t)$ from (2.4) and (2.9), we need the following result.

PROPOSITION 3.2. *Let* $m \in \mathbb{N}_+$ *and* $0 < T < \infty$. *Suppose* $\rho(z,t)$ *satisfies for any* $l \in \mathbb{N}$ *that*

$$\langle z \rangle^l \, \partial_t^k \rho \in L^2(0,T;H_z^{2m-2-2k}), \quad k = 0, 1, \ldots, m-1,$$

and the compatibility conditions up to order $(m-1)$ *for the following problem:*

$$(3.3) \quad \begin{cases} \varphi_t = -\bar{u}\varphi + \varphi_{zz} + \rho, & (z,t) \in (0,\infty) \times (0,T), \\ \varphi(z,0) = 0, \\ \varphi(0,t) = 0. \end{cases}$$

Then there exists a unique solution φ *to* (3.3) *such that for any* $l \in \mathbb{N}$,

$$\langle z \rangle^l \, \partial_t^k \varphi \in L^2(0, T; H_z^{2m-2k}), \quad k = 0, 1, \ldots, m.$$

Proposition 3.2 follows directly from the standard energy method, and we hence omit the proof. We proceed to introduce the following well-known result for later use.

PROPOSITION 3.3 (see [62, Lemma 1.2]). *Let* V, H, V' *be three Hilbert spaces, satisfying* $V \subset H \subset V'$ *with* V' *being the dual of* V. *If a function* u *belongs to* $L^2(0, T; V)$ *and its time derivative* u_t *belongs to* $L^2(0, T; V')$, *then*

$$u \in C([0, T]; H) \quad \text{and} \quad \|u\|_{L^\infty(0, T; H)} \le C(\|u\|_{L^2(0, T; V)} + \|u_t\|_{L^2(0, T; V')}),$$

where the constant C *depends on* T.

Remark 3.1. Let $m \in \mathbb{N}$. Suppose that $u \in L^2(0, T; H^{m+2})$ and $u_t \in L^2(0, T; H^m)$. Then it follows from Proposition 3.3 that

$$u \in C([0, T]; H^{m+1}) \quad \text{and} \quad \|u\|_{L^\infty(0, T; H^{m+1})} \le C(\|u\|_{L^2(0, T; H^{m+2})} + \|u_t\|_{L^2(0, T; H^m)}).$$

Based on the above preliminaries, we can establish the regularities of solutions to (2.3)–(2.12). First for the problem (2.3), the existence of global solution has been available (see Lemma 2.1). We prove the following regularity results.

LEMMA 3.1. *Let* $(u_0, v_0) \in H^3 \times H^3$ *satisfy the assumptions in Theorem* 2.1. *Then the unique solution* $(u^{I,0}, v^{I,0})$ *of* (2.3) *satisfies that*

$$\partial_t^k u^{I,0} \in L^2(0, T; H^{4-2k}), \quad k = 0, 1, 2;$$
$$v^{I,0} \in L^\infty(0, T; H^3); \quad \partial_t^k v^{I,0} \in L^2(0, T; H^{5-2k}), \quad k = 1, 2.$$

Proof. We shall prove this lemma by Proposition 3.1 and Lemma 2.1. Differentiating the first and second equations of (2.3) with respect to t, respectively, and setting $\tilde{u}^{I,0} = u_t^{I,0}$, $\tilde{v}^{I,0} = v_t^{I,0}$, one gets

(3.4)
$$\begin{cases} \tilde{u}_t^{I,0} = (f_1 \tilde{u}^{I,0})_x + (f_2 \tilde{v}^{I,0})_x + \tilde{u}_{xx}^{I,0}, \\ \tilde{v}_t^{I,0} = \tilde{u}_x^{I,0}, \\ (\tilde{u}^{I,0}, \tilde{v}^{I,0})(x, 0) = (\tilde{u}_0, \tilde{v}_0)(x), \\ \tilde{u}^{I,0}(0, t) = \tilde{u}^{I,0}(1, t) = 0, \end{cases}$$

where $f_1 := v^{I,0}$, $f_2 := u^{I,0}$, $\tilde{u}_0 := (u_0 v_0)_x + u_{0xx}$, $\tilde{v}_0 := u_{0x}$ and the first and second equations of (2.3) have been used to determine initial data $\tilde{u}_0$ and $\tilde{v}_0$, respectively. We next verify that $\tilde{u}_0$, $\tilde{v}_0$, f_1, and f_2 fulfill the assumptions in Proposition 3.1 with $m = 1$. First, by the assumptions in Theorem 2.1 and using Lemma 2.1 one finds that

(3.5) $\qquad \|\tilde{u}_0\|_{H^1} + \|\tilde{v}_0\|_{H^1} \le C_0; \qquad \|f_1\|_{L^2(0, T; H^1)} + \|f_2\|_{L^2(0, T; H^1)} \le C.$

Noting that the compatibility condition of order zero for (3.4) is satisfied under assumption (A), thus using (3.5), we apply Proposition 3.1 with $m = 1$ to system (3.4) and conclude that

(3.6) $\qquad \partial_t^k u^{I,0} \in L^2(0, T; H^{4-2k}), \quad k = 1, 2; \qquad \partial_t v^{I,0} \in L^2(0, T; H^1),$

where $\tilde{u}^{I,0} := u_t^{I,0}$ and $\tilde{v}^{I,0} := v_t^{I,0}$ have been used. It only remains to prove

$$(3.7) \qquad u^{I,0} \in L^2(0,T;H^4), \quad v^{I,0} \in L^\infty(0,T;H^3), \quad v_t^{I,0} \in L^2(0,T;H^3).$$

To this end, we apply the differential operator ∂_x^3 to the second equation of (2.3) and use the first equation of (2.3) to get

$$(3.8) \qquad v_{xxxt}^{I,0} = u_{xxxx}^{I,0} = u_{xxt}^{I,0} - (u^{I,0}v^{I,0})_{xxx},$$

which, multiplied by $2v_{xxx}^{I,0}$ in L^2, gives

$$\begin{aligned}
\frac{d}{dt}\|v_{xxx}^{I,0}(t)\|_{L^2}^2 &\le 2\|u_{xxt}^{I,0}(t)\|_{L^2}\|v_{xxx}^{I,0}(t)\|_{L^2} + C_0\|u^{I,0}(t)\|_{H^3}\|v^{I,0}(t)\|_{H^3}^2 \\
&\le C_0(1+\|u^{I,0}(t)\|_{H^3})\|v_{xxx}^{I,0}(t)\|_{L^2}^2 \\
&\quad + C_0(\|u_t^{I,0}(t)\|_{H^2}^2 + \|u^{I,0}(t)\|_{H^3}\|v^{I,0}(t)\|_{H^2}^2).
\end{aligned}$$

Thus it follows from Gronwall's inequality, Lemma 2.1, and (3.6) that

$$(3.9) \qquad \|v_{xxx}^{I,0}\|_{L^\infty(0,T;L^2)}^2 \le C.$$

Furthermore, using (3.8), (3.6), (3.9), and Lemma 2.1, one has

$$(3.10)$$
$$\|u_{xxxx}^{I,0}\|_{L^2(0,T;L^2)} \le \|u_t^{I,0}\|_{L^2(0,T;H^2)} + C_0\|u^{I,0}\|_{L^2(0,T;H^3)}\|v^{I,0}\|_{L^\infty(0,T;H^3)} \le C.$$

Finally, the second equation of (2.3) along with (3.10) and Lemma 2.1 yields

$$(3.11) \qquad \|v_t^{I,0}\|_{L^2(0,T;H^3)} \le \|u^{I,0}\|_{L^2(0,T;H^4)} \le C.$$

Collecting (3.9), (3.10), (3.11) and using Lemma 2.1 we obtain (3.7), which in conjunction with (3.6) finishes the proof. $\square$

LEMMA 3.2. *Let* $(u^{I,0}, v^{I,0})$ *be the solution obtained in Lemma* 3.1. *Then*
$$(3.12)$$
$$v^{B,0}(z,t) := \int_0^t \int_{-\infty}^0 \frac{1}{\sqrt{\pi(t-s)}} e^{-\left(\frac{(z-y)^2}{4(t-s)} + \bar{u}(t-s)\right)} \left[\bar{u}(\bar{v}-v^{I,0}(0,s)) - v_s^{I,0}(0,s)\right] dy\, ds$$

is the unique solution of (2.4). *Moreover, for any* $0 < T < \infty$ *and* $l \in \mathbb{N}$, *it holds that*

$$(3.13) \qquad \langle z \rangle^l \, \partial_t^k v^{B,0} \in L^2(0,T;H_z^{4-2k}) \quad \text{for } k = 0, 1, 2.$$

Consequently it follows from (2.5) *that*

$$\langle z \rangle^l \, \partial_t^k u^{B,1} \in L^2(0,T;H_z^{4-2k}) \quad \text{for } k = 0, 1, 2.$$

Proof. We first prove (3.12) by setting $w(z,t) := e^{\bar{u}t}[v^{B,0}(z,t) - (\bar{v} - v^{I,0}(0,t))]$. Then from (2.4) we derive the heat equation subject to the homogeneous Dirichlet boundary condition,

$$\begin{cases} w_t - w_{zz} = -[e^{\bar{u}t}(\bar{v} - v^{I,0}(0,t))]_t, & (z,t) \in (0,\infty) \times (0,\infty) \\ w(z,0) = 0, \\ w(0,t) = 0, \end{cases}$$

which can be solved explicitly by the reflection method with odd extensions (cf. [33]) as follows:

$$w(z,t) = 2 \int_0^t \int_{-\infty}^0 \Gamma(z-y, t-s)[e^{\bar{u}s}(\bar{v} - v^{I,0}(0,s))]_s \, dy ds - e^{\bar{u}t}(\bar{v} - v^{I,0}(0,t))$$

with the heat kernel $\Gamma(z,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{z^2}{4t}}$. Hence (3.12) follows by substituting the above equality into the definition of $w(z,t)$. We proceed to prove (3.13). Let $\theta(z)$ be a smooth function defined on $[0,\infty)$ satisfying

$$(3.14) \qquad\qquad \theta(0) = 1, \quad \theta(z) = 0 \text{ for } z > 1.$$

Let $b(t) := \bar{v} - v^{I,0}(0,t)$ and $\tilde{v}^{B,0} := v^{B,0} - \theta(z)b(t)$. Then from (2.4) we deduce that $\tilde{v}^{B,0}$ satisfies

$$(3.15) \qquad \begin{cases} \tilde{v}_t^{B,0} = -\bar{u}\tilde{v}^{B,0} + \tilde{v}_{zz}^{B,0} + \tilde{\rho}, \\ \tilde{v}^{B,0}(z,0) = 0, \\ \tilde{v}^{B,0}(0,t) = 0, \end{cases}$$

where $\tilde{\rho}(z,t) := \theta_{zz}(z)b(t) - \bar{u}\theta(z)b(t) - \theta(z)b_t(t)$, and the compatibility condition $\bar{v} = v_0(0)$ has been used to determine the initial value of $\tilde{v}^{B,0}$. We shall apply Proposition 3.2 to (3.15) to derive the desired regularity for $v^{B,0}$. To this end, we need to verify that $\tilde{\rho}$ satisfies the assumptions in Proposition 3.2 with $m = 2$. First, it is easy to check that $\tilde{\rho}$ satisfies the compatibility conditions up to order one for problem (3.15) under assumption (A). Then noticing that for any $G(x,t) \in L^p(0,T;H^1)$ with $1 \le p \le \infty$, it follows from the Sobolev embedding inequality that

$$(3.16) \qquad \|G(0,t)\|_{L^p(0,T)} \le \|G\|_{L^p(0,T;L^\infty)} \le C_0 \|G\|_{L^p(0,T;H^1)}.$$

By (3.16) and Lemma 3.1, one finds that

$$(3.17) \qquad \|\partial_t^k v^{I,0}(0,t)\|_{L^2(0,T)} \le C, \; k = 1, 2; \qquad \|v^{I,0}(0,t)\|_{L^2(0,T)} \le C.$$

Collecting (3.14) and (3.17), one deduces for $k = 0, 1$ and $l \in \mathbb{N}$ that

$$\langle z \rangle^l \, \partial_t^k \tilde{\rho} = \langle z \rangle^l \, \theta_{zz} \, \partial_t^k b - \bar{u}\langle z \rangle^l \, \theta \, \partial_t^k b - \langle z \rangle^l \, \theta \, \partial_t^{k+1} b \in L^2(0,T;H_z^{2-2k}),$$

which along with Proposition 3.2 entails for $k = 0, 1, 2$ and $l \in \mathbb{N}$ that

$$\langle z \rangle^l \, \partial_t^k \tilde{v}^{B,0} \in L^2(0,T;H_z^{4-2k}).$$

Thus (3.13) follows from the definition of $\tilde{v}^{B,0}$, (3.14), and (3.17). Finally by (3.13), we use (2.5) and the Hölder inequality to get for $k = 0, 1, 2$ and $l \in \mathbb{N}$ that

$$(3.18) \qquad \begin{aligned} &\|\langle z \rangle^l \partial_t^k u^{B,1}\|_{L^2(0,T;H_z^{4-2k})}^2 \\ &\le C_0 \bar{u}^2 \Big(1 + \int_0^\infty \int_z^\infty \langle y \rangle^{-4} \, dy dz \Big) \|\langle z \rangle^{l+2} \partial_t^k v^{B,0}\|_{L^2(0,T;H_z^{4-2k})}^2 \le C, \end{aligned}$$

which completes the proof. □

By a similar procedure as proving Lemma 3.2, we have the following results.

