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Global existence and asymptotic behavior of the Boussinesq–Burgers system

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ABSTRACT

This paper is concerned with the Boussinesq–Burgers system which models the propagation of bores by combing the dissipation, dispersion and nonlinearity. We establish the global existence and asymptotical behavior of classical solutions of the initial value boundary problem of the Boussinesq–Burgers system with the help of a Lyapunov functional and the technique of Moser iteration. Particularly we show that the solution converges to the unique constant stationary solution exponentially as time tends to infinity.

1. Introduction

The water flows from the higher to lower elevation are termed bores which occur readily in nature. There are two classes of bores: strong and weak bores. The former refers to the rapid turbulent change of water level, while the latter have a gently sloping or oscillatory transition between the different water levels. Although there is a large number of literature that discusses the propagation of bores (cf. [20] and references therein), little is known mathematically about this phenomenon. While strong bores are hard to deal with mathematically due to the difficulty of modeling the wave breaks/turbulence, weak bores are relatively easier to handle. There are two well-known models describing the propagation of weak bores. One is the Korteweg–de Vries equation (KdV equation for short) which can be expressed in non-dimensional variables as

$$v_t + vv_x + v_{xxx} = 0.$$

The other one is the Boussinesq system expressed as follows (cf. [21])

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$$\begin{cases} \rho_t + w_x + (w\rho)_x = 0, \\ w_t + \rho_x + ww_x - \delta w_{xxt} = 0, \end{cases}$$

where $\rho(x,t)$ and $w(x,t)$ represent the height and the velocity of the free surface of the fluid above the bottom, respectively, and $\delta > 0$ is a parameter measuring the strength of fluid dispersion. These two models contain the nonlinearity and dispersive effect. The Boussinesq system and its variants have been extensively studied in the literature (see [5,6] and references therein). However it was pointed out in [3, 12,13] that the dissipative effects must be included, at least in the laboratory scale, in order to accurately predict the wave propagation. The simplest way of incorporating the dissipation is to append a Burgers-type term to the KdV or Boussinesq system, which then yields the so-called KdV–Burgers equation or the Boussinesq–Burgers system, respectively. The KdV–Burgers equation has been well studied in the literature (see [23] and references therein). In this paper, we consider the Boussinesq–Burgers system which reads as

$$\begin{cases} \rho_t + w_x + (w\rho)_x = \varepsilon \rho_{xx}, \\ w_t + \rho_x + ww_x - w_{xxt} = \mu w_{xx} \end{cases} \tag{1.1}$$

with $\varepsilon, \mu > 0$. Compared to the KdV–Burgers equation, the Boussinesq–Burgers system (1.1) is not so widely studied. There are a few results on its variants (e.g., see [8,18] and references therein) for the whole interval $\mathbb{R}$. As we know, the only result of the Boussinesq–Burgers system (1.1) is the existence of traveling wave solutions obtained in [21] in the whole interval $\mathbb{R}$ with bore-like data, where $\varepsilon = \mu$. The goal of this paper is to study the initial–boundary value problem of the Boussinesq–Burgers system in a bounded interval. To this end, we make a change of variable as in [21] by letting $u(x,t) = 1 + \rho(x,t)$. Then the initial–boundary value problem of the Boussinesq–Burgers system considered in the present paper reads:

$$\begin{cases} u_t + (uw)_x = \varepsilon u_{xx}, & x \in (0,1),\ t > 0, \\ w_t + (u + \frac{w^2}{2})_x = \mu w_{xx} + \delta w_{xxt}, & x \in (0,1),\ t > 0, \\ (u,w)(x,0) = (u_0, w_0)(x), & x \in [0,1], \\ u_x|_{x=0,1} = w|_{x=0,1} = 0, & t > 0 \end{cases} \tag{1.2}$$

where $\varepsilon, \mu, \delta > 0$. It is noted that when the dissipation and dispersion are ignored (i.e. $\varepsilon = \mu = \delta = 0$), the system becomes the well-known water wave equation [9]. As $\delta = 0$ and the nonlinear advection term is $\frac{w^2}{2}$ is replaced by $-\mu w^2$, the model becomes a system derived from the chemotactic movement considered in [16,17,22]. In this paper, we shall establish the global existence and asymptotic behavior of classical solutions to the initial–boundary value problem (1.2). We point out that it is physically meaningful to consider $u(x,t) \geq 0$ since the bore is weak (i.e., $|\rho|$ is small). The main results of this paper are given in the following theorem.

