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Prevention of blow-up by fast diffusion in chemotaxis

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ABSTRACT

In this paper, we study a strongly coupled parabolic system with cross diffusion term which models chemotaxis. The diffusion coefficient goes to infinity when cell density tends to an allowable maximum value. Such 'fast diffusion' leads to global existence of solutions in bounded domains for any given initial data irrespective of the spatial dimension, which is usually the goal of many modifications to the classical Keller–Segel model. The key estimates that make this possible have been obtained by a technique that uses ideas from Moser's iterations.

1. Introduction

We consider a strongly coupled parabolic system with cross diffusion term:

$$\begin{cases} u_t = \nabla \cdot \left(\dfrac{1}{(1-u)^\alpha} \nabla u - \chi(u,v) \nabla v \right), \\ v_t = d\Delta v + f(u) - \beta v, \\ \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0, \quad \dfrac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0, \quad v|_{t=0} = v_0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded $C^{2+\gamma}$ domain for some $0 < \gamma < 1$. Here $\alpha > 0$, $\beta \geqslant 0$ and $d > 0$ are given constants, while positive functions χ and f are given smooth functions representing the cross diffusion coefficient and a source term of the second species, respectively. With $\partial/\partial\nu$ being the directional derivative in the outward normal direction, the boundary conditions correspond to zero flux for both species. We require the initial conditions $0 \leqslant u_0(x) < 1$ and $v_0(x) \geqslant 0$ for all $x \in \overline{\Omega}$. This is necessary because if otherwise, the diffusion coefficient $1/(1-u)^\alpha$ will be undefined. Further assumptions are necessary to establish the global existence of solution to (1.1). They will be clearly stated later.

The system (1.1) is a modified Keller–Segel model which describes directed cell movement in response to chemical concentration gradient (see [12]). The species u represents the cell density and v accounts for the chemical (external signal) concentration. The cross diffusion coefficient χ is known as the chemotactic sensitivity function describing the mechanism of signal detection. It models the migration of species u to location with high concentration of v. A high concentration of u in turn generates more chemical v through the source term f. Such a positive feedback mechanism in the classical minimal

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Keller–Segel model, which corresponds to $\alpha = 0$, $\chi(u, v) = u$ and $f(u) = u$, leads to aggregation patterns of cell density (e.g., see [11]).

When the domain Ω is one-dimensional, it is known that the classical minimal Keller–Segel model allows global existence of solutions for any non-negative initial data u_0 and v_0. On the other hand, blow-up of solution at finite time may occur when the domain Ω is two-dimensional and the initial cell mass is large (see [11]). Since cell density does not blow up in nature, a number of modifications based on either mathematical motivation or biological inspiration have been made to the classical Keller–Segel model to eliminate such finite time blow-up phenomenon. Such exploration of global existence of solutions to these modified models is usually studied by using the two principal methods: (1) finding a priori L^∞-estimate for the chemotactic flux term and (2) constructing a Lyapunov function which provides a bound for some high energy norms (see a review article [10]).

It is natural to allow for an increase in cell diffusivity at high concentration to relieve over-crowding. Some models allow for infinite diffusivity when the cell density approaches a threshold value. Our particular form of diffusion coefficient, namely $1/(1-u)^\alpha$ for some $\alpha > 0$, is found in both reaction–diffusion models [9] and chemotaxis models [15,17]. For example in *volume filling model* extended by Wang and Hillen (see [17]), when the number of elastic cells approach their crowding capacity (which corresponds to a volumetric constraint of how many cells can be accommodated per unit area), their derivation of the governing model equations leads naturally to an infinite diffusivity. Following [15,17], we will refer such phenomenon as *fast diffusion* in this paper to conform with the corresponding terminology in the study of scalar *porous medium equation* [16].

Let the maximum allowable cell density be $u = 1$ after scaling. When u is close to maximum allowable cell density, fast diffusion can relieve such high concentration. This corresponds to making $\alpha > 0$ in (1.1). In order for the model to be well defined, one needs to show that $u < 1$ as time evolves. As the cross diffusion term in (1.1a) does not have a sign when $u = 1$, building comparison functions and applying the maximum principle will not help. An energy estimate coupled with a Moser iteration like technique will be employed [6,2] in this paper to achieve this goal.

The following hypotheses (H) are now prescribed:

(H1) $\chi : [0,\infty)^2 \to [0,\infty)$ is $C^{1+\gamma_1}$ for some $0 < \gamma_1 < 1$ and $\chi(0, v) = 0$ for $v \geqslant 0$;
(H2) $f : [0,\infty) \to [0,\infty)$ is C^1.
(H3) The initial conditions u_0 and v_0 are in $C^{2+\gamma_2}(\overline{\Omega})$ for some $0 < \gamma_2 < 1$ and satisfy $0 \leqslant u_0(x) < 1$ and $v_0(x) \geqslant 0$ for all $x \in \overline{\Omega}$. Moreover $\partial u_0/\partial\nu = \partial v_0/\partial\nu = 0$ at the boundary so that they are compatible with the boundary conditions.

Under such assumptions, we will show that (1.1) allows global existence of solution. In fact we have

Theorem 1.1. *Let assumptions* (H1) *to* (H3) *hold. If $\alpha \geqslant 2$ and $n \geqslant 2$, then there exists a unique global solution (u, v) to the system* (1.1) *such that both u and v are in $C^{(2+\gamma)/2,2+\gamma}([0,\infty) \times \overline{\Omega})$ for some $0 < \gamma \leqslant \min\{\gamma_1, \gamma_2\}$ and remain non-negative. Moreover $u(t, x) < 1$ for all $x \in \overline{\Omega}$ and all finite t.*

In case of a one-dimensional domain, we have a stronger result:

Theorem 1.2. *Let assumptions* (H1) *to* (H3) *hold. If $n = 1$ and $\alpha > 1$, then the same conclusions as in Theorem* 1.1 *can be drawn.*

Remark 1. For initial data which are less smooth than as required in condition (H3), local existence and uniqueness theorem [5] may conclude that solution becomes smooth and compatible with boundary condition for $t > 0$. Then Theorems 1.1 and 1.2 can be employed to yield global existence of solutions.

2. Key lemmas

In this section, we establish some preliminary results which will be used in the sequel. Let $\|\cdot\|_q$ denote the $L^q(\Omega)$ norm for any $1 \leqslant q \leqslant \infty$. First we recall the Gagliardo–Nirenberg inequality for functions that do not vanish at the boundary of Ω (see [14, Theorem 1] and [13, Theorem 2.2, p. 62]).

