Nonlinear stability of phase transition steady states to a hyperbolic–parabolic system modeling vascular networks

Guangyi Hong, Hongyun Peng, Zhi-An Wang and Changjiang Zhu

Abstract

This paper is concerned with the existence and stability of phase transition steady states to a quasi-linear hyperbolic–parabolic system of chemotactic aggregation, which was proposed in [Ambrosi, Bussolino and Preziosi, *J. Theoret. Med.* 6 (2005) 1–19; Gamba *et al.*, *Phys. Rev. Lett.* 90 (2003) 118101.] to describe the coherent vascular network formation observed *in vitro* experiment. Considering the system in the half line $\mathbb{R}_+ = (0, \infty)$ with Dirichlet boundary conditions, we first prove the existence and uniqueness of non-constant phase transition steady states under some structure conditions on the pressure function. Then we prove that this unique phase transition steady state is nonlinearly asymptotically stable against a small perturbation. We prove our results by the method of energy estimates, the technique of *a priori* assumption and a weighted Hardy-type inequality.

1. Introduction

Experiments of *in vitro* blood vessel formation demonstrate that endothelial cells randomly dispersing on a gel substrate (matrix) can spontaneously organize into a coherent vascular network (see [**1, 13**] and figures therein), which is called angiogenesis — a major factor driving the tumor growth. How endothelial cells self-organize geometrically into capillary networks and how separate individual cells cooperate in the formation of coherent patterns remain poorly understood biologically up to date. These networking patterns cannot be explained by the macroscopic aggregation models such as Keller–Segel type chemotaxis models that lead to point-wise blowup or rounded aggregates, nor by the microscopic kinetic models that describe individual cell behaviors, as commented in [**6**]. Strikingly they can be numerically reproduced by a hydrodynamic (hyperbolic-parabolic) models of chemotaxis proposed in [**1, 13**] as follows

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p(\rho) = \mu \rho \nabla \phi - \alpha \rho u, \\ \partial_t \phi - \Delta \phi = a\rho - b\phi, \end{cases} \tag{1.1}$$

where ρ denotes the endothelial cell density, u the cell velocity, and ϕ the concentration of chemoattractant secreted by cells; p is a pressure function accounting for the fact that closely packed cells resist to compression due to the impenetrability of cellular matter, the parameter $\mu > 0$ measures the intensity of cell response to the chemoattractant and $\alpha \rho u$ corresponds to a

damping (friction) force due to the interaction between cells and underlying substratum with a drag coefficient $\alpha > 0$; a and b are positive constants accounting for the growth and death rates of the chemoattractant. The convective term $\nabla \cdot (\rho u \otimes u)$ models the cell persistence (inertia effect). We refer more detailed biological interpretations on the model (1.1) to [**1, 13**]. The hydrodynamic system (1.1) has been (formally) derived from the following kinetic equation in [**7**] via the mean-field approximation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \int_V (T[\phi](v', v) f(t, x, v') - T[\phi](v, v') f(t, x, v)) \mathrm{d}v',$$

where $f(t, x, v)$ denotes cell density distribution at time t, position $x \in \mathbb{R}^d$ moving with a velocity v from a compact set V in $\mathbb{R}^d$, and tumbling kernel $T[\phi](v', v)$ describes the frequency of changing trajectories from velocity v' (anterior) to v (posterior) in response to a chemical concentration ϕ.

At first glance from mathematical point of view, the above hydrodynamic system (1.1) is analogous to the well-known damped Euler–Poisson system where the ϕ-equation is elliptic (that is, $-\Delta\phi = a\rho$) which appears in various important applications depending on the sign of μ, such as the propagation of electrons in semiconductor devices (cf. [**22**]) and the transport of ions in plasma physics (cf. [**8**]) and the collapse of gaseous stars due to self-gravitation [**5**]. However, the parabolic ϕ-equation in (1.1) will bring substantial differences in mathematical analysis and many existing mathematical frameworks developed for the Euler–Poisson system are inapplicable directly to (1.1). Due to the competing interactions between parabolic dissipation and hyperbolic anti-dissipation effect plus nonlinear convection, the global well-posedness and regularity of solutions to (1.1) is very complicated as can be glimpsed from the Euler–Poisson equations for which the understanding of solution behaviors is rather incomplete despite numerous studies attempted. Up to date, there are only few results obtained for (1.1) in the literature. First when the initial value (ρ_0, u_0, ϕ_0) is a small perturbation of a constant ground state $(\bar{\rho}, 0, \bar{\phi})$ in $H^s(\mathbb{R}^d)(s > d/2 + 1)$ with $\bar{\rho} > 0$ sufficiently small, the global existence and stability of solutions with nonvacuum (that is, $\inf_{x \in \mathbb{R}^d} \rho > 0$) to (1.1) was established in [**9, 11**]. The linear stability of constant ground state $(\bar{\rho}, 0, \bar{\phi})$ was obtained under the condition $p'(\bar{\rho}) > \frac{a\mu}{b}\bar{\rho}$ in [**16**] where an additional viscous term Δu is supplied to the second equation of (1.1). The stationary solutions of (1.1) with vacuum (bump solutions) in a bounded interval with zero-flux boundary condition and in $\mathbb{R}$ were constructed in [**2**] and further elaborated in [**3**]. The model (1.1) with $p(\rho) = \rho$ and periodic boundary conditions in one dimension was numerically explored in [**12**]. These appear to be the only results available to the system (1.1) in the literature and further studies are highly in demand. For results on some other types of hyperbolic–parabolic chemotaxis models, we refer to [**10, 17, 18, 24, 27**] and references therein.

Note that the above-mentioned mathematical works on (1.1) prescribe initial data as a small perturbation of constant equilibria and the large-time profile of solutions is also constant, which cannot explain the experimental observations of [**1, 13**] showing prominent phase transition patterns connecting regions clear (or low density) of cells. This motivates us to explore the possible non-constant phase transition profiles of solutions. The aim of this paper is to study the existence and stability of phase transition steady states without vacuum to the system (1.1) in one-dimensional half space $\mathbb{R}_+ = (0, \infty)$. For the convenience of presentation in the sequel, we set $m = \rho u$, namely m denotes the momentum of cells, and recast the one-dimensional system (1.1) in $\mathbb{R}_+$ as

$$\begin{cases} \rho_t + m_x = 0, \quad x \in \mathbb{R}_+, \quad t > 0, & (1.2a) \\[2mm] m_t + \left(\dfrac{m^2}{\rho} + p(\rho)\right)_x = \mu\rho\phi_x - \alpha m, & (1.2b) \\[2mm] \phi_t = \phi_{xx} + a\rho - b\phi. & (1.2c) \end{cases}$$

As in [16], we assume the pressure function p depends only on the density and satisfies

$$p(\rho) \in C^3(0, \infty), \quad p'(\rho) - \frac{a\mu}{b}\rho > 0 \quad \text{for any } \rho > 0. \tag{1.3}$$

A typical form of p is $p(\rho) = \frac{K}{2}\rho^2$ with $K > \frac{a\mu}{b}$. We supplement the system (1.2a)–(1.2c) with the following boundary conditions

$$m(0, t) = 0, \quad \phi(0, t) = \phi_-, \tag{1.4}$$

and the initial value

$$(\rho, m, \phi)(x, 0) = (\rho_0, m_0, \phi_0)(x) \to (\rho_+, 0, \phi_+) \quad \text{as} \quad x \to +\infty, \tag{1.5}$$

where ϕ_-, $\rho_+ > 0$, $\phi_+ > 0$ are constants and $\phi_- \neq \phi_+$.

In this paper, we shall first use delicate analysis to show that the system (1.2a)–(1.2c) has a unique non-constant stationary solution $(\bar{\rho}, 0, \bar{\phi})$ without vacuum satisfying

$$\bar{\phi}(0) = \phi_- \quad \text{and} \quad \lim_{x \to +\infty} (\bar{\rho}, \bar{\phi})(x) = (\rho_+, \phi_+),$$

where $\bar{\rho}$ and $\bar{\phi}$ are monotone. Then we show that the stationary solution $(\bar{\rho}, 0, \bar{\phi})$ is asymptotically stable, namely the solution of (1.2a)–(1.2c) converges to $(\bar{\rho}, 0, \bar{\phi})$ point-wisely as $t \to \infty$ if the initial value (ρ_0, m_0, ϕ_0) is an appropriate small perturbation of $(\bar{\rho}, 0, \bar{\phi})$. The monotonicity of $\bar{\rho}$ and $\bar{\phi}$ indicates that the steady states have phase transition profiles. To the best of our knowledge, there are not such results available in the literature for (1.2a)–(1.2c) and even for the Euler–Poisson equations (cf. [14, 19–21, 26]). To prove our results, we fully capture the structure of (1.2a)–(1.2c) where the stationary solution has exponential decay at far field, which is not enjoyed by the Euler–Poisson (or Euler) equations. Although part of the proof of our results is inspired by some ideas of [14, 15, 25, 26] on Euler–Poisson or Euler equations, we have added lot of extra efforts to deal with complicated couplings and boundary effects. The coupling term $\mu\rho\phi_x$ leads to two linear terms in the linearized system around stationary solutions and how to make these linear terms under control is crucial to the asymptotic stability against small perturbations. We resolve this issue by the structural assumption (1.3) to take up the dissipation and use the exponentially weighted Hardy inequality in half space to compensate for the lack of dissipation in the hyperbolic equations. Due to the couplings and boundary effects, the energy estimates are very sophisticated, where the lower-order estimates involve higher-order estimates and vice versa. We use the delicate energy estimates along with the technique of a *priori* assumptions to unravel these tangles and gradually achieve our results.

The rest of this paper is organized as follows. In Section 2, we state our main results on the existence of non-constant stationary solutions (Theorem 2.1) and stability of stationary solutions (Theorem 2.2). In Section 3, we study the stationary problem and prove Theorem 2.1. The proof of Theorem 2.2 is given in Section 4.

2. Statement of main results

In this section, we shall state the main results of this paper. To be precise, we first introduce some notations used. Throughout the paper, we use $\|\cdot\|_{L^\infty}$, $\|\cdot\|$ and $\|\cdot\|_k$ to denote the norms of usual $L^\infty(\mathbb{R}_+)$, $L^2(\mathbb{R}_+)$ and the standard Sobolev space $H^k(\mathbb{R}_+)$, respectively. We also use $\|(f_1, \ldots, f_n)\|$ (respectively, $\|(f_1, \ldots, f_n)\|_k$) to denote $\|f_1\| + \cdots + \|f_n\|$ (respectively, $\|f_1\|_k + \cdots + \|f_n\|_k$) for some $n \in \mathbb{Z}_+$. We denote by C a generic constant that may vary in the context, and by C_η a constant depending on η. Occasionally, we simply write $f \sim g$ if $C^{-1} \leqslant f \leqslant Cg$ for some constant $C > 0$.

It can be verified that the system (1.2a)–(1.2c) possesses the following energy functional (cf. [2, 7])

$$F[\rho, u, \phi] = \frac{1}{2\mu} \int_{\mathbb{R}_+} \rho u^2 \mathrm{d}x + \frac{1}{\mu} \int_{\mathbb{R}_+} \mathcal{G}(\rho)\mathrm{d}x + \frac{1}{2a} \int_{\mathbb{R}_+} (|\nabla \phi|^2 + b\phi^2)\mathrm{d}x - \int_{\mathbb{R}_+} \rho\phi \mathrm{d}x,$$

which, subject to the boundary condition (1.4), satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} F[\rho, u, \phi] + \frac{\alpha}{\mu} \int_{\mathbb{R}_+} \rho u^2 \mathrm{d}x + \frac{1}{a} \int_{\mathbb{R}_+} |\phi_t|^2 \mathrm{d}x = 0,$$

where $\rho \mathcal{G}''(\rho) = p'(\rho)$. Thus the stationary solution satisfying $\frac{\mathrm{d}}{\mathrm{d}t} F[\rho, u, \phi] = 0$ gives rise to $\rho u = 0$ and $\phi_t = 0$ in $\mathbb{R}_+$. Since we are interested in non-constant profile for ρ, $u \equiv 0$ is the only (physical) stationary profile for the velocity u. Therefore stationary solutions of (1.2a)–(1.2c) without vacuum must possess the form $(\bar{\rho}, 0, \bar{\phi})$, where $(\bar{\rho}, \bar{\phi})$ satisfies

$$\begin{cases} p(\bar{\rho})_x = \mu\bar{\rho}\bar{\phi}_x, \quad x \in \mathbb{R}_+, & \text{(2.1a)} \\[2mm] \bar{\phi}_{xx} + a\bar{\rho} - b\bar{\phi} = 0, & \text{(2.1b)} \\[2mm] \bar{\phi}(0) = \phi_-, & \text{(2.1c)} \\[2mm] \lim_{x \to +\infty} (\bar{\rho}, \bar{\phi}) = (\rho_+, \phi_+). & \text{(2.1d)} \end{cases}$$

Here the pressure p satisfies (1.3) and the constants ρ_+ and $\phi_\pm$ are the same as in (1.4) and (1.5).

Then our first result concerning the existence and uniqueness of solutions to the stationary problem (2.1a)–(2.1d) is given below.

THEOREM 2.1. Let $\rho_+ > 0$ and $\phi_- \neq \phi_+$ such that $a\rho_+ = b\phi_+$. If

$$\phi_- - \phi_+ + \int_0^{\rho_+} \frac{p'(s)}{\mu s} \mathrm{d}s > 0, \tag{2.2}$$

then there is a unique constant $\rho_- > 0$ such that the problem (2.1a)–(2.1d) with (1.3) admits a unique solution $(\bar{\rho}, \bar{\phi})$ satisfying $\bar{\rho}(0) = \rho_-$ and

$$\begin{cases} \bar{\rho}'(x) < 0, \ \bar{\phi}'(x) < 0 & \text{if } \phi_- > \phi_+, \\[2mm] \bar{\rho}'(x) > 0, \ \bar{\phi}'(x) > 0 & \text{if } \phi_- < \phi_+. \end{cases} \tag{2.3}$$

Moreover, if $|\phi_- - \phi_+|$ is small enough, it holds that

$$\sum_{k=1}^{2} \left| \frac{\mathrm{d}^k}{\mathrm{d}x^k} (\bar{\rho}, \bar{\phi}) \right| + |\bar{\rho}(x) - \rho_+| + |\bar{\phi}(x) - \phi_+| \leqslant C\mathrm{e}^{-\lambda x} |\phi_- - \phi_+|, \quad x \geqslant 0 \tag{2.4}$$

for some constants $C > 0$ and $\lambda > 0$ which may depend on ρ_+, a and b, but independent of $\phi_- - \phi_+$.

REMARK 2.1. Under the condition (1.3), the integral $\int_0^{\rho_+} \frac{p'(s)}{\mu s} \mathrm{d}s$ in (2.2) is positive, but not necessarily finite since $\frac{p'(s)}{\mu s} \to +\infty$ as $s \to 0$ is possible. While in the case of $\int_0^{\rho_+} \frac{p'(s)}{\mu s} \mathrm{d}s = +\infty$, the condition (2.2) is free for any given ϕ_- and ϕ_+.

Our second result is the asymptotic stability of the stationary solutions obtained in Theorem 2.1, which is stated in the following theorem.

THEOREM 2.2. *Let the conditions in Theorem* 2.1 *hold and define*

$$\varphi_0 = -\int_x^\infty (\rho_0(y) - \bar\rho(y))\mathrm{d}y, \quad \Phi_0 = \phi_0(x) - \bar\phi.$$

If $\varphi_0 \in H^3 \cap H_0^1(\mathbb{R}_+)$, $m_0 \in H^2 \cap H_0^1(\mathbb{R}_+)$, *and* $\Phi_0 \in H^4 \cap H_0^1(\mathbb{R}_+)$ *with* $\inf_{x\in\mathbb{R}_+} \rho_0(x) > 0$ *(namely* $\inf_{x\in\mathbb{R}_+} \{\varphi_{0x} + \bar\rho\} > 0$*), then there exists a constant* $\delta_0 > 0$ *such that if*

$$\|\varphi_0\|_3 + \|m_0\|_2 + \|\Phi_0\|_4 + |\phi_- - \phi_+| \leqslant \delta_0,$$

the problem (1.2a)–(1.2c) *subject to the initial-boundary conditions* (1.4)–(1.5) *admits a unique classical solution* $(\rho(x,t), m(x,t), \phi(x,t))$ *in* $\mathbb{R}_+ \times (0, \infty)$ *satisfying* $\inf_{x\in\mathbb{R}_+} \rho(x) > 0$ *for any* $t > 0$ *and*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}_+} \left|(\rho, m, \phi)(x,t) - (\bar\rho, 0, \bar\phi)(x)\right| \to 0. \tag{2.5}$$

REMARK 2.2. With the condition $\Phi_0 \in H^4$, we can define the initial values of ϕ_t and ϕ_{tt} through the equation for ϕ. That is

$$\Phi_{t0} := \phi_{t0} = (\Phi_0)_{xx} + a(\varphi_0)_x - b\Phi_0, \tag{2.6}$$

$$\Phi_{tt0} := \phi_{tt0} = (\Phi_{t0})_{xx} - a(m_0)_x - b\Phi_{t0}. \tag{2.7}$$

These initial values of time derivatives are of importance in deriving the higher order estimates in Section 4. Furthermore, we always assume that the initial data are compatible with the boundary conditions at $x = 0$.

3. Stationary problem (*Proof of Theorem* 2.1)

In this section, we shall study the stationary problem (2.1a)–(2.1d) and complete the proof of Theorem 2.1. To this end, we first reformulate our problem (2.1a)–(2.1d), and then prove the existence and uniqueness of solutions. Finally, we derive the monotone and decay properties of solutions.

3.1. *Reformulation of our problem*

We start by proving the following lemma, which plays a key role in the reformulation of our problem.

LEMMA 3.1. *If* f *is a solution to the problem*

$$\begin{cases} f_x = \omega(f(x)), & x \in \mathbb{R}_+, \\ f(+\infty) = k_0, \end{cases} \tag{3.1}$$

where ω *is a continuous function and* k_0 *is a constant, then we have*

$$\lim_{x\to+\infty} f_x = \omega(k_0) = 0.$$

Proof. Since ω is continuous, and $\lim_{x\to+\infty} f(x) = k_0$, we have $\lim_{x\to+\infty} f_x = \omega(k_0)$. It remains to show $\omega(k_0) = 0$. We proof this by contradiction. Supposing that $\omega(k_0) \neq 0$, without loss of generality, we assume $\omega(k_0) > 0$. Due to the continuity of ω and $\lim_{x\to+\infty} f(x) = k_0$,

there exists a constant $X_0 \in \mathbb{R}_+$ such that for any $x \geqslant X_0$, $\omega(f(x)) > \frac{\omega(k_0)}{2} > 0$. This along with $(3.1)_1$ implies

$$f(x) = f(X_0) + \int_{X_0}^{x} \omega(f(x)) \mathrm{d}x \geqslant f(X_0) + \frac{\omega(k_0)}{2}(x - X_0) \to +\infty \ \text{ as } \ x \to +\infty,$$

which contradicts the fact $f(+\infty) = k_0$. Hence, $\omega(k_0) = 0$. The proof of Lemma 3.1 is complete. $\qquad\square$

Define

$$F(s) := \int_{\rho_+}^{s} \frac{p'(\tau)}{\mu\tau} \mathrm{d}\tau \tag{3.2}$$

for any $s > 0$. Clearly, $F(\rho_+) = 0$. We claim that under the conditions (1.3), (2.2) and $\rho_+ > 0$, there exists a unique constant $\rho_- > 0$ such that

$$F(\rho_-) = \phi_- - \phi_+ \tag{3.3}$$

and

$$\phi_- > \phi_+ \Longleftrightarrow \rho_- > \rho_+ \ \ (\text{resp.} \ \phi_- < \phi_+ \Longleftrightarrow \rho_- < \rho_+). \tag{3.4}$$

Indeed, in view of (1.3), we know that

$$F'(s) = \frac{p'(s)}{s\mu} > \frac{a}{b} > 0. \tag{3.5}$$

This implies that the function $F(s)$ is strictly monotonically increasing. Furthermore, we have $F(s) < 0$ if $s < \rho_+$, and $\lim_{s \to +\infty} F(s) = +\infty$. For the case $\phi_- > \phi_+$, since $F(\rho_+) = 0 < \phi_- - \phi_+ < \infty$, then there exists a unique constant $\rho_- \in (\rho_+, +\infty)$ such that (3.3) holds. For the case $\phi_- < \phi_+$, we have $\phi_- - \phi_+ < 0$. If $\int_0^{\rho_+} \frac{p'(\tau)}{\mu\tau} \mathrm{d}\tau = \infty$, we know that $\lim_{s \to 0} F(s) = -\infty < \phi_- - \phi_+ < 0 = F(\rho_+)$, then similar to the case $\phi_- > \phi_+$, there exists a unique constant $\rho_- \in (0, \rho_+)$ such that $F(\rho_-) = \phi_- - \phi_+$. Now it remains to consider the case when $\phi_- < \phi_+$ and $\int_0^{\rho_+} \frac{p'(\tau)}{\mu\tau} \mathrm{d}\tau < \infty$. In this case, since the $F(s)$ is continuous, monotonic, and bounded below, we can extend $F(s)$ by defining $F(0) := \lim_{s \to 0} F(s) > -\infty$. Then the extended function $F(s)$ is continuous on $[0, \rho_+]$. Furthermore, from (2.2), we get $F(0) < \phi_- - \phi_+ < 0 = F(\rho_+)$. Hence, there exists a unique constant $\rho_- \in (0, \rho_+)$ such that $F(\rho_-) = \phi_- - \phi_+$. Then (3.3) is proved. Moreover, with the help of (3.3) and (3.5), we immediately get (3.4). We thus finish the proof of the claim.