LEMMA 3.3. *Let $(u^{I,0}, v^{I,0})$ be the solution obtained in Lemma* 3.1. *Then the unique solution $v^{b,0}(\xi, t)$ of* (2.6) *is as follows:*

(3.19)
$$v^{b,0}(\xi,t) := \int_0^t \int_0^\infty \frac{1}{\sqrt{\pi(t-s)}} e^{-\left(\frac{(\xi-y)^2}{4(t-s)} + \bar{u}(t-s)\right)} \left[\bar{u}(\bar{v} - v^{I,0}(1,s)) - v_s^{I,0}(1,s)\right] dy\, ds.$$

Furthermore, for any $0 < T < \infty$ and $l \in \mathbb{N}$, the following holds true:

$$\langle\xi\rangle^l \partial_t^k v^{b,0}, \quad \langle\xi\rangle^l \partial_t^k u^{b,1} \in L^2(0,T; H_\xi^{4-2k}) \quad \text{for } k = 0, 1, 2.$$

Based on Lemmas 3.2 and 3.3, we proceed to solve (2.8).

LEMMA 3.4. *Let $v^{B,0}$ and $v^{b,0}$ be the solution obtained in Lemmas* 3.2 *and* 3.3, *respectively. Then there exists a unique solution $(u^{I,1}, v^{I,1})$ to* (2.8) *on $[0,T]$ for any $0 < T < \infty$, such that*

$$\partial_t^k u^{I,1} \in L^2(0,T; H^{4-2k}), \quad k = 0, 1, 2;$$
$$v^{I,1} \in L^\infty(0,T; H^3); \quad \partial_t^k v^{I,1} \in L^2(0,T; H^{5-2k}), \quad k = 1, 2.$$

Proof. Let $b_1(t) := \bar{u} \int_0^\infty v^{B,0}(z,t)\, dz$, $b_2(t) := \bar{u} \int_0^{-\infty} v^{b,0}(\xi,t)\, d\xi$, $b(x,t) := xb_2(t) + (1-x)b_1(t)$ and $\tilde{u}^{I,1} := u^{I,1} + b(x,t)$. Then from (2.8), we deduce that $(\tilde{u}^{I,1}, v^{I,1})$ satisfy

(3.20)
$$\begin{cases} \tilde{u}_t^{I,1} = (f_1 \tilde{u}^{I,1})_x + (f_2 v^{I,1})_x + \tilde{u}_{xx}^{I,1} + f, \\ v_t^{I,1} = \tilde{u}_x^{I,1} + g, \\ (\tilde{u}^{I,1}, v^{I,1})(x,0) = (0,0), \\ \tilde{u}^{I,1}(0,t) = \tilde{u}^{I,1}(1,t) = 0, \end{cases}$$

where $f_1 := v^{I,0}$, $f_2 := u^{I,0}$, $f := -(bv^{I,0})_x + b_t$, $g := b_1(t) - b_2(t)$, and $v^{B,0}(z,0) = v^{b,0}(\xi,0) = 0$ has been used in deriving the initial data for $\tilde{u}^{I,1}$. We next verify that f_1, f_2, f, and g fulfill the assumptions in Proposition 3.1 with $m = 2$. Indeed, it follows from Lemma 3.1 that

(3.21)
$$\partial_t^k u^{I,0}, \quad \partial_t^k v^{I,0} \in L^2(0,T; H^{3-2k}), \quad k = 0, 1.$$

Lemma 3.2 gives for $k = 0, 1, 2$ that

(3.22)
$$\|\partial_t^k b_1\|_{L^2(0,T)}^2 \leq \bar{u}^2 \int_0^\infty \langle z\rangle^{-2}\, dy \cdot \|\langle z\rangle \partial_t^k v^{B,0}\|_{L^2(0,T;L_z^2)}^2 \leq C$$

and similarly Lemma 3.3 implies for $k = 0, 1, 2$ that

(3.23)
$$\|\partial_t^k b_2\|_{L^2(0,T)}^2 \leq C.$$

Thus from (3.22), (3.23), and the definition of g, we have

(3.24)
$$\partial_t^k g \in L^2(0,T; H^{3-2k}), \quad k = 0, 1.$$

To estimate f, we use (3.22)–(3.23), Lemma 3.1, and Proposition 3.3 and get for $k = 0, 1$ that

$$\|\partial_t^k (bv^{I,0})_x\|_{L^2(0,T;H^{2-2k})} \leq \sum_{j=0}^k (\|\partial_t^j b_1\|_{L^2(0,T)} + \|\partial_t^j b_2\|_{L^2(0,T)})$$
$$\times \|\partial_t^{k-j} v^{I,0}\|_{L^\infty(0,T;H^{3-2(k-j)})} \leq C,$$

which, in conjunction with the definition of f, (3.22), and (3.23) entails that

$$(3.25) \qquad \partial_t^k f \in L^2(0, T; H^{2-2k}), \quad k = 0, 1.$$

Noting that for (3.20), compatibility conditions up to order one are fulfilled under assumption (A), by (3.21), (3.24), and (3.25), we apply Proposition 3.1 with $m = 2$ to (3.20) and get

$$(3.26) \qquad \begin{aligned} &\partial_t^k \tilde{u}^{I,1} \in L^2(0, T; H^{4-2k}), \quad k = 0, 1, 2; \\ &v^{I,1} \in L^\infty(0, T; H^3); \quad \partial_t^k v^{I,1} \in L^2(0, T; H^{5-2k}), \quad k = 1, 2. \end{aligned}$$

The first estimate in (3.26) along with the definition of $\tilde{u}^{I,1}$, (3.22), and (3.23) gives rise to

$$(3.27) \qquad \partial_t^k u^{I,1} \in L^2(0, T; H^{4-2k}), \quad k = 0, 1, 2.$$

Thus the combination of (3.26) and (3.27) completes the proof. $\qquad\square$

We next turn to the regularity of solutions to (2.9) and (2.10).

LEMMA 3.5. *Let $(u^{I,1}, v^{I,1})$ be the solution obtained in Lemma 3.4. Then there exists a unique solution $v^{B,1}$ to (2.9) on $[0, T]$ for any $0 < T < \infty$ such that for any $l \in \mathbb{N}$,*

$$(3.28) \qquad \langle z \rangle^l \, \partial_t^k v^{B,1} \in L^2(0, T; H_z^{4-2k}), \quad k = 0, 1, 2.$$

Consequently, it follows from (2.10) that

$$(3.29) \qquad \langle z \rangle^l \, \partial_t^k u^{B,2} \in L^2(0, T; H_z^{4-2k}), \quad k = 0, 1.$$

Proof. Let $\tilde{v}^{B,1} := v^{B,1} + \theta(z) v^{I,1}(0, t)$ with θ defined in (3.14). Then from (2.9), we deduce that $\tilde{v}^{B,1}$ satisfies

$$(3.30) \qquad \begin{cases} \tilde{v}_t^{B,1} = -\bar{u} \tilde{v}^{B,1} + \tilde{v}_{zz}^{B,1} + \rho, \\ \tilde{v}^{B,1}(z, 0) = 0, \\ \tilde{v}^{B,1}(0, t) = 0, \end{cases}$$

where $\rho := \bar{u} \theta v^{I,1}(0, t) + \theta v_t^{I,1}(0, t) - \theta_{zz} v^{I,1}(0, t) - 2(v^{I,0}(0, t) + v^{B,0}) v_z^{B,0} + \int_z^\infty \Phi(y, t) \, dy$. We shall apply Proposition 3.2 with $m = 2$ to (3.30) to prove this lemma by verifying that ρ satisfies the assumptions in Proposition 3.2. Let us start by dividing ρ into three parts:

(3.31)

$$\rho = [\bar{u} \theta v^{I,1}(0, t) + \theta v_t^{I,1}(0, t) - \theta_{zz} v^{I,1}(0, t)] - 2(v^{I,0}(0, t) + v^{B,0}) v_z^{B,0} + \int_z^\infty \Phi(y, t) \, dy$$

$$:= I_1 + I_2 + I_3.$$

We next estimate I_1, I_2, and I_3. First it follows from (3.16) and Lemma 3.4 that

$$\|\partial_t^k v^{I,1}(0, t)\|_{L^2(0,T)} \leq C_0 \|\partial_t^k v^{I,1}\|_{L^2(0,T;H^1)} \leq C, \quad k = 0, 1, 2,$$

which, along with the definition of θ in (3.14), implies that

$$(3.32) \qquad \langle z \rangle^l \, \partial_t^k I_1 \in L^2(0, T; H_z^{2-2k}), \quad l \in \mathbb{N}, \ k = 0, 1.$$

Then applying (3.16) to $\partial_t^j v^{I,0}$ and using Lemma 3.1, Lemma 3.2, and Remark 3.1, we have for $k = 0,1$ and $l \in \mathbb{N}$ that

(3.33)
$$\|\langle z \rangle^l \, \partial_t^k I_2\|_{L^2(0,T;H_z^{2-2k})}$$
$$\leq C_0 \sum_{j=0}^{k} (\|\partial_t^j v^{I,0}\|_{L^\infty(0,T;H^{3-2j})} + \|\partial_t^j v^{B,0}\|_{L^\infty(0,T;H_z^{3-2j})}) \|\partial_t^{k-j} v^{B,0}\|_{L^2(0,T;H_z^{4-2(k-j)})}$$
$$\leq C.$$

For I_3, the estimate is a little more complicated, since it involves several terms. The Hölder inequality entails for $k = 0,1$ and $l \in \mathbb{N}$ that

(3.34)
$$\|\langle z \rangle^l \, \partial_t^k I_3\|^2_{L^2(0,T;H_z^{2-2k})} \leq C_0 \Big(1 + \int_0^\infty \int_z^\infty \langle y \rangle^{-4} \, dydz \Big) \|\langle z \rangle^{l+2} \, \partial_t^k \Phi\|^2_{L^2(0,T;H_z^{2-2k})}.$$

Noting that the integration term in parentheses of the above inequality is finite, we only need to estimate the remaining term. By the definition of Φ below (2.10), one gets for $l \in \mathbb{N}$ that

(3.35)
$$\langle z \rangle^l \, \partial_t^k \Phi = \langle z \rangle^l \, \partial_t^k [u^{I,1}(0,t) v_z^{B,0}] + \langle z \rangle^l \, \partial_t^k [u^{B,1} v_z^{B,0}] + \langle z \rangle^l \, \partial_t^k [u_x^{I,0}(0,t) v^{B,0}]$$
$$+ \langle z \rangle^l \, \partial_t^k [v^{I,0}(0,t) u_z^{B,1}] + \langle z \rangle^l \, \partial_t^k [u_z^{B,1} v^{B,0}] + z \langle z \rangle^l \, \partial_t^k [u_x^{I,0}(0,t) v_z^{B,0}]$$
$$:= M_1 + M_2 + M_3 + M_4 + M_5 + M_6.$$

Applying (3.16) to $\partial_t^j u^{I,1}$, by Lemma 3.2, Lemma 3.4, and Remark 3.1, we get for $k = 0,1$ that

$$\|M_1\|_{L^2(0,T;H_z^{2-2k})} \leq C_0 \sum_{j=0}^{k} \|\partial_t^j u^{I,1}\|_{L^\infty(0,T;H^{3-2j})} \|\langle z \rangle^l \, \partial_t^{k-j} v^{B,0}\|_{L^2(0,T;H_z^{3-2(k-j)})} \leq C.$$