Lemma 2.1. *Let Ω be a bounded smooth domain* (*which satisfies a uniform cone property*) *in $\mathbb{R}^n$ with $n \geqslant 2$, and assume that $q \in [1,\infty)$ if $n = 2$ and $q \in [1, 2n/(n-2))$ if $n \geqslant 3$. Then there exists a positive constant C_q, which depends on n, q, Ω, such that for all $u \in W^{1,2}(\Omega)$,*

$$\|u\|_q \leqslant C_q \left(\|\nabla u\|_2^a \|u\|_1^{1-a} + \|u\|_1\right) \tag{2.1}$$

where $a = (1 - \frac{1}{q})/(\frac{1}{n} + \frac{1}{2})$ and $0 \leqslant a < 1$.

If q is such that a is being bounded away from 1, it is known that there is a uniform bound on the constants C_q. As this fact is needed and many texts do not document how C_q depends on q, we include the following simple lemma.

Lemma 2.2. *Let the conditions in Lemma 2.1 hold, $a_0 = n/(n+2)$ and $k \geqslant \alpha$. Then there exists a constant M_1, independent of k, such that for all $u \in W^{1,2}(\Omega)$,*

$$\|u\|_{\frac{2k}{k+\alpha}} \leqslant M_1\big(\|\nabla u\|_2^{a_0(k-\alpha)/k}\|u\|_1^{1-a_0(k-\alpha)/k} + \|u\|_1\big). \tag{2.2}$$

Proof. Note that $1 \leqslant \frac{2k}{k+\alpha} < 2$ for $k \geqslant \alpha$. Using an interpolation inequality and Lemma 2.1, we obtain

$$\begin{aligned}
\|u\|_{\frac{2k}{k+\alpha}} &\leqslant \|u\|_2^{\frac{k-\alpha}{k}}\|u\|_1^{\frac{\alpha}{k}} \\
&\leqslant \big[C_2\big(\|\nabla u\|_2^{a_0}\|u\|_1^{1-a_0} + \|u\|_1\big)\big]^{\frac{k-\alpha}{k}}\|u\|_1^{\frac{\alpha}{k}} \\
&\leqslant C_2^{\frac{k-\alpha}{k}} 2^{-\alpha/k}\big(\|\nabla u\|_2^{a_0(k-\alpha)/k}\|u\|_1^{(1-a_0)(k-\alpha)/k} + \|u\|_1^{\frac{k-\alpha}{k}}\big)\|u\|_1^{\frac{\alpha}{k}} \\
&\leqslant M_1\big(\|\nabla u\|_2^{a_0(k-\alpha)/k}\|u\|_1^{1-a_0(k-\alpha)/k} + \|u\|_1\big),
\end{aligned}$$

where we have used the inequality $(a+b)^r \leqslant 2^{r-1}(a^r + b^r)$ (see [1]) for $a, b, r \geqslant 0$. □

Lemma 2.3. *Let a non-negative numerical sequence $\{X_m\}_{m=0}^{\infty}$ satisfy $X_{m+1} \leqslant a_m X_m^{1+\beta_m}$ with $a_m \geqslant 1$ and $\beta_m \geqslant 0$ for $m \geqslant 0$. Assume $X_0 \geqslant 1$ and $\prod_{m=1}^{\infty}(1+\beta_m) = M_2 < \infty$. Then*

$$X_m \leqslant \left(\prod_{i=0}^{\infty} a_i\right)^{M_2} X_0^{M_2} \quad \textit{for any } m.$$

Proof. Since $M_2 \geqslant 1$, it is straightforward to see that

$$\begin{aligned}
X_m &\leqslant a_{m-1}X_{m-1}^{1+\beta_{m-1}} \\
&\leqslant a_{m-1}\big(a_{m-2}X_{m-2}^{1+\beta_{m-2}}\big)^{1+\beta_{m-1}} \\
&\leqslant a_{m-1}^{M_2}a_{m-2}^{M_2}X_{m-2}^{(1+\beta_{m-2})(1+\beta_{m-1})} \\
&\ \ \vdots \\
&\leqslant \left(\prod_{i=0}^{\infty} a_i\right)^{M_2} X_0^{M_2}. \qquad \square
\end{aligned}$$

Recall that $\vec{\nu}$ represents the unit outward normal vector at the boundary of Ω. We now study the auxiliary scalar equation

$$\begin{cases}
u_t = \nabla \cdot \left(\dfrac{1}{(1-u)^\alpha}\nabla u\right) - \nabla \cdot \vec{b}, \\
\dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0, \quad \vec{b}\cdot\vec{\nu}|_{\partial\Omega} = 0, \\
u|_{t=0} = u_0,
\end{cases} \tag{2.3}$$

where $\vec{b}$ is a given function in $L^\infty([0,\infty)\times\Omega)$ with $\|\vec{b}\|_\infty \equiv M$ and $u_0(x) < 1$ for all $x \in \overline{\Omega}$. Though we will ultimately identify the u in (2.3) with the u in (1.1), the following lemma can be useful in other circumstances as well. The goal is to show that as long as solution to (2.3) exists, u will be bounded away from 1. Thus $1/(1-u)^\alpha$ will not blow up and solution can be continued beyond any fixed time T.

Lemma 2.4. *Let $\alpha \geqslant 2$ and $u_0(x) < 1$ for all $x \in \overline{\Omega}$. Assume smooth solution u to Eq. (2.3) exists on $Q_T \equiv [0,T)\times\overline{\Omega}$ with $u < 1$. Then for any $T > 0$, there exists a constant $\delta_T > 0$ such that the solution $u(t,x) \leqslant 1-\delta_T$ for all $x \in \overline{\Omega}$ and $t \in [0,T)$. Here δ_T depends only on $M \equiv \|\vec{b}\|_\infty$, $\delta \equiv \|1-u_0\|_\infty$ and T.*

Remark 2. The above lemma does not exclude the possibility that $\delta_T \to 0$ as $T \to \infty$.

Proof of Lemma 2.4. Take $\delta > 0$ such that $u_0(x) \leqslant 1-\delta$ for all $x \in \overline{\Omega}$. Without loss of generality let $|\Omega| = 1$ by scaling x and t in the governing equation. We divide the proof into three steps.

Step 1. We claim that for any $1 \leqslant p < \infty$, $\frac{1}{1-u} \in L^p(\Omega)$.