To proceed, assume that $(\bar{\rho}, \bar{\phi})$ is a classical solution to (2.1a)–(2.1d) with $\bar{\rho} > 0$. Dividing (2.1a) by $\bar{\rho}$ and integrating the resulting equation over $(x, +\infty)$, we get

$$\bar{\phi}(x) = F(\bar{\rho}(x)) - F(\rho_+) + \phi_+, \tag{3.6}$$

where $F(s)$ is as in (3.2). Sending $x \to 0^+$ along with (2.1d), (3.3), and the fact $F(\rho_+) = 0$, we get

$$\lim_{x \to 0^+} F(\bar{\rho}(x)) = F(\rho_-).$$

By using the monotonicity and continuity of $F(s)$, we further have that

$$\bar{\rho}(0) = \lim_{x \to 0^+} \bar{\rho}(x) = \lim_{x \to 0^+} F^{-1}(F(\bar{\rho}(x))) = F^{-1}(F(\rho_-)) = \rho_-.$$

Inserting (3.6) into (2.1b), we get

$$[F(\bar{\rho})]_{xx} = b[F(\bar{\rho}) - F(\rho_+)] + b\phi_+ - a\bar{\rho}. \tag{3.7}$$

Multiplying (3.7) by $2F'(\bar\rho)\bar\rho_x$, it follows that

$$2F'(\bar\rho)\bar\rho_x[F'(\bar\rho)\bar\rho_x]_x = 2\{b[F(\bar\rho) - F(\rho_+)] + b\phi_+ - a\bar\rho\} =: 2H(\bar\rho)F'(\bar\rho)\bar\rho_x, \qquad (3.8)$$

with

$$H(s) := b[F(s) - F(\rho_+)] + b\phi_+ - as. \qquad (3.9)$$

Thus,

$$[F'(\bar\rho)\bar\rho_x]^2 = G(\bar\rho) + C_0 \geqslant 0, \quad x \in \mathbb{R}_+ \qquad (3.10)$$

for some constant C_0 and some function $G(s)$ with

$$G'(s) = 2F'(s)H(s). \qquad (3.11)$$

By virtue of (1.3) and (3.9), we get

$$H'(s) = bF'(s) - a > 0 \qquad (3.12)$$

for any $s > 0$. Due to the condition $a\rho_+ = b\phi_+$, it holds that $H(\rho_+) = 0$. This along with (3.12) yields that

$$H(s) > H(\rho_+) = 0 \text{ if } s > \rho_+ \text{ and } H(s) < H(\rho_+) = 0 \text{ if } s < \rho_+. \qquad (3.13)$$

Then by (3.5), (3.11), and (3.13), we get $G'(s) > 0$ if $s > \rho_+$ and $G'(s) < 0$ if $s < \rho_+$. This gives

$$G(s) - G(\rho_+) > 0 \qquad (3.14)$$

for any $s \neq \rho_+$. We claim that $C_0 = -G(\rho_+)$. Otherwise, we have $C_0 < -G(\rho_+)$ or $C_0 > -G(\rho_+)$. If $C_0 < -G(\rho_+)$, by the continuity of G and $\bar\rho$, there exists a constant $K_0 > 0$ such that if $x \geqslant K_0$,

$$G(\bar\rho) < \frac{G(\rho_+) - C_0}{2} < -C_0.$$

Then $G(\bar\rho) + C_0 < 0$ for $x \geqslant K_0$. This contradicts to (3.10). If $C_0 > -G(\rho_+)$, using (3.14), we get $\bar\rho_x \neq 0$ for any $x \in \mathbb{R}_+$. Therefore, for any $x \in \mathbb{R}_+$, it holds that

$$\bar\rho_x = -\frac{\sqrt{G(\bar\rho) + C_0}}{F'(\bar\rho)} \text{ if } \rho_- > \rho_+ \text{ and } \bar\rho_x = \frac{\sqrt{G(\bar\rho) + C_0}}{F'(\bar\rho)} \text{ if } \rho_- < \rho_+.$$

With the fact $\bar\rho(+\infty) = \rho_+$ and Lemma 3.1, we have $C_0 = -G(\rho_+)$. This is a contradiction. Hence, we have $C_0 = -G(\rho_+)$ and

$$\begin{cases} [F'(\bar\rho)\bar\rho_x]^2 = G(\bar\rho) - G(\rho_+), \quad x \in \mathbb{R}_+, \\ \bar\rho(0) = \rho_-, \quad \bar\rho(+\infty) = \rho_+. \end{cases} \qquad (3.15)$$

This together with (3.5) and (3.14) implies that $\bar\rho_x \leqslant 0$ if $\rho_- > \rho_+$ and $\bar\rho_x \geqslant 0$ if $\rho_+ > \rho_-$, and that

$$\bar\rho_x(x) = 0 \text{ if and only if } \bar\rho(x) = \rho_+ \qquad (3.16)$$

for any $x \in \mathbb{R}_+$. Hence, we can solve $\bar\rho_x$ from (3.15) that

$$\bar\rho_x = -\frac{\sqrt{G(\bar\rho) - G(\rho_+)}}{F'(\bar\rho)} \text{ if } \phi_- > \phi_+ \text{ and } \bar\rho_x = \frac{\sqrt{G(\bar\rho) - G(\rho_+)}}{F'(\bar\rho)} \text{ if } \phi_- < \phi_+,$$

where we have used (3.4).

Summing up, we have the following lemma.

LEMMA 3.2.   *Under the conditions of Theorem 2.1, if $(\bar{\rho}, \bar{\phi})$ is a classical solution to the problem (2.1a)–(2.1d) satisfying $\bar{\rho}(x) > 0$ for any $x \in \mathbb{R}_+$, then $(\bar{\rho}, \bar{\phi})$ is also a solution to the following problem:*

$$
\begin{cases}
\bar{\rho}_x = -\dfrac{\sqrt{G(\bar{\rho}) - G(\rho_+)}}{F'(\bar{\rho})} \ \text{if } \rho_- > \rho_+ \ \ (\text{reps. } \bar{\rho}_x = \dfrac{\sqrt{G(\bar{\rho}) - G(\rho_+)}}{F'(\bar{\rho})} \ \text{if } \rho_- < \rho_+), \\[2mm]
\bar{\phi}(x) = F(\bar{\rho}(x)) - F(\rho_+) + \phi_+, \\[2mm]
\bar{\rho}(0) = \rho_-, \ \ \bar{\rho}(+\infty) = \rho_+.
\end{cases}
\tag{3.17}
$$

Here $F(s)$ and $G(s)$ are given in (3.2) and (3.11), respectively, and $\rho_- > 0$ is determined by (3.3).

In the following, we shall show that the problem (3.17) is indeed equivalent to (2.1a)–(2.1d).

LEMMA 3.3 (Reformulation).   *Suppose that the conditions of Theorem 2.1 hold. Then $(\bar{\rho}(x), \bar{\phi}(x))$ is a classical solution to the problem (2.1a)–(2.1d) satisfying $\bar{\rho} > 0$, if and only if it is a classical solution to the problem (3.17).*

Proof.   In view of Lemma 3.2, it remains to show that if $(\bar{\rho}, \bar{\phi})$ is a solution to the problem (3.17), then $(\bar{\rho}, \bar{\phi})$ solves the problem (2.1a)–(2.1d). By using (3.17)$_3$ and (3.5), one can easily derive (2.1a). Due to (3.3) and (3.17)$_3$, we have $\bar{\phi}(0) = \phi_-$ and $\bar{\phi}(+\infty) = \phi_+$. To show (2.1b), by (3.8), (3.15), and (3.17)$_3$, it suffices to show that $\bar{\rho}_x \neq 0$ for any $x \in \mathbb{R}_+$. We prove this for the case $\rho_- > \rho_+$ (that is, $\phi_- > \phi_+$), and the proof for the case $\rho_- < \rho_+$ (that is, $\phi_- < \phi_+$) is similar. Since $\rho_- > \rho_+$, we have $\bar{\rho}_x \leqslant 0$ for any $x \in \mathbb{R}_+$. Denote

$$
\mathcal{D}(\bar{\rho}) := -\frac{\sqrt{G(\bar{\rho}) - G(\rho_+)}}{F'(\bar{\rho})}.
$$

We claim that $\mathcal{D}(\bar{\rho})$ is Lipschitz continuous on $[\rho_+, \rho_-]$. With this claim, we can prove that $\bar{\rho}_x < 0$ for $x \in \mathbb{R}_+$, and hence finish the proof. Indeed, if there exists a point $x_0 \in \mathbb{R}_+$ such that $\bar{\rho}_x(x_0) = 0$, then from (3.16), we have $\bar{\rho}(x_0) = \rho_+$. This implies that $\bar{\rho}$ is a solution to the following problem

$$
\begin{cases}
\rho_x^* = \mathcal{D}(\rho^*), \ \ 0 \leqslant x \leqslant x_0, \\
\rho^*(x_0) = \rho_+.
\end{cases}
\tag{3.18}
$$

Since $\mathcal{D}(\bar{\rho})$ is Lipschitz continuous on $[\rho_+, \rho_-]$, the problem (3.18) admits a unique solution on $[0, x_0]$. While $\rho^* \equiv \rho_+$ is also a solution to (3.18), and obviously, $\bar{\rho} \not\equiv \rho_+$. This is a contradiction. Therefore, $\bar{\rho}_x < 0$ for any $x \in \mathbb{R}_+$. Now it remains to prove the claim that $\mathcal{D}(\bar{\rho})$ is Lipschitz continuous on $[\rho_+, \rho_-]$. With the help of (1.3), (3.5), and (3.14), we know that $\mathcal{D}(\bar{\rho})$ is differentiable if $\bar{\rho} \neq \rho_+$. Furthermore, a direct computation gives

$$
\mathcal{D}'(\bar{\rho}) = \frac{\sqrt{G(\bar{\rho}) - G(\rho_+)}}{[F'(\bar{\rho})]^2} F''(\bar{\rho}) - \frac{H(\bar{\rho})}{\sqrt{G(\bar{\rho}) - G(\rho_+)}}.
\tag{3.19}
$$

By using (3.11), (3.12), and L'Hôpital's rule, we have

$$
\lim_{\bar{\rho} \to \rho_+^+} \frac{H^2(\bar{\rho})}{G(\bar{\rho}) - G(\rho_+)} = \lim_{\bar{\rho} \to \rho_+^+} \frac{2H'(\bar{\rho})H(\bar{\rho})}{G'(\bar{\rho})} = \lim_{\bar{\rho} \to \rho_+^+} \frac{H'(\bar{\rho})}{F'(\bar{\rho})} = \frac{bF'(\rho_+) - a}{F'(\rho_+)} > 0.
\tag{3.20}
$$

From (3.13) and (3.14), we get $\frac{H(\bar{\rho})}{\sqrt{G(\bar{\rho})-G(\rho_+)}} > 0$ for $\bar{\rho} > \rho_+$. This along with (3.20) yields

$$\lim_{\bar{\rho}\to\rho_+^+} \frac{H(\bar{\rho})}{\sqrt{G(\bar{\rho})-G(\rho_+)}} = \left(\lim_{\bar{\rho}\to\rho_+^+} \frac{H^2(\bar{\rho})}{G(\bar{\rho})-G(\rho_+)}\right)^{\frac{1}{2}} = \sqrt{\frac{aF'(\rho_+)-b}{F'(\rho_+)}}.$$

Therefore,

$$\lim_{\bar{\rho}\to\rho_+^+} \mathcal{D}'(\bar{\rho}) = -\lim_{\bar{\rho}\to\rho_+^+} \frac{H(\bar{\rho})}{\sqrt{G(\bar{\rho})-G(\rho_+)}} = -\sqrt{\frac{aF'(\rho_+)-b}{F'(\rho_+)}}.$$

Note that $\frac{\mathcal{D}(\bar{\rho})-\mathcal{D}(\rho_+)}{\bar{\rho}-\rho_+} = \frac{\mathcal{D}(\bar{\rho})}{\bar{\rho}-\rho_+} < 0$ for $\bar{\rho} > \rho_+$, and that

$$\lim_{\bar{\rho}\to\rho_+} \frac{|\mathcal{D}(\bar{\rho})-\mathcal{D}(\rho_+)|^2}{|\bar{\rho}-\rho_+|^2} = \frac{1}{[F'(\rho_+)]^2} \lim_{\bar{\rho}\to\rho_+} \frac{G(\bar{\rho})-G(\rho_+)}{|\bar{\rho}-\rho_+|^2} = \lim_{\bar{\rho}\to\rho_+} \frac{H(\bar{\rho})}{\bar{\rho}-\rho_+}$$

$$= \lim_{\bar{\rho}\to\rho_+} \frac{H'(\bar{\rho})}{F'(\rho_+)} = \frac{aF'(\rho_+)-b}{F'(\rho_+)} > 0,$$

due to (3.11), (3.12), and L'Hôpital's rule, we have

$$\mathcal{D}'_+(\rho_+) = \lim_{\bar{\rho}\to\rho_+^+} \frac{\mathcal{D}(\bar{\rho})-\mathcal{D}(\rho_+)}{\bar{\rho}-\rho_+} = -\left(\lim_{\bar{\rho}\to\rho_+} \frac{|\mathcal{D}(\bar{\rho})-\mathcal{D}(\rho_+)|^2}{|\bar{\rho}-\rho_+|^2}\right)^{\frac{1}{2}} = -\sqrt{\frac{bF'(\rho_+)-a}{F'(\rho_+)}},$$

where $\mathcal{D}'_+(\rho_+)$ is the right derivative of $\mathcal{D}(\bar{\rho})$ at ρ_+. We thus have $\lim_{\bar{\rho}\to\rho_+^+} \mathcal{D}'(\bar{\rho}) = \mathcal{D}'_+(\rho_+)$. This in combination with (3.19) yields that $\mathcal{D}'(\bar{\rho})$ is continuous on $[\rho_+, \rho_-]$, and thus $|\mathcal{D}'(\bar{\rho})| \leqslant C(\rho_-, \rho_+)$ for some constant $C(\rho_-, \rho_+) > 0$ depending on ρ_- and ρ_+. This implies that $\mathcal{D}(\bar{\rho})$ is Lipschitz continuous on $[\rho_+, \rho_-]$. The proof of the present lemma is complete. $\qquad\square$

3.2. Existence and uniqueness of solutions

In this section, we will prove that the problem (2.1a)–(2.1d) admits a unique solution $(\bar{\rho}, \bar{\phi})$ with $\bar{\rho} > 0$. Due to Lemma 3.3, it now suffices to consider the problem (3.17). As before, we focus only on the case $\rho_- > \rho_+$ (that is, $\phi_- > \phi_+$), the proof for the case $\rho_- < \rho_+$ (that is, $\phi_- < \phi_+$) is similar and so omitted. Let us begin with the following ODE problem

$$\begin{cases} \bar{\rho}_x = \mathcal{D}(\bar{\rho}), & x > 0, \\ \bar{\rho}(0) = \rho_-. \end{cases} \tag{3.21}$$

By the Lipschitz continuity of $\mathcal{D}(\bar{\rho})$ on $[\rho_+, \rho_-]$, we conclude that the problem (3.21) admits a unique solution on $[0, X_*)$ for some $X_* \in \mathbb{R}_+$. Then by the contradiction argument and discussions in Step 1 on the uniqueness of solutions to (3.18), we get $\bar{\rho}(x) > \rho_+$ for any $x \in [0, X_*)$. This, along with the standard extension theorem for ordinary differential equations, implies that the solution $\bar{\rho}$ to the problem (3.21) exists globally in $\mathbb{R}_+$, and for any $x \in \mathbb{R}_+$, $\bar{\rho}(x) > \rho_+$. In addition, note from (3.16) that $\bar{\rho}_x(x) = 0$ if and only if $\bar{\rho}(x) = \rho_+$, we have that

$$\bar{\rho}_x < 0 \quad \text{for any } x \in \mathbb{R}_+, \tag{3.22}$$

and that $\lim_{x\to+\infty} \bar{\rho}(x)$ exists. Denoting $\bar{\rho}(+\infty) := \lim_{x\to+\infty} \bar{\rho}(x)$, from Lemma 3.1, we obtain $G(\bar{\rho}(+\infty)) - G(\rho_+) = 0$. This combined with (3.14) give rise to $\bar{\rho}(+\infty) = \rho_+$. With $\bar{\rho}(x)$ at hand, we can define $\bar{\phi}(x)$ from (3.17)$_2$. Clearly, $(\bar{\rho}, \bar{\phi})$ is a solution to (3.17) for $\rho_- > \rho_+$. Finally, since $\rho_+ \leqslant \bar{\rho}(x) \leqslant \rho_-$, the uniqueness of solutions can be proved by the Lipschitz continuity of $\mathcal{D}(\bar{\rho})$ on $[\rho_+, \rho_-]$.