Similar arguments further give the estimate for $\{M_i\}_{2 \leq i \leq 6}$:

$$\|M_i\|_{L^2(0,T;H_z^{2-2k})} \leq C, \quad 2 \leq i \leq 6, \;\; k = 0,1.$$

Plugging the above estimates into (3.35), we conclude for any $l \in \mathbb{N}$ that

(3.36)
$$\langle z \rangle^l \, \partial_t^k \Phi \in L^2(0,T;H_z^{2-2k}), \quad k = 0,1,$$

which along with (3.34) gives rise to

(3.37)
$$\langle z \rangle^l \, \partial_t^k I_3 \in L^2(0,T;H_z^{2-2k}), \quad k = 0,1.$$

Then it follows from (3.31), (3.32), (3.33), and (3.37) that

(3.38)
$$\langle z \rangle^l \, \partial_t^k \rho \in L^2(0,T;H_z^{2-2k}), \quad k = 0,1, \;\; l \in \mathbb{N}.$$

Moreover for (3.30) it is easy to check that ρ fulfills the compatibility conditions up to order one under assumption (A). Thus by (3.38), we apply Proposition 3.2 with $m = 2$ to (3.30) and have

(3.39)
$$\langle z \rangle^l \, \partial_t^k \tilde{v}^{B,1} \in L^2(0,T;H_z^{4-2k}), \qquad k = 0,1,2, \;\; l \in \mathbb{N}.$$

To convert the result in (3.39) back to $v^{B,1}$, we note that

$$(3.40) \qquad \langle z \rangle^l \, \partial_t^k v^{B,1} = \langle z \rangle^l \, \partial_t^k \tilde{v}^{B,1} - \langle z \rangle^l \, \theta(z) \partial_t^k v^{I,1}(0,t),$$

where the second term on the right-hand side is estimated by the definition of θ, (3.16), and Lemma 3.4 for $k = 1, 2$ and $l \in \mathbb{N}$ as

$$\|\langle z \rangle^l \, \theta(z) \partial_t^k v^{I,1}(0,t)\|_{L^2(0,T;H_z^{4-2k})} \leq C_0 \|\partial_t^k v^{I,1}(0,t)\|_{L^2(0,T)}$$
$$\leq C_0 \|\partial_t^k v^{I,1}\|_{L^2(0,T;H^{5-2k})} \leq C$$

and for $k = 0$ and $l \in \mathbb{N}$ as

$$\|\langle z \rangle^l \, \theta(z) v^{I,1}(0,t)\|_{L^2(0,T;H_z^4)} \leq C_0 \|v^{I,1}(0,t)\|_{L^2(0,T)} \leq C \|v^{I,1}\|_{L^\infty(0,T;H^1)} \leq C.$$

Inserting the above two estimates with (3.39) into (3.40), we derive (3.28). It remains to estimate $u^{B,2}$. Indeed, (2.10) implies for $l \in \mathbb{N}$ that

$$(3.41) \qquad \langle z \rangle^l \, \partial_t^k u^{B,2} = \bar{u} \, \langle z \rangle^l \int_z^\infty \partial_t^k v^{B,1}(y,t) \, dy - \langle z \rangle^l \int_z^\infty \int_y^\infty \partial_t^k \Phi(s,t) \, ds dy$$
$$:= I_4 + I_5.$$

By (3.28) and a similar argument in deriving (3.18), one gets $\|I_4\|_{L^2(0,T;H_z^{4-2k})}^2 \leq C$ for $k = 0, 1, 2$. Noting that I_5 is a double integral of $\partial_t^k \Phi$, we employ (3.36) and have for $k = 0, 1$ that

$$\|I_5\|_{L^2(0,T;H_z^{4-2k})}^2 \leq C_0 \Big(1 + \int_0^\infty \int_z^\infty \langle y \rangle^{-4} \, dy dz + \int_0^\infty \Big\{ \int_z^\infty \Big[\int_y^\infty \langle s \rangle^{-6} \, ds \Big]^{\frac{1}{2}} \, dy \Big\}^2 \, dz \Big)$$
$$\times \|\langle z \rangle^{l+3} \partial_t^k \Phi\|_{L^2(0,T;H_z^{2-2k})}^2$$
$$\leq C.$$

Substituting the above estimates for I_4 and I_5 into (3.41) one gets (3.29). The proof is completed. $\qquad \square$

Noticing the similarity between (2.9) and (2.11), by analogous arguments as proving Lemma 3.5, one gets the following.

LEMMA 3.6. *Let* $(u^{I,1}, v^{I,1})$ *be the solution obtained in Lemma* 3.4. *Then there exists a unique solution* $v^{b,1}$ *to* (2.11) *on* $[0, T]$ *for any* $0 < T < \infty$ *such that for any* $l \in \mathbb{N}$,

$$\langle \xi \rangle^l \, \partial_t^k v^{b,1} \in L^2(0,T;H_\xi^{4-2k}), \quad k = 0, 1, 2,$$

and

$$\langle \xi \rangle^l \, \partial_t^k u^{b,2} \in L^2(0,T;H_\xi^{4-2k}), \quad k = 0, 1.$$

4. Stability of boundary layers (Proof of Theorem 2.1).

4.1. Reformulation of the problem. To prove Theorem 2.1, if we decompose the solution $(u^\varepsilon, v^\varepsilon)$ as

$$(4.1) \qquad u^\varepsilon(x,t) = u^{I,0}(x,t) + R_1^\varepsilon(x,t),$$
$$v^\varepsilon(x,t) = v^{I,0}(x,t) + v^{B,0}\Big(\frac{x}{\sqrt{\varepsilon}}, t\Big) + v^{b,0}\Big(\frac{x-1}{\sqrt{\varepsilon}}, t\Big) + R_2^\varepsilon(x,t),$$

then it remains to derive the equations satisfied by $R_i^\varepsilon(x,t)$ $(i=1,2)$ and to show

$$\|R_i^\varepsilon\|_{L^\infty([0,1]\times[0,T])} = \mathcal{O}(\varepsilon^{1/2}).$$

But if we substitute (4.1) into (1.5), we shall find that the equations of R_i^ε have source terms containing a singular quantity of order $\varepsilon^{-1/2}$, which brings difficulties to deriving the uniform-in-ε boundedness of $\|R_i^\varepsilon\|_{L^\infty([0,1]\times[0,T])}$ $(i=1,2)$. Therefore we invoke the higher-order terms in the expansion of $(u^\varepsilon, v^\varepsilon)$ to overcome this difficulty motivated by the work [46]. To this end, we employ (2.1)–(2.2) to write $R_i^\varepsilon(x,t)(i=1,2)$ as

$$\begin{aligned}
R_1^\varepsilon(x,t) &= \varepsilon^{1/2}[u^{I,1}(x,t) + u^{B,1}(z,t) + u^{b,1}(\xi,t)] + \varepsilon[u^{B,2}(z,t) + u^{b,2}(\xi,t)] \\
&\quad + b_1^\varepsilon(x,t) + \varepsilon^{1/2}U^\varepsilon(x,t), \\
R_2^\varepsilon(x,t) &= \varepsilon^{1/2}[v^{I,1}(x,t) + v^{B,1}(z,t) + v^{b,1}(\xi,t)] \\
&\quad + b_2^\varepsilon(x,t) + \varepsilon^{1/2}V^\varepsilon(x,t),
\end{aligned}$$

where the perturbation functions $(U^\varepsilon, V^\varepsilon)(x,t)$ are to be determined, and the auxiliary functions $b_i(x,t)$ $(i=1,2)$ are constructed as follows to homogenize the boundary conditions of $(U^\varepsilon, V^\varepsilon)(x,t)$:

$$\begin{aligned}
b_1^\varepsilon(x,t) &= -(1-x)[\varepsilon^{1/2}u^{b,1}(-\varepsilon^{-1/2},t) + \varepsilon u^{B,2}(0,t) + \varepsilon u^{b,2}(-\varepsilon^{-1/2},t)] \\
&\quad - x[\varepsilon^{1/2}u^{B,1}(\varepsilon^{-1/2},t) + \varepsilon u^{b,2}(0,t) + \varepsilon u^{B,2}(\varepsilon^{-1/2},t)], \\
b_2^\varepsilon(x,t) &= -(1-x)[v^{b,0}(-\varepsilon^{-1/2},t) + \varepsilon^{1/2}v^{b,1}(-\varepsilon^{-1/2},t)] \\
&\quad - x[v^{B,0}(\varepsilon^{-1/2},t) + \varepsilon^{1/2}v^{B,1}(\varepsilon^{-1/2},t)].
\end{aligned}$$

We should remark that the term $u^{I,2}$ has been intentionally omitted in the expression of $R_1^\varepsilon(x,t)$ since we find it is unnecessary for our purpose. Indeed if we include the term $u^{I,2}$ in $R_1^\varepsilon(x,t)$, then a higher regularity $L^2(0,T;H^4)$ will be required on $u^{I,2}$ in the proof of Lemma 4.1 when estimating f^ε. This demands a higher regularity on initial data (u_0, v_0) so that $(u_0, v_0) \in H^5 \times H^5$. Therefore, to reduce the regularity of (u_0, v_0), we deliberately omit $u^{I,2}$ in $R_1^\varepsilon(x,t)$, which is a trick we employed.

For simplicity of presentation, with z and ξ given in (2.2) we define new functions

$$\begin{aligned}
\tilde{U}^\varepsilon(x,t) &:= u^{I,0}(x,t) + \varepsilon^{1/2}[u^{I,1}(x,t) + u^{B,1}(z,t) + u^{b,1}(\xi,t)] \\
&\quad + \varepsilon[u^{B,2}(z,t) + u^{b,2}(\xi,t)] + b_1^\varepsilon(x,t), \\
\tilde{V}^\varepsilon(x,t) &:= v^{I,0}(x,t) + v^{B,0}(z,t) + v^{b,0}(\xi,t) \\
&\quad + \varepsilon^{1/2}[v^{I,1}(x,t) + v^{B,1}(z,t) + v^{b,1}(\xi,t)] + b_2^\varepsilon(x,t),
\end{aligned}$$

and then the perturbation functions $(U^\varepsilon, V^\varepsilon)(x,t)$ can be written as

(4.2) $$U^\varepsilon = \varepsilon^{-1/2}(u^\varepsilon - \tilde{U}^\varepsilon), \qquad V^\varepsilon = \varepsilon^{-1/2}(v^\varepsilon - \tilde{V}^\varepsilon).$$

Substituting (4.2) into (1.5)–(1.6) and using the initial-boundary conditions in (2.3)–(2.11), one finds that $(U^\varepsilon, V^\varepsilon)$ satisfies

(4.3) $$\begin{cases}
U_t^\varepsilon = \varepsilon^{1/2}(U^\varepsilon V^\varepsilon)_x + (U^\varepsilon \tilde{V}^\varepsilon)_x + (V^\varepsilon \tilde{U}^\varepsilon)_x + U_{xx}^\varepsilon + \varepsilon^{-1/2}f^\varepsilon, \\
V_t^\varepsilon = -2\varepsilon^{3/2}V^\varepsilon V_x^\varepsilon - 2\varepsilon(V^\varepsilon \tilde{V}^\varepsilon)_x + U_x^\varepsilon + \varepsilon V_{xx}^\varepsilon + \varepsilon^{-1/2}g^\varepsilon, \\
(U^\varepsilon, V^\varepsilon)(x,0) = (0,0), \\
(U^\varepsilon, V^\varepsilon)(0,t) = (U^\varepsilon, V^\varepsilon)(1,t) = (0,0),
\end{cases}$$

with

(4.4) $\qquad f^\varepsilon = \tilde{U}^\varepsilon_{xx} + (\tilde{U}^\varepsilon \tilde{V}^\varepsilon)_x - \tilde{U}^\varepsilon_t, \quad g^\varepsilon = \varepsilon \tilde{V}^\varepsilon_{xx} + \tilde{U}^\varepsilon_x - \tilde{V}^\varepsilon_t - 2\varepsilon \tilde{V}^\varepsilon \tilde{V}^\varepsilon_x.$

Now the key is to give the L^∞ estimates for the solution $(U^\varepsilon, V^\varepsilon)$ of (4.3)–(4.4), which will be gradually achieved in what follows by the method of energy estimates.