Theorem 1.1. *Assume that $(u_0, w_0) \in W^{2,p}(0,1)$ with $p > 3$ and $u_0 \geq 0, u_0 \not\equiv 0$. Then, for any $\varepsilon, \mu, \delta > 0$, the problem (1.2) has a unique classical solution (u,w) in $(0,1) \times (0,\infty)$ with $u > 0$ such that $(u,w) \in C^0([0,1] \times [0,\infty)) \cap C^{2,1}([0,1] \times (0,\infty))$. Moreover there is a constant $\beta > 0$ such that for all $t > 0$:*

$$\|u - \bar{u}_0\|_{L^\infty(0,1)} + \|w\|_{L^\infty(0,1)} \leq Ce^{-\beta t}$$

where $\bar{u}_0 = \int_0^1 u_0 dx$ denotes the average of $u_0(x)$ over $(0,1)$.

2. Local existence

To deal with the nonlinear term $(w^2/2)_x$ and prove the existence of local solutions of (1.2), we need some regularity assumptions on the initial data. Since the dispersion term w_{xxt} contains the temporal derivative, the proof of local existence will be somewhat different from the standard argument for the parabolic system. Here we shall employ the fact that the dispersion term w_{xxt} has a stronger dissipative effect than the diffusion w_{xx} to construct a contracting mapping to prove the local existence. Inspired by a result from [15], we depart with a linear problem

$$\begin{cases} w_t - \delta w_{xxt} = f(x,t), & x \in (0,1),\ t > 0, \\ w(0,t) = w(1,t) = 0, & t > 0 \\ w(x,0) = w_0(x), & x \in [0,1]. \end{cases} \tag{2.1}$$

For this linear problem, we have the following result.

Lemma 2.1. *Assume that $w_0 \in W^{2,p}(0,1)$ and $f \in L^p(0,1)$ for $p \geq 1$. Then the problem (2.1) has a unique solution in the cylinder $Q_T = (0,1) \times (0,T)$ for some $T > 0$, which satisfies*

$$\|w\|_{C([0,T];W^{2,p}(0,1))} \leq \|w_0\|_{W^{2,p}(0,1)} + c_1 T \|f\|_{C([0,T];L^p(0,1))}.$$

Proof. By a change of variable $v(x,t) = w_t(x,t)$, the linear problem (2.1) becomes an elliptic problem with the parameter $t \in (0,T)$

$$\begin{cases} v_x - \delta v_{xx} = f(x,t), & x \in (0,1),\ t \in (0,T) \\ v(0,t) = v(1,t) = 0, & t \in (0,T). \end{cases} \tag{2.2}$$

If $f \in L^p(0,1)$, then by the Agmon–Douglas–Nirenberg theorem [1,2], the problem (2.2) has a unique solution $v \in W^{2,p}(0,1)$ such that $\|v\|_{W^{2,p}(0,1)} \leq c_1 \|f\|_{L^p(0,1)}$ for some $c_1 > 0$, which implies that the solution of (2.1) satisfies

$$\big\|w_t(\cdot,t)\big\|_{W^{2,p}(0,1)} \leq c_1 \|f\|_{L^p(0,1)}. \tag{2.3}$$

Noticing that

$$w(x,t) = w_0(x) + \int_0^t w_s(x,s)ds$$

we have

$$\|w\|_{C([0,T];W^{2,p}(0,1))} \leq \|w_0\|_{W^{2,p}(0,1)} + T\|w_t\|_{C([0,T];L^p(0,1))}.$$

Then the lemma is proved by applying (2.3) into the above inequality. □

Using the above results, we can prove the following local existence theorem.

Lemma 2.2 *(Local existence). Assume that $(u_0, w_0) \in W^{2,p}(0,1)$ with $p > 3$ and $u_0 \geq 0, u_0 \not\equiv 0$. Then there exists $T_{max} \in (0,\infty]$ such that (1.2) has a unique classical solution $(u,w) \in C^0([0,1] \times [0,T_{max})) \cap C^{2,1}([0,1] \times (0,T_{max}))$. Moreover, $u > 0$ in $(0,1) \times (0,T_{max})$ and*

$$\text{if } T_{max} < \infty, \quad \text{then } \big\|u(\cdot,t)\big\|_{L^\infty(0,1)} + \big\|w(\cdot,t)\big\|_{L^\infty(0,1)} \to \infty \quad \text{as } t \nearrow T_{max}.$$

Proof. Let $T \in (0,1)$ to be specified below and denote $Q_T := (0,1) \times (0,T)$. In the Banach space

$$X := C\big([0,T]; W^{2,p}(0,1)\big) \times C\big([0,T]; W^{2,p}(0,1)\big),$$

we define

$$X_T := \big\{(u,w) \in X \;\big|\; \big\|u(\cdot,t)\big\|_{C^{1,0}(\bar{Q}_T)} \le R \text{ and } \big\|w(\cdot,t)\big\|_{C^{1,0}(\bar{Q}_T)} \le R\big\}$$

where

$$R := \|u_0\|_{W^{2,p_0}(0,1)} + \|w_0\|_{W^{2,p_0}(0,1)} + \|u_0\|_{C^1[0,1]} + \|w_0\|_{C^1[0,1]} + 1.$$