3.3. *Monotonicity and decay properties*

Recalling (3.4) and (3.22), we get $\bar{\rho}'(x) < 0$ if $\phi_- > \phi_+$. In a manner similar to the derivation of (3.22), we have $\bar{\rho}'(x) > 0$ if $\phi_- < \phi_+$. With the help of (3.5), (3.6), and the properties of $\bar{\rho}_x$, we have

$$\bar{\phi}_x < 0 \text{ if } \phi_- > \phi_+ \text{ and } \bar{\phi}_x > 0 \text{ if } \phi_- < \phi_+.$$

This gives (2.3). Now let us turn to the decay properties of the solution. By using (3.3) and (3.5), we get

$$|\phi_- - \phi_+| = \left| \int_{\rho_-}^{\rho_+} F'(s)\mathrm{d}s \right| \geqslant \frac{a}{b}|\rho_- - \rho_+|. \tag{3.23}$$

Thus $|\phi_- - \phi_+| \ll 1$ implies $|\rho_- - \rho_+| \ll 1$. In the following, without loss of generality, we assume that $[\rho_+, \rho_-] \subset [\rho_+, \rho_+ + 1]$, and thus for any continuous function f defined on $\mathbb{R}_+$, $\sup_{x \in [\rho_+, \rho_-]} f(x)$ depends only on ρ_+. If $\phi_- > \phi_+$ (that is, $\rho_- > \rho_+$), recalling (2.3) and (3.17), we have

$$\rho_- > \bar{\rho} > \rho_+ \text{ and } \bar{\rho}_x = -\frac{\sqrt{G(\bar{\rho}) - G(\rho_+)}}{F'(\bar{\rho})} \tag{3.24}$$

for any $x \in \mathbb{R}_+$. Owing to (1.3), (3.5), (3.11), and the condition $a\rho_+ = b\phi_+$, we get

$$G''(s) = 2[F'(s)H(s)]' = 2F'(s)H'(s) + 2F''(s)H(s)$$

$$= 2F'(s)(bF'(s) - a) + \frac{2}{\mu s^2}(sp''(s) - p'(s))(b[F(s) - F(\rho_+)] + b\phi_+ - as)$$

$$\geqslant 2F'(s)(bF'(s) - a) - |s - \rho_+| \sup_{\iota \in [\rho_+, \rho_-]} \frac{2}{\mu \iota^2}|(|\iota p''(\iota)| + p'(\iota))| \left(b \sup_{\iota \in [\rho_+, \rho_-]} F'(\iota) + a \right)$$

$$\geqslant 2F'(s)(bF'(s) - a) - C(\rho_+)|s - \rho_+| \tag{3.25}$$

for any $s \in [\rho_+, \rho_-]$, where $C(\rho_+) > 0$ is a constant depending on ρ_+. From (3.11) and (3.13), we get $G'(\rho_+) = 0$. This combined with (3.5), (3.25), and the Taylor expansion implies

$$G(\bar{\rho}) - G(\rho_+) = G(\bar{\rho}) - G(\rho_+) - G'(\rho_+)(\bar{\rho} - \rho_+) = \int_0^1 \int_0^s G''(\tau(\bar{\rho} - \rho_+) + \rho_+)\mathrm{d}\tau\mathrm{d}s|\bar{\rho} - \rho_+|^2$$

$$\geqslant \frac{1}{2}[2F'(s)(bF'(s) - a) - C(\rho_+)|\bar{\rho} - \rho_+|]|\bar{\rho} - \rho_+|^2$$

$$\geqslant C(\rho_+)|\bar{\rho} - \rho_+|^2, \tag{3.26}$$

provided $|\rho_- - \rho_+|$ is suitably small, where $C(\rho_+)$ is a positive constant depending on ρ_+. Combining (3.26) with (3.24), we get

$$(\bar{\rho} - \rho_+)_x = -\frac{\sqrt{G(\bar{\rho}) - G(\rho_+)}}{F'(\bar{\rho})} \leqslant -\frac{\sqrt{G(\bar{\rho}) - G(\rho_+)}}{\sup_{x \in \mathbb{R}_+} F'(\bar{\rho})} \leqslant -\lambda_1(\bar{\rho} - \rho_+)$$

for some constant $\lambda_1 > 0$ depending on ρ_+, provided $|\rho_- - \rho_+|$ is suitably small, where we have used (1.3) and (3.5). Consequently, with (3.23), we have the following decay estimate:

$$|\bar{\rho} - \rho_+| = \bar{\rho} - \rho_+ \leqslant (\rho_- - \rho_+)\mathrm{e}^{-\lambda_1 x} = |\rho_- - \rho_+|\mathrm{e}^{-\lambda_1 x} \leqslant \frac{b}{a}|\phi_- - \phi_+|\mathrm{e}^{-\lambda_1 x}, \quad x \geqslant 0. \tag{3.27}$$

For the case $\phi_- < \phi_+$ (that is, $\rho_- < \rho_+$), it holds that

$$\rho_- < \bar{\rho} < \rho_+ \text{ and } \bar{\rho}_x = \frac{\sqrt{G(\bar{\rho}) - G(\rho_+)}}{F'(\bar{\rho})}. \tag{3.28}$$

Using (3.5), (3.26), and (3.28), we get

$$\bar{\rho}_x \geqslant \frac{\sqrt{C(\rho_+)}|\bar{\rho} - \rho_+|}{\sup_{x \in [\rho_-, \rho_+]} F'(\bar{\rho})} \geqslant -\lambda_2(\bar{\rho} - \rho_+),$$

that is,

$$(\rho_+ - \bar{\rho})_x + \lambda_2(\rho_+ - \bar{\rho}) \leqslant 0$$

for some constant $\lambda_2 > 0$ depending on ρ_+, provided $|\rho_- - \rho_+|$ is suitably small. It thus holds that

$$|\rho_+ - \bar{\rho}| = \rho_+ - \bar{\rho} \leqslant (\rho_+ - \rho_-)\mathrm{e}^{-\lambda_2 x} \leqslant \frac{b}{a}|\phi_- - \phi_+|\mathrm{e}^{-\lambda_2 x}, \quad x \geqslant 0.$$

Finally, by (2.1b), (3.3), (3.6), (3.7), and (3.27), we get (2.4). The proof is complete. $\qquad\square$

4. Global existence and asymptotic stability

In this section, we are devoted to studying the asymptotic stability of the unique stationary solution to (1.2a)–(1.2a) obtained in Section 3. To this end, we first reformulate the problem with the technique of taking anti-derivative for ρ.

4.1. Reformulation of problem

Combining (1.2a)–(1.2c) with (2.1a)–(2.1c), we have

$$\begin{cases} (\rho - \bar{\rho})_t + m_x = 0, & \text{(4.1a)} \\[2mm] m_t + \left(\dfrac{m^2}{\rho}\right)_x + [p(\rho) - p(\bar{\rho})]_x = \mu\rho\phi_x - \mu\bar{\rho}\bar{\phi}_x - \alpha m, & \text{(4.1b)} \\[2mm] (\phi - \bar{\phi})_t = \left(\phi - \bar{\phi}\right)_{xx} + a(\rho - \bar{\rho}) - b(\phi - \bar{\phi}). & \text{(4.1c)} \end{cases}$$

It follows from (4.1a) and $m(0,t) = 0$ that

$$\int_{\mathbb{R}_+} (\rho - \bar{\rho})\mathrm{d}x = \int_{\mathbb{R}_+} (\rho_0 - \bar{\rho})\mathrm{d}x = \varphi_0(0),$$

which, together with the condition $\varphi_0 \in H^3(\mathbb{R}_+) \cap H_0^1(\mathbb{R}_+)$ in Theorem 2.2, gives

$$\int_{\mathbb{R}_+} (\rho - \bar{\rho})\mathrm{d}x = 0.$$

Defining the perturbation function (φ, ψ, Φ)

$$\varphi = -\int_x^\infty (\rho - \bar{\rho})\mathrm{d}y, \quad \psi = m, \quad \Phi = \phi - \bar{\phi}, \tag{4.2}$$

with

$$(\varphi_0, \psi_0, \Phi_0) := \left(-\int_x^\infty (\rho_0 - \bar{\rho})\mathrm{d}y, m_0, \phi_0 - \bar{\phi}\right),$$

we get the reformulated problem:

$$\varphi_t + \psi = 0,$$

$$\psi_t + \left(\frac{\psi^2}{\varphi_x + \bar{\rho}}\right)_x + p(\varphi_x + \bar{\rho})_x - p(\bar{\rho})_x = \mu\left(\rho\phi_x - \bar{\rho}\bar{\phi}_x\right) - \alpha\psi,$$

$$\Phi_t = \Phi_{xx} + a\varphi_x - b\Phi,$$

$$(\varphi, \psi, \Phi)|_{t=0} = (\varphi_0, \psi_0, \Phi_0),$$

$$(\varphi, \psi, \Phi)|_{x=0} = (0, 0, 0),$$

and its linearized problem around $\bar{\rho}$ is

$$\begin{cases} \varphi_{tt} - (p'(\bar{\rho})\varphi_x)_x + \alpha\varphi_t = \mathcal{F} + \mathcal{H}, & \text{(4.3a)} \\[2mm] \Phi_t - \Phi_{xx} + b\Phi = a\varphi_x, & \text{(4.3b)} \\[2mm] (\varphi, \varphi_t, \Phi)|_{t=0} = (\varphi_0, -\psi_0, \Phi_0), & \text{(4.3c)} \\[2mm] (\varphi, \varphi_t, \Phi)|_{x=0} = (0, 0, 0), & \text{(4.3d)} \\[2mm] \psi = -\varphi_t, & \text{(4.3e)} \end{cases}$$

where

$$\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2, \quad \mathcal{H} = \left(\frac{\varphi_t^2}{\varphi_x + \bar{\rho}}\right)_x, \tag{4.4}$$

and

$$\mathcal{F}_1 = [p(\varphi_x + \bar{\rho}) - p(\bar{\rho}) - p'(\bar{\rho})\varphi_x]_x, \quad \mathcal{F}_2 = -\mu[\varphi_x\Phi_x + \varphi_x\bar{\phi}_x + \bar{\rho}\Phi_x]. \tag{4.5}$$

To proceed, we define the solution space of the problem (4.3a)–(4.3d) as follows:

$$X(0, T) = \{(\varphi, \psi, \Phi)| \; \varphi \in C([0, T]; H^3) \cap C^1([0, T]; H^2), \; \psi \in C([0, T]; H^2) \cap C^1([0, T]; H^1),$$

$$\Phi \in C([0, T]; H^4) \cap C^1([0, T]; H^2)\}$$

for any $T \in (0, +\infty)$.

Since we are interested in the case where the solution has no vacuum, naturally we require that $\inf_{x \in \mathbb{R}_+} \rho_0(x) > 0$, namely

$$\inf_{x \in \mathbb{R}_+} \{\varphi_{0x} + \bar{\rho}\} > 0. \tag{4.6}$$

For simplicity, we denote

$$\mathcal{N}_0 := \|\varphi_0\|_3^2 + \|\psi_0\|_2^2 + \|\Phi_0\|_4^2.$$

Then by the standard parabolic theory and fixed point theorem (cf. [**23**]), we have the following local existence result.

PROPOSITION 4.1 (Local existence). *Let the conditions of Theorem* 2.1 *hold. Assume* $\varphi_0 \in H^3 \cap H_0^1(\mathbb{R}_+)$, $\psi_0 \in H^2 \cap H_0^1(\mathbb{R}_+)$, *and* $\Phi_0 \in H^4 \cap H_0^1(\mathbb{R}_+)$ *such that* (4.6) *holds. Then there exists a positive constant* T_0 *depending on* $\mathcal{N}_0$ *such that the initial-boundary value problem* (4.3a)–(4.3d) *admits a unique solution* $(\varphi(x, t), \psi(x, t), \Phi(x, t)) \in X(0, T_0)$ *such that* $\inf_{x \in \mathbb{R}_+} \{\varphi_x + \bar{\rho}\} > 0$ *for an* $0 \leqslant t \leqslant T_0$ *and*

$$\sup_{t \in [0, T_0]} \left(\|\varphi\|_3^2 + \|\psi\|_2^2 + \|\Phi\|_4^2\right) \leqslant 2\mathcal{N}_0.$$

In what follows, we are devoted to proving the following theorem on the global existence and uniqueness of solutions to the problem (4.3a)–(4.3d).

PROPOSITION 4.2. *Let the conditions in Theorem 2.1 hold and assume* $\varphi_0 \in H^3$, $\psi_0 \in H^2$, *and* $\Phi_0 \in H^4$ *satisfying* (4.6). *Then there exists a suitably small constant* $\delta_1 > 0$ *independent of* t *such that if*

$$\|\varphi_0\|_3 + \|\psi_0\|_2 + \|\Phi_0\|_4 + |\phi_- - \phi_+| \leqslant \delta_1,$$

the problem (4.3a)–(4.3d) *admits a unique global solution* $(\varphi(x,t), \psi(x,t), \Phi(x,t)) \in X(0,\infty)$ *such that for any* $t \geqslant 0$ *there holds that*

$$\|\varphi\|_3 + \|\psi\|_2 + \|\Phi\|_4 \leqslant C\delta_1 \tag{4.7}$$

and

$$\int_0^t \left(\|\Phi\|_3^2 + \|(\varphi_x, \varphi_\tau, \psi, \Phi_\tau)\|_2^2 + \|(\varphi_{\tau\tau}, \psi_\tau, \Phi_{\tau\tau})\|_1^2 \right) \mathrm{d}\tau \leqslant C\delta_1^2, \tag{4.8}$$

where C *is a constant independent of* t.

Theorem 2.2 will be proved by Propositions 4.1 and 4.2. Next, we are devoted to proving Proposition 4.2.

4.2. Some preliminaries

The proof of Proposition 4.2 is based on the combination of the local existence result in Proposition 4.1 with the *a priori* estimates given in (4.7) and (4.8). In the sequel, we assume that $(\varphi, \psi, \Phi) \in X(0,T)$ is a solution to the problem (4.3a)–(4.3d) obtained in Proposition 4.1 for some $T > 0$ and derive the *a priori* estimates (4.7) and (4.8) based on the technique of *a priori* assumption. That is we first assume that the solution (φ, ψ, Φ) of (4.3a)–(4.3d) satisfies

$$\sup_{0 \leqslant t < T} \left\{ \|(\varphi, \Phi)(\cdot, t)\|_3^2 + \|\psi(\cdot, t)\|_2^2 \right\} \leqslant \varepsilon^2, \tag{4.9}$$

where $\varepsilon > 0$ is a constant to be determined later, and then derive the *a priori* estimates to obtain the global existence of solutions. Finally we justify that the global solutions obtained satisfy the above *a priori* assumption and thus close our argument.

Using the fact $\varphi_t = -\psi$ from (4.3e) and the Sobolev inequality, we have

$$\sum_{k=0}^2 \|\partial_x^k(\varphi, \Phi)(\cdot, t)\|_{L^\infty(\mathbb{R}_+)} + \sum_{k=0}^1 \|\partial_x^k(\psi, \varphi_t)(\cdot, t)\|_{L^\infty(\mathbb{R}_+)} \leqslant C\varepsilon. \tag{4.10}$$

Denote $\delta := |\phi_- - \phi_+|$ by (2.4), one can find a constant $c_1 > 0$ depending on ρ_+, a, and b such that

$$c_1^{-1} \leqslant \bar{\rho}(x) \leqslant c_1, \tag{4.11}$$

provided δ is suitably small. Combining (4.11) with (4.10), we get

$$\frac{1}{c} \leqslant \rho = \varphi_x + \bar{\rho} \leqslant c, \tag{4.12}$$

for some constant $c > 0$ depending on ρ_+, a and b, provided ε and δ are small enough. The boundary condition (4.3d) together with equation (4.3a) leads to the following boundary conditions on higher order derivatives:

$$(\varphi_t, \varphi_{tt}, \Phi_t, \Phi_{tt})|_{x=0} = 0, \quad ((p'(\bar{\rho})\varphi_x)_x + \mathcal{F})|_{x=0} = 0, \quad ((p'(\bar{\rho})\varphi_x)_x + \mathcal{F})_t\big|_{x=0} = 0. \tag{4.13}$$

Moreover, the following Hardy inequality plays a key role in deriving the *a priori* estimates.

LEMMA 4.1. *Let $k > 0$ be a constant, it holds that*

$$\int_{\mathbb{R}_+} \mathrm{e}^{-kx} f^2 \mathrm{d}x \leqslant C_k \int_{\mathbb{R}_+} f_x^2 \mathrm{d}x \tag{4.14}$$

for any $f \in H_0^1(\mathbb{R}_+)$, where $C_k > 0$ is a constant depending on k but independent of f.

Proof. From [**4**, Lemma 3.4], we get

$$\int_{\mathbb{R}_+} (1+x)^{-2} f^2 \mathrm{d}x \leqslant 4 \int_{\mathbb{R}_+} f_x^2 \mathrm{d}x$$

for any $f \in H_0^1(\mathbb{R}_+)$. This along with the basic fact $\mathrm{e}^{-kx}(1+x)^2 \leqslant C_k$ with some positive constant $C_k > 0$ for any $x \in \mathbb{R}_+$ implies

$$\int_{\mathbb{R}_+} \mathrm{e}^{-kx} f^2 \mathrm{d}x \leqslant C_k \int_{\mathbb{R}_+} (1+x)^{-2} f^2 \mathrm{d}x \leqslant C_k \int_{\mathbb{R}_+} f_x^2 \mathrm{d}x.$$

We thus get (4.14). $\square$

4.3. *Energy estimates*

In this section, we will derive some estimates for the solution (φ, Φ) of (4.3a)–(4.3d) under the *a priori* assumption (4.9) by the method of energy estimates. The estimates for ψ follows from the fact $\psi = -\varphi_t$.

We begin with the lower order estimates.

LEMMA 4.2. *Let the assumptions in Proposition 4.2 hold. If ε and $\delta := |\phi_- - \phi_+|$ are sufficiently small, then the solution (φ, Φ) of (4.3a)–(4.3d) satisfies*

$$\|(\varphi, \Phi)\|_1^2 + \|\varphi_t\|^2 + \int_0^t \|(\varphi_x, \varphi_\tau, \Phi, \Phi_x, \Phi_\tau)\|^2 \mathrm{d}\tau \leqslant C(\|(\varphi_0, \Phi_0)\|_1^2 + \|\psi_0\|^2) \tag{4.15}$$

for any $t \in (0, T)$, where $C > 0$ is a constant independent of T.

Proof. Multiplying (4.3a) by φ and integrating the resulting equation over $\mathbb{R}_+$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}_+} \left(\varphi \varphi_t + \frac{\alpha}{2} \varphi^2 \right) \mathrm{d}x + \int_{\mathbb{R}_+} p'(\bar{\rho}) \varphi_x^2 \mathrm{d}x$$

$$= \|\varphi_t\|^2 + \int_{\mathbb{R}_+} \mathcal{F}_1 \varphi \mathrm{d}x + \int_{\mathbb{R}_+} \mathcal{F}_2 \varphi \mathrm{d}x + \int_{\mathbb{R}_+} \mathcal{H} \varphi \mathrm{d}x. \tag{4.16}$$

By the Taylor expansion, we get

$$p(\varphi_x + \bar{\rho}) - p(\bar{\rho}) - p'(\bar{\rho}) \varphi_x = p''(\bar{\rho} + \vartheta_1 \varphi_x) \varphi_x^2$$

for some $\vartheta_1 \in (0, 1)$. Then it follows from (1.3), (4.5), (4.10), and (4.11) that

$$\int_{\mathbb{R}_+} \mathcal{F}_1 \varphi \mathrm{d}x = -\int_{\mathbb{R}_+} [p(\varphi_x + \bar{\rho}) - p(\bar{\rho}) - p'(\bar{\rho}) \varphi_x] \varphi_x \mathrm{d}x \leqslant C\varepsilon \|\varphi_x\|^2, \tag{4.17}$$

provided ε and δ are suitably small. Integrating by parts, we have

$$\int_{\mathbb{R}_+} \mathcal{F}_2 \varphi \mathrm{d}x = -\mu \int_{\mathbb{R}_+} \varphi_x \Phi_x \varphi \mathrm{d}x - \mu \int_{\mathbb{R}_+} \varphi_x \bar{\phi}_x \varphi \mathrm{d}x - \mu \int_{\mathbb{R}_+} \bar{\rho} \Phi_x \varphi \mathrm{d}x$$

$$= -\mu \int_{\mathbb{R}_+} \varphi_x \Phi_x \varphi \mathrm{d}x - \mu \int_{\mathbb{R}_+} \varphi_x \bar{\phi}_x \varphi \mathrm{d}x + \mu \int_{\mathbb{R}_+} \bar{\rho}_x \Phi \varphi \mathrm{d}x + \mu \int_{\mathbb{R}_+} \bar{\rho} \Phi \varphi_x \mathrm{d}x. \tag{4.18}$$

In view of (4.10) and the Cauchy–Schwarz inequality, we deduce

$$-\mu \int_{\mathbb{R}_+} \varphi_x \Phi_x \varphi \mathrm{d}x \leqslant C\|\varphi\|_{L^\infty} \|\varphi_x\| \|\Phi_x\| \leqslant C\varepsilon \|(\varphi_x, \Phi_x)\|^2. \tag{4.19}$$

By the fact $|(\bar{\rho}_x, \bar{\phi}_x)| \leqslant C\delta e^{-\lambda x}$ from (2.4) and the Hardy inequality (4.14), it holds that

$$- \mu \int_{\mathbb{R}_+} \varphi_x \bar{\phi}_x \varphi \mathrm{d}x + \mu \int_{\mathbb{R}_+} \bar{\rho}_x \Phi \varphi \mathrm{d}x$$

$$\leqslant C\delta \|\varphi_x\| \|e^{-\lambda x}\varphi\| + C\delta \|e^{-\frac{\lambda}{2}x}\varphi\| \|e^{-\frac{\lambda}{2}x}\Phi\|$$

$$\leqslant C\delta \|(\varphi_x, \Phi_x)\|^2, \tag{4.20}$$

where the Cauchy–Schwarz inequality has been used. Inserting (4.19) and (4.20) into (4.18) leads to

$$\int_{\mathbb{R}_+} \mathcal{F}_2 \varphi \mathrm{d}x \leqslant C(\delta + \varepsilon) \|(\varphi_x, \Phi_x)\|^2 + \mu \int_{\mathbb{R}_+} \bar{\rho} \Phi \varphi_x \mathrm{d}x. \tag{4.21}$$

For the last term on the right-hand side of (4.16), from (4.4), (4.10), (4.12), integration by parts and Cauchy–Schwarz inequality, we have

$$\int_{\mathbb{R}_+} \mathcal{H}\varphi \mathrm{d}x = -\int_{\mathbb{R}_+} \frac{\varphi_t^2}{\varphi_x + \bar{\rho}} \varphi_x \mathrm{d}x \leqslant C \int_{\mathbb{R}_+} \varphi_t^2 |\varphi_x| \mathrm{d}x \leqslant C\|\varphi_t\|_{L^\infty} \|\varphi_t\| \|\varphi_x\| \leqslant C\varepsilon \|(\varphi_t, \varphi_x)\|^2. \tag{4.22}$$

Substituting (4.17), (4.21), and (4.22) into (4.16), we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}_+} \left(\varphi\varphi_t + \frac{\alpha}{2}\varphi^2 \right)\mathrm{d}x + \int_{\mathbb{R}_+} p'(\bar{\rho})\varphi_x^2 \mathrm{d}x$$

$$\leqslant C(\delta + \varepsilon) \|(\varphi_x, \Phi_x)\|^2 + C\|\varphi_t\|^2 + \mu \int_{\mathbb{R}_+} \bar{\rho} \Phi \varphi_x \mathrm{d}x. \tag{4.23}$$