4.2. Energy estimates. We shall develop various delicate energy estimates in this subsection to attain the L^∞ estimates of $(U^\varepsilon, V^\varepsilon)$ to (4.3)–(4.4). Before proceeding, we introduce some basic facts for later use. First for any $G_1(z,t) \in H_z^m$ and $G_2(\xi,t) \in H_\xi^m$ with $m \in \mathbb{N}$, we have from the change of variables in (2.2) that

(4.5) $\qquad \left\| \partial_x^m G_1\left(\dfrac{x}{\sqrt{\varepsilon}}, t\right) \right\|_{L^2} = \varepsilon^{\frac{1}{4} - \frac{m}{2}} \|\partial_z^m G_1(z,t)\|_{L_z^2}$

and

(4.6) $\qquad \left\| \partial_x^m G_2\left(\dfrac{x-1}{\sqrt{\varepsilon}}, t\right) \right\|_{L^2} = \varepsilon^{\frac{1}{4} - \frac{m}{2}} \|\partial_\xi^m G_2(\xi,t)\|_{L_\xi^2}.$

For $h(\cdot, t) \in H^1$ with $h|_{x=0,1} = 0$, we have

$$h^2(x,t) = 2 \int_0^x h h_y \, dy \le 2\|h(\cdot,t)\|_{L^2} \|h_x(\cdot,t)\|_{L^2}.$$

Thus
(4.7)
$$\|h(\cdot,t)\|_{L^\infty} \le \sqrt{2} \|h(\cdot,t)\|_{L^2}^{1/2} \|h_x(\cdot,t)\|_{L^2}^{1/2} \quad \text{and} \quad \|h(\cdot,t)\|_{L^\infty} \le C_0 \|h_x(\cdot,t)\|_{L^2},$$

thanks to the Poincaré inequality $\|h(\cdot,t)\|_{L^2} \le C_0 \|h_x(\cdot,t)\|_{L^2}$.

We start with estimating f^ε and g^ε.

LEMMA 4.1. *Let $0 < T < \infty$, $0 < \varepsilon < 1$, and f^ε be as defined in (4.4). Then there is a constant C independent of ε such that*

(4.8) $\qquad \|f^\varepsilon\|_{L^2(0,T;L^2)} \le C\varepsilon^{3/4}.$

Proof. First applying the definitions of $\tilde{U}^\varepsilon$ and $\tilde{V}^\varepsilon$ into the expression of f^ε in (4.4) and using the first equations in (2.3) and in (2.8), we end up with

$$\begin{aligned}
f^\varepsilon =\ & \varepsilon^{1/2} u_{xx}^{B,1} + \varepsilon^{1/2} u_{xx}^{b,1} + \varepsilon u_{xx}^{B,2} + \varepsilon u_{xx}^{b,2} + \varepsilon (u^{I,1} v^{I,1})_x \\
& + \left[(u^{I,0} + \varepsilon^{1/2} u^{I,1})(v^{B,0} + v^{b,0} + \varepsilon^{1/2} v^{B,1} + \varepsilon^{1/2} v^{b,1}) \right]_x \\
(4.9) \qquad & + [(\varepsilon^{1/2} u^{B,1} + \varepsilon^{1/2} u^{b,1} + \varepsilon u^{B,2} + \varepsilon u^{b,2}) \\
& \quad \times (v^{I,0} + v^{B,0} + v^{b,0} + \varepsilon^{1/2} v^{I,1} + \varepsilon^{1/2} v^{B,1} + \varepsilon^{1/2} v^{b,1})]_x \\
& - \varepsilon^{1/2} u_t^{B,1} - \varepsilon^{1/2} u_t^{b,1} - \varepsilon u_t^{B,2} - \varepsilon u_t^{b,2} + F^\varepsilon,
\end{aligned}$$

where

(4.10)
$$\begin{aligned}
F^\varepsilon :=\ & [b_1^\varepsilon (v^{I,0} + v^{B,0} + v^{b,0} + \varepsilon^{1/2} v^{I,1} + \varepsilon^{1/2} v^{B,1} + \varepsilon^{1/2} v^{b,1})]_x \\
& + [b_2^\varepsilon (u^{I,0} + \varepsilon^{1/2} u^{I,1} + \varepsilon^{1/2} u^{B,1} + \varepsilon^{1/2} u^{b,1} + \varepsilon u^{B,2} + \varepsilon u^{b,2})]_x + (b_1^\varepsilon b_2^\varepsilon)_x - b_{1t}^\varepsilon.
\end{aligned}$$

By the transformation (2.2), one gets from (A.8), (A.9), (A.12), and (A.13) (see the appendix) that

$$\varepsilon^{1/2} u_{xx}^{B,1} = - u^{I,0}(0,t) v_x^{B,0},$$
$$\varepsilon u_{xx}^{B,2} = - x u_x^{I,0}(0,t) v_x^{B,0} - \varepsilon^{1/2} u^{I,1}(0,t) v_x^{B,0} - u_x^{I,0}(0,t) v^{B,0}$$
$$- \varepsilon^{1/2} u^{I,0}(0,t) v_x^{B,1} - \varepsilon^{1/2} u_x^{B,1} v^{I,0}(0,t) - \varepsilon^{1/2} (u^{B,1} v^{B,0})_x,$$

and

$$\varepsilon^{1/2} u_{xx}^{b,1} = - u^{I,0}(1,t) v_x^{b,0},$$
$$\varepsilon u_{xx}^{b,2} = - (x-1) u_x^{I,0}(1,t) v_x^{b,0} - \varepsilon^{1/2} u^{I,1}(1,t) v_x^{b,0} - u_x^{I,0}(1,t) v^{b,0}$$
$$- \varepsilon^{1/2} u^{I,0}(1,t) v_x^{b,1} - \varepsilon^{1/2} u_x^{b,1} v^{I,0}(1,t) - \varepsilon^{1/2} (u^{b,1} v^{b,0})_x.$$

Then feeding (4.9) on the above four expressions and rearranging the results, we have

$$
\begin{aligned}
f^\varepsilon =& [(u^{I,0}(x,t) - u^{I,0}(0,t) - x u_x^{I,0}(0,t)) v_x^{B,0}] \\
&+ [(u^{I,0}(x,t) - u^{I,0}(1,t) - (x-1) u_x^{I,0}(1,t)) v_x^{b,0}] \\
&+ [(u_x^{I,0}(x,t) - u_x^{I,0}(0,t)) v^{B,0} + (u_x^{I,0}(x,t) - u_x^{I,0}(1,t)) v^{b,0}] \\
&+ \varepsilon^{1/2} [(u^{I,0}(x,t) - u^{I,0}(0,t)) v_x^{B,1} + (u^{I,1}(x,t) - u^{I,1}(0,t)) v_x^{B,0} \\
&\qquad + u_x^{B,1}(v^{I,0}(x,t) - v^{I,0}(0,t))] \\
&+ \varepsilon^{1/2} [(u^{I,0}(x,t) - u^{I,0}(1,t)) v_x^{b,1} + (u^{I,1}(x,t) - u^{I,1}(1,t)) v_x^{b,0} \\
&\qquad + u_x^{b,1}(v^{I,0}(x,t) - v^{I,0}(1,t))] \\
&+ \varepsilon^{1/2} [u_x^{I,0}(v^{B,1} + v^{b,1}) + u_x^{I,1}(v^{B,0} + v^{b,0}) + (u^{B,1} + u^{b,1}) v_x^{I,0}] \\
&+ \varepsilon^{1/2} [u_x^{B,1} v^{b,0} + u^{B,1} v_x^{b,0} + u_x^{b,1} v^{B,0} + u^{b,1} v_x^{B,0}] \\
&+ \varepsilon[(u^{I,1} + u^{B,1} + u^{b,1})(v^{I,1} + v^{B,1} + v^{b,1})]_x \\
&+ \varepsilon[(u^{B,2} + u^{b,2})(v^{I,0} + v^{B,0} + v^{b,0} + \varepsilon^{1/2} v^{I,1} + \varepsilon^{1/2} v^{B,1} + \varepsilon^{1/2} v^{b,1})]_x \\
&- [\varepsilon^{1/2} u_t^{B,1} + \varepsilon^{1/2} u_t^{b,1} + \varepsilon u_t^{B,2} + \varepsilon u_t^{b,2}] + F^\varepsilon \\
:=& \sum_{i=1}^{10} K_i + F^\varepsilon.
\end{aligned}
$$

(4.11)

We proceed to estimate K_i $(1 \leq i \leq 10)$. Recalling that $x = \varepsilon^{1/2} z$, then by Taylor's formula, (4.5), and Lemmas 3.1–3.2, K_1 is estimated as follows:

$$
\begin{aligned}
\|K_1\|_{L^2(0,T;L^2)} &= \varepsilon \left\| \frac{u^{I,0}(x,t) - u^{I,0}(0,t) - x u_x^{I,0}(0,t)}{x^2} \cdot z^2 v_x^{B,0} \right\|_{L^2(0,T;L^2)} \\
&\leq \varepsilon \|u_{xx}^{I,0}\|_{L^2(0,T;L^\infty)} \|z^2 v_x^{B,0}\|_{L^\infty(0,T;L^2)} \\
&\leq C_0 \varepsilon^{3/4} \|u^{I,0}\|_{L^2(0,T;H^3)} \|z^2 v_z^{B,0}\|_{L^\infty(0,T;L_z^2)} \\
&\leq C \varepsilon^{3/4}.
\end{aligned}
$$

Similarly, by using (4.6) we have

$$\|K_2\|_{L^2(0,T;L^2)} \leq \varepsilon \left\| \frac{u^{I,0}(x,t) - u^{I,0}(1,t) - (x-1) u_x^{I,0}(1,t)}{(x-1)^2} \cdot \xi^2 v_x^{b,0} \right\|_{L^2(0,T;L^2)} \leq C \varepsilon^{3/4}.$$

Similar arguments further give

$$\|K_i\|_{L^2(0,T;L^2)} \leq C\varepsilon^{3/4}, \quad i = 3,4,5.$$

By the Sobolev embedding inequality, (4.5)–(4.6), and Lemmas 3.1–3.6 we obtain

$$
\begin{aligned}
\|K_6\|_{L^2(0,T;L^2)} &\leq C_0\varepsilon^{3/4}\|u^{I,0}\|_{L^\infty(0,T;H^2)}(\|v^{B,1}\|_{L^2(0,T;L^2_z)} + \|v^{b,1}\|_{L^2(0,T;L^2_\xi)}) \\
&\quad + C_0\varepsilon^{3/4}\|u^{I,1}\|_{L^\infty(0,T;H^2)}(\|v^{B,0}\|_{L^2(0,T;L^2_z)} + \|v^{b,0}\|_{L^2(0,T;L^2_\xi)}) \\
&\quad + C_0\varepsilon^{3/4}\|v^{I,0}\|_{L^\infty(0,T;H^2)}(\|u;^{B,1}\|_{L^2(0,T;L^2_z)} + \|u^{b,1}\|_{L^2(0,T;L^2_\xi)}) \\
&\leq C\varepsilon^{3/4}.
\end{aligned}
$$

Then using a similar argument as estimating K_6 and recalling $0 < \varepsilon < 1$, one infers that

$$\|K_i\|_{L^2(0,T;L^2)} \leq C\varepsilon^{3/4}, \quad 8 \leq i \leq 10.$$

We proceed to bound each term in K_7. Indeed for $0 < x < 1/2$, it follows that $-\infty < \xi = \frac{x-1}{\sqrt{\varepsilon}} < -\frac{1}{2\sqrt{\varepsilon}}$. Thus, by transformation (2.2) and the Sobolev embedding inequality, one deduces for fixed $t \in [0,T]$ and $m \in \mathbb{N}_+$ that

$$
\begin{aligned}
&\int_0^{\frac{1}{2}} (u_x^{B,1} v^{b,0})^2 \, dx \\
&= \int_0^{\frac{1}{2}} \left[u_x^{B,1}\left(\frac{x}{\sqrt{\varepsilon}},t\right) v^{b,0}\left(\frac{x-1}{\sqrt{\varepsilon}},t\right) \right]^2 dx \\
&\leq \int_0^{\frac{1}{2}} \left[u_x^{B,1}\left(\frac{x}{\sqrt{\varepsilon}},t\right) \right]^2 dx \cdot (2\sqrt{\varepsilon})^{2m} \left\|\left(-\frac{1}{2\sqrt{\varepsilon}}\right)^m v^{b,0}(\xi,t)\right\|^2_{L^\infty_\xi(-\infty,-\frac{1}{2\sqrt{\varepsilon}})} \\
&\leq \varepsilon^{-1/2} \int_0^{\frac{1}{2\sqrt{\varepsilon}}} [u_z^{B,1}(z,t)]^2 \, dz \cdot (2\sqrt{\varepsilon})^{2m} \|\langle\xi\rangle^m v^{b,0}(\xi,t)\|^2_{L^\infty_\xi(-\infty,0)} \\
&\leq C_0\varepsilon^{(2m-1)/2}\|u_z^{B,1}(z,t)\|^2_{L^2_z}\|\langle\xi\rangle^m v^{b,0}(\xi,t)\|^2_{H^1_\xi} \\
&\leq C\varepsilon^{1/2},
\end{aligned}
$$

where Lemmas 3.2 and 3.3 have been used. Similarly, for $\frac{1}{2} < x < 1$ one has that $\frac{1}{2\sqrt{\varepsilon}} < z = \frac{x}{\sqrt{\varepsilon}} < \infty$ and for $m \in \mathbb{N}_+$ that

$$
\begin{aligned}
&\int_{\frac{1}{2}}^1 (u_x^{B,1} v^{b,0})^2 \, dx \\
&\leq \varepsilon^{1/2} \int_{-\infty}^0 [v^{b,0}(\xi,t)]^2 \, d\xi \cdot \varepsilon^{-1} \left\|u_z^{B,1}(z,t)\right\|^2_{L^\infty_z(\frac{1}{2\sqrt{\varepsilon}},\infty)} \\
&\leq \varepsilon^{1/2}\|v^{b,0}(\xi,t)\|^2_{L^2_\xi} \cdot \varepsilon^{-1}(2\sqrt{\varepsilon})^{2m}\left\|z^m u_z^{B,1}(z,t)\right\|^2_{L^\infty_z(0,\infty)} \\
&\leq C\varepsilon^{1/2}.
\end{aligned}
$$