Indeed from (2.3), we have

$$\frac{d}{dt}\int_\Omega (1-u)^{-p}\,dx = p\int_\Omega (1-u)^{-p-1}u_t\,dx$$
$$= -p(1+p)\int_\Omega \left(\frac{|\nabla u|^2}{(1-u)^{\alpha+p+2}} - \frac{\nabla u\cdot\vec{b}}{(1-u)^{p+2}}\right)dx.$$

If we set $w_p = (1-u)^{-\frac{\alpha+p}{2}}$, then the above equation becomes

$$\begin{aligned}\frac{d}{dt}\int_\Omega (1-u)^{-p}\,dx &= \frac{p(1+p)}{\alpha+p}\left(-\frac{4}{\alpha+p}\int_\Omega |\nabla w_p|^2\,dx + 2\int_\Omega \frac{\nabla w_p\cdot\vec{b}}{(1-u)^{(p+2-\alpha)/2}}\,dx\right)\\ &\leqslant \frac{p(1+p)}{\alpha+p}\left(-\frac{4}{\alpha+p}\int_\Omega |\nabla w_p|^2\,dx + 2M\int_\Omega \frac{|\nabla w_p|}{(1-u)^{(p+2-\alpha)/2}}\,dx\right)\\ &\leqslant \frac{p(1+p)}{\alpha+p}\left(-\frac{2}{\alpha+p}\int_\Omega |\nabla w_p|^2\,dx + \frac{M^2(p+\alpha)}{\varepsilon}\int_\Omega \frac{1}{(1-u)^{p+2-\alpha}}\,dx\right),\end{aligned} \tag{2.4}$$

where we have used the Young's inequality and ε is a small number which is independent of p.

Now let $\alpha \geqslant 2$ and then $p+2-\alpha \leqslant p$. Note that $\frac{1}{1-u} > 1$. Then it follows from the above inequality that

$$\frac{d}{dt}\int_\Omega (1-u)^{-p}\,dx \leqslant \frac{p(1+p)}{\alpha+p}\left(-\frac{2}{\alpha+p}\int_\Omega |\nabla w_p|^2\,dx + \frac{M^2(p+\alpha)}{\varepsilon}\int_\Omega \frac{1}{(1-u)^{p}}\,dx\right). \tag{2.5}$$

For brevity, denote $J_p = \int_\Omega (1-u)^{-p}\,dx$. Thus for some constant $C > 0$, one has

$$\frac{dJ_p}{dt} \leqslant Cp(p+1)J_p,$$

by which we conclude that

$$J_p^{\frac{1}{p}} \leqslant \big(J_p(0)\big)^{\frac{1}{p}} e^{C(p+1)t} \leqslant \frac{|\Omega|^{1/p}}{\delta} e^{C(p+1)t} \leqslant \widetilde{C} e^{C(p+1)t}$$

for some positive constant $\widetilde{C}$, which is independent of p. So for any $0 \leqslant T < \infty$, we have

$$\big\|(1-u)^{-1}\big\|_p \leqslant \widetilde{C} e^{C(p+1)T}.$$

In other words, $\frac{1}{1-u} \in L^p(\Omega)$ for any $1 \leqslant p < \infty$ and $0 \leqslant T < +\infty$. In fact $\|(1-u)^{-1}\|_p$ depends only on $M \equiv \|\vec{b}\|_\infty$, $\delta \equiv \|1-u_0\|_\infty$ and T.

Step 2. Let $C \geqslant 1$ be a generic positive constant, which is independent of p, that can change from one equation to the next in the following calculations. Restrict our attention to $p \geqslant 3\alpha$ so that $p(1+p)/(\alpha+p)^2 \geqslant 1/2$. It then follows from (2.5), Lemma 2.2 and the Young's inequality ($ab \leqslant \varepsilon a^r/r + \varepsilon^{-s/r}b^s/s$ when $1/r + 1/s = 1$) that

$$\begin{aligned}\frac{d}{dt}\int_\Omega w_p^{\frac{2p}{p+\alpha}}\,dx &\leqslant -\int_\Omega |\nabla w_p|^2\,dx + Cp^2\int_\Omega w_p^{\frac{2p}{p+\alpha}}\,dx\\ &\leqslant -\|\nabla w_p\|_2^2 + CM_1^{2p/(p+\alpha)}p^2\big(\|\nabla w_p\|_2^{a_0(1-\alpha/p)}\|w_p\|_1^{1-a_0(1-\alpha/p)} + \|w_p\|_1\big)^{2p/(p+\alpha)}\\ &\leqslant -\|\nabla w_p\|_2^2 + Cp^2\left(\left(\frac{\sqrt{\varepsilon_0}}{p}\right)^{(p+\alpha)/p}\|\nabla w_p\|_2 + \left[1+\left(\frac{p}{\sqrt{\varepsilon_0}}\right)^{\beta_p}\right]\|w_p\|_1\right)^{2p/(p+\alpha)}\end{aligned}$$

where $\beta_p \equiv \frac{p+\alpha}{p}\frac{a_0(1-\alpha/p)}{1-a_0(1-\alpha/p)}$. It can be checked that β_p is an increasing function in p for $p \geqslant 3\alpha$, and hence $\beta_p \leqslant a_0/(1-a_0) = n/2$. Observing that

$$\|\nabla w_p\|_2^{2p/(p+\alpha)} \leqslant \|\nabla w_p\|_2^2 + 1 \leqslant \|\nabla w_p\|_2^2 + \|w_p\|_1$$

and choosing ε_0 to be sufficiently small, we have

$$\frac{d}{dt}\int_\Omega w_p^{\frac{2p}{p+\alpha}}\,dx \leqslant -\frac{1}{2}\|\nabla w_p\|_2^2 + Cp^{2+np/(p+\alpha)}\|w_p\|_1^{\frac{2p}{p+\alpha}}$$
$$\leqslant -\frac{1}{2}\|\nabla w_p\|_2^2 + Cp^{2+n}\|w_p\|_1^{\frac{2p}{p+\alpha}}. \tag{2.6}$$

Another similar calculation using Lemma 2.2 and the Young's inequality gives

$$\|w_p\|_{\frac{2p}{p+\alpha}} \leqslant 2M_1\big(\|\nabla w_p\|_2 + \|w_p\|_1\big),$$

and hence

$$\|w_p\|^2_{\frac{2p}{p+\alpha}} \leqslant 8M_1^2\big(\|\nabla w_p\|_2^2 + \|w_p\|_1^2\big). \tag{2.7}$$

Since $w_p \geqslant 1$ and $2p/(p+\alpha) \leqslant 2$, a substitution of the above inequality into (2.6) yields

$$\frac{d}{dt}\int_\Omega w_p^{\frac{2p}{p+\alpha}}\,dx \leqslant -m\|w_p\|^2_{\frac{2p}{p+\alpha}} + Cp^{2+n}\|w_p\|_1^{\frac{2p}{p+\alpha}} + \frac{1}{2}\|w_p\|_1^2$$
$$\leqslant -m\|w_p\|^{\frac{2p}{p+\alpha}}_{\frac{2p}{p+\alpha}} + Cp^{2+n}\|w_p\|_1^2 \tag{2.8}$$

for some positive constants m and C.