Multiplying (4.3b) by $\frac{\mu}{a}\bar{\rho}\Phi$ and integrating the resulting equation over $\mathbb{R}_+$, one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}_+} \frac{\mu}{a}\bar{\rho}\Phi^2 \mathrm{d}x + \frac{\mu}{a} \int_{\mathbb{R}_+} \left(b\bar{\rho}\Phi^2 + \bar{\rho}\Phi_x^2 \right)\mathrm{d}x$$

$$= -\int_{\mathbb{R}_+} \frac{\mu}{a}\bar{\rho}_x \Phi \Phi_x \mathrm{d}x + \mu \int_{\mathbb{R}_+} \bar{\rho}\Phi\varphi_x \mathrm{d}x,$$

where, due to the fact $|\bar{\rho}_x| \leqslant C\delta e^{-\lambda x}$ from (2.4) and the Hardy inequality (4.14), the following inequality holds

$$-\int_{\mathbb{R}_+} \frac{\mu}{a}\bar{\rho}_x \Phi \Phi_x \mathrm{d}x \leqslant C\delta \|\Phi_x\| \|e^{-\lambda x}\Phi\| \leqslant C\delta \|\Phi_x\|^2.$$

Therefore,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}_+} \frac{\mu}{a}\bar{\rho}\Phi^2 \mathrm{d}x + \frac{\mu}{a} \int_{\mathbb{R}_+} \left(b\bar{\rho}\Phi^2 + \bar{\rho}\Phi_x^2 \right)\mathrm{d}x \leqslant \mu \int_{\mathbb{R}_+} \bar{\rho}\Phi\varphi_x \mathrm{d}x + C\delta \|\Phi_x\|^2. \tag{4.24}$$

Combining (4.24) with (4.23), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}_+} \left(\varphi\varphi_t + \frac{\alpha}{2}\varphi^2 + \frac{\mu}{2a}\bar{\rho}\Phi^2 \right)\mathrm{d}x + \int_{\mathbb{R}_+} \left[\frac{\mu}{2a}\bar{\rho}\Phi_x^2 + \left(p'(\bar{\rho})\varphi_x^2 - 2\mu\bar{\rho}\Phi\varphi_x + \frac{\mu b}{a}\bar{\rho}\Phi^2 \right) \right]\mathrm{d}x$$

$$\leqslant C(\delta + \varepsilon)\|\varphi_x\|^2 + C\|\varphi_t\|^2 \tag{4.25}$$

for suitably small ε and δ. By (1.3) and (4.11), we have

$$p'(\bar{\rho})\varphi_x^2 - 2\mu\bar{\rho}\Phi\varphi_x + \frac{\mu b}{a}\bar{\rho}\Phi^2 \geqslant C\big(\varphi_x^2 + \Phi^2\big)$$

for some constant $C > 0$ independent of t. Then, for sufficiently small ε and δ, we have from (4.25) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\Big(\varphi\varphi_t + \frac{\alpha}{2}\varphi^2 + \frac{\mu}{a}\bar{\rho}\Phi^2\Big)\mathrm{d}x + C\|(\varphi_x, \Phi, \Phi_x)\|^2 \leqslant C_1\|\varphi_t\|^2 \qquad (4.26)$$

for some constant $C_1 > 0$, where we have used (4.11). Now let us turn to the estimate for φ_t. Multiplying (4.3a) by φ_t and integrating the resulting equation over $\mathbb{R}_+$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\big[\varphi_t^2 + p'(\bar{\rho})\varphi_x^2\big]\mathrm{d}x + \int_{\mathbb{R}_+}\alpha\varphi_t^2\mathrm{d}x = \int_{\mathbb{R}_+}\mathcal{F}_1\varphi_t\mathrm{d}x + \int_{\mathbb{R}_+}\mathcal{F}_2\varphi_t\mathrm{d}x + \int_{\mathbb{R}_+}\mathcal{H}\varphi_t\mathrm{d}x. \quad (4.27)$$

Next, we estimate the terms on the right-hand side of (4.27). First, it follows from a direct computation that

$$\int_{\mathbb{R}_+}\mathcal{F}_1\varphi_t\mathrm{d}x = -\int_{\mathbb{R}_+}\big[p(\varphi_x + \bar{\rho}) - p(\bar{\rho}) - p'(\bar{\rho})\varphi_x\big]\varphi_{xt}\mathrm{d}x$$

$$= -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\Big(\int_{\bar{\rho}}^{\bar{\rho}+\varphi_x}p(s)\mathrm{d}y - p(\bar{\rho})\varphi_x - \frac{1}{2}p'(\bar{\rho})\varphi_x^2\Big)\mathrm{d}x.$$

Second, due to (4.10), Cauchy–Schwarz inequality, and the fact $|(\bar{\rho}_x, \bar{\phi}_x)| \leqslant \delta\mathrm{e}^{-\lambda x}$ from (2.4), we have

$$\int_{\mathbb{R}_+}\mathcal{F}_2\varphi_t\mathrm{d}x = -\mu\int_{\mathbb{R}_+}\varphi_x\Phi_x\varphi_t\mathrm{d}x - \mu\int_{\mathbb{R}_+}\varphi_x\bar{\phi}_x\varphi_t\mathrm{d}x + \mu\int_{\mathbb{R}_+}\bar{\rho}_x\Phi\varphi_t\mathrm{d}x + \mu\int_{\mathbb{R}_+}\bar{\rho}\Phi\varphi_{xt}\mathrm{d}x,$$

$$\leqslant C\|\varphi_x\|_{L^\infty}\|\Phi_x\|\|\varphi_t\| + \|\bar{\phi}_x\|_{L^\infty}\|\varphi_x\|\|\varphi_t\| + C\delta\|\mathrm{e}^{-\lambda x}\Phi\|\|\varphi_t\|$$

$$+ \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\mu\bar{\rho}\Phi\varphi_x\mathrm{d}x - \mu\int_{\mathbb{R}_+}\bar{\rho}\Phi_t\varphi_x\mathrm{d}x$$

$$\leqslant C(\varepsilon + \delta)\|(\varphi_x, \varphi_t, \Phi_x)\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\mu\bar{\rho}\Phi\varphi_x\mathrm{d}x - \mu\int_{\mathbb{R}_+}\bar{\rho}\Phi_t\varphi_x\mathrm{d}x, \qquad (4.28)$$

where we have used the fact $\|\mathrm{e}^{-\lambda x}\Phi\|^2 \leqslant C\|\Phi_x\|^2$ by the Hardy inequality (4.14). Finally, using (4.10), (4.12), and integration by parts, one has

$$\int_{\mathbb{R}_+}\mathcal{H}\varphi_t\mathrm{d}x = -\int_{\mathbb{R}_+}\frac{\varphi_t^2}{\varphi_x + \bar{\rho}}\varphi_{xt}\mathrm{d}x \leqslant C\|\varphi_{xt}\|_{L^\infty}\|\varphi_t\|^2 \leqslant C\varepsilon\|\varphi_t\|^2. \qquad (4.29)$$

Hence, for suitably small δ and ε, we find from (4.27) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\big[\varphi_t^2 + p'(\bar{\rho})\varphi_x^2\big]\mathrm{d}x + \frac{\alpha}{2}\int_{\mathbb{R}_+}\varphi_t^2\mathrm{d}x$$

$$\leqslant \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\mu\bar{\rho}\Phi\varphi_x\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\Big(\int_{\bar{\rho}}^{\bar{\rho}+\varphi_x}p(s)\mathrm{d}y - p(\bar{\rho})\varphi_x - \frac{1}{2}p'(\bar{\rho})\varphi_x^2\Big)\mathrm{d}x$$

$$- \mu\int_{\mathbb{R}_+}\bar{\rho}\Phi_t\varphi_x\mathrm{d}x + C(\delta + \varepsilon)\|(\varphi_x, \Phi_x)\|^2, \qquad (4.30)$$

where the terms $\|\varphi_t\|^2$ on the right-hand side of (4.28) and (4.29) have been absorbed. To proceed, we multiply (4.3b) by $\frac{\mu}{a}\bar{\rho}\Phi_t$ and integrate the resulting equation over $\mathbb{R}_+$ to get

$$\frac{\mu}{2a}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\left(b\bar{\rho}\Phi^2 + \bar{\rho}\Phi_x^2\right)\mathrm{d}x + \frac{\mu}{a}\int_{\mathbb{R}_+}\bar{\rho}\Phi_t^2\mathrm{d}x$$

$$= \int_{\mathbb{R}_+}\frac{\mu}{a}\bar{\rho}_x\Phi_t\Phi_x\mathrm{d}x + \mu\int_{\mathbb{R}_+}\bar{\rho}\Phi_t\varphi_x\mathrm{d}x$$

$$\leqslant \mu\int_{\mathbb{R}_+}\bar{\rho}\Phi_t\varphi_x\mathrm{d}x + \delta\|(\Phi_t,\Phi_x)\|^2, \tag{4.31}$$

where (2.4) and the Cauchy–Schwarz inequality have been used. Combining (4.31) with (4.30) gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\left(\frac{b\mu}{a}\bar{\rho}\Phi^2 - 2\mu\bar{\rho}\Phi\varphi_x + p'(\bar{\rho})\varphi_x^2 + \frac{\mu}{a}\bar{\rho}\Phi_x^2 + \varphi_t^2\right)\mathrm{d}x + \frac{\alpha}{2}\|\varphi_t\|^2 + \frac{\mu}{a}\int_{\mathbb{R}_+}\bar{\rho}\Phi_t^2\mathrm{d}x$$

$$\leqslant -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\left(\int_{\bar{\rho}}^{\bar{\rho}+\varphi_x}p(s)\mathrm{d}y - p(\bar{\rho})\varphi_x - \frac{1}{2}p'(\bar{\rho})\varphi_x^2\right)\mathrm{d}x + C(\delta+\varepsilon)\|(\varphi_x,\Phi_x,\Phi_t)\|^2, \tag{4.32}$$

where

$$\frac{b\mu}{a}\bar{\rho}\Phi^2 - 2\mu\bar{\rho}\Phi\varphi_x + p'(\bar{\rho})\varphi_x^2 \sim \Phi^2 + \varphi_x^2,$$

due to (1.3) and (4.11). Given any constant $K_0 > 0$, adding (4.26) with (4.32) multiplied by K_0 leads to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\left[\alpha\varphi^2 + 2\varphi\varphi_t + K_0\varphi_t^2 + K_0\left(\frac{b\mu}{a}\bar{\rho}\Phi^2 - 2\mu\bar{\rho}\Phi\varphi_x + p'(\bar{\rho})\varphi_x^2 + \frac{\mu}{a}\bar{\rho}\Phi_x^2\right)\right]\mathrm{d}x$$

$$+ C\|(\varphi_x,\Phi,\Phi_x)\|^2 + \left(\frac{\alpha K_0}{2} - C_1\right)\|\varphi_t\|^2 + \frac{\mu K_0}{a}\int_{\mathbb{R}_+}\bar{\rho}\Phi_t^2\mathrm{d}x$$

$$\leqslant -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}K_0\left(\int_{\bar{\rho}}^{\bar{\rho}+\varphi_x}p(s)\mathrm{d}y - p(\bar{\rho})\varphi_x - \frac{1}{2}p'(\bar{\rho})\varphi_x^2\right)\mathrm{d}x + CK_0(\delta+\varepsilon)\|(\varphi_x,\Phi_x,\Phi_t)\|^2, \tag{4.33}$$

where C_1 is as in (4.26). From (4.11), it holds that

$$\frac{\mu K_0}{a}\int_{\mathbb{R}_+}\bar{\rho}\Phi_t^2\mathrm{d}x \leqslant CK_0\|\Phi_t\|^2$$

for some constant $C > 0$ which depends on ρ_+, μ, and a. Taking K_0 large enough such that $\frac{\alpha K_0}{2} > C_1$ and

$$\frac{\alpha}{2}\varphi^2 + \varphi\varphi_t + \frac{K_0}{2}\varphi_t^2 \geqslant C\left(\varphi^2 + \varphi_t^2\right)$$

for some constant $C > 0$, then for suitably small δ and ε, we have from (4.33) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\left[\alpha\varphi^2 + 2\varphi\varphi_t + K_0\varphi_t^2 + K_0\left(\frac{b\mu}{a}\bar{\rho}\Phi^2 - 2\mu\bar{\rho}\Phi\varphi_x + p'(\bar{\rho})\varphi_x^2 + \frac{\mu}{a}\bar{\rho}\Phi_x^2\right)\right]\mathrm{d}x$$

$$+ \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\left(\int_{\bar{\rho}}^{\bar{\rho}+\varphi_x}p(s)\mathrm{d}y - p(\bar{\rho})\varphi_x - \frac{1}{2}p'(\bar{\rho})\varphi_x^2\right)\mathrm{d}x + C\left(\|\Phi\|_1^2 + \|(\varphi_x,\varphi_t,\Phi_t)\|^2\right) \leqslant 0, \tag{4.34}$$

where

$$\alpha\varphi^2 + 2\varphi\varphi_t + K_0\left(\varphi_t^2 + \frac{b\mu}{a}\bar{\rho}\Phi^2 - 2\mu\bar{\rho}\Phi\varphi_x + p'(\bar{\rho})\varphi_x^2 + \frac{\mu}{a}\bar{\rho}\Phi_x^2\right) \sim \varphi^2 + \varphi_x^2 + \varphi_t^2 + \Phi^2 + \Phi_x^2.$$

Applying the Taylor expansion to the function $h(s) := \int_{\bar{\rho}(x)}^s p(s)\mathrm{d}s$ along with (4.10) leads to

$$\left|\int_{\bar{\rho}}^{\bar{\rho}+\varphi_x} p(s)\mathrm{d}y - p(\bar{\rho})\varphi_x - \frac{1}{2}p'(\bar{\rho})\varphi_x^2\right| = \frac{1}{6}\left|p''(\bar{\rho} + \vartheta_2\varphi_x)\varphi_x^3\right| \leqslant C\varepsilon\varphi_x^2 \tag{4.35}$$

for some constant $\vartheta_2 \in (0, 1)$. With (4.35), integrating (4.34) with respect to t, by taking δ and ε suitably small, we get (4.15) and hence complete the proof. □

LEMMA 4.3. *Let the assumptions in Proposition* 4.2 *hold. If ε and δ are sufficiently small, then for any $t \in (0, T)$, the solution (φ, Φ) of (4.3a)–(4.3d) satisfies*

$$\|(\varphi_x, \Phi_x)\|_1^2 + \|\varphi_{xt}\|^2 + \int_0^t \|(\varphi_{xx}, \Phi_x, \Phi_{xx}, \varphi_{x\tau}, \Phi_{x\tau})\|^2\mathrm{d}\tau$$

$$\leqslant C(\|(\varphi_{0x}, \Phi_{0x})\|_1^2 + \|\psi_{0x}\|^2) + C\int_0^t \left(\|\varphi_x\|^2 + \|\varphi_x\|\|\Phi_{xx\tau}\|\right)\mathrm{d}\tau. \tag{4.36}$$

Proof. Differentiating (4.3a)–(4.3b) with respect to x, we get

$$\begin{cases} \varphi_{xtt} + \alpha\varphi_{xt} - (p'(\bar{\rho})\varphi_x)_{xx} = \mathcal{F}_x + \mathcal{H}_x, & \text{(4.37a)} \\ \Phi_{xt} = \Phi_{xxx} + a\varphi_{xx} - b\Phi_x. & \text{(4.37b)} \end{cases}$$

Multiplying (4.37a) by φ_x, and integrating it over $\mathbb{R}_+$, we get, due to (4.13), that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} \left(\frac{\alpha}{2}\varphi_x^2 + \varphi_x\varphi_{xt}\right)\mathrm{d}x + \int_{\mathbb{R}_+} p'(\bar{\rho})\varphi_{xx}^2\mathrm{d}x - \int_{\mathbb{R}_+} \varphi_{xt}^2\mathrm{d}x$$

$$= -\int_{\mathbb{R}_+} p''(\bar{\rho})\bar{\rho}_x\varphi_x\varphi_{xx}\mathrm{d}x - \int_{\mathbb{R}_+} \mathcal{F}_1\varphi_{xx}\mathrm{d}x - \int_{\mathbb{R}_+} \mathcal{F}_2\varphi_{xx}\mathrm{d}x + \int_{\mathbb{R}_+} \mathcal{H}_x\varphi_x\mathrm{d}x. \tag{4.38}$$

Recalling the definitions of $\mathcal{F}_1$ and $\mathcal{F}_2$ in (4.5), using (1.3), (4.11), (4.10), the fact $|(\bar{\rho}_x, \bar{\phi}_x)| \leqslant C\delta e^{-\lambda x}$ from (2.4), and Cauchy–Schwarz inequality, we have

$$-\int_{\mathbb{R}_+} p''(\bar{\rho})\bar{\rho}_x\varphi_x\varphi_{xx}\mathrm{d}x - \int_{\mathbb{R}_+} \mathcal{F}_2\varphi_{xx}\mathrm{d}x$$

$$= -\int_{\mathbb{R}_+} \left(p''(\bar{\rho})\bar{\rho}_x - \mu\bar{\phi}_x\right)\varphi_x\varphi_{xx}\mathrm{d}x + \mu\int_{\mathbb{R}_+} \varphi_x\Phi_x\varphi_{xx}\mathrm{d}x + \mu\int_{\mathbb{R}_+} \bar{\rho}\Phi_x\varphi_{xx}\mathrm{d}x$$

$$\leqslant C\delta\|\varphi_x\|\|\varphi_{xx}\| + C\|\varphi_x\|\|\Phi_x\|_{L^\infty}\|\varphi_{xx}\| + \mu\int_{\mathbb{R}_+} \bar{\rho}\Phi_x\varphi_{xx}\mathrm{d}x$$

$$\leqslant C(\delta + \varepsilon)\|(\varphi_x, \varphi_{xx})\|^2 + \mu\int_{\mathbb{R}_+} \bar{\rho}\Phi_x\varphi_{xx}\mathrm{d}x \tag{4.39}$$

and

$$\int_{\mathbb{R}_+} \mathcal{F}_1\varphi_{xx}\mathrm{d}x = -\int_{\mathbb{R}_+} \int_0^1 \int_0^s p'''(\bar{\rho} + \tau\varphi_x)\mathrm{d}\tau\mathrm{d}s\varphi_x^2\bar{\rho}_x\varphi_{xx}\mathrm{d}x$$

$$- \int_{\mathbb{R}_+} \int_0^1 p''(\bar{\rho} + s\varphi_x)\mathrm{d}s\varphi_x\varphi_{xx}^2\mathrm{d}x$$

$$\leqslant C\|\varphi_x\|_{L^\infty}\|\varphi_x\|\|\varphi_{xx}\| + C\|\varphi_x\|_{L^\infty}\|\varphi_{xx}\|^2$$

$$\leqslant C\varepsilon\|(\varphi_x, \varphi_{xx})\|^2,$$

where we have used the following identity

$$\mathcal{F}_1 = [p'(\varphi_x + \bar{\rho}) - p'(\bar{\rho}) - p''(\bar{\rho})\varphi_x]\bar{\rho}_x + [p'(\varphi_x + \bar{\rho}) - p'(\bar{\rho})]\varphi_{xx}$$

$$= \int_0^1 \int_0^s p'''(\bar{\rho} + \tau\varphi_x)\mathrm{d}\tau\mathrm{d}s\varphi_x^2\bar{\rho}_x + \int_0^1 p''(\bar{\rho} + s\varphi_x)\mathrm{d}s\varphi_x\varphi_{xx} \qquad (4.40)$$

due to the Taylor expansion. For the last term on the right hand of (4.38), due to (2.4), (4.4), (4.10), (4.12), Cauchy–Schwarz inequality, and the Hardy inequality (4.14), it holds for suitably small ε and δ that

$$\int_{\mathbb{R}_+} \mathcal{H}_x\varphi_x\mathrm{d}x = -\int_{\mathbb{R}_+} \left(\frac{\varphi_t^2}{\varphi_x + \bar{\rho}}\right)_x \varphi_{xx}\mathrm{d}x$$

$$= \int_{\mathbb{R}_+} \left(\frac{2\varphi_t\varphi_{tx}}{\varphi_x + \bar{\rho}} - \frac{\varphi_t^2(\varphi_{xx} + \bar{\rho}_x)}{(\varphi_x + \bar{\rho})^2}\right)\varphi_{xx}\mathrm{d}x$$

$$\leqslant C\int_{\mathbb{R}_+} |\varphi_t||\varphi_{tx}||\varphi_{xx}|\mathrm{d}x + C\int_{\mathbb{R}_+} \varphi_t^2(|\varphi_{xx}| + |\bar{\rho}_x|)|\varphi_{xx}|\mathrm{d}x$$

$$\leqslant C\|\varphi_t\|_{L^\infty}\|\varphi_{tx}\|\|\varphi_{xx}\| + C\|\varphi_t\|_{L^\infty}^2\|\varphi_{xx}\|^2 + C\delta\|\varphi_t\|_{L^\infty}\|\mathrm{e}^{-\lambda x}\varphi_t\|\|\varphi_{xx}\|$$

$$\leqslant C\varepsilon\|\varphi_{tx}\|\|\varphi_{xx}\| + C\varepsilon^2\|\varphi_{xx}\|^2 + C\varepsilon\delta\|\varphi_{tx}\|\|\varphi_{xx}\|$$

$$\leqslant C(\varepsilon + \delta)\|(\varphi_{tx}, \varphi_{xx})\|^2. \qquad (4.41)$$