Combining the above two estimates, we end up with $\|u_x^{B,1} v^{b,0}\|_{L^2(0,T;L^2)} \leq C\varepsilon^{1/4}$. By similar arguments, one derives that

$$\|u^{B,1} v_x^{b,0}\|_{L^2(0,T;L^2)} + \|u_x^{b,1} v^{B,0}\|_{L^2(0,T;L^2)} + \|u^{b,1} v_x^{B,0}\|_{L^2(0,T;L^2)} \leq C\varepsilon^{1/4}.$$

Thus $\|K_7\|_{L^2(0,T;L^2)} \leq C\varepsilon^{3/4}$.

For the last term F^ε, we first note for any integer $m \geq 2$ that

$$
\begin{aligned}
\|u^{b,1}(-\varepsilon^{-1/2}, t)\|_{L^\infty(0,T)} &= \varepsilon^{m/2} \|(-\varepsilon^{-1/2})^m u^{b,1}(-\varepsilon^{-1/2}, t)\|_{L^\infty(0,T)} \\
&\leq \varepsilon^{m/2} \|\langle \xi \rangle^m u^{b,1}(\xi, t)\|_{L^\infty(0,T;L^\infty_\xi)} \\
&\leq C_0 \varepsilon^{m/2} \|\langle \xi \rangle^m u^{b,1}(\xi, t)\|_{L^\infty(0,T;H^1_\xi)} \\
&\leq C \varepsilon^{m/2}
\end{aligned}
$$

and that $\|u^{B,2}(0, t)\|_{L^\infty(0,T)} \leq C_0 \|u^{B,2}\|_{L^\infty(0,T;H^1_z)} \leq C$. By similar arguments, we can estimate other terms in b^ε_1, b^ε_2 and conclude that

$$
(4.12) \qquad \|b^\varepsilon_1\|_{L^\infty(0,T;H^1)} + \|b^\varepsilon_2\|_{L^\infty(0,T;H^1)} \leq C(\varepsilon^{m/2} + \varepsilon) \leq C\varepsilon.
$$

Similar arguments further entail that

$$
(4.13) \qquad \|b^\varepsilon_{1t}\|_{L^2(0,T;H^1)} + \|b^\varepsilon_{2t}\|_{L^2(0,T;H^1)} \leq C(\varepsilon^{m/2} + \varepsilon) \leq C\varepsilon.
$$

Then substituting (4.12)–(4.13) into the definition of F^ε in (4.10) and using $0 < \varepsilon < 1$ and (4.5)–(4.6), one has

$$
\begin{aligned}
&\|F^\varepsilon\|_{L^2(0,T;L^2)} \\
&\leq C_0 \|b^\varepsilon_1\|_{L^\infty(0,T;H^1)} \Big\{ \|v^{I,0}\|_{L^2(0,T;H^1)} + \varepsilon^{-1/4} \|v^{B,0}\|_{L^2(0,T;H^1_z)} + \varepsilon^{-1/4} \|v^{b,0}\|_{L^2(0,T;H^1_\xi)} \\
&\qquad\qquad + \varepsilon^{1/2} \|v^{I,1}\|_{L^2(0,T;H^1)} + \varepsilon^{1/4} \|v^{B,1}\|_{L^2(0,T;H^1_z)} \\
&\qquad\qquad + \varepsilon^{1/4} \|v^{b,1}\|_{L^2(0,T;H^1_\xi)} \Big\} \\
&\quad + C_0 \|b^\varepsilon_2\|_{L^\infty(0,T;H^1)} \Big\{ \|u^{I,0}\|_{L^2(0,T;H^1)} + \varepsilon^{1/2} \|u^{I,1}\|_{L^2(0,T;H^1)} + \varepsilon^{1/4} \|u^{B,1}\|_{L^2(0,T;H^1_z)} \\
&\qquad\qquad + \varepsilon^{1/4} \|u^{b,1}\|_{L^2(0,T;H^1_\xi)} + \varepsilon^{3/4} \|u^{B,2}\|_{L^2(0,T;H^1_z)} \\
&\qquad\qquad + \varepsilon^{3/4} \|u^{b,2}\|_{L^2(0,T;H^1_\xi)} \Big\} \\
&\quad + T^{1/2} \|b^\varepsilon_1\|_{L^\infty(0,T;H^1)} \|b^\varepsilon_2\|_{L^\infty(0,T;H^1)} + \|b^\varepsilon_{1t}\|_{L^2(0,T;L^2)} \leq C\varepsilon^{3/4}.
\end{aligned}
$$

Collecting the above estimates for K_i $(1 \leq i \leq 10)$ and F^ε, from (4.11) one derives (4.8) and finishes the proof. $\qquad\square$

LEMMA 4.2. *Let $0 < T < \infty$, $0 < \varepsilon < 1$, and g^ε be as defined in* (4.4). *Then*

$$
\|g^\varepsilon\|_{L^2(0,T;L^2)} \leq C\varepsilon^{3/4}.
$$

Proof. Substituting the definition for $\tilde{U}^\varepsilon$ and $\tilde{V}^\varepsilon$ into g^ε in (4.4), then using the second equations in (2.3), (2.8), (A.16), and (A.17) (see the appendix), we have

$$
\begin{aligned}
(4.14) \\
g^\varepsilon &= [\varepsilon v^{I,0}_{xx} + \varepsilon^{3/2} v^{I,1}_{xx}] + [\varepsilon^{3/2}(v^{B,1}_{xx} + v^{b,1}_{xx}) + \varepsilon(u^{B,2}_x + u^{b,2}_x) - \varepsilon^{1/2}(v^{B,1}_t + v^{b,1}_t)] \\
&\quad - [2\varepsilon \tilde{V}^\varepsilon \tilde{V}^\varepsilon_x] + [b^\varepsilon_{1x} - b^\varepsilon_{2t}] \\
&:= \sum_{i=11}^{14} K_i.
\end{aligned}
$$

We next estimate $K_i(11 \leq i \leq 14)$. Lemmas 3.1 and 3.4 imply that $\|K_{11}\|_{L^2(0,T;L^2)} \leq C\varepsilon$. Using (2.2), (4.5)–(4.6), and Lemmas 3.5–3.6, we estimate K_{12} as follows:

$$
\begin{aligned}
\|K_{12}\|_{L^2(0,T;L^2)} &\leq \varepsilon^{3/4}(\|v_{zz}^{B,1}\|_{L^2(0,T;L_z^2)} + \|v_{\xi\xi}^{b,1}\|_{L^2(0,T;L_\xi^2)} + \|u_z^{B,2}\|_{L^2(0,T;L_z^2)} \\
&\quad + \|u_\xi^{b,2}\|_{L^2(0,T;L_\xi^2)} + \|v_t^{B,1}\|_{L^2(0,T;L_z^2)} + \|v_t^{b,1}\|_{L^2(0,T;L_\xi^2)}) \\
&\leq C\varepsilon^{3/4}.
\end{aligned}
$$

To bound K_{13}, we first estimate $\|\tilde{V}^\varepsilon\|_{L^\infty([0,1]\times[0,T])}$ and $\|\tilde{V}_x^\varepsilon\|_{L^\infty(0,T;L^2)}$. For any $G_1(z,t) \in L^p(0,T;H_z^1)$, $G_2(\xi,t) \in L^p(0,T;H_\xi^1)$ with $1 \leq p \leq \infty$, it follows from the Sobolev embedding inequality that

$$
(4.15)
\begin{aligned}
\left\|G_1\Big(\frac{x}{\sqrt{\varepsilon}},t\Big)\right\|_{L^p(0,T;L^\infty)} &\leq \|G_1(z,t)\|_{L^p(0,T;L_z^\infty)} \leq C_0\|G_1\|_{L^p(0,T;H_z^1)} \leq C, \\
\left\|G_2\Big(\frac{x-1}{\sqrt{\varepsilon}},t\Big)\right\|_{L^p(0,T;L^\infty)} &\leq \|G_2(\xi,t)\|_{L^p(0,T;L_\xi^\infty)} \leq C_0\|G_2\|_{L^p(0,T;H_\xi^1)} \leq C.
\end{aligned}
$$

Then by the definition of $\tilde{V}^\varepsilon$, (4.15), Lemmas 3.1–3.6, and (4.13), we deduce that
(4.16)
$$
\begin{aligned}
&\|\tilde{V}^\varepsilon\|_{L^\infty([0,1]\times[0,T])} \\
&\leq \|v^{I,0}\|_{L^\infty([0,1]\times[0,T])} + \|v^{B,0}\|_{L^\infty(0,T;L_z^\infty)} + \|v^{b,0}\|_{L^\infty(0,T;L_\xi^\infty)} + C_0\|b_2^\varepsilon\|_{L^\infty(0,T;H^1)} \\
&\quad + \varepsilon^{1/2}(\|v^{I,1}\|_{L^\infty([0,1]\times[0,T])} + \|v^{B,1}\|_{L^\infty(0,T;L_z^\infty)} + \|v^{b,1}\|_{L^\infty(0,T;L_\xi^\infty)}) \\
&\leq C(1 + \varepsilon^{1/2} + \varepsilon^{m/2}) \leq C,
\end{aligned}
$$

where the assumption $0 < \varepsilon < 1$ has been used. Moreover (4.5), (4.6), and (4.13) lead to

$$
(4.17)
\begin{aligned}
&\|\tilde{V}_x^\varepsilon\|_{L^\infty(0,T;L^2)} \\
&\leq \|v_x^{I,0}\|_{L^\infty(0,T;L^2)} + \varepsilon^{-1/4}(\|v_z^{B,0}\|_{L^\infty(0,T;L_z^2)} + \|v_\xi^{b,0}\|_{L^\infty(0,T;L_\xi^2)}) + C\varepsilon^{m/2} \\
&\quad + \varepsilon^{1/2}\|v_x^{I,1}\|_{L^\infty(0,T;L^2)} + \varepsilon^{1/4}(\|v_z^{B,1}\|_{L^\infty(0,T;L_z^2)} + \|v_\xi^{b,1}\|_{L^\infty(0,T;L_\xi^2)}) \\
&\leq C\varepsilon^{-1/4}.
\end{aligned}
$$

Thus the above two estimates indicate that

$$
\|K_{13}\|_{L^2(0,T;L^2)} \leq C\varepsilon\|\tilde{V}^\varepsilon\|_{L^\infty([0,1]\times[0,T])}\|\tilde{V}_x^\varepsilon\|_{L^\infty(0,T;L^2)} \leq C\varepsilon^{3/4}.
$$

Finally, the estimate for K_{14} follows from (4.12), (4.13), and the assumption $0 < \varepsilon < 1$ that

$$
\|K_{14}\|_{L^2(0,T;L^2)} \leq \|b_1^\varepsilon\|_{L^2(0,T;H^1)} + \|b_{2t}^\varepsilon\|_{L^2(0,T;L^2)} \leq C\varepsilon \leq C\varepsilon^{3/4}.
$$

Then inserting the above estimates for K_i ($11 \leq i \leq 14$) into (4.14) yields the desired estimate for g^ϵ. □

The next lemma gives the estimate for U^ε, V^ε in $L^\infty(0,T;L^2)$.