Step 3. Recall $|\Omega| = 1$. For $p \geqslant 3\alpha$, define

$$U_p = \max\left\{\left\|\frac{1}{1-u(0,\cdot)}\right\|_\infty, \sup_{0\leqslant t<T}\left(\int_\Omega w_p\,dx\right)^{\frac{2}{p+\alpha}}\right\}$$
$$= \max\left\{\left\|\frac{1}{1-u(0,\cdot)}\right\|_\infty, \sup_{0\leqslant t<T}\left(\int_\Omega \left(\frac{1}{1-u}\right)^{\frac{p+\alpha}{2}} dx\right)^{\frac{2}{p+\alpha}}\right\}$$
$$\geqslant 1.$$

First, due to the result in Step 1, U_p is finite, well defined, and depends only on M, δ and T. It is also immediate that U_p is a nondecreasing function in p, since the function $(\frac{1}{|\Omega|}\int_\Omega |u|^q\,dx)^{\frac{1}{q}}$ is a nondecreasing function with respect to q for $q \geqslant 1$ [7, p. 146]. Then it follows from (2.8) that

$$\frac{d}{dt}\int_\Omega w_p^{\frac{2p}{p+\alpha}}\,dx \leqslant -m\|w_p\|^{\frac{2p}{p+\alpha}}_{\frac{2p}{p+\alpha}} + Cp^{2+n}U_p^{p+\alpha},$$

which is equivalent to

$$\frac{d}{dt}\left(e^{mt}\int_\Omega w_p^{\frac{2p}{p+\alpha}}\,dx\right) \leqslant Cp^{2+n}e^{mt}U_p^{p+\alpha}.$$

An integration leads to

$$\int_\Omega w_p^{\frac{2p}{p+\alpha}}\,dx \leqslant e^{-mt}\left\|\frac{1}{1-u(0,\cdot)}\right\|_\infty^p + \frac{C}{m}p^{2+n}U_p^{p+\alpha}$$
$$\leqslant U_p^p + \frac{C}{m}p^{2+n}U_p^{p+\alpha}$$
$$\leqslant Cp^{2+n}U_p^{p+\alpha}. \tag{2.9}$$

Letting $p = 2k-\alpha$ in the definition of U_p, it can be readily checked that

$$U_{2k-\alpha} = \max\left\{\left\|\frac{1}{1-u(0,\cdot)}\right\|_\infty, \sup_{0\leqslant t<T}\left(\int_\Omega \left(\frac{1}{1-u}\right)^k dx\right)^{\frac{1}{k}}\right\}.$$

With

$$\int_\Omega \left(\frac{1}{1-u}\right)^k dx = \int_\Omega w_k^{\frac{2k}{k+\alpha}} dx,$$

one infers from (2.9) that

$$\begin{aligned} U_{2k-\alpha} &\leqslant \max\left\{ \left\| \frac{1}{1-u(0,\cdot)} \right\|_\infty, \left(Ck^{2+n}U_k^{k+\alpha}\right)^{1/k} \right\} \\ &\leqslant \max\left\{U_k, \left(Ck^{2+n}U_k^{k+\alpha}\right)^{1/k}\right\} \\ &\leqslant \left(Ck^{2+n}\right)^{1/k} U_k^{1+\alpha/k}. \end{aligned}$$

Since $2k-\alpha \geqslant 3k/2$ for $k \geqslant 3\alpha$ and $U_k > 1$ is an increasing function in k, it follows that

$$U_{3k/2} \leqslant \left(Ck^{2+n}\right)^{1/k} U_k^{1+\alpha/k}.$$

Now let $k = (\frac{3}{2})^\mu$ for some positive integer μ, and rewrite the above inequality as

$$U_{(\frac{3}{2})^{\mu+1}} \leqslant C^{\frac{1}{(\frac{3}{2})^\mu}} \left(\frac{3}{2}\right)^{\frac{(2+n)\mu}{(\frac{3}{2})^\mu}} [U_{(\frac{3}{2})^\mu}]^{1+\frac{\alpha}{(\frac{3}{2})^\mu}}.$$

Take a μ_0 so that $(\frac{3}{2})^{\mu_0} \geqslant 3\alpha$ and consider the sequence $\{U_{(\frac{3}{2})^\mu}\}$ for $\mu = \mu_0, \mu_0+1, \ldots.$ Observe that $U_{(\frac{3}{2})^{\mu_0}} < \infty$ for any finite time T by using Step 1. Since $\sum_\mu \frac{1}{(3/2)^\mu}$, $\sum_\mu \frac{\mu}{(3/2)^\mu}$ and $\prod_{\mu=0}^\infty (1+\frac{\alpha}{(\frac{3}{2})^\mu})$ are convergent (the last claim is easily seen by noting $\sum_\mu \log(1+\frac{\alpha}{(\frac{3}{2})^\mu}) \leqslant \sum_\mu \frac{\alpha}{(\frac{3}{2})^\mu} < \infty$), the sequences $\{U_{(\frac{3}{2})^\mu}\}_{\mu=\mu_0}^\infty$ is bounded from above by Lemma 2.3, which implies that as $\mu \to \infty$,

$$\frac{1}{1-u} \in L^\infty(\Omega).$$

Hence u is bounded away from 1. That is, there exists a constant $\delta_T > 0$ such that $u \leqslant 1-\delta_T$. From the above calculations, it is clear that δ_T depends only on M, δ and T. (In fact, one expects δ_T decreases as M increases, δ decreases and T increases.) Thus the proof is complete. □

3. Proof of Theorem 1.1

First we give the local existence and uniqueness of solutions to system (1.1). One can easily establish this directly, or appeal to the following general theorem by Amann on strongly coupled parabolic equations [3,4], [5, Theorem 14.6, Corollary 14.7]. For simplicity, we have in fact put in more stringent conditions on the initial conditions than those required in Amann's theorem.

Lemma 3.1. *Let Ω be a bounded $C^{2+\gamma}$ domain in $\mathbb{R}^n$ with $0 < \gamma < 1$ and hypotheses* (H1)–(H3) *hold. Then*:

(1) *There exists a positive constant T_0 depending on initial data (u_0, v_0) and a constant $\delta_1 > 0$ such that the initial-boundary problem* (1.1) *has a unique maximal solution (u, v) defined on $[0, T_0) \times \Omega$ satisfying $(u, v) \in C^{(2+\gamma)/2, 2+\gamma}([0, T_0) \times \overline{\Omega}; \mathbb{R}^2)$ with $u \geqslant 0$ and $v \geqslant 0$.*

(2) *As long as solution u is bounded above away from 1 and v is bounded for each finite time t, then $T_0 = \infty$, namely, the solution (u, v) obtained in* (1) *is a global classical solution of the system* (1.1)*.*

Proof. Let $\omega = (u, v) \in \mathbb{R}^2$. Then the system (1.1) can be reformulated as

$$\begin{cases} \omega_t = \nabla \cdot \left(a(\omega)\nabla\omega\right) + \mathcal{F}(\omega), \\ \dfrac{\partial w}{\partial \nu} = 0 \quad \text{on } [0, +\infty) \times \partial\Omega, \\ \omega(0, \cdot) = (u_0, v_0) \quad \text{in } \Omega, \end{cases} \tag{3.1}$$

where

$$a(\omega) = \begin{pmatrix} \frac{1}{(1-u)^\alpha} & -\chi(u, v) \\ 0 & d \end{pmatrix}, \qquad \mathcal{F}(\omega) = \begin{pmatrix} 0 \\ f(u) - \beta v \end{pmatrix}.$$

Since the given initial conditions satisfy $u_0 \leqslant 1-\delta$ for some $\delta > 0$, the eigenvalues of the matrix $a(\omega)$ are positive at $t = 0$. Hence the system (3.1) is normally parabolic and local existence of solution follows from Theorem 7.3 of [3], i.e. there exists a $T_0 > 0$ such that unique solution $(u, v) \in C([0, T_0) \times \overline{\Omega}; \mathbb{R}^2) \cap C^{1,2}((0, T_0) \times \overline{\Omega}; \mathbb{R}^2)$ exists. Typical regularity bootstrap enables one to obtain the improved smoothness as stated in the lemma.