We thus conclude from (4.38)–(4.41) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} \left(\frac{\alpha}{2}\varphi_x^2 + \varphi_x\varphi_{xt}\right)\mathrm{d}x + \int_{\mathbb{R}_+} p'(\bar{\rho})\varphi_{xx}^2\mathrm{d}x$$

$$\leqslant C(\delta + \varepsilon)\|(\varphi_x, \varphi_{xt}, \varphi_{xx})\|^2 + \mu\int_{\mathbb{R}_+} \bar{\rho}\Phi_x\varphi_{xx}\mathrm{d}x + \|\varphi_{xt}\|^2. \qquad (4.42)$$

Multiplying (4.37b) by $\frac{\mu}{a}\bar{\rho}\Phi_x$, and integrating it to get

$$\frac{\mu}{2a}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} \bar{\rho}\Phi_x^2\mathrm{d}x + \frac{\mu}{a}\int_{\mathbb{R}_+} \bar{\rho}\Phi_{xx}^2\mathrm{d}x + \frac{b\mu}{a}\int_{\mathbb{R}_+} \bar{\rho}\Phi_x^2\mathrm{d}x$$

$$= -\frac{\mu}{a}\bar{\rho}(0)\Phi_{xx}\Phi_x\Big|_{x=0} - \int_{\mathbb{R}_+} \bar{\rho}_x\Phi_{xx}\Phi_x\mathrm{d}x + \mu\int_{\mathbb{R}_+} \bar{\rho}\varphi_{xx}\Phi_x\mathrm{d}x$$

$$\leqslant C\delta\|(\Phi_x, \Phi_{xx})\|^2 - \frac{\mu}{a}\rho_-\Phi_{xx}\Phi_x\Big|_{x=0} + \mu\int_{\mathbb{R}_+} \bar{\rho}\Phi_x\varphi_{xx}\mathrm{d}x, \qquad (4.43)$$

where (2.4), (4.11), and Cauchy–Schwarz inequality have been used. Since $\Phi = 0$ at $x = 0$ and hence $\Phi_t = 0$ at $x = 0$, recalling (4.3b), we have

$$\Phi_{xx} = -a\varphi_x \text{ at } x = 0. \qquad (4.44)$$

This along with the Sobolev inequality $\|f\|_{L^\infty} \leqslant C\|f\|^{\frac{1}{2}}\|f_x\|^{\frac{1}{2}}$ and Young's inequality implies

$$-\frac{\mu}{a}\bar{\rho}(0)\Phi_{xx}\Phi_x\Big|_{x=0} = \mu\bar{\rho}(0)\varphi_x\Phi_x|_{x=0} \leqslant C\|\varphi_x\|_{L^\infty}\|\Phi_x\|_{L^\infty}$$

$$\leqslant C\|\varphi_x\|^{\frac{1}{2}}\|\varphi_{xx}\|^{\frac{1}{2}}\|\Phi_x\|^{\frac{1}{2}}\|\Phi_{xx}\|^{\frac{1}{2}}$$

$$\leqslant C\|\varphi_x\|^{\frac{1}{2}}\|\varphi_{xx}\|^{\frac{1}{2}}(\|\Phi_x\| + \|\Phi_{xx}\|)$$

$$\leqslant \eta\|(\Phi_x, \Phi_{xx}, \varphi_{xx})\|^2 + C_\eta\|\varphi_x\|^2 \tag{4.45}$$

for any $\eta > 0$. Substituting (4.45) into (4.43), we get

$$\frac{\mu}{2a}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\bar{\rho}\Phi_x^2\mathrm{d}x + \frac{\mu}{a}\int_{\mathbb{R}_+}\bar{\rho}\Phi_{xx}^2\mathrm{d}x + \frac{b\mu}{a}\int_{\mathbb{R}_+}\bar{\rho}\Phi_x^2\mathrm{d}x$$

$$\leqslant C(\eta + \delta)\|(\Phi_x, \Phi_{xx})\|^2 + \eta\|\varphi_{xx}\|^2 + C_\eta\|\varphi_x\|^2 + \mu\int_{\mathbb{R}_+}\bar{\rho}\varphi_{xx}\Phi_x\mathrm{d}x. \tag{4.46}$$

Combining (4.46) with (4.42), we get after taking δ, ε, and η suitably small that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\left(\frac{\alpha}{2}\varphi_x^2 + \varphi_x\varphi_{xt} + \frac{\mu}{2a}\bar{\rho}\Phi_x^2\right)\mathrm{d}x + C\|(\varphi_{xx}, \Phi_x, \Phi_{xx})\|^2$$

$$\leqslant [1 + C(\delta + \varepsilon)]\|\varphi_{xt}\|^2 + C\|\varphi_x\|^2, \tag{4.47}$$

where we have used (4.11) and the following inequality

$$p'(\bar{\rho})\varphi_{xx}^2 - 2\mu\bar{\rho}\varphi_{xx}\Phi_x + \frac{b\mu}{a}\bar{\rho}\Phi_x^2 \geqslant C\left(\varphi_{xx}^2 + \Phi_x^2\right) \tag{4.48}$$

due to (1.3). Next, we integrate (4.37a) multiplied by φ_{xt} over $\mathbb{R}_+$ to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}(\varphi_{xt}^2 + p'(\bar{\rho})\varphi_{xx}^2)\mathrm{d}x + \alpha\|\varphi_{xt}\|^2$$

$$= -\int_{\mathbb{R}_+}p''(\bar{\rho})\bar{\rho}_x\varphi_x\varphi_{xxt}\mathrm{d}x - \int_{\mathbb{R}_+}\mathcal{F}_1\varphi_{xxt}\mathrm{d}x - \int_{\mathbb{R}_+}\mathcal{F}_2\varphi_{xxt}\mathrm{d}x + \int_{\mathbb{R}_+}\mathcal{H}_x\varphi_{xt}\mathrm{d}x. \tag{4.49}$$

A direct computation along with (2.4) and Cauchy–Schwarz inequality gives

$$-\int_{\mathbb{R}_+}p''(\bar{\rho})\bar{\rho}_x\varphi_x\varphi_{xxt}\mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}p''(\bar{\rho})\bar{\rho}_x\varphi_x\varphi_{xx}\mathrm{d}x + \int_{\mathbb{R}_+}p''(\bar{\rho})\bar{\rho}_x\varphi_{xt}\varphi_{xx}\mathrm{d}x$$

$$\leqslant -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}p''(\bar{\rho})\bar{\rho}_x\varphi_x\varphi_{xx}\mathrm{d}x + C\delta\|(\varphi_{xt}, \varphi_{xx})\|^2. \tag{4.50}$$

Recalling (4.40), we arrive at

$$-\int_{\mathbb{R}_+}\mathcal{F}_1\varphi_{xxt}\mathrm{d}x = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}[p'(\varphi_x + \bar{\rho}) - p'(\bar{\rho})]\varphi_{xx}^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}_+}p''(\varphi_x + \bar{\rho})\varphi_{xt}\varphi_{xx}^2\mathrm{d}x$$

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}[p'(\varphi_x + \bar{\rho}) - p'(\bar{\rho}) - p''(\bar{\rho})\varphi_x]\bar{\rho}_x\varphi_{xx}\mathrm{d}x$$

$$+\int_{\mathbb{R}_+}[p''(\varphi_x + \bar{\rho}) - p''(\bar{\rho})]\varphi_{xt}\varphi_{xx}\bar{\rho}_x\mathrm{d}x. \tag{4.51}$$

From (1.3), (2.4), (4.10), and (4.12), it holds that

$$\frac{1}{2}\int_{\mathbb{R}_+}p''(\varphi_x + \bar{\rho})\varphi_{xt}\varphi_{xx}^2\mathrm{d}x + \int_{\mathbb{R}_+}[p''(\varphi_x + \bar{\rho}) - p''(\bar{\rho})]\varphi_{xt}\varphi_{xx}\bar{\rho}_x\mathrm{d}x$$

$$\leqslant C\|\varphi_{xx}\|_{L^\infty}\|\varphi_{tx}\|\|\varphi_{xx}\| + C\|\bar{\rho}_x\|_{L^\infty}\|\varphi_{xt}\|\|\varphi_{xx}\|$$

$$\leqslant C(\delta + \varepsilon)\|(\varphi_{xx}, \varphi_{xt})\|^2.$$

Therefore, we have

$$-\int_{\mathbb{R}_+} \mathcal{F}_1 \varphi_{xxt} \mathrm{d}x \leqslant C(\delta+\varepsilon)\|(\varphi_{xx}, \varphi_{xt})\|^2 - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}[p'(\varphi_x+\bar\rho)-xp'(\bar\rho)]\varphi_{xx}^2\mathrm{d}x$$

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}[p'(\varphi_x+\bar\rho)-p'(\bar\rho)-p''(\bar\rho)\varphi_x]\bar\rho_x\varphi_{xx}\mathrm{d}x. \tag{4.52}$$

Similar to (4.51)–(4.52), the third term on the right-hand side of (4.49) can be estimated as follows:

$$-\int_{\mathbb{R}_+}\mathcal{F}_2\varphi_{xxt}\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\mu(\varphi_x\Phi_x+\varphi_x\bar\phi_x+\bar\rho\Phi_x)\varphi_{xx}\mathrm{d}x - \mu\int_{\mathbb{R}_+}\bar\rho\Phi_{xt}\varphi_{xx}\mathrm{d}x$$

$$-\mu\int_{\mathbb{R}_+}\left(\varphi_{xt}\Phi_x+\varphi_x\Phi_{xt}+\varphi_{xt}\bar\phi_x\right)\varphi_{xx}\mathrm{d}x$$

$$\leqslant \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\mu(\varphi_x\Phi_x+\varphi_x\bar\phi_x+\bar\rho\Phi_x)\varphi_{xx}\mathrm{d}x - \mu\int_{\mathbb{R}_+}\bar\rho\Phi_{xt}\varphi_{xx}\mathrm{d}x$$

$$+C\|\Phi_x\|_{L^\infty}\|\varphi_{xt}\|\|\varphi_{xx}\| + C\|\varphi_x\|_{L^\infty}\|\Phi_{xt}\|\|\varphi_{xx}\| + C\|\bar\phi_x\|_{L^\infty}\|\varphi_{xt}\|\|\varphi_{xx}\|$$

$$\leqslant \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\mu(\varphi_x\Phi_x+\varphi_x\bar\phi_x+\bar\rho\Phi_x)\varphi_{xx}\mathrm{d}x - \mu\int_{\mathbb{R}_+}\bar\rho\Phi_{xt}\varphi_{xx}\mathrm{d}x$$

$$+C(\varepsilon+\delta)\|(\varphi_{xx}, \varphi_{xt}, \Phi_{xt})\|^2. \tag{4.53}$$

Noticing

$$\mathcal{H} = \left(\frac{\varphi_t^2}{\varphi_x+\bar\rho}\right)_x = \frac{2\varphi_t\varphi_{tx}}{\varphi_x+\bar\rho} - \frac{\varphi_t^2(\varphi_{xx}+\bar\rho_x)}{(\varphi_x+\bar\rho)^2},$$

we get, due to integration by parts and the boundary condition $\varphi_t = 0$ at $x = 0$, that

$$\int_{\mathbb{R}_+}\mathcal{H}_x\varphi_{xt}\mathrm{d}x = -\int_{\mathbb{R}_+}\varphi_{xt}\left(\frac{\varphi_t^2(\varphi_{xx}+\bar\rho_x)}{(\varphi_x+\bar\rho)^2}\right)_x\mathrm{d}x + 2\int_{\mathbb{R}_+}\varphi_{xt}\left(\frac{\varphi_t\varphi_{tx}}{\varphi_x+\bar\rho}\right)_x\mathrm{d}x$$

$$=\int_{\mathbb{R}_+}\left[\left(\frac{\varphi_{xx}^2}{2}\right)_t+\bar\rho_x\varphi_{xxt}\right]\frac{\varphi_t^2}{(\varphi_x+\bar\rho)^2}\mathrm{d}x + \int_{\mathbb{R}_+}\varphi_{xt}^2\left(\frac{\varphi_t}{\varphi_x+\bar\rho}\right)_x\mathrm{d}x$$

$$=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\frac{\varphi_t^2}{(\varphi_x+\bar\rho)^2}\left(\frac{1}{2}\varphi_{xx}^2+\bar\rho_x\varphi_{xx}\right)\mathrm{d}x + \int_{\mathbb{R}_+}\varphi_{xt}^2\left(\frac{\varphi_t}{\varphi_x+\bar\rho}\right)_x\mathrm{d}x$$

$$-\int_{\mathbb{R}_+}\left(\frac{1}{2}\varphi_{xx}^2+\bar\rho_x\varphi_{xx}\right)\left(\frac{\varphi_t^2}{(\varphi_x+\bar\rho)^2}\right)_t\mathrm{d}x$$

$$=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\frac{\varphi_t^2}{(\varphi_x+\bar\rho)^2}\left(\frac{1}{2}\varphi_{xx}^2+\bar\rho_x\varphi_{xx}\right)\mathrm{d}x + \mathcal{D}_1 + \mathcal{D}_2. \tag{4.54}$$

Next, we estimate $\mathcal{D}_1$ and $\mathcal{D}_2$. First, we utilize (2.4), (4.10), and (4.12) to get

$$\mathcal{D}_1 = \int_{\mathbb{R}_+}\varphi_{xt}^2\left(\frac{\varphi_{xt}}{\varphi_x+\bar\rho} - \frac{\varphi_t(\varphi_{xx}+\bar\rho_x)}{(\varphi_x+\bar\rho)^2}\right)\mathrm{d}x$$

$$\leqslant C\|\varphi_{xt}\|_{L^\infty}\|\varphi_{xt}\|^2 + C\|\varphi_t\|_{L^\infty}(\|\varphi_{xx}\|_{L^\infty}+\|\bar\rho_x\|_{L^\infty})\|\varphi_{xt}\|^2$$

$$\leqslant C(\varepsilon+\delta)\|\varphi_{xt}\|^2, \tag{4.55}$$

provided ε and δ are suitably small. For $\mathcal{D}_2$, by using (2.4), (4.10), (4.12), Cauchy–Schwarz inequality, and the Hardy inequality (4.14), we obtain

$$\mathcal{D}_2 = -\int_{\mathbb{R}_+} \left(\frac{1}{2}\varphi_{xx}^2 + \bar{\rho}_x\varphi_{xx}\right)\left(\frac{2\varphi_t\varphi_{tt}}{(\varphi_x + \bar{\rho})^2} - 2\frac{\varphi_t^2\varphi_{xt}}{(\varphi_x + \bar{\rho})^3}\right)\mathrm{d}x$$

$$\leqslant C\|\varphi_{xx}\|^2\left(\|\varphi_t\|_{L^\infty}\|\varphi_{tt}\|_{L^\infty} + \|\varphi_t\|_{L^\infty}^2\|\varphi_{xt}\|_{L^\infty}\right)$$

$$+ C\delta\|\varphi_{xx}\|\|\mathrm{e}^{-\lambda x}\varphi_t\|_{L^2}\|\varphi_{tt}\|_{L^\infty} + C\|\bar{\rho}\|_{L^\infty}\|\varphi_t\|_{L^\infty}^2\|\varphi_{xt}\|$$

$$\leqslant C(\delta + \varepsilon)\|(\varphi_{xt}, \varphi_{xx})\|^2, \tag{4.56}$$

where we have used the following inequality

$$\|\varphi_{tt}\|_{L^\infty} \leqslant C\|(p'(\bar{\rho})\varphi_x)_x\|_{L^\infty} + C\|\varphi_t\|_{L^\infty} + \|\mathcal{F}_1\|_{L^\infty} + \|\mathcal{F}_2\|_{L^\infty} + C\|\mathcal{H}\|_{L^\infty} \leqslant C\varepsilon, \tag{4.57}$$

due to (2.4), (4.3a), (4.4), (4.5), (4.10), and (4.12). With (4.55) and (4.56), we update (4.54) as

$$\int_{\mathbb{R}_+} \mathcal{H}_x\varphi_{xt}\mathrm{d}x \leqslant \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} \frac{\varphi_t^2}{(\varphi_x + \bar{\rho})^2}\left(\frac{1}{2}\varphi_{xx}^2 + \bar{\rho}_x\varphi_{xx}\right)\mathrm{d}x + C(\delta + \varepsilon)\|(\varphi_{xt}, \varphi_{xx})\|^2. \tag{4.58}$$

Substituting (4.50), (4.52), (4.53), and (4.58) into (4.49), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} (\varphi_{xt}^2 + p'(\bar{\rho})\varphi_{xx}^2 - 2\mu\bar{\rho}\Phi_x\varphi_{xx})\mathrm{d}x + \alpha\|\varphi_{xt}\|^2$$

$$\leqslant -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} [p'(\varphi_x + \bar{\rho}) - p'(\bar{\rho})]\varphi_{xx}^2\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} [p'(\varphi_x + \bar{\rho}) - p'(\bar{\rho}) - p''(\bar{\rho})\varphi_x]\bar{\rho}_x\varphi_{xx}\mathrm{d}x$$

$$+ \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} \mu(\varphi_x\Phi_x + \varphi_x\bar{\phi}_x)\varphi_{xx}\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} \frac{\varphi_t^2}{(\varphi_x + \bar{\rho})^2}\left(\frac{1}{2}\varphi_{xx}^2 + \bar{\rho}_x\varphi_{xx}\right)\mathrm{d}x$$

$$+ C(\delta + \varepsilon)\|(\varphi_{xt}, \varphi_{xx}, \Phi_{xt})\|^2 - \mu\int_{\mathbb{R}_+} \bar{\rho}\Phi_{xt}\varphi_{xx}\mathrm{d}x. \tag{4.59}$$

Multiplying (4.37b) by $\frac{\mu}{a}\bar{\rho}\Phi_{xt}$, and integrating it to get

$$\frac{\mu b}{2a}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} \bar{\rho}\Phi_x^2\mathrm{d}x + \frac{\mu}{2a}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} \bar{\rho}\Phi_{xx}^2\mathrm{d}x + \frac{\mu}{a}\int_{\mathbb{R}_+} \bar{\rho}\Phi_{xt}^2\mathrm{d}x$$

$$= -\frac{\mu}{a}\bar{\rho}\Phi_{xx}\Phi_{xt}\Big|_{x=0} + \mu\int_{\mathbb{R}_+} \bar{\rho}\varphi_{xx}\Phi_{xt}\mathrm{d}x - \int_{\mathbb{R}_+} \Phi_{xx}\Phi_{xt}\bar{\rho}_x\mathrm{d}x, \tag{4.60}$$

where, in view of (2.4), (4.44), the Sobolev inequality $\|f\|_{L^\infty} \leqslant C\|f\|^{\frac{1}{2}}\|f_x\|^{\frac{1}{2}}$, and Cauchy–Schwarz inequality, the following inequalities hold:

$$-\int_{\mathbb{R}_+} \Phi_{xx}\Phi_{xt}\bar{\rho}_x\mathrm{d}x \leqslant C\|\bar{\rho}_x\|_{L^\infty}\|\Phi_{xx}\|\|\Phi_{xt}\| \leqslant C\delta\|(\Phi_{xt}, \Phi_{xx})\|^2,$$

$$-\frac{\mu}{a}\bar{\rho}\Phi_{xx}\Phi_{xt}\Big|_{x=0} \leqslant C\|\varphi_x\|_{L^\infty}\|\Phi_{xt}\|_{L^\infty}$$

$$\leqslant C\|\varphi_x\|^{\frac{1}{2}}\|\varphi_{xx}\|^{\frac{1}{2}}\|\Phi_{xt}\|^{\frac{1}{2}}\|\Phi_{xxt}\|^{\frac{1}{2}}$$

$$\leqslant C\|\varphi_x\|^{\frac{1}{2}}\|\Phi_{xxt}\|^{\frac{1}{2}}(\|\Phi_{xt}\| + \|\varphi_{xx}\|)$$

$$\leqslant \eta\|(\Phi_{xt}, \varphi_{xx})\|^2 + C_\eta\|\varphi_x\|\|\Phi_{xxt}\|$$

for any $\eta > 0$. Then, combining (4.60) with (4.59) gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\left(\varphi_{xt}^2 + p'(\bar{\rho})\varphi_{xx}^2 - 2\mu\bar{\rho}\Phi_x\varphi_{xx} + \frac{\mu b}{a}\bar{\rho}\Phi_x^2 + \frac{\mu}{a}\bar{\rho}\Phi_{xx}^2\right)\mathrm{d}x + C\|\Phi_{xt}\|^2 + \alpha\|\varphi_{xt}\|^2$$

$$\leqslant -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}[p'(\varphi_x + \bar{\rho}) - p'(\bar{\rho})]\varphi_{xx}^2\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}[p'(\varphi_x + \bar{\rho}) - p'(\bar{\rho}) - p''(\bar{\rho})\varphi_x]\bar{\rho}_x\varphi_{xx}\mathrm{d}x$$

$$+ \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\mu(\varphi_x\Phi_x + \varphi_x\bar{\phi}_x)\varphi_{xx}\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\frac{\varphi_t^2}{(\varphi_x + \bar{\rho})^2}\left(\frac{1}{2}\varphi_{xx}^2 + \bar{\rho}_x\varphi_{xx}\right)\mathrm{d}x$$

$$+ C(\delta + \varepsilon)\|(\varphi_{xt}, \varphi_{xx}, \Phi_{xt})\|^2 + \eta\|(\Phi_{xt}, \varphi_{xx})\|^2 + C_\eta\|\varphi_x\|\|\Phi_{xxt}\| \tag{4.61}$$

for any $\eta > 0$, provided ε and δ are suitably small, where we have used (4.11).