With this R, we introduce a mapping $\Phi : X_T \longmapsto X_T$ such that given $(\tilde{u}, \tilde{w}) \in X_T$, $\Phi(\tilde{u}, \tilde{w}) = (u,w)$ where u is the solution of

$$\begin{cases} u_t - \varepsilon u_{xx} + w u_x + w_x u = 0, & x \in (0,1),\ t \in (0,T), \\ u_x|_{x=0,1} = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,1], \end{cases} \tag{2.4}$$

and w is the solution of

$$\begin{cases} w_t - \delta w_{xxt} = \mu \tilde{w}_{xx} + (\tilde{w}^2/2 + \tilde{u})_x, & x \in (0,1), t \in (0,T), \\ w|_{x=0,1} = 0, & t \in (0,T), \\ w(x,0) = w_0(x), & x \in [0,1]. \end{cases} \tag{2.5}$$

We shall show that for T small enough Φ has a unique fixed point.

For consistency, throughout the remainder of this section we denote

$$W^{2,1,p}(Q_T) = \big\{u \;\big|\; u, u_x, u_{xx}, u_t \in L^p(Q_T)\big\}$$

for $p \ge 1$, equipped with the norm

$$\|u\|_{W^{2,1,p}(Q_T)} = \|u\|_{L^p(Q_T)} + \|u_x\|_{L^p(Q_T)} + \|u_{xx}\|_{L^p(Q_T)} + \|u_t\|_{L^p(Q_T)}.$$

Since (2.5) is an elliptic problem for w_t, the solvability of this problem follows from Lemma 2.1. Indeed, since $(\tilde{u}, \tilde{w}) \in X_T$, then $\tilde{w}_{xx}(\cdot,t) \in L^p(0,1)$ for all $t \in [0,T]$. That is there is a $c_2 > 0$ such that $\sup_{t\in[0,T]} \|w_{xx}(\cdot,t)\|_{L^p(0,1)} \le c_2$. Furthermore $(\tilde{u}, \tilde{w}) \in X_T$, along with the Sobolev embedding theorem: $W^{2,p}(0,1) \hookrightarrow C^{1,0}(0,1)$, implies that $\tilde{u}_x \in L^\infty(0,1)$ and $(\tilde{w}^2/2)_x = \tilde{w}\tilde{w}_x \in L^\infty(0,1)$ such that $\|(\tilde{w}^2/2 + \tilde{u})_x\|_{L^\infty(0,1)} \le R(1+R)$. Then by Lemma 2.1, we obtain a unique solution $w \in W^{2,1,p}(Q_T)$ to (2.5) such that

$$\begin{aligned} \|w\|_{W^{2,1,p}(Q_T)} &\le \|w\|_{C([0,T];W^{2,p}(0,1))} + \|w_t\|_{C([0,T];W^{2,p}(0,1))} \\ &\le \|w_0\|_{W^{2,p}(0,1)} + c_1(1+T)\big[c_2 + R(1+R)\big] \\ &\le R + 2c_1\big[c_2 + R(1+R)\big] =: c_3(R) \end{aligned}$$

where we have used the fact $T \in (0,1)$ and $\|w_0\|_{W^{2,p}(0,1)} \le R$. This, combined with the Sobolev embedding theorem [14, Lemma II.3.3], upgrades the regularity of the solution such that

$$\|w\|_{C^{1+\theta,(1+\theta)/2}(\bar{Q}_T)} \le c_4 \|w\|_{W^{2,1,p}(Q_T)} \le c_5(R) := c_4 \cdot c_3(R) \tag{2.6}$$

where $\theta := 1 - \frac{3}{p}$ $(p > 3)$. Thus, we have

$$\begin{aligned}\|w\|_{C^{1,0}(\bar{Q}_T)} &\le \big\|w(x,t) - w(x,0)\big\|_{C^{1,0}(\bar{Q}_T)} + \big\|w(x,0)\big\|_{C^{1,0}(\bar{Q}_T)} \\ &\le T^{\frac{1+\theta}{2}}\|w\|_{C^{0,(1+\theta)/2}(\bar{Q}_T)} + \|w_0\|_{C^{1,0}(\bar{Q}_T)} \\ &\le c_5(R)T^{\frac{1+\theta}{2}} + \|w_0\|_{C^{1,0}(\bar{Q}_T)}.\end{aligned}$$

If we let T be small such that $T \le (\frac{1}{c_5(R)})^{\frac{2}{1+\theta}}$, then it follows that

$$\|w\|_{C^{1,0}(\bar{Q}_T)} \le \|w_0\|_{C^{1,0}(\bar{Q}_T)} + 1 = R. \tag{2.7}$$