Now rewrite (1.1a) as

$$u_t = \frac{1}{(1-u)^\alpha}\Delta u + \left(\frac{\alpha}{(1-u)^{\alpha+1}}\nabla u - \chi_u \nabla v\right)\cdot \nabla u - \left(\chi \Delta v + \chi_v |\nabla v|^2\right). \tag{3.2}$$

Treat this as a scalar linear equation in u. Assumption (H1) requires that $\chi(0, v) = 0$ for all $v \geqslant 0$. This implies $\chi_v(0, v) = 0$ for all $v \geqslant 0$. Thus if $u = 0$ at some (x_0, t_0), the source term $\chi \Delta v + \chi_v |\nabla v|^2$ is zero there. We can therefore apply the maximum principle to (1.1a) to infer that $u \geqslant 0$ whenever $v \geqslant 0$. Similarly we show $v \geqslant 0$ from (1.1b) whenever $u \geqslant 0$. The proof of statement (1) is complete.

Since the system (3.1) is an upper triangular system, statement (2) follows from Theorem 5.2 in [4]. The proof of the lemma is complete. □

Now we are in the position to show Theorem 1.1.

Proof of Theorem 1.1. Assume the maximal time of a $C^{(2+\gamma)/2,2+\gamma}$ solution to be $T_0 < \infty$. Then by Lemma 2.4, $0 \leqslant u \leqslant 1$. Treating u as a source term in (1.1b), L^p estimate for parabolic equation [13, p. 351] yields $\|v\|_{W_p^{2,1}(Q_T)} \leqslant C_{p,T_0}\|1\|_{L^p([0,T_0)\times\Omega)} \leqslant C_{p,T_0}$ for any $p > 1$. (Note that the two constants C_{p,T_0} may be different, though we are using the same symbol.) By taking sufficiently large p, Sobolev type estimate [13, p. 80] gives an L^∞ norm bound on both v and $|\nabla v|$. From statement (2) in Lemma 3.1, we have T_0 being infinite. This contradicts our original assumption that T_0 is finite. Hence the maximal time of existence of smooth solution must be infinite. □

Remark 3. One can prove global existence of solution without the use of Amann's theorem in this simple case. From the established estimate on v in the above proof, $\chi_u \nabla v \in L^\infty$, $\chi_v |\nabla v|^2 \in L^\infty$, $\chi \Delta v \in L^p$ for any $p > 1$, and gradient of u grows quadratically in the form of $|\nabla u|^2$ in (3.2), then [13, Theorem 7.2 on p. 486] gives an L^∞ norm bound for ∇u. Now Schauder type estimates on individual equation in (1.1) give $C^{(2+\gamma)/2,2+\gamma}$ norm bounds for both u and v, and hence solution can be continued beyond $t = T_0$. Thus T_0 has to be infinite.

4. Proof of Theorem 1.2

It suffices to extend Lemma 2.4 to cover the cases $1 < \alpha < 2$. In other words for any $T > 0$, we like to show that $\sup_{0\leqslant t\leqslant T} \frac{1}{1-u(t,\cdot)}$ is bounded. Once this is done, the same proof in Section 3 gives Theorem 1.2.

Step 1. As in Step 2 of Lemma 2.4, we restrict our attention to $p \geqslant 3\alpha$ so that $p(1+p)/(p+\alpha)^2 > 1/2$. From (2.4), there exists a constant $C > 0$, which is independent of p, such that

$$\frac{d}{dt}\int_\Omega (1-u)^{-p}\,dx \leqslant -\int_\Omega \left|\frac{\partial w_p}{\partial x}\right|^2 dx + Cp^2 \int_\Omega \frac{1}{(1-u)^{p+2-\alpha}}\,dx,$$

where $w_p = (1-u)^{-\frac{p+\alpha}{2}}$. This is equivalent to

$$\frac{d}{dt}\int_\Omega w_p^{\frac{2p}{p+\alpha}}\,dx \leqslant -\int_\Omega \left|\frac{\partial w_p}{\partial x}\right|^2 dx + Cp^2 \int_\Omega w_p^{\frac{2(p+2-\alpha)}{p+\alpha}}\,dx. \tag{4.1}$$

We will allow the constant C to change from one equation to the next in the following calculations, so long as it is independent of p.

Recall that without loss of generality, we let $|\Omega| = 1$. Thus we take $n = 1$, $\Omega = [0, 1]$ and $1 < \alpha < 2$. First, an integration of (1.1a) leads immediately to the conservation condition

$$\int_0^1 u(t,\cdot)\,dx = \int_0^1 u_0(x)\,dx = m_0,$$

where $m_0 \geqslant 0$ is a constant representing the total initial cell mass. By the mean value theorem, there exists an $x_0 \in [0, 1]$, which can depend on t, such that

$$u(t, x_0) = \int_0^1 u(t,\cdot)\,dx = m_0.$$

Thus we have

$$w_p(t,x_0) = \big(1-u(t,x_0)\big)^{-\frac{p+\alpha}{2}} = (1-m_0)^{-\frac{p+\alpha}{2}}.$$