Finally, similar to the proof of Lemma 4.2, adding (4.47) with (4.61) multiplied by a constant $K_1 > \frac{2}{\alpha}$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}_1(t) + C\|(\varphi_{xx}, \Phi_x, \Phi_{xx}, \Phi_{xt})\|^2 + (K_1\alpha - 1)\|\varphi_{xt}\|^2$$

$$\leqslant C(\delta + \varepsilon)\|(\varphi_{xt}, \varphi_{xx}, \Phi_{xt})\|^2 + \eta\|(\Phi_{xt}, \varphi_{xx})\|^2 + C_\eta\|\varphi_x\|\|\Phi_{xxt}\| + C_\eta\|\varphi_x\|^2, \tag{4.62}$$

where $K_1\alpha - 1 > 1$, $\mathcal{G}_1(t)$ is given by

$$\mathcal{G}_1(t) := \int_{\mathbb{R}_+}\left\{\frac{K_1}{2}\varphi_{xt}^2 + \frac{\alpha}{2}\varphi_x^2 + \varphi_x\varphi_{xt} + \frac{\mu\bar{\rho}}{2a}\Phi_x^2\right.$$

$$\left. + \frac{K_1}{2}\left[p'(\bar{\rho})\varphi_{xx}^2 - 2\mu\bar{\rho}\Phi_x\varphi_{xx} + \frac{\mu\bar{\rho}}{a}(b\Phi_x^2 + \Phi_{xx}^2)\right]\right\}\mathrm{d}x$$

$$+ \frac{K_1}{2}\int_{\mathbb{R}_+}[p'(\varphi_x + \bar{\rho}) - p'(\bar{\rho})]\varphi_{xx}^2\mathrm{d}x - \int_{\mathbb{R}_+}K_1[p'(\varphi_x + \bar{\rho}) - p'(\bar{\rho}) - p''(\bar{\rho})\varphi_x]\bar{\rho}_x\varphi_{xx}\mathrm{d}x$$

$$- \int_{\mathbb{R}_+}\mu K_1(\varphi_x\Phi_x + \varphi_x\bar{\phi}_x)\varphi_{xx}\mathrm{d}x + K_1\int_{\mathbb{R}_+}\frac{\varphi_t^2}{(\varphi_x + \bar{\rho})^2}\left(\frac{1}{2}\varphi_{xx}^2 + \bar{\rho}_x\varphi_{xx}\right)\mathrm{d}x$$

$$= \mathcal{G}_{1,0} + \mathcal{G}_{1,1} + \mathcal{G}_{1,2} + \mathcal{G}_{1,3} + \mathcal{G}_{1,4}. \tag{4.63}$$

Taking K_1 large enough such that

$$\frac{K_1}{2}\varphi_{xt}^2 + \frac{\alpha}{2}\varphi_x^2 + \varphi_x\varphi_{xt} \geqslant C\left(\varphi_{xt}^2 + \varphi_x^2\right)$$

for some constant $C > 0$ independent of t, recalling (4.48), we have

$$\mathcal{G}_{1,0} \sim \|(\varphi_x, \varphi_{xx}, \varphi_{xt}, \Phi_x, \Phi_{xx})\|^2.$$

By (1.3), (2.4), (4.10), (4.12), Cauchy–Schwarz inequality, and the Taylor expansion, we have

$$\mathcal{G}_{1,1} \leqslant C\|\bar{\rho}_x\|_{L^\infty}\|\varphi_x\|\|\varphi_{xx}\| \leqslant C\delta\|(\varphi_x, \varphi_{xx})\|^2,$$

$$\mathcal{G}_{1,2} \leqslant C\|\varphi_{xx}\|\|\bar{\rho}_x\|_{L^\infty}\|\varphi_x\|^2 \leqslant C\delta\|(\varphi_x, \varphi_{xx})\|^2,$$

$$\mathcal{G}_{1,3} \leqslant C\|\varphi_{xx}\|\left(\|\varphi_x\|\|\Phi_x\|_{L^\infty} + \|\bar{\phi}_x\|_{L^\infty}\|\varphi_x\|\right) \leqslant C(\varepsilon + \delta)\|(\varphi_x, \varphi_{xx})\|^2,$$

$$\mathcal{G}_{1,4} \leqslant C\|\varphi_{xx}\|\left(\|\varphi_x\|\|\Phi_x\|_{L^\infty} + \|\bar{\phi}_x\|_{L^\infty}\|\varphi_x\|\right) \leqslant C(\varepsilon + \delta)\|(\varphi_x, \varphi_{xx})\|^2.$$

Therefore, for sufficiently small ε and δ, we have from (4.63) that

$$\mathcal{G}_1 \sim \|(\varphi_x, \varphi_{xx}, \varphi_{xt}, \Phi_x, \Phi_{xx})\|^2. \tag{4.64}$$

With (4.64), after integrating (4.62) over $(0, t)$ and taking ε, δ, η sufficiently small, we get (4.36) and thus finish the proof of Lemma 4.3. $\qquad\square$

To close the a priori assumption (4.9), some higher order estimates of solutions are needed. Let us begin with the estimates on $(\varphi_{xxx}, \varphi_{xxt})$.

LEMMA 4.4. Let the assumptions in Proposition 4.2 hold. If ε and δ are sufficiently small, then the solution (φ, Φ) of (4.3a)–(4.3d) satisfies

$$\|(\varphi_{xxx}, \varphi_{xxt})\|^2 + \int_0^t \|\varphi_{xx\tau}\|^2 \mathrm{d}\tau$$

$$\leqslant C(\delta + \varepsilon) \int_0^t \|(\Phi_{xx\tau}, \varphi_{xxx})\|^2 \mathrm{d}\tau + C \int_0^t \left(\|(\varphi_x, \varphi_\tau)\|_1^2 + \|(\Phi_{x\tau}, \Phi_{xx})\|^2 + \|\Phi_{x\tau\tau}\|^2 \right) \mathrm{d}\tau$$

$$+ C \left(\|(\varphi_{0x}, \psi_0)\|_2^2 + \|\Phi_{0xx}\|^2 \right) + C \left(\|(\varphi_x, \varphi_\tau)\|_1^2 + \|(\Phi_{xx}, \Phi_{x\tau})\|^2 \right) \tag{4.65}$$

for any $t \in (0, T)$, where the constant $C > 0$ is independent of T.

Proof. Multiplying (4.37a) by $-((p'(\bar\rho)\varphi_x)_{xx} + \mathcal{F}_x)_t$ followed by an integration over $\mathbb{R}_+$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} p'(\bar\rho)\varphi_{xxt}^2 \mathrm{d}x + \alpha \int_{\mathbb{R}_+} p'(\bar\rho)\varphi_{xxt}^2 \mathrm{d}x$$

$$= -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}(\varphi_{xxt} + \alpha\varphi_{xx})\mathcal{F}_t \mathrm{d}x - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}[(p'(\bar\rho)\varphi_x)_{xx} + \mathcal{F}_x]^2 \mathrm{d}x$$

$$- \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\varphi_{xxt}p'(\bar\rho)\bar\rho_x\varphi_{xt} \mathrm{d}x + \underbrace{\int_{\mathbb{R}_+}\varphi_{xxt}p'(\bar\rho)\bar\rho_x(\varphi_{xtt} - \varphi_{xt}) \mathrm{d}x}_{\mathcal{I}_1}$$

$$+ \underbrace{\int_{\mathbb{R}_+}(\varphi_{xxt} + \alpha\varphi_{xx})\mathcal{F}_{tt} \mathrm{d}x}_{\mathcal{I}_2} - \underbrace{\int_{\mathbb{R}_+}\mathcal{H}_x((p'(\bar\rho)\varphi_x)_{xx} + \mathcal{F}_x)_t \mathrm{d}x}_{\mathcal{I}_3}. \tag{4.66}$$

Denote

$$\mathcal{G}_2(t) := \frac{1}{2}\int_{\mathbb{R}_+} p'(\bar\rho)\varphi_{xxt}^2 \mathrm{d}x + \int_{\mathbb{R}_+}(\varphi_{xxt} + \alpha\varphi_{xx})\mathcal{F}_t \mathrm{d}x$$

$$+ \frac{1}{2}\int_{\mathbb{R}_+}[(p'(\bar\rho)\varphi_x)_{xx} + \mathcal{F}_x]^2 \mathrm{d}x + \int_{\mathbb{R}_+}\varphi_{xxt}p'(\bar\rho)\bar\rho_x\varphi_{xt} \mathrm{d}x,$$

then (4.66) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}_2(t) + \alpha \int_{\mathbb{R}_+} p'(\bar\rho)\varphi_{xxt}^2 \mathrm{d}x = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \tag{4.67}$$

Noting that

$$\mathcal{F}_t = [p'(\varphi_x + \bar\rho) - p'(\bar\rho)]\varphi_{xxt} + [p''(\varphi_x + \bar\rho) - p''(\bar\rho)]\varphi_{xt}\bar\rho_x + p''(\varphi_x + \bar\rho)\varphi_{xt}\varphi_{xx}$$

$$- \mu\left(\varphi_{xt}\Phi_x + \varphi_x\Phi_{xt} + \varphi_{xt}\bar\phi_x + \bar\rho\Phi_{xt}\right),$$

we utilize (1.3), (2.4), (4.10), (4.11), and the mean value theorem to get

$$|\mathcal{F}_t| \leqslant C\varepsilon|\varphi_{xxt}| + C(\varepsilon + \delta)(|\varphi_{xt}| + |\Phi_{xt}|) + C|\Phi_{xt}| \leqslant C\varepsilon|\varphi_{xxt}| + C(|\varphi_{xt}| + |\Phi_{xt}|).$$

Then by (2.4) and Cauchy–Schwarz inequality, we have

$$\left| \int_{\mathbb{R}_+} \varphi_{xxt} p'(\bar{\rho}) \bar{\rho}_x \varphi_{xt} \mathrm{d}x \right| + \left| \int_{\mathbb{R}_+} (\varphi_{xxt} + \alpha \varphi_{xx}) \mathcal{F}_t \mathrm{d}x \right|$$

$$\leqslant C(\varepsilon + \delta) \|\varphi_{xxt}\|^2 + C\|(\varphi_{xx}, \varphi_{xt}, \Phi_{xt})\|^2. \tag{4.68}$$

A direct computation leads to

$$\mathcal{F}_x = [p'(\varphi_x + \bar{\rho}) - p'(\bar{\rho}) - p''(\bar{\rho})\varphi_x]\bar{\rho}_{xx} + [p'(\varphi_x + \bar{\rho}) - p'(\bar{\rho})]\varphi_{xxx}$$

$$+ [p''(\varphi_x + \bar{\rho}) - p''(\bar{\rho}) - p'''(\bar{\rho})\varphi_x]\bar{\rho}_x^2 + 2[p''(\varphi_x + \bar{\rho}) - p''(\bar{\rho})]\varphi_{xx}\bar{\rho}_x$$

$$+ p''(\varphi_x + \bar{\rho})\varphi_{xx}^2 + \varphi_{xx}\Phi_x + \varphi_x\Phi_{xx} + \varphi_{xx}\bar{\phi}_x + \bar{\rho}\Phi_{xx}, \tag{4.69}$$

$$(p'(\bar{\rho})\varphi_x)_{xx} = p'(\bar{\rho})\varphi_{xxx} + 2p''(\bar{\rho})\bar{\rho}_x\varphi_{xx} + p''(\bar{\rho})\bar{\rho}_{xx}\varphi_x + p'''(\bar{\rho})\bar{\rho}_x^2\varphi_x. \tag{4.70}$$

Combining the above identities with (1.3), (2.4), (4.10)–(4.12), and the Taylor expansion yields

$$|\mathcal{F}_x| \leqslant C\varepsilon|\varphi_{xxx}| + C(\varepsilon + \delta)(|\varphi_{xx}| + |\varphi_x|) + C|\Phi_{xx}|, \tag{4.71}$$

$$|(p'(\bar{\rho})\varphi_x)_{xx} - p'(\bar{\rho})\varphi_{xxx}| \leqslant C\delta(|\varphi_x| + |\varphi_{xx}|). \tag{4.72}$$

We thus deduce that

$$\int_{\mathbb{R}_+} [(p'(\bar{\rho})\varphi_x)_{xx} + \mathcal{F}_x]^2 \mathrm{d}x = \int_{\mathbb{R}_+} |p'(\bar{\rho})\varphi_{xxx} + (p'(\bar{\rho})\varphi_x)_{xx} - p'(\bar{\rho})\varphi_{xxx} + \mathcal{F}_x|^2 \mathrm{d}x$$

$$\geqslant \int_{\mathbb{R}_+} |p'(\bar{\rho})\varphi_{xxx}|^2 \mathrm{d}x - 2\int_{\mathbb{R}_+} |p'(\bar{\rho})\varphi_{xxx}|(|\mathcal{F}_x| + |(p'(\bar{\rho})\varphi_x)_x - p'(\bar{\rho})\varphi_{xxx}|)\mathrm{d}x$$

$$\geqslant \frac{1}{2}\int_{\mathbb{R}_+} |p'(\bar{\rho})\varphi_{xxx}|^2 \mathrm{d}x - C\|(\varphi_x, \varphi_{xx}, \Phi_{xx})\|^2 \tag{4.73}$$

for suitably small ε and δ, and that

$$\int_{\mathbb{R}_+} [(p'(\bar{\rho})\varphi_x)_{xx} + \mathcal{F}_x]^2 \mathrm{d}x \leqslant C(\|\varphi_x\|_2^2 + \|\Phi_{xx}\|^2). \tag{4.74}$$

Here (1.3), (4.11), and Cauchy–Schwarz inequality have been used. Due to (4.68), (4.73), and (4.74), it follows that

$$\begin{cases} \mathcal{G}_2(t) \leqslant C(\|\varphi_{xt}\|_1^2 + \|\varphi_x\|_2^2 + \|(\Phi_{xx}, \Phi_{xt})\|^2), \\ \mathcal{G}_2(t) \geqslant C\|(\varphi_{xxt}, \varphi_{xxx})\|^2 - C(\|\varphi_x\|_1^2 + \|(\Phi_{xx}, \Phi_{xt}, \varphi_{xt})\|^2), \end{cases} \tag{4.75}$$

where we have used (1.3) and the bounds of $\bar{\rho}$. Now let us turn to the estimates of $\mathcal{I}_i$. From (4.3a), we get

$$\varphi_{xtt} = (p'(\bar{\rho})\varphi_x)_{xx} - \alpha\varphi_{xt} + \mathcal{F}_x + \mathcal{H}_x. \tag{4.76}$$

A direct computation leads to

$$\mathcal{H}_x = \frac{2\varphi_{tx}^2 + 2\varphi_t\varphi_{txx}}{\varphi_x + \bar{\rho}} - \frac{4\varphi_t\varphi_{tx}(\varphi_{xx} + \bar{\rho}_x)}{(\varphi_x + \bar{\rho})^2} - \frac{\varphi_t^2(\varphi_{xxx} + \bar{\rho}_{xx})}{(\varphi_x + \bar{\rho})^2} + \frac{2\varphi_t^2(\varphi_{xx} + \bar{\rho}_x)^2}{(\varphi_x + \bar{\rho})^3}. \tag{4.77}$$

This along with (1.3), (2.4), (4.10), and (4.12) implies

$$|\mathcal{H}_x| \leqslant C\varepsilon(|\varphi_{xxx}| + |\varphi_{xxt}|) + C(|\varphi_{xt}| + |\varphi_t|) \tag{4.78}$$

for sufficiently small ε and δ. Therefore, it holds that

$$\|\varphi_{xtt}\| \leqslant C\|(p'(\bar{\rho})\varphi_x)_{xx}\| + \|\alpha\varphi_{xt}\| + \|\mathcal{F}_x\| + \|\mathcal{H}_x\| \leqslant C(\|(\varphi_x, \varphi_t)\|_2 + \|\Phi_{xx}\|), \tag{4.79}$$

where we have used (1.3), (2.4), (4.11), (4.71), and (4.78). Resorting to (2.4), (4.79), and Cauchy–Schwarz inequality, we get

$$\mathcal{I}_1 \leqslant C\delta\|\varphi_{xxt}\|\|(\varphi_{xtt}, \varphi_{xt})\| \leqslant C\delta\|(\varphi_{xxx}, \varphi_{xxt})\|^2 + C\Big(\|(\varphi_x, \varphi_t)\|_1^2 + \|\Phi_{xx}\|^2\Big).$$

For $\mathcal{I}_2$, a direct computation gives

$$\begin{aligned}
\mathcal{F}_{tt} &= [p'(\varphi_x + \bar{\rho}) - p'(\bar{\rho})]\varphi_{xxtt} + 2p''(\varphi_x + \bar{\rho})\varphi_{xt}\varphi_{xxt} + [p''(\varphi_x + \bar{\rho}) - p''(\bar{\rho})]\varphi_{xtt}\bar{\rho}_x \\
&\quad + p'''(\varphi_x + \bar{\rho})\varphi_{xt}^2\bar{\rho}_x + p'''(\varphi_x + \bar{\rho})\varphi_{xt}^2\varphi_{xx} + p''(\varphi_x + \bar{\rho})\varphi_{xtt}\varphi_{xx} \\
&\quad - \mu\big(\varphi_{xtt}\Phi_x + 2\varphi_{xt}\Phi_{xt} + \varphi_x\Phi_{xtt} + \varphi_{xtt}\bar{\phi}_x + \bar{\rho}\Phi_{xtt}\big) \\
&= [p'(\varphi_x + \bar{\rho}) - p'(\bar{\rho})]\varphi_{xxtt} + \mathcal{J},
\end{aligned}$$

where, due to (1.3), (2.4), and (4.10), $\mathcal{J}$ can be estimated as follows

$$|\mathcal{J}| \leqslant C(\varepsilon + \delta)(|\varphi_{xtt}| + |\varphi_{xt}| + |\Phi_{xt}|) + C\varepsilon|\varphi_{xxt}| + C|\Phi_{xtt}|.$$