LEMMA 4.3. *Let $0 < T < \infty$ and $0 < \varepsilon < 1$. Then there exists a constant C, independent of ε, such that*

$$\|U^\varepsilon\|^2_{L^\infty(0,T;L^2)} + \|V^\varepsilon\|^2_{L^\infty(0,T;L^2)} + \|U^\varepsilon_x\|^2_{L^2(0,T;L^2)} + \varepsilon\|V^\varepsilon_x\|^2_{L^2(0,T;L^2)} \le C\varepsilon^{1/2}.$$

Proof. Taking the L^2 inner product of the first equation of (4.3) with $2U^\varepsilon$, then using integration by parts we have

(4.18)
$$\frac{d}{dt}\|U^\varepsilon(t)\|^2_{L^2} + 2\|U^\varepsilon_x(t)\|^2_{L^2} = -2\varepsilon^{1/2}\int_0^1 U^\varepsilon V^\varepsilon U^\varepsilon_x\,dx - 2\int_0^1 (U^\varepsilon\tilde{V}^\varepsilon + V^\varepsilon\tilde{U}^\varepsilon)U^\varepsilon_x\,dx$$
$$+ 2\varepsilon^{-1/2}\int_0^1 f^\varepsilon\,U^\varepsilon\,dx$$
$$:= M_1(t) + M_2(t) + M_3(t).$$

We next estimate $M_i(t)$ $(i = 1, 2, 3)$. First, (4.7) gives

$$M_1(t) \le C_0\varepsilon^{1/2}\|U^\varepsilon(t)\|^{1/2}_{L^2}\|U^\varepsilon_x(t)\|^{3/2}_{L^2}\|V^\varepsilon(t)\|_{L^2}$$
$$\le \frac{1}{4}\|U^\varepsilon_x(t)\|^2_{L^2} + C_0\varepsilon^2\|V^\varepsilon(t)\|^4_{L^2}\|U^\varepsilon(t)\|^2_{L^2}.$$

For the term $\|V^\varepsilon(t)\|^4_{L^2}$, we use the definition of V^ε, Lemma 2.1, and (4.16) to get

(4.19) $$\|V^\varepsilon\|_{L^\infty(0,T;L^2)} \le \varepsilon^{-1/2}(\|v^\varepsilon\|_{L^\infty(0,T;L^2)} + \|\tilde{V}^\varepsilon\|_{L^\infty(0,T;L^2)}) \le C\varepsilon^{-1/2},$$

which, substituted into the above estimate for $M_1(t)$, gives rise to $M_1(t) \le \frac{1}{4}\|U^\varepsilon_x(t)\|^2_{L^2} + C\|U^\varepsilon(t)\|^2_{L^2}$. By a similar argument as deriving (4.16), one infers that

(4.20) $$\|\tilde{U}^\varepsilon\|_{L^\infty([0,1]\times[0,T])} \le C,$$

which along with (4.16) leads to

$$M_2(t) \le \frac{1}{4}\|U^\varepsilon_x(t)\|^2_{L^2} + 8\|U^\varepsilon(t)\|^2_{L^2}\|\tilde{V}^\varepsilon(t)\|^2_{L^\infty} + 8\|V^\varepsilon(t)\|^2_{L^2}\|\tilde{U}^\varepsilon(t)\|^2_{L^\infty}$$
$$\le \frac{1}{4}\|U^\varepsilon_x(t)\|^2_{L^2} + C(\|U^\varepsilon(t)\|^2_{L^2} + \|V^\varepsilon(t)\|^2_{L^2}).$$

For the last term $M_3(t)$, we have by the Cauchy–Schwarz inequality that $M_3(t) \le \|U^\varepsilon(t)\|^2_{L^2} + \varepsilon^{-1}\|f^\varepsilon(t)\|^2_{L^2}$. Substituting the above estimates of $M_i(t)$ $(1 \le i \le 3)$ into (4.18), we arrive at

(4.21) $$\frac{d}{dt}\|U^\varepsilon(t)\|^2_{L^2} + \frac{3}{2}\|U^\varepsilon_x(t)\|^2_{L^2} \le C(\|U^\varepsilon(t)\|^2_{L^2} + \|V^\varepsilon(t)\|^2_{L^2}) + \varepsilon^{-1}\|f^\varepsilon(t)\|^2_{L^2}.$$

We turn to estimate V^ε. Multiplying the second equation of (4.3) by $2V^\varepsilon$ in L^2 and using the integration by parts we derive

$$\frac{d}{dt}\|V^\varepsilon(t)\|^2_{L^2} + 2\varepsilon\|V^\varepsilon_x(t)\|^2_{L^2} = -4\varepsilon^{3/2}\int_0^1 V^\varepsilon V^\varepsilon V^\varepsilon_x\,dx + 4\varepsilon\int_0^1 V^\varepsilon\tilde{V}^\varepsilon V^\varepsilon_x\,dx$$
(4.22)
$$+ 2\int_0^1 U^\varepsilon_x V^\varepsilon\,dx + 2\varepsilon^{-1/2}\int_0^1 g^\varepsilon V^\varepsilon\,dx$$
$$:= M_4(t) + M_5(t) + M_6(t) + M_7(t).$$

We proceed to bound $M_i(t)$ ($4 \leq i \leq 7$). Applying (4.7) to V^ε together with (4.19) leads to

$$M_4(t) \leq C_0 \varepsilon^{3/2} \|V_x^\varepsilon(t)\|_{L^2}^{3/2} \|V^\varepsilon(t)\|_{L^2}^{3/2}$$

$$\leq \frac{1}{4}\varepsilon \|V_x^\varepsilon(t)\|_{L^2}^2 + C_0 \varepsilon^3 \|V^\varepsilon(t)\|_{L^2}^6$$

$$\leq \frac{1}{4}\varepsilon \|V_x^\varepsilon(t)\|_{L^2}^2 + C\varepsilon \|V^\varepsilon(t)\|_{L^2}^2.$$

We employ the Cauchy–Schwarz inequality and (4.16) to deduce that

$$M_5(t) \leq \frac{1}{4}\varepsilon \|V_x^\varepsilon(t)\|_{L^2}^2 + 16\varepsilon \|\tilde{V}^\varepsilon(t)\|_{L^\infty}^2 \|V^\varepsilon(t)\|_{L^2}^2 \leq \frac{1}{4}\varepsilon \|V_x^\varepsilon(t)\|_{L^2}^2 + C\varepsilon \|V^\varepsilon(t)\|_{L^2}^2.$$

Finally, the estimates for $M_6(t)$ and $M_7(t)$ follow from the Cauchy–Schwarz inequality that

$$M_6(t) \leq \frac{1}{4}\|U_x^\varepsilon(t)\|_{L^2}^2 + 4\|V^\varepsilon(t)\|_{L^2}^2, \qquad M_7(t) \leq \|V^\varepsilon(t)\|_{L^2}^2 + \varepsilon^{-1}\|g^\varepsilon(t)\|_{L^2}^2.$$

Plugging the above estimates for $M_i(t)$ ($4 \leq i \leq 7$) into (4.22) and using $0 < \varepsilon < 1$ give us

$$\frac{d}{dt}\|V^\varepsilon(t)\|_{L^2}^2 + \varepsilon \|V_x^\varepsilon(t)\|_{L^2}^2 \leq \frac{1}{4}\|U_x^\varepsilon(t)\|_{L^2}^2 + C\|V^\varepsilon(t)\|_{L^2}^2 + \varepsilon^{-1}\|g^\varepsilon(t)\|_{L^2}^2,$$

which added to (4.21) yields

$$\frac{d}{dt}(\|U^\varepsilon(t)\|_{L^2}^2 + \|V^\varepsilon(t)\|_{L^2}^2) + \|U_x^\varepsilon(t)\|_{L^2}^2 + \varepsilon \|V_x^\varepsilon(t)\|_{L^2}^2$$

$$\leq C(\|U^\varepsilon(t)\|_{L^2}^2 + \|V^\varepsilon(t)\|_{L^2}^2) + \varepsilon^{-1}\|f^\varepsilon(t)\|_{L^2}^2 + \varepsilon^{-1}\|g^\varepsilon(t)\|_{L^2}^2.$$

Applying Gronwall's inequality to the above inequality along with Lemmas 4.1 and 4.2, one gets the desired estimates. The proof is completed. $\square$

LEMMA 4.4. *Let* $0 < T < \infty$ *and* $0 < \varepsilon < 1$. *Then there is a constant* C, *independent of* ε, *such that*

$$\|U_x^\varepsilon\|_{L^\infty(0,T;L^2)}^2 + \|V_x^\varepsilon\|_{L^\infty(0,T;L^2)}^2 + \|U_{xx}^\varepsilon\|_{L^2(0,T;L^2)}^2 + \varepsilon \|V_{xx}^\varepsilon\|_{L^2(0,T;L^2)}^2 \leq C\varepsilon^{-1/2}.$$

Proof. Taking the L^2 inner product of the second equation of (4.3) with $-2\varepsilon V_{xx}^\varepsilon$, and using integration by parts, we obtain

(4.23)
$$\frac{d}{dt}(\varepsilon \|V_x^\varepsilon(t)\|_{L^2}^2) + 2\varepsilon^2 \|V_{xx}^\varepsilon(t)\|_{L^2}^2 = 4\varepsilon^{5/2}\int_0^1 V^\varepsilon V_x^\varepsilon V_{xx}^\varepsilon \, dx + 4\varepsilon^2 \int_0^1 (V^\varepsilon \tilde{V}^\varepsilon)_x V_{xx}^\varepsilon \, dx$$

$$- 2\varepsilon \int_0^1 U_x^\varepsilon V_{xx}^\varepsilon \, dx - 2\varepsilon^{1/2}\int_0^1 g^\varepsilon V_{xx}^\varepsilon \, dx$$

$$:= R_1(t) + R_2(t) + R_3(t) + R_4(t).$$

We proceed to estimate $R_i(t)$ for $1 \leq i \leq 4$. By (4.7) we deduce that

$$R_1(t) \leq C_0 \varepsilon^{5/2}\|V_x^\varepsilon(t)\|_{L^2}^2 \|V_{xx}^\varepsilon(t)\|_{L^2} \leq \frac{1}{4}\varepsilon^2 \|V_{xx}^\varepsilon(t)\|_{L^2}^2 + C_0 \varepsilon^3 \|V_x^\varepsilon(t)\|_{L^2}^4.$$

Similarly, it follows from (4.7), (4.16), and (4.17) that

$$R_2(t) \leq C_0\varepsilon^2 \|V_x^\varepsilon(t)\|_{L^2}\|\tilde{V}^\varepsilon(t)\|_{H^1}\|V_{xx}^\varepsilon(t)\|_{L^2}$$
$$\leq \frac{1}{4}\varepsilon^2\|V_{xx}^\varepsilon(t)\|_{L^2}^2 + C_0\varepsilon^2(\|\tilde{V}^\varepsilon(t)\|_{L^2}^2 + \|\tilde{V}_x^\varepsilon(t)\|_{L^2}^2)\|V_x^\varepsilon(t)\|_{L^2}^2$$
$$\leq \frac{1}{4}\varepsilon^2\|V_{xx}^\varepsilon(t)\|_{L^2}^2 + C(\varepsilon^2 + \varepsilon^{3/2})\|V_x^\varepsilon(t)\|_{L^2}^2.$$

For $R_3(t)$ and $R_4(t)$, we employ the Cauchy–Schwarz inequality to have

$$R_3(t) \leq \frac{1}{4}\varepsilon^2\|V_{xx}^\varepsilon(t)\|_{L^2}^2 + 4\|U_x^\varepsilon(t)\|_{L^2}^2 \quad \text{and} \quad R_4(t) \leq \frac{1}{4}\varepsilon^2\|V_{xx}^\varepsilon(t)\|_{L^2}^2 + 4\varepsilon^{-1}\|g^\varepsilon(t)\|_{L^2}^2.$$

Collecting the above estimates of $R_i(t)$ ($1 \leq i \leq 4$) and using (4.23), we end up with

$$\frac{d}{dt}(\varepsilon\|V_x^\varepsilon(t)\|_{L^2}^2) + \varepsilon^2\|V_{xx}^\varepsilon(t)\|_{L^2}^2$$
$$\leq C(\varepsilon^2\|V_x^\varepsilon(t)\|_{L^2}^2 + \varepsilon + \varepsilon^{1/2})(\varepsilon\|V_x^\varepsilon(t)\|_{L^2}^2) + 4(\|U_x^\varepsilon(t)\|_{L^2}^2 + \varepsilon^{-1}\|g^\varepsilon(t)\|_{L^2}^2),$$

which, along with Gronwall's inequality, Lemmas 4.2–4.3, and $0 < \varepsilon < 1$, yields

$$(4.24) \qquad \varepsilon\|V_x^\varepsilon\|_{L^\infty(0,T;L^2)}^2 + \varepsilon^2\|V_{xx}^\varepsilon\|_{L^2(0,T;L^2)}^2 \leq C\varepsilon^{1/2}.$$