Now we turn to the problem (2.4). Note that (2.6) yields a constant $c_6(R) > 0$ such that $\|w\|_{L^\infty(0,1)} + \|w_x\|_{L^\infty(0,1)} \le c_6(R)$. Due to $\|u_0\|_{W^{2,p}(0,1)} \le R$, from the linear parabolic L^p-theory [10, Theorem 2.3] and [14, Theorem IV.9.1], we conclude that the problem (2.4) has a unique solution $u(x,t) \in W^{2,1,p}(Q_T)$ such that

$$\|u\|_{W^{2,1,p}(Q_T)} \le c_6(R)\|u_0\|_{W^{2,p}(0,1)} \le c_6(R) \cdot R =: c_7(R).$$

Using the same argument as deriving (2.6), we can find some constant $c_8(R) > 0$ such that

$$\|u\|_{C^{1+\theta,(1+\theta)/2}(\bar{Q}_T)} \le c_8(R).$$

Then by the same idea used for w, if we let T be small such that $T \le (\frac{1}{c_8(R)})^{\frac{2}{1+\theta}}$, we obtain

$$\|u\|_{C^{1,0}(\bar{Q}_T)} \le \|u_0\|_{C^{1,0}(\bar{Q}_T)} + 1 = R \tag{2.8}$$

which, along with (2.7), asserts that $(u,w) \in X_T$ for some $T > 0$. Hence the function Φ maps X_T into itself. By a direct adaptation of the above derivation, one can easily deduce that if T is further diminished then Φ in fact becomes a contraction on X_T. For such T we therefore conclude from the contraction mapping principle [11, Theorem 5.1] that there exists a unique fixed point $(u,w) \in X_T$ such that $\Phi(u,w) = (u,w)$. This unique fixed point in X_T corresponds to a unique solution of (1.2) in X_T. This solution may be further prolonged in the interval $[0, T_{max})$ with either $T_{max} = \infty$ or $T_{max} < \infty$, where in the latter case

$$\big\|u(\cdot,t)\big\|_{L^\infty(0,1)} + \big\|w(\cdot,t)\big\|_{L^\infty(0,1)} \to \infty \quad \text{as } t \nearrow T_{max},$$

because T_0 depends only on R. Now it remains to derive the regularity of solutions to finish the proof. Indeed by (2.4), $w, w_x \in C^{\theta,\theta/2}(\bar{Q}_T)$ and the classical regularity of parabolic equations [14, Theorem V.6.1], we obtain

$$u(x,t) \in C^{2+\theta,(1+\theta)/2}\big([0,1] \times [\eta, T]\big) \quad \text{for all } \eta \in (0, T_0].$$

Similar argument leads to

$$w(x,t) \in C^{2+\theta,(1+\theta)/2}\big([0,1] \times [\eta, T]\big) \quad \text{for all } \eta \in (0, T_0].$$

This proves the regularity of the solution (u,w) to (1.2). Finally, the positivity of u results from the strong parabolic maximum principle, because $u_0 \not\equiv 0$ ensures that $u \not\equiv 0$. This completes the proof of Lemma 2.2. $\square$

3. Global dynamics

3.1. Global boundedness

Hereafter for simplicity, the norm of the space $L^p(0,1)$, $1 \leq p \leq \infty$ will be denoted simply by $\|\cdot\|_{L^p}$ by omitting the interval $(0,1)$. The key in our analysis is the following Lyapunov functional

$$\mathcal{F}(u,w) := \int_0^1 \left(u \ln u + \frac{w^2}{2} + \frac{\delta}{2} w_x^2\right) dx, \tag{3.1}$$

for which we have the following result.

Lemma 3.1. *The classical solution (u,w) to (1.2) satisfies the equality*

$$\frac{d}{dt}\mathcal{F}\big(u(t),w(t)\big) = -\int_0^1 \left(\frac{\varepsilon u_x^2}{u} + \mu w_x^2\right) dx \quad \textit{for all } t \in (0, T_{max}). \tag{3.2}$$

Proof. From the first two equations in (1.2), we have with the integration by parts

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}\big(u(t),w(t)\big) &= \int_0^1 \big((\ln u + 1)u_t + w w_t\big) dx + \delta \int_0^1 w_x w_{xt} dx \\
&= -\varepsilon \int_0^1 \frac{u_x^2}{u} dx + \int_0^1 u_x w dx - \mu \int_0^1 w_x^2 dx - \delta \int_0^1 w_x w_{xt} dx \\
&\quad + \frac{1}{6} \int_0^1 \left(w^3\right)_x dx - \int_0^1 u_x w dx + \delta \int_0^1 w_x w_{xt} dx \\
&= -\int_0^1 \left(\frac{\varepsilon u_x^2}{u} + \mu w_x^2\right) dx,
\end{aligned}$$

where we have used the boundary conditions $u_x|_{x=0,1} = w|_{x=0,1} = 0$. This completes the proof of Lemma 3.1. $\square$

Then the following result is an immediate consequence of Lemma 3.1.