Substituting $w_p(t,x) = w_p(t,x_0) + \int_{x_0}^{x} \frac{\partial w_p(t,\xi)}{\partial \xi}\, d\xi$ into Eq. (4.1), this gives

$$\begin{aligned}
\frac{d}{dt}\int_0^1 w_p^{\frac{2p}{p+\alpha}}\,dx &\leqslant -\int_0^1 \left|\frac{\partial w_p}{\partial x}\right|^2 dx + Cp^2\int_0^1 \left[\left((1-m_0)^{-\frac{p+\alpha}{2}} + \int_0^1 \left|\frac{\partial w_p}{\partial x}\right| dx\right)^{\frac{2(p+2-\alpha)}{p+\alpha}}\right] dx \\
&\leqslant -\int_0^1 \left|\frac{\partial w_p}{\partial x}\right|^2 dx + Cp^2\left[(1-m_0)^{-(p+2-\alpha)} + \left(\int_0^1 \left|\frac{\partial w_p}{\partial x}\right| dx\right)^{\frac{2(p+2-\alpha)}{p+\alpha}}\right] \\
&\leqslant -\int_0^1 \left|\frac{\partial w_p}{\partial x}\right|^2 dx + Cp^2\left[(1-m_0)^{-(p+2-\alpha)} + \left(\int_0^1 \left|\frac{\partial w_p}{\partial x}\right|^2 dx\right)^{\frac{(p+2-\alpha)}{p+\alpha}}\right]
\end{aligned}$$

by using Jensen's inequality in the last calculation. With $(p+2-\alpha)/(p+\alpha) < 1$, we can apply the Young's inequality to the last term in the above inequality to deduce

$$\begin{aligned}
\frac{d}{dt}\int_0^1 w_p^{\frac{2p}{p+\alpha}}\,dx &\leqslant -\int_0^1 \left|\frac{\partial w_p}{\partial x}\right|^2 dx + Cp^2(1-m_0)^{-(p+2-\alpha)} \\
&\quad + Cp^2\left\{\frac{\varepsilon_0}{p^2}\int_0^1 \left|\frac{\partial w_p}{\partial x}\right|^2 dx + \left(\frac{p^2}{\varepsilon_0}\right)^{(p+2-\alpha)/2(\alpha-1)} \frac{1}{p}\right\} \\
&\leqslant -\frac{1}{2}\int_0^1 \left|\frac{\partial w_p}{\partial x}\right|^2 dx + Cp^2(1-m_0)^{-(p+2-\alpha)} + Cp^{1+(p+2-\alpha)/(\alpha-1)} \\
&\leqslant Cp^{1+(p+2-\alpha)/(\alpha-1)}(1-m_0)^{-(p+2-\alpha)} \qquad (4.2)
\end{aligned}$$

by taking ε_0 sufficiently small. Hence for any $0 \leqslant t \leqslant T$, we solve the above inequality and obtain

$$\int_0^1 (1-u)^{-p}\,dx = \int_0^1 w_p^{\frac{2p}{p+\alpha}}\,dx \leqslant \left\|\frac{1}{1-u_0}\right\|_\infty^p + Cp^{1+(p+2-\alpha)/(\alpha-1)}(1-m_0)^{-(p+2-\alpha)}T,$$

which yields

$$\begin{aligned}
\left\|\frac{1}{1-u(t,\cdot)}\right\|_p &\leqslant \left\|\frac{1}{1-u_0}\right\|_\infty + \left[Cp^{1+(p+2-\alpha)/(\alpha-1)}(1-m_0)^{-(p+2-\alpha)}T\right]^{1/p} \\
&\leqslant \left\|\frac{1}{1-u_0}\right\|_\infty + \frac{p^{1/(\alpha-1)}}{1-m_0}\left[Cp^{1+(2-\alpha)/(\alpha-1)}(1-m_0)^{-(2-\alpha)}T\right]^{1/p} \\
&\leqslant \left\|\frac{1}{1-u_0}\right\|_\infty + \frac{Cp^{1/(\alpha-1)}}{1-m_0} \\
&\leqslant \left\|\frac{1}{1-u_0}\right\|_\infty + Cp^{1/(\alpha-1)}.
\end{aligned}$$

Therefore for any fixed $T > 0$ and $0 \leqslant t \leqslant T$, $1/(1-u(t,\cdot)) \in L^p(0,1)$ for $3\alpha \leqslant p < \infty$. Since the L^p norm is an increasing function in p when $|\Omega| = 1$,

$$\frac{1}{1-u(t,\cdot)} \in L^p(0,1) \quad \text{for } 1 \leqslant p < \infty.$$

Step 2. Using a similar proof as for Lemma 2.2, it can be shown that for $n = 1$, there exists a constant $C > 0$, which is independent of p, such that

$$\|w\|_{2(p+2-\alpha)/(p+\alpha)} \leqslant C\left\{\|w_x\|_2^{\theta}\|w\|_1^{1-\theta} + \|w\|_1\right\} \leqslant C\left\{\left(\frac{\varepsilon_0}{p^2}\right)^{\frac{p+\alpha}{2(p+2-\alpha)}}\|w_x\|_2 + \left(\frac{p^2}{\varepsilon_0}\right)^{\frac{p+\alpha}{2(p+2-\alpha)}\frac{\theta}{1-\theta}}\|w\|_1\right\} \tag{4.3}$$

where $\theta = (p+4-3\alpha)/3(p+2-\alpha)$. Setting $w = w_p$ and using this inequality in (4.1), we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega w_p^{\frac{2p}{p+\alpha}}\,dx &\leqslant -\int_\Omega \left|\frac{\partial w_p}{\partial x}\right|^2 dx + Cp^2\|w_p\|_{2(p+2-\alpha)/(p+\alpha)}^{2(p+2-\alpha)/(p+\alpha)} \\ &\leqslant -\left\|\frac{\partial w_p}{\partial x}\right\|_2^2 + Cp^2\left\{\frac{\varepsilon_0}{p^2}\left\|\frac{\partial w_p}{\partial x}\right\|_2^{\frac{2(p+2-\alpha)}{p+\alpha}} + \left(\frac{p^2}{\varepsilon_0}\right)^{\theta/(1-\theta)}\|w_p\|_1^{2(p+2-\alpha)/(p+\alpha)}\right\}.\end{aligned}$$

Since $w_p \geqslant 1$,

$$\left\|\frac{\partial w_p}{\partial x}\right\|_2^{\frac{2(p+2-\alpha)}{p+\alpha}} \leqslant \left\|\frac{\partial w_p}{\partial x}\right\|_2^2 + 1 \leqslant \left\|\frac{\partial w_p}{\partial x}\right\|_2^2 + \|w_p\|_1.$$

By setting ε_0 to be sufficiently small, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega w_p^{\frac{2p}{p+\alpha}}\,dx &\leqslant -\frac{1}{2}\left\|\frac{\partial w_p}{\partial x}\right\|_2^2 + Cp^2\left\{p^{2\theta/(1-\theta)}\|w_p\|_1^{2(p+2-\alpha)/(p+\alpha)}\right\} \\ &\leqslant -\frac{1}{2}\int_\Omega\left|\frac{\partial w_p}{\partial x}\right|^2 dx + Cp^{2+2\theta/(1-\theta)}\|w_p\|_1^2.\end{aligned}$$

Let $p \geqslant 5$ so that $\frac{\theta}{1-\theta} \leqslant 1$. Thus the above inequality becomes

$$\frac{d}{dt}\int_\Omega w_p^{\frac{2p}{p+\alpha}}\,dx \leqslant -\frac{1}{2}\left\|\frac{\partial w_p}{\partial x}\right\|_2^2 + Cp^4\|w_p\|_1^2. \tag{4.4}$$