We next turn to estimate U_x^ε. Taking the L^2 inner product of the first equation of (4.3) against $-2U_{xx}^\varepsilon$ and using integration by parts we get

$$(4.25)$$
$$\frac{d}{dt}\|U_x^\varepsilon(t)\|_{L^2}^2 + 2\|U_{xx}^\varepsilon(t)\|_{L^2}^2 = -2\varepsilon^{1/2}\int_0^1 (U^\varepsilon V^\varepsilon)_x U_{xx}^\varepsilon \, dx - 2\int_0^1 (U^\varepsilon \tilde{V}^\varepsilon)_x U_{xx}^\varepsilon \, dx$$
$$- 2\int_0^1 (V^\varepsilon \tilde{U}^\varepsilon)_x U_{xx}^\varepsilon \, dx - 2\varepsilon^{-1/2}\int_0^1 f^\varepsilon U_{xx}^\varepsilon \, dx$$
$$:= R_5(t) + R_6(t) + R_7(t) + R_8(t).$$

By (4.7) and (4.24), we estimate $R_5(t)$ as

$$R_5(t) \leq \frac{1}{4}\|U_{xx}^\varepsilon(t)\|_{L^2}^2 + C_0\varepsilon\|V_x^\varepsilon(t)\|_{L^2}^2\|U_x^\varepsilon(t)\|_{L^2}^2 \leq \frac{1}{4}\|U_{xx}^\varepsilon(t)\|_{L^2}^2 + C\varepsilon^{1/2}\|U_x^\varepsilon(t)\|_{L^2}^2.$$

Similarly, we estimate $R_6(t)$ from (4.7), (4.16), and (4.17) as

$$R_6(t) \leq \frac{1}{4}\|U_{xx}^\varepsilon(t)\|_{L^2}^2 + C_0(\|\tilde{V}^\varepsilon(t)\|_{L^2}^2 + \|\tilde{V}_x^\varepsilon(t)\|_{L^2}^2)\|U_x^\varepsilon(t)\|_{L^2}^2$$
$$\leq \frac{1}{4}\|U_{xx}^\varepsilon(t)\|_{L^2}^2 + C(1 + \varepsilon^{-1/2})\|U_x^\varepsilon(t)\|_{L^2}^2.$$

To bound $R_7(t)$, we use the definition of $\tilde{U}^\varepsilon$ and a similar argument as deriving (4.17) to get

$$\|\tilde{U}_x^\varepsilon\|_{L^\infty(0,T;L^2)} \leq C(1 + \varepsilon^{1/2} + \varepsilon^{1/4} + \varepsilon^{3/4} + \varepsilon) \leq C,$$

where $0 < \varepsilon < 1$ has been used. The above estimate in conjunction with (4.20) and (4.24) gives

$$R_7(t) \leq \frac{1}{4}\|U_{xx}^\varepsilon(t)\|_{L^2}^2 + C_0(\|\tilde{U}^\varepsilon(t)\|_{L^2}^2 + \|\tilde{U}_x^\varepsilon(t)\|_{L^2}^2)\|V_x^\varepsilon(t)\|_{L^2}^2$$
$$\leq \frac{1}{4}\|U_{xx}^\varepsilon(t)\|_{L^2}^2 + C\varepsilon^{-1/2}.$$

Last, the Cauchy–Schwarz inequality yields $R_8(t) \le \frac{1}{4}\|U_{xx}^\varepsilon(t)\|_{L^2}^2 + 4\varepsilon^{-1}\|f^\varepsilon(t)\|_{L^2}^2$. Feeding (4.25) on the above estimates of $R_i(t)$ $(5 \le i \le 8)$ leads to

$$\frac{d}{dt}\|U_x^\varepsilon(t)\|_{L^2}^2 + \|U_{xx}^\varepsilon(t)\|_{L^2}^2 \le C(\varepsilon^{-1/2}+1+\varepsilon^{1/2})\|U_x^\varepsilon(t)\|_{L^2}^2 + C\varepsilon^{-1/2} + 4\varepsilon^{-1}\|f^\varepsilon(t)\|_{L^2}^2,$$

which, upon integration over $(0,t)$ with $t \le T$, gives rise to

$$\|U_x^\varepsilon\|_{L^\infty(0,T;L^2)}^2 + \|U_{xx}^\varepsilon\|_{L^2(0,T;L^2)}^2 \le C\varepsilon^{-1/2},$$

where Lemma 4.1, Lemma 4.3, and $0 < \varepsilon < 1$ have been used. The above estimate along with (4.24) completes the proof. □

4.3. Proof of Theorem 2.1. To prove Theorem 2.1, it suffices to estimate $\|R_1^\varepsilon\|_{L^\infty([0,1]\times[0,T])}$ and $\|R_2^\varepsilon\|_{L^\infty([0,1]\times[0,T])}$. For this, we first estimate U^ε and V^ε in $L^\infty([0,1]\times[0,T])$ by (4.7), Lemmas 4.3, and 4.4 and get

$$(4.26) \qquad \begin{aligned} \|U^\varepsilon\|_{L^\infty([0,1]\times[0,T])} &\le C_0\|U^\varepsilon\|_{L^\infty(0,T;L^2)}^{1/2}\|U_x^\varepsilon\|_{L^\infty(0,T;L^2)}^{1/2} \le C, \\ \|V^\varepsilon\|_{L^\infty([0,1]\times[0,T])} &\le C_0\|V^\varepsilon\|_{L^\infty(0,T;L^2)}^{1/2}\|V_x^\varepsilon\|_{L^\infty(0,T;L^2)}^{1/2} \le C. \end{aligned}$$

Then the estimate for R_1^ε follows from (4.15), Lemmas 3.2–3.6, (4.12), and (4.26) that

$$\begin{aligned} &\|R_1^\varepsilon\|_{L^\infty([0,1]\times[0,T])} \\ &\le C_0\varepsilon^{1/2}\big(\|u^{I,1}\|_{L^\infty(0,T;H^1)} + \|u^{B,1}\|_{L^\infty(0,T;H_z^1)} + \|u^{b,1}\|_{L^\infty(0,T;H_\xi^1)}\big) \\ &\quad + C_0\varepsilon\big(\|u^{B,2}\|_{L^\infty(0,T;H_z^1)} + \|u^{b,2}\|_{L^\infty(0,T;H_\xi^1)}\big) + C_0\|b_1^\varepsilon\|_{L^\infty(0,T;H^1)} \\ &\quad + \varepsilon^{1/2}\|U^\varepsilon\|_{L^\infty([0,1]\times[0,T])} \\ &\le C\varepsilon^{1/2}, \end{aligned}$$

where $0 < \varepsilon < 1$ has been used. Similarly, by (4.15), Lemmas 3.4–3.6, (4.13), (4.26), and $0 < \varepsilon < 1$, we have

$$\begin{aligned} \|R_2^\varepsilon\|_{L^\infty([0,1]\times[0,T])} &\le C_0\varepsilon^{1/2}\big(\|v^{I,1}\|_{L^\infty(0,T;H^1)} + \|v^{B,1}\|_{L^\infty(0,T;H_z^1)} + \|v^{b,1}\|_{L^\infty(0,T;H_\xi^1)}\big) \\ &\quad + C_0\|b_2^\varepsilon\|_{L^\infty(0,T;H^1)} + \varepsilon^{1/2}\|V^\varepsilon\|_{L^\infty([0,1]\times[0,T])} \\ &\le C\varepsilon^{1/2}. \end{aligned}$$

The above two estimates along with (4.1) imply (2.13) and complete the proof of Theorem 2.1. □

5. Proof of Theorem 2.2. We are now in a position to prove Theorem 2.2 by converting the result of Theorem 2.1 to the pretransformed chemotaxis model (2.15).

Proof of Theorem 2.2. Let $(u^\varepsilon, c^\varepsilon)$ and (u^0, c^0) be solutions of (2.15) with $\varepsilon > 0$ and $\varepsilon = 0$, respectively. The convergence rate in (2.17) between u^ε and u^0 is a direct consequence of Theorem 2.1. We are left to prove the convergence for c^ε in (2.17) and for c_x^ε in (2.18). Indeed from the second equation of (2.15) one deduces that

$$\begin{cases} (\ln c^\varepsilon)_t = \varepsilon(v^\varepsilon)^2 - \varepsilon v_x^\varepsilon - u^\varepsilon, \\ (\ln c^0)_t = -u^0, \end{cases}$$

$v^\varepsilon = -(\ln c^\varepsilon)_x$. We consider the difference of the two equations:

$$(\ln c^\varepsilon - \ln c^0)_t = \varepsilon(v^\varepsilon)^2 - \varepsilon v_x^\varepsilon - (u^\varepsilon - u^0),$$

which, upon integration with respect to t, gives rise to

$$\frac{c^\varepsilon(x,t)}{c^0(x,t)} = \frac{c^\varepsilon(x,0)}{c^0(x,0)} \exp\left\{ \int_0^t [-(u^\varepsilon - u^0) + \varepsilon(v^\varepsilon)^2 - \varepsilon v_x^\varepsilon]\, d\tau \right\}.$$

It follows from the initial condition $c^\varepsilon(x,0) = c^0(x,0) = c_0(x)$ that

$$|c^\varepsilon(x,t) - c^0(x,t)| = |c^0(x,t)| \cdot \left| \exp\left\{ \int_0^t [-(u^\varepsilon - u^0) + \varepsilon(v^\varepsilon)^2 - \varepsilon v_x^\varepsilon]\, d\tau \right\} - 1 \right|$$

$$(5.1) \qquad\qquad = |c^0(x,t)| \cdot \left| \exp\left\{ G_1^\varepsilon(x,t) + G_2^\varepsilon(x,t) + G_3^\varepsilon(x,t) \right\} - 1 \right|,$$

with $G_1^\varepsilon(x,t) := -\int_0^t (u^\varepsilon - u^0)\, d\tau$, $G_2^\varepsilon(x,t) := \varepsilon \int_0^t (v^\varepsilon)^2\, d\tau$ and $G_3^\varepsilon(x,t) := -\varepsilon \int_0^t v_x^\varepsilon\, d\tau$. We next estimate $G_1^\varepsilon(x,t)$, $G_2^\varepsilon(x,t)$, and $G_3^\varepsilon(x,t)$. First, Theorem 2.1 gives

$$(5.2) \qquad\qquad |G_1^\varepsilon(x,t)| \leq T\|u^\varepsilon - u^0\|_{L^\infty([0,1]\times[0,T])} \leq C\varepsilon^{1/2}.$$

Using Theorem 2.1, (4.15), Lemmas 3.1–3.3, and $0 < \varepsilon < 1$, we estimate $G_2^\varepsilon(x,t)$ as

$$(5.3)$$
$$|G_2^\varepsilon(x,t)| \leq T\varepsilon\left(\|v^{I,0}\|^2_{L^\infty(0,T;H^1)} + \|v^{B,0}\|^2_{L^\infty(0,T;H^1_z)} + \|v^{b,0}\|^2_{L^\infty(0,T;H^1_\xi)} + C\varepsilon^{1/2}\right) \leq C\varepsilon.$$

For any integer $m \geq 2$, similar arguments as deriving (4.13) entail that $\|b_{2x}^\varepsilon\|_{L^2(0,T;L^\infty)} \leq C\varepsilon^{m/2} \leq C\varepsilon$, which along with the definition of V^ε in (4.2), (2.2), the Sobolev embedding inequality, and Lemmas 3.1–3.6 and Lemma 4.4, leads to

$$(5.4)$$
$$|G_3^\varepsilon(x,t)| \leq T^{1/2}\varepsilon(\|v_x^{I,0}\|_{L^2(0,T;L^\infty)} + \varepsilon^{-1/2}\|v_z^{B,0}\|_{L^2(0,T;L^\infty_z)} + \varepsilon^{-1/2}\|v_\xi^{b,0}\|_{L^2(0,T;L^\infty_\xi)})$$
$$\qquad\qquad + T^{1/2}\varepsilon(\varepsilon^{1/2}\|v_x^{I,1}\|_{L^2(0,T;L^\infty)} + \|v_z^{B,1}\|_{L^2(0,T;L^\infty_z)} + \|v_\xi^{b,1}\|_{L^2(0,T;L^\infty_\xi)})$$
$$\qquad\qquad + T^{1/2}\varepsilon(\|b_{2x}^\varepsilon\|_{L^2(0,T;L^\infty)} + \varepsilon^{1/2}\|V_x^\varepsilon\|_{L^2(0,T;H^1)})$$
$$\qquad\quad \leq C\varepsilon^{3/4}.$$

Collecting (5.2)–(5.4) and noticing that $0 < \varepsilon < 1$, we end up with

$$|G_1^\varepsilon(x,t) + G_2^\varepsilon(x,t) + G_3^\varepsilon(x,t)| \leq C\varepsilon^{1/2}$$

for some positive constant C independent of ε (but dependent on T). Thus it follows from the Taylor expansion and $0 < \varepsilon < 1$ that

$$(5.5) \quad |e^{G_1^\varepsilon(x,t)+G_2^\varepsilon(x,t)+G_3^\varepsilon(x,t)} - 1| \leq \sum_{k=1}^\infty \frac{1}{k!}|G_1^\varepsilon(x,t) + G_2^\varepsilon(x,t) + G_3^\varepsilon(x,t)|^k \leq C\varepsilon^{1/2}.$$

We proceed by employing (2.16) and find that

$$(5.6) \qquad\qquad 0 < c^0(x,t) = c_0(x)e^{-\int_0^t u^0(x,\tau)\, d\tau} \leq c_0(x) \leq C_0,$$

subject to the fact $u^0(x,t) \geq 0$ for $(x,t) \in [0,1] \times [0,T]$. The combination of (5.1), (5.5), and (5.6) yields (2.17).