Lemma 3.2. *The classical solution (u,w) to (1.2) has the following properties for all $t \in (0, T_{max})$:*

$$\int_0^1 \left(w^2 + \delta w_x^2\right) dx \leq 2\mathcal{F}(u_0, w_0) + 2/e,$$

$$\int_0^t \int_0^1 \left(\frac{\varepsilon u_x^2}{u} + \mu w_x^2\right) dx ds \leq \mathcal{F}(u_0, w_0) + 1/e.$$

Proof. Integrating (3.2) over $t \in (0, T_{max})$ we obtain

$$\int_0^1 \left(\frac{w^2}{2} + \frac{\delta}{2} w_x^2\right) dx + \int_0^t \int_0^1 \left(\frac{\varepsilon u_x^2}{u} + \mu w_x^2\right) dx ds = \mathcal{F}(u_0, w_0) - \int_0^1 u(x,t) \ln u(x,t) dx$$

for all $t \in (0, T_{max})$. The fact that $-\xi \ln \xi \le \frac{1}{e}$ for all $\xi > 0$ completes the proof. □

To elongate the local solutions to the global ones, it suffices to show that $\|u\|_{L^\infty}$ and $\|w\|_{L^\infty}$ are bounded in time by the extension criterion in Lemma 2.2. Next we shall employ the method of Moser iteration to derive the *a priori* L^∞-norm of solutions of the problem (1.2). Before embarking on this, we remark that the L^1-norm of u is conserved by integrating the first equation of (1.2) with the boundary condition:

$$\bar{u} = \int_0^1 u dx = \int_0^1 u_0 dx =: \bar{u}_0, \tag{3.3}$$

which will be essentially applied in our analysis. Moreover the following interpolation inequality will be used later.

Gagliardo–Nirenberg inequality [19]: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $p, q \ge 1$ satisfy $(n-q)p \le nq$ and let $r \in (0, p)$. Then, for any $u(x) \in W^{1,q}(\Omega) \cap L^r(\Omega)$, there exists a constant $c_1 > 0$ such that

$$\|u\|_{L^p(\Omega)} \le c_1 \|\nabla u\|^a_{L^q(\Omega)} \|u\|^{1-a}_{L^r(\Omega)} + c_2 \|u\|_{L^r(\Omega)} \tag{3.4}$$

with $a \in (0,1)$ satisfying

$$\frac{n}{p} = a\left(\frac{n}{q} - 1\right) + \frac{n}{r}(1-a).$$

Then we are ready to prove the following global estimates on the solution component u.

Lemma 3.3. *Assume that $u_0 \in L^1 \cap L^\infty$. Then there is some constant $c(\varepsilon, \mu, \delta) > 0$ such that the classical solution (u, w) of (1.2) satisfies*

$$\|u\|_{L^\infty} \le c \quad \textit{for all } t \in (0, T_{max}).$$

Proof. Multiplying the first equation in (1.2) by pu^{p-1} and integrating the result over $[0,1]$, we obtain with the Hölder inequality that

$$\begin{aligned}
\frac{d}{dt}\int_0^1 u^p dx &= -p(p-1)\int_0^1 u^{p-2}|\nabla u|^2 dx - p(p-1)\int_0^1 u^{p-1} u_x w dx \\
&= -\frac{4(p-1)}{p}\int_0^1 \left|\left(u^{\frac{p}{2}}\right)_x\right|^2 dx + (p-1)\int_0^1 u^p w_x dx \\
&\le -\frac{4(p-1)}{p}\int_0^1 \left|\left(u^{\frac{p}{2}}\right)_x\right|^2 dx + (p-1)\left(\int_0^1 u^{2p} dx\right)^{\frac{1}{2}} \cdot \left(\int_0^1 w_x^2 dx\right)^{\frac{1}{2}} \\
&\le -\frac{4(p-1)}{p}\int_0^1 \left|\left(u^{\frac{p}{2}}\right)_x\right|^2 dx + \sqrt{\frac{C_0}{\delta}}(p-1)\left(\int_0^1 u^{2p} dx\right)^{\frac{1}{2}}
\end{aligned} \tag{3.5}$$

for all $t \in (0, T_{max})$, where we have used the fact that $w|_{x=0,1} = 0$ and the first inequality in Lemma 3.2. The Gagliardo–Nirenberg inequality (3.4) with inequality $(a+b)^2 \le 2(a^2+b^2)$ yields that $\|f\|_{L^4}^2 \le 2c_1^2(\|f_x\| \cdot \|f\|_{L^1} + \|f\|_{L^1}^2)$, which entails that

$$\left(\int_0^1 u^{2p} dx\right)^{\frac{1}{2}} = \left[\left(\int_0^1 (u^{\frac{p}{2}})^4\right)^{\frac{1}{4}}\right]^2 = \|u^{\frac{p}{2}}\|_{L^4}^2 \le 2c_1^2\left(\|(u^{\frac{p}{2}})_x\| \cdot \|u^{\frac{p}{2}}\|_{L^1} + \|u^{\frac{p}{2}}\|_{L^1}^2\right).$$