Now we use (2.7) to obtain

$$\frac{d}{dt}\int_\Omega w_p^{\frac{2p}{p+\alpha}}\,dx \leqslant -m\|w_p\|_{\frac{2p}{p+\alpha}}^2 + Cp^4\|w_p\|_1^2 \leqslant -m\|w_p\|_{\frac{2p}{p+\alpha}}^{\frac{2p}{p+\alpha}} + Cp^4\|w_p\|_1^2.$$

This equation is the same as (2.8) when p^{2+n} is replaced by p^4. Hence a repetition of Step 3 in the proof of Lemma 2.4 will yield $1/(1-u(t,\cdot)) \in L^\infty(0,1)$. The proof of Theorem 1.2 is complete. □

5. Numerical experiments and future works

We have established the global existence of solutions to the fast diffusion chemotaxis model (1.1), which is a modification of classical (minimal) Keller–Segel model. Does solutions to this model inherit features associated with the Keller–Segel model such as its aggregation patterns? In this section, we will give numerical evidence of such patterns when the physical parameters in (1.1) are in the proper regime. It will be interesting to validate the existence of such patterns using qualitative analysis.

Take any constant $u^* > 0$ and $v^* \equiv f(u^*)/\beta$. Then (u^*, v^*) is a spatially homogeneous steady state solution to (1.1). Since transient solutions exist for all time, it is likely that they will converge to a steady state solution. Naturally we investigate the range of physical parameters which give rise to unstable constant steady state solutions. This may indicate the existence of non-homogeneous steady state solutions which correspond to aggregation patterns.

A standard linearization of the system (1.1) about the steady state (u^*, v^*) leads to

$$\begin{cases} U_t = \dfrac{1}{(1-u^*)^\alpha}\Delta U - \chi(u^*, v^*)\Delta V, \\ V_t = d\Delta V + f'(u^*)U - \beta V \end{cases} \tag{5.1}$$

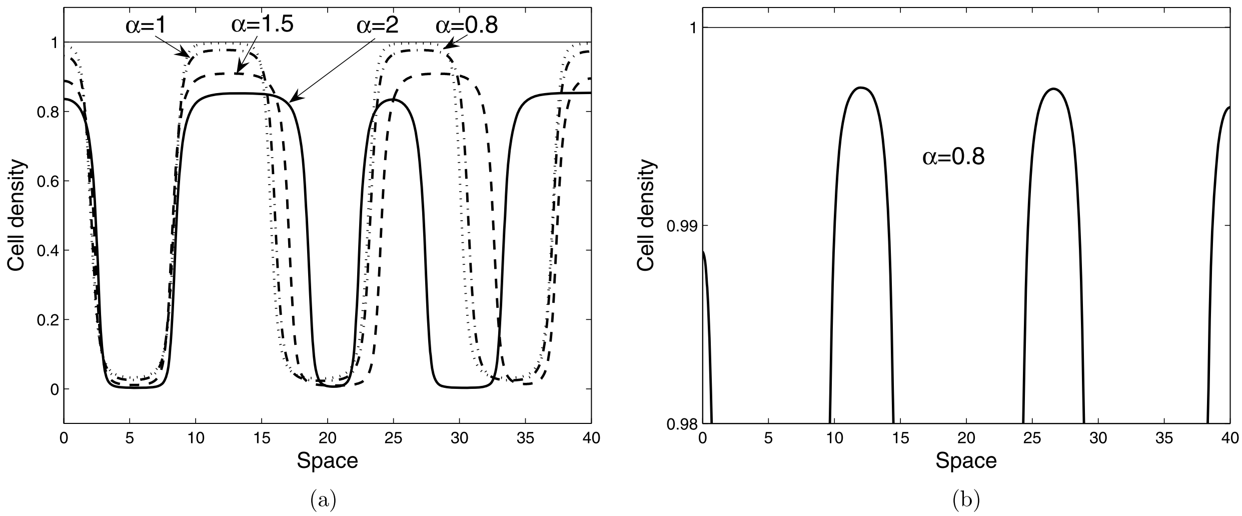

Fig. 1. Peak-like solutions for the model (1.1) at final time $t = 200$. We employ $\beta = 2, d = 1$, $u^* = 0.5$, $\kappa = 2\beta/(u^*(1-u^*)^\alpha)$, initial data $u(0,x) = 0.5 + 0.1\exp(-10x^2)$, $v(0,x) = 0.25 + 0.1\exp(-10x^2)$, and discretize the domain $[0, 40]$ with 800 grid points. (a) Numerical solutions of (1.1) for a range of α. (b) A magnified view of the solution when $\alpha = 0.8$.

with zero Neumann boundary conditions on both U and V, which can be thought of as small perturbations from (u^*, v^*). Let λ_k and φ_k be the kth eigenvalue and the corresponding eigenfunction of the Laplacian operator with zero Neumann boundary conditions in the domain Ω, respectively, i.e. $\Delta\varphi_k + \lambda_k\varphi_k = 0$, $k = 0, 1, 2, \ldots$. It is known that $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ with $\varphi_0 = 1$. Restrict ourselves to solutions of the form

$$\begin{pmatrix} U \\ V \end{pmatrix} = \varphi_k e^{\delta t} \overrightarrow{W_k}, \quad k = 0, 1, 2, \ldots, \tag{5.2}$$

where $\overrightarrow{W_k} \in \mathbb{R}^2$ is a non-zero constant vector. For a given initial condition, $\int_\Omega u(t,\cdot)$ is conserved in time so that only $\overrightarrow{W_0}$ of the form $(0,1)^T$ is allowed. Using this fact one can check the stability of the zeroth mode corresponding to $\lambda_0 = 0$. Thus we can focus on (5.2) for $k = 1, 2, \ldots$ only.

Substituting this into (5.1), one can deduce that δ is an eigenvalue of the stability matrix

$$M_k = \begin{pmatrix} -\lambda_k/(1-u^*)^\alpha & \lambda_k \chi(u^*, v^*) \\ f'(u^*) & -\beta - d\lambda_k \end{pmatrix},$$

provided we take $\overrightarrow{W_k}$ to be its eigenvector. If δ has positive real part for some λ_k, the homogeneous steady state is linearly unstable. It is an easy calculation to show that in our case this condition is equivalent to $\det M_k < 0$, which simplifies to

$$\chi(u^*, v^*) f'(u^*)(1-u^*)^\alpha > \beta + d\lambda_k, \quad k = 1, 2, \ldots. \tag{5.3}$$

In our numerical simulation, we let $f(u) = u$, $\chi(u,v) = \kappa u$ for some positive constant κ, and $\Omega = [0, L]$ so that $\lambda_k = k^2\pi^2/L^2$. Then the instability parameter region is governed by

$$\kappa u^*(1-u^*)^\alpha > \beta + d\pi^2/L^2.$$

We choose the parameters in this region and experiment with various initial data which include both u being close to 1 in some spatial region and small spatial perturbations of the homogeneous steady state (u^*, v^*). A summary of the numerical results is given in Figs. 1–3. As expected, using initial data which are incompatible to the boundary conditions yields the same conclusion.