To prove (2.18), we use the transformation $v^\varepsilon = -\frac{c_x^\varepsilon}{c^\varepsilon}$, Theorem 2.1, and (2.17) and get

$$
\begin{aligned}
c_x^\varepsilon - c_x^0 &= - [v^\varepsilon c^\varepsilon - v^0 c^0] \\
&= - [(v^\varepsilon - v^0)c^\varepsilon + v^0(c^\varepsilon - c^0)] \\
&= - [(v^{B,0} + v^{b,0} + \mathcal{O}(\varepsilon^{1/2}))(c^0 + \mathcal{O}(\varepsilon^{1/2})) + v^0\,\mathcal{O}(\varepsilon^{1/2})] \\
&= - c^0(v^{B,0} + v^{b,0}) + \mathcal{O}(\varepsilon^{1/2}),
\end{aligned}
$$

which implies (2.18) and completes the proof of Theorem 2.2. $\qquad\square$

Appendix A. In this section, we shall show the derivation of (2.3)–(2.12) by the method of matched asymptotic expansions. The same approach has been used in the appendix of [26] to determine the thickness of boundary layers, where for the boundary layer profiles, only the equations on the leading-order left boundary layer profile $(v^{B,0}, u^{B,1})$ have been obtained. Here we carry out further procedures to deduce (2.6)–(2.12) for $(v^{b,0}, u^{b,1})$ and the higher-order profiles. For brevity, we shall just outline the procedures that have not been demonstrated in [26].

Step 1. *Initial-boundary conditions.* Upon the substitution of (2.1) into the initial and boundary conditions in (1.5) and following the arguments in [26, Appendix, Step 2], one gets the initial conditions

(A.1)
$$
\begin{aligned}
u^{I,0}(x,0) = u_0(x), \quad u^{B,0}(z,0) = u^{b,0}(\xi,0) = 0, \\
v^{I,0}(x,0) = v_0(x), \quad v^{B,0}(z,0) = v^{b,0}(\xi,0) = 0,
\end{aligned}
$$

and for $j \geq 1$

(A.2)
$$
\begin{aligned}
u^{I,j}(x,0) = u^{B,j}(z,0) = u^{b,j}(\xi,0) = 0, \\
v^{I,j}(x,0) = v^{B,j}(z,0) = v^{b,j}(\xi,0) = 0.
\end{aligned}
$$

The boundary conditions are given by

(A.3)
$$
\begin{aligned}
\bar{u} = u^{I,0}(0,t) + u^{B,0}(0,t), \qquad \bar{u} = u^{I,0}(1,t) + u^{b,0}(0,t), \\
\bar{v} = v^{I,0}(0,t) + v^{B,0}(0,t), \qquad \bar{v} = v^{I,0}(1,t) + v^{b,0}(0,t),
\end{aligned}
$$

and with $j \geq 1$

(A.4)
$$
\begin{aligned}
u^{I,j}(0,t) + u^{B,j}(0,t) = 0, \qquad u^{I,j}(1,t) + u^{b,j}(0,t) = 0, \\
v^{I,j}(0,t) + v^{B,j}(0,t) = 0, \qquad v^{I,j}(1,t) + v^{b,j}(0,t) = 0.
\end{aligned}
$$

Step 2. *Equations for $u^{I,j}$, $u^{B,j}$, and $u^{b,j}$.* For profiles of $u^{I,j}$, $u^{B,j}$ we first employ the argument of [26, Appendix, Step 3] to derive

(A.5)
$$
u_t^{I,j} - \sum_{k=0}^{j}(u^{I,k}v^{I,j-k})_x = u_{xx}^{I,j} \quad \text{for } j \geq 0
$$

and

(A.6)
$$
\sum_{j \geq -2} \varepsilon^{j/2}\tilde{G}_j(z,t) = 0
$$

with

$$
\begin{cases}
\tilde{G}_{-2} = -u_{zz}^{B,0}, \\
\tilde{G}_{-1} = -u^{I,0}(0,t)v_z^{B,0} - v^{I,0}(0,t)u_z^{B,0} - (u^{B,0}v^{B,0})_z - u_{zz}^{B,1}, \\
\tilde{G}_0 = u_t^{B,0} - u^{B,0}v_x^{I,0}(0,t) - (u^{I,0}(0,t) + u^{B,0})v_z^{B,1} - (u^{I,1}(0,t) + u^{B,1})v_z^{B,0} \\
\qquad - u_x^{I,0}(0,t)v^{B,0} \\
\qquad - u_z^{B,0}(v^{I,1}(0,t) + v^{B,1}) - u_z^{B,1}(v^{I,0}(0,t) + v^{B,0}) - u_{zz}^{B,2} - zu_x^{I,0}(0,t)v_z^{B,0} \\
\qquad - zv_x^{I,0}(0,t)u_z^{B,0}, \\
\quad \cdots \cdots,
\end{cases}
$$

where $\tilde{G}_j = 0$ for $j \geq -2$. In particular $\tilde{G}_{-2} = 0$, $\tilde{G}_{-1} = 0$, and $\tilde{G}_0 = 0$ along with integration over (z, ∞) entail that

(A.7) $$u^{B,0}(z,t) = 0 \quad \text{for } (z,t) \in [0,\infty) \times [0,T],$$

(A.8) $$u_z^{B,1} = -u^{I,0}(0,t)v^{B,0} = -\bar{u}v^{B,0},$$

and

(A.9) $$
\begin{aligned}
u_{zz}^{B,2} = &-u^{I,0}(0,t)v_z^{B,1} - (u^{I,1}(0,t) + u^{B,1})v_z^{B,0} \\
&- u_x^{I,0}(0,t)v^{B,0} - u_z^{B,1}(v^{I,0}(0,t) + v^{B,0}) - zu_x^{I,0}(0,t)v_z^{B,0}.
\end{aligned}
$$

Then integrating (A.9) with respect to z twice, one finds that

(A.10) $$u^{B,2} = \bar{u}\int_z^\infty v^{B,1}(y,t)\,dy - \int_z^\infty\int_y^\infty \Phi(s,t)\,dsdy$$

with $\Phi(z,t) := (u^{I,1}(0,t) + u^{B,1})v_z^{B,0} + u_x^{I,0}(0,t)v^{B,0} + u_z^{B,1}(v^{I,0}(0,t) + v^{B,0}) + zu_x^{I,0}(0,t)v_z^{B,0}$.

For the right boundary layer profiles $u^{b,j}$, we modify the approach (detailed in [26, Appendix, Step 3]) in deriving (A.6) by neglecting the left boundary layer profiles $u^{B,j}$, $v^{B,j}$ in (2.1) and substituting the remaining terms into the first equation of (1.5), then subtracting (A.5) and applying the Taylor expansion (at $x = 1$) to the remaining $u^{I,j}$, $v^{I,j}$ in the resulting equation, to derive an expression similar to (A.6),

$$\sum_{j \geq -2} \varepsilon^{j/2}\tilde{F}_j(\xi,t) = 0,$$

where $\tilde{F}_j$ is defined as $\tilde{G}_j$ in (A.6) by replacing $(u^{B,k}, v^{B,k})$ with $(u^{b,k}, v^{b,k})$, $(u^{I,k}, v^{I,k})(0,t)$ with $(u^{I,k}, v^{I,k})(1,t)$, and z with ξ, for $k \in \mathbb{N}$. Hence, we deduce from $\tilde{F}_{-2} = 0$, $\tilde{F}_{-1} = 0$, and $\tilde{F}_0 = 0$ that

(A.11) $$u^{b,0}(\xi,t) = 0 \quad \text{for } (\xi,t) \in (-\infty,0] \times [0,T],$$

(A.12) $$u_\xi^{b,1} = -\bar{u}v^{b,0},$$

and

(A.13) $$
\begin{aligned}
u_{\xi\xi}^{b,2} = &-u^{I,0}(1,t)v_\xi^{b,1} - (u^{I,1}(1,t) + u^{b,1})v_\xi^{b,0} \\
&- u_x^{I,0}(1,t)v^{b,0} - u_\xi^{b,1}(v^{I,0}(1,t) + v^{b,0}) - \xi u_x^{I,0}(1,t)v_\xi^{b,0}.
\end{aligned}
$$

Thus

$$(A.14) \qquad u^{b,2} = \bar{u} \int_\xi^{-\infty} v^{b,1}(y,t)\,dy - \int_\xi^{-\infty}\int_y^{-\infty} \Psi(s,t)\,dsdy$$

with $\Psi(\xi,t) := (u^{I,1}(1,t)+u^{b,1})v_\xi^{b,0}+u_x^{I,0}(1,t)v^{b,0}+u_\xi^{b,1}(v^{I,0}(1,t)+v^{b,0})+\xi u_x^{I,0}(1,t)v_\xi^{b,0}$.

Step 3. *Equations for $v^{I,j}$, $v^{B,j}$, and $v^{b,j}$.* Applying the above arguments in Step 2 to the second equation of (1.5), we have for the outer layer profiles $v^{I,j}$ that

$$(A.15) \qquad \begin{cases} v_t^{I,0} - u_x^{I,0} = 0, \\ v_t^{I,1} - u_x^{I,1} = 0, \\ v_t^{I,j} + 2\displaystyle\sum_{k=0}^{j-2} v^{I,k} v_x^{I,j-2-k} - u_x^{I,j} - v_{xx}^{I,j-2} = 0 \quad \text{for } j \geq 2 \end{cases}$$

and for the left boundary layer profiles $v^{B,j}$ that

$$(A.16) \qquad \begin{cases} -u_z^{B,0} = 0, \\ v_t^{B,0} - u_z^{B,1} - v_{zz}^{B,0} = 0, \\ v_t^{B,1} + 2(v^{I,0}(0,t)+v^{B,0})v_z^{B,0} - u_z^{B,2} - v_{zz}^{B,1} = 0, \\ \quad \cdots\cdots. \end{cases}$$

Moreover, the right boundary layer profiles $v^{b,j}$ satisfy that

$$(A.17) \qquad \begin{cases} -u_\xi^{b,0} = 0 \\ v_t^{b,0} - u_\xi^{b,1} - v_{\xi\xi}^{b,0} = 0, \\ v_t^{b,1} + 2(v^{I,0}(1,t)+v^{b,0})v_\xi^{b,0} - u_\xi^{b,2} - v_{\xi\xi}^{b,1} = 0, \\ \quad \cdots\cdots. \end{cases}$$

Finally, we collect the results obtained in Steps 1 to 3 to derive the initial boundary value problems (2.3)–(2.12) given in section 2. First, from (A.5) with $j = 0$, (A.15), (A.1), (A.3), (A.7), and (A.11), we get (2.3). Combining (A.8), (A.16), (A.1), and (A.3), one gets (2.4)–(2.5). Similarly (A.12), (A.17), (A.1), and (A.3) lead to (2.6)–(2.7). Moreover, (A.5) with $j = 1$, (A.15), (A.2), and (A.4) give rise to (2.8), and (2.9)–(2.10) come from (A.10), (A.16), (A.2), and (A.4). Finally (2.11)–(2.12) follow from (A.14), (A.17), (A.2), and (A.4).

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