Then the above inequality with the Cauchy–Schwarz inequality gives rise to

$$\left(\int_0^1 u^{2p} dx\right)^{\frac{1}{2}} \le \frac{2}{c_2 p}\left\|(u^{\frac{p}{2}})_x\right\|^2 + 2c_1^2(1 + c_1^2 c_2 p)\left\|u^{\frac{p}{2}}\right\|_{L^1}^2 \tag{3.6}$$

where we choose $c_2 = \sqrt{\frac{C_0}{\delta}}$. Then substituting (3.6) into (3.5) yields

$$\frac{d}{dt}\int_0^1 u^p dx \le -\frac{2(p-1)}{p}\int_0^1 \left|(u^{\frac{p}{2}})_x\right|^2 dx + c_3\left(\int_0^1 u^{\frac{p}{2}} dx\right)^2 \tag{3.7}$$

with $c_3 = \sqrt{\frac{C_0}{\delta}}(p-1)2c_1^2(1 + c_1^2 c_2 p)$. Now adding the term $\int_0^1 u^p dx$ on both sides of (3.7), we get

$$\frac{d}{dt}\int_0^1 u^p dx + \int_0^1 u^p dx \le -\frac{2(p-1)}{p}\int_0^1 \left|(u^{\frac{p}{2}})_x\right|^2 dx + c_4\left(\int_0^1 u^{\frac{p}{2}} dx\right)^2 + \int_0^1 u^p dx. \tag{3.8}$$

Based on (3.8), we shall next use the Moser iteration procedure to derive that $\|u(\cdot,t)\|_{L^\infty}$ is bounded uniformly in time. To this end, we need the following interpolation inequality [14, p. 63]: for any $f \in W^{1,2}(\Omega)$, it holds

$$\|f - \bar{f}\|_{L^2(\Omega)}^2 \le c_4\|\nabla f\|_{L^2(\Omega)}^{2\alpha}\|f\|_{L^1(\Omega)}^{2(1-\alpha)},$$

where $\bar{f} = \frac{1}{|\Omega|}\int_\Omega f dx$, $\alpha = n/(n+2)$, and c_4 is a constant depending only on n and Ω. Then applying the Young inequality: $ab \le \epsilon a^p + (\epsilon p)^{-q/p}q^{-1}b^q$, $a, b, \epsilon, p, q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ into above inequality and using the fact $\|\bar{f}\|_{L^2} = \int_\Omega f dx = \|f\|_{L^1}$ gives

$$\|f\|_{L^2(\Omega)}^2 \le \epsilon\|\nabla f\|_{L^2(\Omega)}^2 + c_5\left(1 + \epsilon^{-\frac{n}{2}}\right)\|f\|_{L^1(\Omega)}^2 \quad \text{for any } \epsilon > 0, \tag{3.9}$$

where $c_5 > 0$ depends on n and Ω, but is independent of ϵ. Then employing (3.9) with $f = u^{\frac{p}{2}}$, $\epsilon = \frac{2(p-1)}{p}$, $n = 1$, $\Omega = (0,1)$, we have for $p \ge 2$

$$\int_0^1 u^p dx = \left\|u^{\frac{p}{2}}\right\|_{L^2}^2 \le \frac{2(p-1)}{p}\left\|(u^{\frac{p}{2}})_x\right\|_{L^2}^2 + c_6(1+p)\left\|u^{\frac{p}{2}}\right\|_{L^1}^2, \tag{3.10}$$

with some constant $c_6 > 0$. This, along with (3.8), yields

$$\frac{d}{dt}\int_0^1 u^p dx + \int_0^1 u^p dx \le c_6(1+p)\left(\int_0^1 u^{\frac{p}{2}} dx\right)^2$$

which leads to

$$\frac{d}{dt}\left(e^t\int_0^1 u^p dx\right) \le c_6 e^t(1+p)\left(\int_0^1 u^{\frac{p}{2}} dx\right)^2.$$

Then the integration of above inequality over the time interval $[0,t]$ for $0 < t < T_{max}$ gives

$$\int_0^1 u^p dx \le \int_0^1 u_0^p dx + c_6(1+p)\sup_{0\le t\le T_{max}}\left(\int_0^1 u^{\frac{p}{2}} dx\right)^2. \tag{3.11}$$