Then, owing to (1.3), (4.10)–(4.12), (4.79), Cauchy–Schwarz inequality, and the mean value theorem, we have

$$\begin{aligned}
\mathcal{I}_2 &= \int_{\mathbb{R}_+} (\varphi_{xxt} + \alpha\varphi_{xx})([p'(\varphi_x + \bar{\rho}) - p'(\bar{\rho})]\varphi_{xxtt} + \mathcal{J})\mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}_+} [p'(\varphi_x + \bar{\rho}) - p'(\bar{\rho})]\Big(\frac{1}{2}\varphi_{xxt}^2 + \varphi_{xx}\varphi_{xxt}\Big)\mathrm{d}x - \alpha\int_{\mathbb{R}_+} [p'(\varphi_x + \bar{\rho}) - p'(\bar{\rho})]\varphi_{xxt}^2\mathrm{d}x \\
&\quad - \int_{\mathbb{R}_+} p''(\varphi_x + \bar{\rho})\varphi_{xt}\Big(\frac{1}{2}\varphi_{xxt}^2 + \varphi_{xx}\varphi_{xxt}\Big)\mathrm{d}x + \int_{\mathbb{R}_+} (\varphi_{xxt} + \alpha\varphi_{xx})\mathcal{J}\mathrm{d}x \\
&\leqslant \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}_+} [p'(\varphi_x + \bar{\rho}) - p'(\bar{\rho})]\Big(\frac{1}{2}\varphi_{xxt}^2 + \varphi_{xx}\varphi_{xxt}\Big)\mathrm{d}x + C\|\varphi_x\|_{L^\infty}\|\varphi_{xxt}\|^2 \\
&\quad + C\|\varphi_{xt}\|_{L^\infty}\Big(\|\varphi_{xxt}\|^2 + \|\varphi_{xx}\|\|\varphi_{xxt}\|\Big) + C\|(\varphi_{xxt}, \varphi_{xx})\|\|\mathcal{J}\| \\
&\leqslant \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}_+} [p'(\varphi_x + \bar{\rho}) - p'(\bar{\rho})]\Big(\frac{1}{2}\varphi_{xxt}^2 + \varphi_{xx}\varphi_{xxt}\Big)\mathrm{d}x + C(\varepsilon + \delta)\|(\varphi_{xxt}, \varphi_{xxx})\|^2 \\
&\quad + C\big(\|(\varphi_x, \varphi_t)\|_1^2 + \|(\Phi_{xt}, \Phi_{xx})\|^2\big) + \eta\|\varphi_{xxt}\|^2 + C_\eta\|\Phi_{xtt}\|^2
\end{aligned}$$

for any $\eta > 0$. To deal with $\mathcal{I}_3$, we rearrange $\mathcal{H}_x$ in (4.77) as follows:

$$\begin{aligned}
\mathcal{H}_x &= \frac{2\varphi_t\varphi_{txx}}{\varphi_x + \bar{\rho}} - \frac{\varphi_t^2(\varphi_{xxx} + \bar{\rho}_{xx})}{(\varphi_x + \bar{\rho})^2} + \left[\frac{2\varphi_{tx}^2}{\varphi_x + \bar{\rho}} - \frac{4\varphi_t\varphi_{tx}(\varphi_{xx} + \bar{\rho}_x)}{(\varphi_x + \bar{\rho})^2} + \frac{2\varphi_t^2(\varphi_{xx} + \bar{\rho}_x)^2}{(\varphi_x + \bar{\rho})^3}\right] \\
&= \frac{2\varphi_t\varphi_{txx}}{\varphi_x + \bar{\rho}} - \frac{\varphi_t^2(\varphi_{xxx} + \bar{\rho}_{xx})}{(\varphi_x + \bar{\rho})^2} + \mathcal{L}_1, \tag{4.80}
\end{aligned}$$

with

$$|\partial_x\mathcal{L}_1| \leqslant C(\varepsilon + \delta)|\varphi_{xxt}| + C\varepsilon|\varphi_{xxx}| + C(\varepsilon + \delta)(|\varphi_t| + |\varphi_{xt}|) \tag{4.81}$$

for suitably small ε and δ, due to (2.4), (4.10) and (4.12). Based on (4.80), we split $\mathcal{I}_3$ into three parts:

$$\mathcal{I}_3 = \int_{\mathbb{R}_+} \frac{2\varphi_t \varphi_{txx}}{\varphi_x + \bar{\rho}}((p'(\bar{\rho})\varphi_x)_{xx} + \mathcal{F}_x)_t \mathrm{d}x - \int_{\mathbb{R}_+} \frac{\varphi_t^2(\varphi_{xxx} + \bar{\rho}_{xx})}{(\varphi_x + \bar{\rho})^2}((p'(\bar{\rho})\varphi_x)_{xx} + \mathcal{F}_x)_t \mathrm{d}x$$

$$+ \int_{\mathbb{R}_+} ((p'(\bar{\rho})\varphi_x)_{xx} + \mathcal{F}_x)_t \mathcal{L}_1 \mathrm{d}x = \mathcal{I}_{3,1} + \mathcal{I}_{3,2} + \mathcal{I}_{3,3}.$$

Next, we estimate $\mathcal{I}_{3,i}$. Recalling (4.69) and (4.70), we have

$$((p'(\bar{\rho})\varphi_x)_{xx} + \mathcal{F}_x)_t = p'(\varphi_x + \bar{\rho})\varphi_{xxxt} + [((p'(\bar{\rho})\varphi_x)_{xx} + \mathcal{F}_x)_t - p'(\varphi_x + \bar{\rho})\varphi_{xxxt}]$$

$$= p'(\varphi_x + \bar{\rho})\varphi_{xxxt} + \mathcal{L}_2,$$

where, in view of (1.3), (2.4), (4.10), and (4.11), the following inequality holds

$$|\mathcal{L}_2| \leqslant C(\varepsilon + \delta)(|\varphi_{xxt}| + |\varphi_{xxx}|) + C|\Phi_{xxt}| + C(|\varphi_{xt}| + |\Phi_{xt}|).$$

We thus have

$$\mathcal{I}_{3,1} = \int_{\mathbb{R}_+} \frac{2\varphi_t \varphi_{txx}}{\varphi_x + \bar{\rho}}(p'(\varphi_x + \bar{\rho})\varphi_{xxxt} + \mathcal{L}_2)\mathrm{d}x$$

$$\leqslant -\int_{\mathbb{R}_+} \varphi_{xxt}^2 \left(\frac{p'(\varphi_x + \bar{\rho})\varphi_t}{\varphi_x + \bar{\rho}}\right)_x \mathrm{d}x + C\|\varphi_t\|_{L^\infty}\|\varphi_{xxt}\|\|\mathcal{L}_2\|$$

$$\leqslant \int_{\mathbb{R}_+} \varphi_{xxt}^2(|\varphi_{xt}| + |\varphi_t(\varphi_{xx} + \bar{\rho}_x)|)\mathrm{d}x + C\varepsilon\|\varphi_{xxt}\|\|\mathcal{L}_2\|$$

$$\leqslant C(\delta + \varepsilon)\|(\varphi_{xxt}, \varphi_{xxx}, \varphi_{tx})\|^2 + C\varepsilon\|\Phi_{xt}\|_1^2$$

and

$$\mathcal{I}_{3,2} = -\int_{\mathbb{R}_+} \frac{\varphi_t^2(\varphi_{xxx} + \bar{\rho}_{xx})}{(\varphi_x + \bar{\rho})^2}(p'(\varphi_x + \bar{\rho})\varphi_{xxxt} + \mathcal{L}_2)\mathrm{d}x$$

$$= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} \frac{p'(\varphi_x + \bar{\rho})\varphi_t^2(\varphi_{xxx} + \bar{\rho}_{xx})^2}{(\varphi_x + \bar{\rho})^2}\mathrm{d}x + C(\varepsilon + \delta)(\|\varphi_{xxx}\| + \|\varphi_t\|)\|\mathcal{L}_2\|$$

$$+ \frac{1}{2}\int_{\mathbb{R}_+} (\varphi_{xxx} + \bar{\rho}_{xx})^2 \left(\frac{p''(\varphi_x + \bar{\rho})\varphi_{xt}\varphi_t^2}{(\varphi_x + \bar{\rho})^2} + \frac{2p'(\varphi_x + \bar{\rho})\varphi_t\varphi_{tt}}{(\varphi_x + \bar{\rho})^2} - 2\frac{p'(\varphi_x + \bar{\rho})\varphi_{xt}\varphi_t^2}{(\varphi_x + \bar{\rho})^3}\right)\mathrm{d}x$$

$$= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} \frac{p'(\varphi_x + \bar{\rho})\varphi_t^2(\varphi_{xxx} + \bar{\rho}_{xx})^2}{(\varphi_x + \bar{\rho})^2}\mathrm{d}x + C(\varepsilon + \delta)(\|\varphi_t\|_1^2 + \|\Phi_{xt}\|_1^2 + \|\varphi_{xxx}\|^2),$$

where we have used (1.3), (2.4), (4.10), (4.12), (4.57), Cauchy–Schwarz inequality, and the integration by parts. For $\mathcal{I}_{3,3}$, the integration by parts leads to

$$\mathcal{I}_{3,3} = -\int_{\mathbb{R}_+} ((p'(\bar{\rho})\varphi_x)_x + \mathcal{F})_t \partial_x \mathcal{L}_1 \mathrm{d}x,$$

which combined with (4.4), (4.5), (4.10), (4.81), and Cauchy–Schwarz inequality implies

$$\mathcal{I}_{3,3} \leqslant C\int_{\mathbb{R}_+} (|\varphi_{xt}| + |\Phi_{xt}|)[C(\varepsilon + \delta)(|\varphi_{xxt}| + |\varphi_t| + |\varphi_{xt}|) + C\varepsilon|\varphi_{xxx}|]\mathrm{d}x$$

$$\leqslant C(\varepsilon + \delta)\|(\varphi_{xxx}, \varphi_{xxt})\|^2 + C(\|\varphi_t\|_1^2 + \|\Phi_{xt}\|^2).$$

Hence, we have

$$\mathcal{I}_3 \leqslant -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\frac{p'(\varphi_x+\bar{\rho})\varphi_t^2(\varphi_{xxx}+\bar{\rho}_{xx})^2}{(\varphi_x+\bar{\rho})^2}\mathrm{d}x$$

$$+ C(\varepsilon+\delta)\|(\varphi_{xxx},\varphi_{xxt},\Phi_{xxt})\|^2 + C\big(\|\varphi_t\|_1^2+\|\Phi_{xt}\|^2\big).$$

Plugging $\mathcal{I}_i$ $(i=1,2,3)$ into (4.67), we now reach

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}_2(t) + \alpha\int_{\mathbb{R}_+}p'(\bar{\rho})\varphi_{xxt}^2\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\frac{p'(\varphi_x+\bar{\rho})\varphi_t^2(\varphi_{xxx}+\bar{\rho}_{xx})^2}{(\varphi_x+\bar{\rho})^2}\mathrm{d}x$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}[p'(\varphi_x+\bar{\rho})-p'(\bar{\rho})]\Big(\frac{1}{2}\varphi_{xxt}^2+\varphi_{xx}\varphi_{xxt}\Big)\mathrm{d}x + C(\delta+\varepsilon)\|(\varphi_{xxx},\varphi_{xxt},\Phi_{xxt})\|^2$$

$$+ \eta\|\varphi_{xxt}\|^2 + C_\eta\|\Phi_{xtt}\|^2 + C\|(\varphi_x,\varphi_t)\|_1^2 + C\|(\Phi_{xx},\Phi_{xt})\|^2 \tag{4.82}$$

for any $\eta>0$. Consequently, due to (1.3), (4.10)–(4.12), (4.75), and the mean value theorem, we obtain (4.65) after integrating (4.82) over $(0,t)$ and taking δ, ε, and η small enough. The proof of Lemma 4.4 is complete. $\qquad\square$

In the next lemma, we shall estimate the higher order terms on the right-hand side of (4.65).

LEMMA 4.5. *Let the assumptions in Proposition 4.2 hold. If ε and δ are sufficiently small, then the solution (φ,Φ) of (4.3a)–(4.3d) satisfies*

$$\|(\Phi_{xt},\Phi_{tt},\Phi_{xxx})\|^2 + \int_0^t\|\Phi_{xx\tau}\|^2\mathrm{d}\tau + \int_0^t\|(\Phi_{\tau\tau},\Phi_{x\tau},\Phi_{x\tau\tau},\Phi_{xx\tau})\|^2\mathrm{d}\tau$$

$$\leqslant C(\|\Phi_0\|_4^2+\|\varphi_{0x}\|_2^2+\|\psi_{0x}\|^2) + C\|(\varphi_{xx},\Phi_x)\|^2 + C\int_0^t(\|(\varphi_\tau,\varphi_x)\|_1^2+\|\Phi_{xx}\|^2)\mathrm{d}\tau$$

$$+ C\int_0^t\big(\varepsilon\|(\varphi_{xx\tau},\varphi_{xxx})\|^2+\|\varphi_{xx\tau}\|\|\Phi_{x\tau}\|\big)\mathrm{d}\tau, \tag{4.83}$$

and that

$$\int_0^t\int_{\mathbb{R}_+}\varphi_{xxx}^2\mathrm{d}x\mathrm{d}\tau \leqslant C\|(\varphi_x,\varphi_{xt},\Phi_x)\|_1^2 + C\big(\|\varphi_{0x}\|_2^2+\|\psi_{0x}\|^2\big)$$

$$+ C(\delta+\varepsilon)\int_0^t\|\Phi_{xx\tau}\|^2\mathrm{d}\tau + C\int_0^t\big(\|\varphi_\tau\|_2^2+\|(\Phi_{x\tau},\Phi_{xx})\|^2+\|\varphi_x\|_1^2\big)\mathrm{d}\tau. \tag{4.84}$$

Proof. We divide the proof into two steps.

Step 1: Estimates on $(\Phi_{xt},\Phi_{xxt},\Phi_{xtt})$. Differentiating (4.37b) with respect to t, we have

$$\Phi_{xtt} = \Phi_{xxxt} + a\varphi_{xxt} - b\Phi_{xt}. \tag{4.85}$$

Multiplying (4.85) by Φ_{xt} and integrating the resulting equation over $\mathbb{R}_+$, we get by the Hölder's inequality that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\Phi_{xt}^2\mathrm{d}x + b\int_{\mathbb{R}_+}\Phi_{xt}^2\mathrm{d}x + \int_{\mathbb{R}_+}\Phi_{xxt}^2\mathrm{d}x$$

$$= -\Phi_{xxt}\Phi_{xt}|_{x=0} + a\int_{\mathbb{R}_+}\varphi_{xxt}\Phi_{xt}\mathrm{d}x$$

$$\leqslant -\Phi_{xxt}\Phi_{xt}|_{x=0} + C\|\Phi_{xt}\|\|\varphi_{xxt}\|. \tag{4.86}$$

In view of (4.44), for smooth solution (φ, Φ), we have $\Phi_{xxt} = a\varphi_{xt}$ at $x = 0$. Then it holds that

$$
\begin{aligned}
-\Phi_{xxt}\Phi_{xt}|_{x=0} &\leqslant \|\varphi_{xt}\|_{L^\infty}\|\Phi_{xt}\|_{L^\infty} \\
&\leqslant C\|\varphi_{xt}\|^{\frac{1}{2}}\|\varphi_{xxt}\|^{\frac{1}{2}}\|\Phi_{xt}\|^{\frac{1}{2}}\|\Phi_{xxt}\|^{\frac{1}{2}} \\
&\leqslant C\|\varphi_{xxt}\|^{\frac{1}{2}}\|\Phi_{xt}\|^{\frac{1}{2}}(\|\varphi_{xt}\| + \|\Phi_{xxt}\|) \\
&\leqslant \eta\|\Phi_{xxt}\|^2 + C_\eta\|\Phi_{xt}\|\|\varphi_{xxt}\| + C\|\varphi_{xt}\|^2
\end{aligned} \tag{4.87}
$$

for any $\eta > 0$, where the Sobolev inequality $\|f\|_{L^\infty} \leqslant C\|f\|^{\frac{1}{2}}\|f_x\|^{\frac{1}{2}}$ and Cauchy–Schwarz inequality have been used. Plugging (4.87) into (4.86), we get after taking η suitably small that

$$
\begin{aligned}
&\|\Phi_{xt}\|^2 + \int_0^t \|(\Phi_{xx\tau}, \Phi_{x\tau})\|^2\mathrm{d}\tau \\
&\leqslant C\int_0^t \left(\|\varphi_{xx\tau}\|\|\Phi_{x\tau}\| + \|\varphi_{x\tau}\|^2\right)\mathrm{d}\tau + C(\|\Phi_0\|_3^2 + \|\varphi_{0xx}\|^2),
\end{aligned} \tag{4.88}
$$

where we have used (2.6). By (4.37b) and (4.88), we have

$$
\begin{aligned}
\|\Phi_{xxx}\|^2 &\leqslant C\|\Phi_{xt}\|^2 + C\|(\varphi_{xx}, \Phi_x)\|^2 \\
&\leqslant C\int_0^t \left(\|\varphi_{xx\tau}\|\|\Phi_{x\tau}\| + \|\varphi_{x\tau}\|^2\right)\mathrm{d}\tau + C\|(\varphi_{xx}, \Phi_x)\|^2 + C(\|\Phi_0\|_3^2 + \|\varphi_{0xx}\|^2). \tag{4.89}
\end{aligned}
$$

Differentiating in (4.3b) with respect to t twice gives

$$
\Phi_{ttt} = \Phi_{xxtt} + a\varphi_{xtt} - b\Phi_{tt}. \tag{4.90}
$$

Multiplying (4.90) by Φ_{tt}, we have

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\Phi_{tt}^2\mathrm{d}x + b\int_{\mathbb{R}_+}\Phi_{tt}^2\mathrm{d}x + \int_{\mathbb{R}_+}\Phi_{xtt}^2\mathrm{d}x = a\int_{\mathbb{R}_+}\varphi_{xtt}\Phi_{tt}\mathrm{d}x. \tag{4.91}
$$

Due to (1.3), (2.4), (4.11), (4.71), (4.72), (4.76), (4.78), Cauchy–Schwarz inequality, and integration by parts, one has

$$
\begin{aligned}
&a\int_{\mathbb{R}_+}\varphi_{xtt}\Phi_{tt}\mathrm{d}x \\
&= a\int_{\mathbb{R}_+}p'(\bar{\rho})\varphi_{xxx}\Phi_{tt}\mathrm{d}x + a\int_{\mathbb{R}_+}[((p'(\bar{\rho})\varphi_x)_{xx} - p'(\bar{\rho})\varphi_{xxx}) - \varphi_{xt} + \mathcal{F}_x + \mathcal{H}_x]\Phi_{tt}\mathrm{d}x \\
&\leqslant -a\int_{\mathbb{R}_+}p'(\bar{\rho})\varphi_{xx}\Phi_{xtt}\mathrm{d}x - a\int_{\mathbb{R}_+}p''(\bar{\rho})\bar{\rho}_x\varphi_{xx}\Phi_{tt}\mathrm{d}x + C\|\varphi_{xt}\|\|\Phi_{tt}\| \\
&\quad + C\|(\mathcal{F}_x, \mathcal{H}_x)\|\|\Phi_{tt}\| + C\|(p'(\bar{\rho})\varphi_x)_{xx} - p'(\bar{\rho})\varphi_{xxx}\|\|\Phi_{tt}\| \\
&\leqslant C(\eta + \varepsilon + \delta)\|\Phi_{tt}\|^2 + \eta\|\Phi_{xtt}\|^2 + C\varepsilon\|(\varphi_{xxt}, \varphi_{xxx})\|^2 + C_\eta(\|(\varphi_t, \varphi_x)\|_1^2 + \|\Phi_{xx}\|^2) \tag{4.92}
\end{aligned}
$$

for any $\eta > 0$. Substituting (4.92) into (4.91), we get after taking η, ε, δ small enough that

$$\|\Phi_{tt}\|^2 + \int_0^t \|(\Phi_{\tau\tau}, \Phi_{x\tau\tau})\|^2 d\tau$$

$$\leqslant C(\|\Phi_0\|_4^2 + \|\varphi_{0x}\|_2^2 + \|\psi_{0x}\|^2) + C\varepsilon \int_0^t \|(\varphi_{xx\tau}, \varphi_{xxx})\|^2 d\tau$$

$$+ C \int_0^t (\|(\varphi_\tau, \varphi_x)\|_1^2 + \|\Phi_{xx}\|^2) d\tau, \tag{4.93}$$

where we have used (2.7). Combining (4.93) with (4.88) and (4.89), then we get (4.83).