Fig. 1(a) shows the final "steady state" at large time for a range of α. In particular a magnified view of the solution when $\alpha = 0.8$ in Fig. 1(b) suggests that the solution exists globally for $0 < \alpha \leqslant 1$. This case is not covered in our theorems.

Both Figs. 2 and 3 illustrate a typical transient solution as time evolves. Fig. 2(b) shows the transient in a longer time scale than that in Fig. 2(a). The monotone initial datum quickly develops multiple spatial peaks, which then merge to form larger aggregations. Such patterns are typical for chemotaxis models (see [10]). Very often solutions will eventually congregate to a single peak sitting either on the boundary or the interior of the domain. This agrees with the fact that the most unstable mode as predicted in (5.3) is the one that associates with λ_1. It is an open question in chemotaxis models and will be interesting if a rigorous analysis can be performed. Fig. 3 demonstrates the evolutions of merging peaks in a 3D plot.

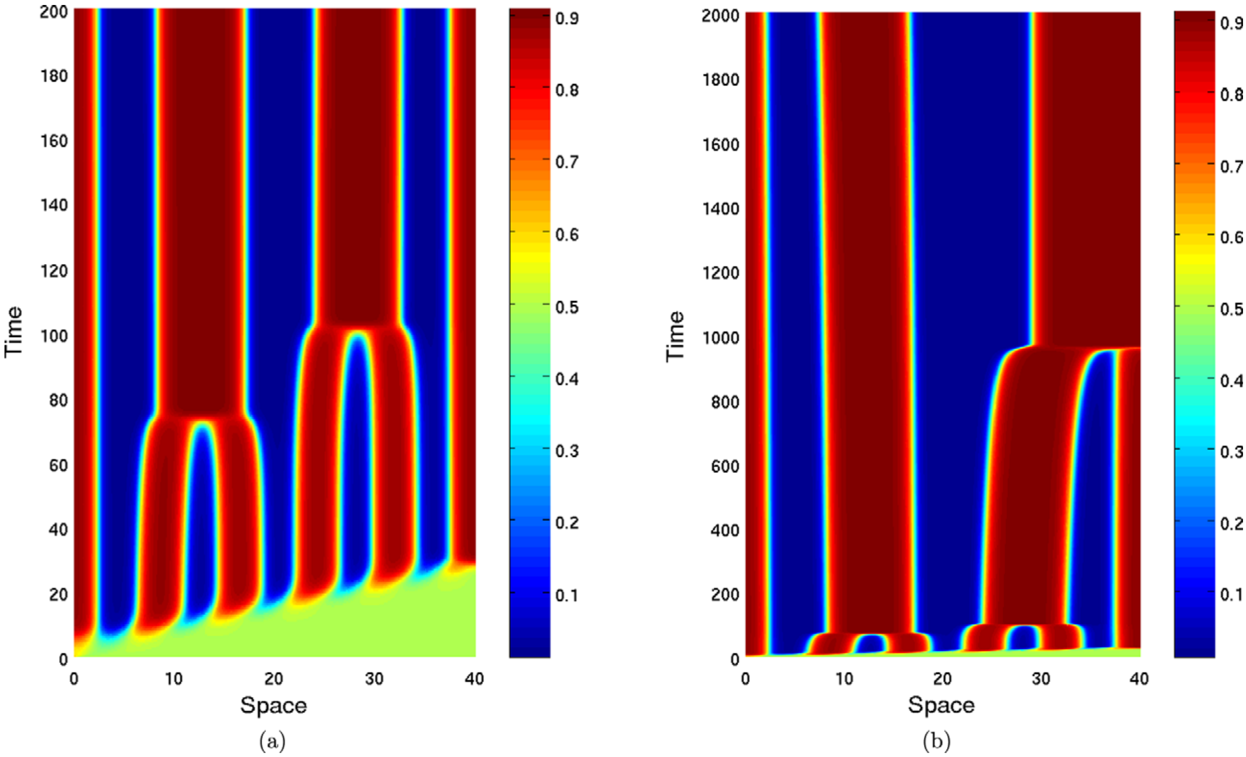

Fig. 2. Pattern formation during transience for the model (1.1) when $\alpha = 1.5$. Other parameters and initial conditions are the same as in Fig. 1. (a) Shorter time scale. (b) Longer time scale.

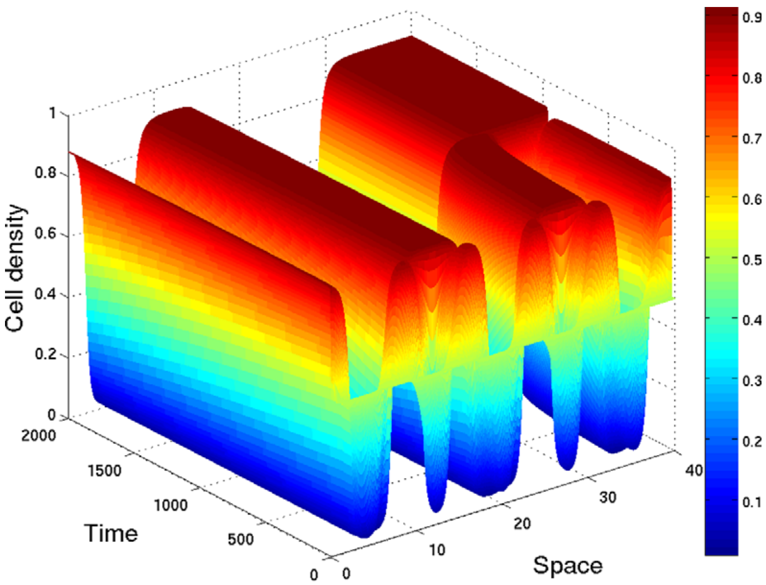

Fig. 3. A 3D plot of the situation in Fig. 2. It illustrates the evolution of merging peaks for the model (1.1).

This paper opens a door for engaging fast diffusion in chemotaxis models. There are many open questions, for example:

(a) The global existence of solutions of the model (1.1) for $0 < \alpha \leqslant 1$ in one dimension and for $0 < \alpha < 2$ in higher dimensions.
(b) The global uniform boundedness of the solutions for large time. Once this is proved, with Lyapunov functionals being known for some special forms of χ and f in (1.1) (e.g. see [8,18]), it is then natural to study (multiple) steady state solutions of (1.1) and their local and global stability.

(c) Establishing the merging and aggregation patterns rigorously when the physical parameters in (1.1) are suitably restricted.

Finally we conclude that (very) fast diffusion in chemotaxis model leads to global existence of solutions in time. This is a complement to the mechanisms of preventing blow-up in chemotaxis models summarized in the paper [10].

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