Now we define

$$A_p = \max\left\{\|u_0\|_{L^\infty}, \sup_{0\le t\le T_{max}}\left(\int_0^1 u^p dx\right)^{\frac{1}{p}}\right\} \quad \text{for all } p \ge 2.$$

Then it follows from (3.11) that

$$A_p \le \left[c_7(1+p)\right]^{\frac{1}{p}} A_{\frac{p}{2}}$$

for some constant $c_7 > 0$. Now taking $p = 2^k,\ k = 1, 2, \cdots$, one obtains

$$\begin{aligned}
A_{2^k} &\le c_7^{2^{-k}}\left(1+2^k\right)^{2^{-k}} A_{2^{k-1}} \\
&\le c_7^{2^{-k}+2^{-(k-1)}}\left(1+2^k\right)^{2^{-k}}\left(1+2^{k-1}\right)^{2^{-(k-1)}} A_{2^{k-2}} \\
&\vdots \\
&\le c_7^{2^{-k}+2^{-(k-1)}+\cdots+2^{-1}}\left(1+2^k\right)^{2^{-k}}\left(1+2^{k-1}\right)^{2^{-(k-1)}}\cdots(1+2)^{2^{-1}} A_1.
\end{aligned} \tag{3.12}$$

Noticing that $2^{-k} + 2^{-(k-1)} + \cdots + 2^{-1} \le 1$ and the series $\frac{k}{2^k} + \frac{k-1}{2^{k-1}} + \cdots + \frac{1}{2}$ is convergent, we can find a constant $c_8 > 0$ such that

$$\begin{aligned}
&\left(1+2^k\right)^{2^{-k}}\left(1+2^{k-1}\right)^{2^{-(k-1)}}\cdots(1+2)^{2^{-1}} \\
&= 2^{k2^{-k}}\left(2^{-k}+1\right)^{\frac{1}{2^k}}\cdot 2^{(k-1)2^{-(k-1)}}\left(2^{-(k-1)}+1\right)^{\frac{1}{2^{k-1}}}\cdots 2^{2^{-1}}\left(2^{-1}+1\right)^{2^{-1}} \\
&\le 2^{\frac{k}{2^k}+\frac{k-1}{2^{k-1}}+\cdots+\frac{1}{2}}\cdot 2^{\frac{1}{2^k}+\frac{1}{2^{k-1}}+\cdots+\frac{1}{2}} \\
&\le c_8.
\end{aligned}$$

Thus letting $k \to \infty$ in (3.12), we have

$$\begin{aligned}
\left\|u(\cdot,t)\right\|_{L^\infty} &\le c_9 A_1 = c_9\max\left\{\|u_0\|_{L^\infty}, \sup_{0\le t\le T_{max}}\left(\int_0^1 u dx\right)\right\} \\
&= c_9\max\left\{\|u_0\|_{L^\infty}, \|u_0\|_{L^1}\right\}
\end{aligned}$$

where $c_9 = c_7 c_8$. The proof is completed. □

3.2. Constant stationary solution

In this section, we shall employ the Lyapunov functional (3.1) to prove that the system (1.2) has only a unique constant stationary solution. The result is the following:

Lemma 3.4. *The only classical stationary solution of (1.2) is the constant pairs $(\bar{u}, 0)$ for $\bar{u} \in (0, \infty)$, where $\bar{u}$ denotes the average of u given in (3.3).*

Proof. By noting that the stationary solution (u_s, w_s) of (1.2) is also a solution to the time-dependent problem, we have

$$0 = \frac{d}{dt}\mathcal{F}(u_s, w_s) = -\int_0^1 \left(\frac{\varepsilon[(u_s)_x]^2}{u_s} + \mu\big[(w_s)_x\big]^2\right)dx$$

which indicates that $u_s = C_1$, and $w_s = C_2$ since $u_s > 0$, where C_1 and C_2 are both constants. The boundary condition of w_s immediately implies that $C_2 = 0$ and the average $\bar{u} = \int_0^1 u_s dx$ determines that $C_1 = \bar{u}$. This completes the proof. □

3.3. Decay property

From the results derived above, we know that the problem (1.2) has only a constant stationary solution $(\bar{u}, 0)$. The existence of the Lyapunov functional (3.1) indicates that the time-dependent solution of (1.2) may converge to the constant stationary solution $(\bar{u}, 0)$. To this end, we first derive some decay properties of the solution (u, w) of (1.2). By modifying the Lyapunov functional (3.1), we define

$$G(u, w) := \int_0^1 \left(u \ln\frac{u}{\bar{u}} + \frac{w^2}{2} + \frac{\delta}{2}w_x^2\right)dx$$

where $\bar{u}$ is given in (3.3). Since the function $\ln s$ is convex for $s > 0$, $\bar{u} = \bar{u}_0$ and hence $\int_0^1 \frac{u}{\bar{u}}dx = 1$, it follows from the Jensen's inequality [9, p. 621] that

$$\int_0^1 u\ln\frac{u}{\bar{u}}dx = \bar{u}_0 \cdot \int_0^1 \frac{u}{\bar{u}}\ln\frac{u}{\bar{u}}dx \geq \bar{u}_0 \cdot \left(\int_0^1 \frac{u}{\bar{u}}dx\right)\ln\left(\int_0^1 \frac{u}{\bar{u}}dx\right) = 0.$$