Step 2: *Estimates on φ_{xxx}.* Multiplying (4.37a) by $-(\varphi_{xxx} - \frac{\mu\rho_-}{p'(\rho_-)}\Phi_{xx})$ with ρ_- as in (3.3), and integrating the resulting equation over $\mathbb{R}_+ \times (0, t)$ for any $t \in (0, T)$, we get

$$\int_0^t \int_{\mathbb{R}_+} p'(\bar\rho)\varphi_{xxx}^2 dx d\tau$$

$$= -\int_0^t \int_{\mathbb{R}_+} ((p'(\bar\rho)\varphi_x)_{xx} - p'(\bar\rho)\varphi_{xxx})\varphi_{xxx} dx d\tau + \int_0^t \int_{\mathbb{R}_+} (p'(\bar\rho)\varphi_x)_{xx} \frac{\mu\rho_-}{p'(\rho_-)}\Phi_{xx} dx d\tau$$

$$- \int_{\mathbb{R}_+} (\varphi_{xt} + \varphi_x)\left(\varphi_{xxx} - \frac{\mu\rho_-\Phi_{xx}}{p'(\rho_-)}\right)(x, 0) dx + \int_{\mathbb{R}_+} (\varphi_{xt} + \varphi_x)\left(\varphi_{xxx} - \frac{\mu\rho_-\Phi_{xx}}{p'(\rho_-)}\right)(x, t) dx$$

$$- \int_0^t \int_{\mathbb{R}_+} (\varphi_{xt} + \alpha\varphi_x)\left(\varphi_{xxx} - \frac{\mu\rho_-\Phi_{xx}}{p'(\rho_-)}\right)_\tau dx d\tau$$

$$- \int_0^t \int_{\mathbb{R}_+} (\mathcal{H}_x + \mathcal{F}_x)\left(\varphi_{xxx} - \frac{\mu\rho_-\Phi_{xx}}{p'(\rho_-)}\right) dx d\tau$$

$$= \sum_{i=1}^6 \mathcal{P}_i. \tag{4.94}$$

Due to (1.3) and (4.11), one can find a constant $C > 0$ such that

$$C \int_0^t \int_{\mathbb{R}_+} \varphi_{xxx}^2 dx d\tau \leqslant \int_0^t \int_{\mathbb{R}_+} p'(\bar\rho)\varphi_{xxx}^2 dx d\tau.$$

Now let us estimate $\mathcal{P}_i$ $(1 \leqslant i \leqslant 6)$ term by term. Recalling (4.72), it holds that

$$\mathcal{P}_1 \leqslant C\delta \int_0^t \int_{\mathbb{R}_+} (|\varphi_x| + |\varphi_{xx}|)|\varphi_{xxx}| dx d\tau \leqslant C\delta \int_0^t \|\varphi_x\|_2^2 d\tau,$$

and that

$$\mathcal{P}_2 = \int_0^t \int_{\mathbb{R}_+} [(p'(\bar\rho)\varphi_x)_{xx} - p'(\bar\rho)\varphi_{xxx} + p'(\bar\rho)\varphi_{xxx}]\frac{\mu\rho_-}{p'(\rho_-)}\Phi_{xx} dx d\tau$$

$$\leqslant \int_0^t \int_{\mathbb{R}_+} (\delta(|\varphi_x| + |\varphi_{xx}|) + |\varphi_{xxx}|)|\Phi_{xx}| dx d\tau$$

$$\leqslant \int_0^t \|\varphi_{xxx}\|\|\Phi_{xx}\| d\tau + C\delta \int_0^t \|(\varphi_x, \varphi_{xx}, \Phi_{xx})\|^2 d\tau$$

$$\leqslant \eta \int_0^t \|\varphi_{xxx}\|^2 d\tau + C_\eta \int_0^t \|\Phi_{xx}\|^2 d\tau + C\delta \int_0^t \|(\varphi_x, \varphi_{xx})\|^2 d\tau$$

for any $\eta > 0$, where the Cauchy–Schwarz has been used. Again, by Cauchy–Schwarz inequality, it holds that

$$\mathcal{P}_3 \leqslant C \int_{\mathbb{R}_+} (|\psi_{0x}| + |\varphi_{0x}|)(|\varphi_{0xxx}| + |\Phi_{0xx}|)\mathrm{d}x$$

$$\leqslant C\big(\|\varphi_{0x}\|_2^2 + \|(\psi_{0x}, \Phi_{0xx})\|^2\big).$$

From the boundary conditions in (4.13), we get

$$\left(\varphi_{xx} - \frac{\mu\rho_-}{p'(\rho_-)}\Phi_x\right)\bigg|_{x=0} = \left\{\mu\bar{\rho}\left(\frac{1}{p'(\varphi_x + \bar{\rho})} - \frac{1}{p'(\bar{\rho})} + \frac{\varphi_x}{p'(\varphi_x + \bar{\rho})}\right)\Phi_x\right.$$

$$\left. -\frac{1}{p'(\varphi_x + \bar{\rho})}(p'(\varphi_x + \bar{\rho}) - p'(\bar{\rho}))\bar{\rho}_x + \frac{\varphi_x\bar{\phi}_x}{p'(\varphi_x + \bar{\rho})}\right\}\bigg|_{x=0},$$

$$\left(\varphi_{xx} - \frac{\mu\rho_-}{p'(\rho_-)}\Phi_x\right)_t\bigg|_{x=0} = \left\{\mu\bar{\rho}\left(\frac{1}{p'(\varphi_x + \bar{\rho})} - \frac{1}{p'(\bar{\rho})} + \frac{\varphi_x}{p'(\varphi_x + \bar{\rho})}\right)\Phi_x\right.$$

$$\left. -\frac{1}{p'(\varphi_x + \bar{\rho})}(p'(\varphi_x + \bar{\rho}) - p'(\bar{\rho}))\bar{\rho}_x + \frac{\varphi_x\bar{\phi}_x}{p'(\varphi_x + \bar{\rho})}\right\}_t\bigg|_{x=0}.$$

Due to (1.3), (2.4), and (4.10)–(4.12), we further have that

$$\begin{cases} \left|\left(\varphi_{xx} - \dfrac{\mu\rho_-}{p'(\rho_-)}\Phi_x\right)\bigg|_{x=0}\right| \leqslant C(\varepsilon + \delta)(\|\Phi_x\|_{L^\infty} + \|\varphi_x\|_{L^\infty}), \\[4mm] \left|\left(\varphi_{xx} - \dfrac{\mu\rho_-}{p'(\rho_-)}\Phi_x\right)_t\bigg|_{x=0}\right| \leqslant C(\varepsilon + \delta)(\|\Phi_{xt}\|_{L^\infty} + \|\varphi_{xt}\|_{L^\infty}). \end{cases} \tag{4.95}$$

By (4.95), Cauchy–Schwarz inequality, the Sobolev inequality $\|f\|_{L^\infty} \leqslant C\|f\|_1$, and the integration by parts, we have

$$\mathcal{P}_4 = \int_{\mathbb{R}_+} (\varphi_{xt} + \varphi_x)(\varphi_{xxx} - \frac{\mu\rho_-}{p'(\rho_-)}\Phi_{xx})\mathrm{d}x$$

$$= -\int_{\mathbb{R}_+} (\varphi_{xxt} + \varphi_{xx})(\varphi_{xx} - \frac{\mu\rho_-}{p'(\rho_-)}\Phi_x)\mathrm{d}x - (\varphi_{xt} + \varphi_x)\left(\varphi_{xx} - \frac{\mu\rho_-}{p'(\rho_-)}\Phi_x\right)\bigg|_{x=0}$$

$$\leqslant C\|\varphi_{xxt}\|\|(\varphi_{xx}, \Phi_x)\| + C\|(\varphi_{xx}, \Phi_x)\|^2 + C(\varepsilon + \delta)\|\varphi_{xt}\|_{L^\infty}(\|\Phi_x\|_{L^\infty} + \|\varphi_x\|_{L^\infty})$$

$$+ C(\varepsilon + \delta)\|\varphi_x\|_{L^\infty}(\|\Phi_x\|_{L^\infty} + \|\varphi_x\|_{L^\infty})$$

$$\leqslant C\|(\varphi_x, \varphi_{xt}, \Phi_x)\|_1^2$$

and

$$\mathcal{P}_5 = \int_0^t \int_{\mathbb{R}_+} (\varphi_{xx\tau} + \alpha\varphi_{xx})\left(\varphi_{xx} - \frac{\mu\rho_-\Phi_x}{p'(\rho_-)}\right)_\tau \mathrm{d}x\mathrm{d}\tau - \int_0^t (\varphi_{x\tau} + \varphi_x)\left(\varphi_{xx} - \frac{\mu\rho_-\Phi_x}{p'(\rho_-)}\right)_\tau\bigg|_{x=0} \mathrm{d}\tau$$

$$\leqslant C \int_0^t \|(\varphi_{xx\tau}, \varphi_{xx}, \Phi_{x\tau})\|^2\mathrm{d}\tau + C(\varepsilon + \delta) \int_0^t (\|\varphi_{x\tau}\|_{L^\infty} + \|\varphi_x\|_{L^\infty})(\|\Phi_{x\tau}\|_{L^\infty} + \|\varphi_{x\tau}\|_{L^\infty})\mathrm{d}\tau$$

$$\leqslant C(\varepsilon + \delta) \int_0^t \|\Phi_{xx\tau}\|^2\mathrm{d}\tau + C \int_0^t \big(\|(\varphi_x, \varphi_{x\tau})\|_1^2 + \|\Phi_{x\tau}\|^2\big)\mathrm{d}\tau.$$

For $\mathcal{P}_6$, we utilize (4.71), (4.78), and Cauchy–Schwarz inequality to get

$$\mathcal{P}_6 \leqslant C \int_0^t \|(\mathcal{H}_x, \mathcal{F}_x)\| \|(\varphi_{xxx}, \Phi_{xx})\| \mathrm{d}\tau$$

$$\leqslant C(\varepsilon + \delta) \int_0^t \|(\varphi_{xxx}, \varphi_{xx\tau})\|^2 \mathrm{d}\tau + C \int_0^t \left(\|(\varphi_\tau, \varphi_x)\|_1^2 + \|\Phi_{xx}\|^2 \right) \mathrm{d}\tau.$$

Inserting the estimates for $\mathcal{P}_i$ $(i = 1, \ldots, 6)$ into (4.94), we get

$$\int_0^t \int_{\mathbb{R}_+} \varphi_{xxx}^2 \mathrm{d}x \mathrm{d}\tau \leqslant C \|(\varphi_x, \varphi_{xt}, \Phi_x)\|_1^2 + C \left(\|\varphi_{0x}\|_2^2 + \|\psi_{0x}\|^2 \right)$$

$$+ C(\delta + \varepsilon) \int_0^t \|\Phi_{xx\tau}\|^2 \mathrm{d}\tau + C \int_0^t \left(\|\varphi_\tau\|_2^2 + \|(\Phi_{x\tau}, \Phi_{xx})\|^2 + \|\varphi_x\|_1^2 \right) \mathrm{d}\tau,$$

provided ε and δ are suitably small. The proof of Lemma 4.5 is complete. $\qquad\square$

From Lemmas 4.4 and 4.5, we have the following higher order estimates.

LEMMA 4.6.   *Under the conditions of Proposition* 4.2, *for any* $t \in (0, T)$, *we have*

$$\|(\varphi_{xxx}, \varphi_{xxt}, \Phi_{xt}, \Phi_{tt}, \Phi_{xxx})\|^2 + \int_0^t \left(\|(\varphi_{xx\tau}, \varphi_{xxx})\|^2 + \|(\varphi_{x\tau}, \Phi_{\tau\tau})\|_1^2 \right) \mathrm{d}\tau$$

$$\leqslant C \left(\|\Phi_0\|_4^2 + \|(\varphi_{0x}, \psi_0)\|_2^2 \right) + C \|(\varphi_x, \varphi_t, \Phi_x)\|_1^2 + C \int_0^t \left(\|(\varphi_\tau, \varphi_x)\|_1^2 + \|(\Phi_{x\tau}, \Phi_{xx})\|^2 \right) \mathrm{d}\tau,$$

$$(4.96)$$

provided ε *and* δ *are sufficiently small.*

Proof.   First, to control the terms related to $(\Phi_{xt}, \Phi_{xxt}, \Phi_{xtt})$ on the right-hand side of (4.65), we add (4.65) with (4.83) multiplied by a large positive constant to get

$$\|(\varphi_{xxx}, \varphi_{xxt}, \Phi_{xt}, \Phi_{tt}, \Phi_{xxx})\|^2 + \int_0^t \|(\Phi_{x\tau}, \Phi_\tau, \Phi_{x\tau\tau}, \Phi_{xx\tau}, \varphi_{xx\tau})\|^2 \mathrm{d}\tau$$

$$\leqslant C(\|\Phi_0\|_4^2 + \|(\varphi_{0x}, \psi_0)\|_2^2) + C \left(\|(\varphi_x, \varphi_t)\|_1^2 + \|\Phi_{xx}\|^2 \right)$$

$$+ C(\delta + \varepsilon) \int_0^t \|\varphi_{xxx}\|^2 \mathrm{d}\tau + C \int_0^t (\|(\varphi_\tau, \varphi_x)\|_1^2 + \|(\Phi_{x\tau}, \Phi_{xx})\|^2) \mathrm{d}\tau, \qquad (4.97)$$

provided ε and δ are suitably small, where the Cauchy–Schwarz inequality has been used. Combining (4.97) with (4.84), for sufficiently small ε and δ, we have

$$\|(\varphi_{xxx}, \varphi_{xxt}, \Phi_{xt}, \Phi_{tt}, \Phi_{xxx})\|^2 + \int_0^t \|(\Phi_{x\tau}, \Phi_{\tau\tau}, \Phi_{x\tau\tau}, \Phi_{xx\tau}, \varphi_{xx\tau}, \varphi_{xxx})\|^2 \mathrm{d}\tau$$

$$\leqslant C(\|\Phi_0\|_4^2 + \|(\varphi_{0x}, \psi_0)\|_2^2) + C \|(\varphi_x, \varphi_t, \Phi_x)\|_1^2 + C \int_0^t (\|(\varphi_\tau, \varphi_x)\|_1^2 + \|(\Phi_{x\tau}, \Phi_{xx})\|^2) \mathrm{d}\tau.$$

This gives rise to (4.96) and thus finishes the proof of Lemma 4.6. $\qquad\square$

4.4.   *Proof of Proposition* 4.2

By the local existence result in Proposition 4.1 and the standard extension criterion, it suffices to show the estimates (4.7) and (4.8) to prove Proposition 4.2. We first close the *a priori*

assumption (4.9). To this end, we add (4.96) to (4.36) multiplied by a suitable positive constant and get

$$\|(\varphi_x, \Phi_x)\|_2^2 + \|\varphi_{xt}\|_1^2 + \|(\Phi_{xt}, \Phi_{tt})\|^2 + \int_0^t \|(\varphi_{xx}, \varphi_{x\tau}, \Phi_x, \Phi_{x\tau}, \Phi_{\tau\tau})\|_1^2 \mathrm{d}\tau$$

$$\leqslant C\big(\|\Phi_0\|_4^2 + \|(\varphi_{0x}, \psi_0)\|_2^2\big) + C\|\varphi_t\|^2 + C \int_0^t \|(\varphi_x, \varphi_\tau)\|^2 \mathrm{d}\tau + C \int_0^t \|\varphi_x\| \|\Phi_{xx\tau}\| \mathrm{d}\tau,$$

where the smallness of ε and δ has been used. This along with the Cauchy–Schwarz inequality gives

$$\|(\varphi_x, \Phi_x)\|_2^2 + \|\varphi_{xt}\|_1^2 + \|(\Phi_{xt}, \Phi_{tt})\|^2 + \int_0^t \|(\varphi_{xx}, \varphi_{x\tau}, \Phi_x, \Phi_{x\tau}, \Phi_{\tau\tau})\|_1^2 \mathrm{d}\tau$$

$$\leqslant C\big(\|\Phi_0\|_4^2 + \|(\varphi_{0x}, \psi_0)\|_2^2\big) + C\|\varphi_t\|^2 + C \int_0^t \|(\varphi_x, \varphi_\tau)\|^2 \mathrm{d}\tau. \tag{4.98}$$

Furthermore, multiplying (4.15) by a suitable positive constant, and adding the resulting inequality to (4.98), we obtain

$$\|(\varphi, \Phi)\|_3^2 + \|\varphi_t\|_2^2 + \|(\Phi_{xt}, \Phi_{tt})\|^2 + \int_0^t \big(\|\varphi_x\|_2^2 + \|(\varphi_\tau, \Phi_\tau, \Phi)\|_2^2 + \|\Phi_{\tau\tau}\|_1^2\big) \mathrm{d}\tau$$

$$\leqslant C\big(\|\varphi_0\|_3^2 + \|\psi_0\|_2^2 + \|\Phi_0\|_4^2\big). \tag{4.99}$$

Then by setting

$$\varepsilon^2 = 2C\big(\|\varphi_0\|_3^2 + \|\psi_0\|_2^2 + \|\Phi_0\|_4^2\big)$$

and taking $\|\varphi_0\|_3 + \|\psi_0\|_2 + \|\Phi_0\|_4$ suitably small, we have

$$\sup_{0 \leqslant t < T} \Big\{ \|(\varphi, \Phi)(\cdot, t)\|_3^2 + \|\psi(\cdot, t)\|_2^2 \Big\} < \varepsilon^2$$

which hence closes the a priori assumption (4.9). To complete the proof of (4.7) and (4.8), now it remains to show the following

$$\|\Phi_{xxxx}\|^2 + \int_0^t \big(\|\Phi_{xxx}\|^2 + \|(\varphi_{\tau\tau}, \varphi_{\tau\tau x})\|^2\big) \mathrm{d}\tau \leqslant C\big(\|\varphi_0\|_3^2 + \|\psi_0\|_2^2 + \|\Phi_0\|_4^2\big).$$

Collecting (4.37b), (4.57), (4.79), and (4.99), one immediately has

$$\int_0^t \big(\|\Phi_{xxx}\|^2 + \|(\varphi_{\tau\tau}, \varphi_{\tau\tau x})\|^2\big) \mathrm{d}\tau \leqslant C\big(\|\varphi_0\|_3^2 + \|\psi_0\|_2^2 + \|\Phi_0\|_4^2\big).$$

To derive the estimate for Φ_{xxxx}, we first deduce from (4.3b) and (4.99) that

$$\|\Phi_t\|^2 \leqslant C\|(\Phi_{xx}, \varphi_x, \Phi)\|^2 \leqslant C\big(\|\varphi_0\|_3^2 + \|\psi_0\|_2^2 + \|\Phi_0\|_4^2\big). \tag{4.100}$$

Next, differentiating (4.3b) with respect to t leads to

$$\Phi_{tt} = \Phi_{xxt} + a\varphi_{xt} + b\Phi_t.$$

This along with (4.100) and (4.99) yields

$$\|\Phi_{xxt}\|^2 \leqslant C\|(\Phi_{tt}, \Phi_t, \varphi_{xt})\|^2 \leqslant C\big(\|\varphi_0\|_3^2 + \|\psi_0\|_2^2 + \|\Phi_0\|_4^2\big).$$

Finally, differentiating (4.37b) with respect to x, we have

$$\Phi_{xxxx} = \Phi_{xxt} - a\varphi_{xxx} + b\Phi_{xx},$$

and thus

$$\|\Phi_{xxxx}\|^2 \leqslant C\|(\Phi_{xx}, \Phi_{xxt}, \varphi_{xxx})\|^2 \leqslant C\big(\|\varphi_0\|_3^2 + \|\psi_0\|_2^2 + \|\Phi_0\|_4^2\big).$$

The proof of Proposition 4.2 is complete. $\square$

4.5. *Proof of Theorem* 2.2

In view of Proposition 4.2, the problem (1.2a)–(1.2c), (1.4)–(1.5) admits a unique classical solution (ρ, m, ϕ) in $\mathbb{R}_+ \times (0, \infty)$. Moreover, due to (4.2), (4.7), and (4.8), it holds that

$$\|(\rho - \bar{\rho}, m)\|_2^2 + \|\phi - \bar{\phi}\|_4^2 \leqslant C\big(\|\varphi_0\|_3^2 + \|\psi_0\|_2^2 + \|\Phi_0\|_4^2\big),$$

and that

$$\int_0^t \big(\|(\rho - \bar{\rho}, m)\|_2^2 + \|\phi - \bar{\phi}\|_3^2 + \|(\rho_\tau, m_\tau, \phi_\tau)\|_1^2\big)\mathrm{d}\tau \leqslant C\big(\|\varphi_0\|_3^2 + \|\psi_0\|_2^2 + \|\Phi_0\|_4^2\big) \quad (4.101)$$

for any $t > 0$. In the following, we shall prove the large time behavior of (ρ, m, ϕ) as in (2.5). For this, recalling the Sobolev inequality $\|f\|_{L^\infty} \leqslant C\|f\|^{\frac{1}{2}}\|f_x\|^{\frac{1}{2}}$, it suffices to show that

$$\lim_{t\to\infty} \|(\rho - \bar{\rho}, m, \phi - \bar{\phi})(\cdot, t)\|^2 \to 0. \quad (4.102)$$

In fact, with the help of (4.101) and Cauchy–Schwarz inequality, we get

$$\int_0^{+\infty} \left|\frac{\mathrm{d}}{\mathrm{d}t}\|(\rho - \bar{\rho}, m, \phi - \bar{\phi})(\cdot, t)\|^2\right|\mathrm{d}t$$

$$\leqslant C \int_0^{+\infty} \big(\|(\rho - \bar{\rho}, m, \phi - \bar{\phi})\|^2 + \|(\rho_t, m_t, \phi_t)\|^2\big)\mathrm{d}t < \infty. \quad (4.103)$$

The estimate (4.101) in combination with (4.103) gives (4.102). Then (2.5) is proved and we complete the proof of Theorem 2.2. $\square$

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Guangyi Hong and Zhi-An Wang
Department of Applied Mathematics
Hong Kong Polytechnic University
Hung Hom
Kowloon Hong Kong
P.R. China

gyhmath05@outlook.com
mawza@polyu.edu.hk

Hongyun Peng
School of Applied Mathematics
Guangdong University of Technology
Guangzhou 510006
P.R. China

penghy010@163.com

Changjiang Zhu
School of Mathematics
South China University of Technology
Guangzhou 510641
P.R. China

machjzhu@scut.edu.cn