Thus, $G(u, w) \geq 0$. Except for the non-negativity, the following property of $G(u, w)$ can be proved inspired by the ideas of [7,22].

Lemma 3.5. *Suppose that (u, w) is the classical solution to (1.2). Then the functional $G(u, w)$ satisfies the following decay property*

$$0 \leq G\big(u(t), w(t)\big) \leq G(u_0, w_0)e^{-\alpha t} \quad \textit{for all } t \in (0, T_{max}),$$

where the positive constant α depends only on u_0, ε, μ and δ.

Proof. Using the first equation of (1.2) and the boundary condition, we obtain with a simple calculation that

$$\frac{d}{dt}G\big(u(t),w(t)\big) = \frac{d}{dt}\mathcal{F}\big(u(t),w(t)\big) - \ln\bar{u}\int_0^1 u_t dx$$
$$= \frac{d}{dt}\mathcal{F}\big(u(t),w(t)\big) = -\int_0^1\left(\frac{\varepsilon u_x^2}{u} + \mu w_x^2\right)dx. \tag{3.13}$$

It can be readily verified that $s\ln s \le s-1+\frac{1}{2}(s-1)^2$ for all $s \ge 0$. Then with $s = u/\bar{u}$, noting $\int_0^1(u/\bar{u}-1)dx \equiv 0$ and using the Poincaré inequality [9, p. 275], we find a constant $c_{10} > 0$ such that

$$\int_0^1 u(t)\ln\frac{u(t)}{\bar{u}}dx = \bar{u}\int_0^1\left[\frac{u(t)}{\bar{u}}\ln\frac{u(t)}{\bar{u}} - \left(\frac{u(t)}{\bar{u}} - 1\right)\right]dx$$
$$\le \bar{u}\int_0^1 \frac{1}{2}\left(\frac{u(t)}{\bar{u}} - 1\right)^2 dx = \frac{1}{2\bar{u}}\int_0^1 (u-\bar{u})^2 dx$$
$$\le c_{10}\int_0^1\big[(u-\bar{u})_x\big]^2 dx = c_{10}\int_0^1 u_x^2 dx.$$

By Lemma 3.3, for all $t \in (0, T_{max})$ one has

$$\int_0^1 u\ln\frac{u}{\bar{u}}dx \le c_{10}\int_0^1 u_x^2 dx \le c_{10}\|u\|_{L^\infty}\int_0^1\frac{u_x^2}{u}dx \le c_{11}(\varepsilon,\mu,\delta)\int_0^1\frac{u_x^2}{u}dx. \tag{3.14}$$

With the boundary condition $w|_{x=0,1} = 0$, the Poincaré inequality provides some $c_{12} > 0$ such that $\int_0^1 w^2 dx \le c_{12}\int_0^1 w_x^2 dx$ for all $t \in (0, T_{max})$, which, combined with (3.14), gives

$$G(u,w) \le \int_0^1\left[u\ln\frac{u}{\bar{u}} + \left(\frac{c_{12}}{2} + \frac{\delta}{2}\right)w_x^2\right]dx$$
$$\le \frac{c_{11}}{\varepsilon}\int_0^1\frac{\varepsilon u_x^2}{u} + \frac{c_{12}+\delta}{2\mu}\int_0^1 \mu w_x^2 dx \le \gamma\int_0^1\left(\frac{\varepsilon u_x^2}{u} + \mu w_x^2\right)dx$$

where $\gamma = \frac{c_{11}}{\varepsilon} + \frac{c_{12}+\delta}{2\mu}$. Then with the above inequality, the integration of (3.13) yields

$$\frac{d}{dt}G(u,w) \le -\int_0^1\left(\frac{\varepsilon u_x^2}{u} + \mu w_x^2\right)dx \le -\frac{1}{\gamma}G(u,w)$$

which, upon the integration, gives

$$G(u,w) \le G(u_0,w_0)e^{-\frac{1}{\gamma}t} =: G(u_0,w_0)e^{-\alpha t}.$$

This completes the proof of Lemma 3.5. □

3.4. Proof of Theorem 1.1

The first inequality in Lemma 3.2 along with the Sobolev embedding: $W^{1,2}(0,1) \hookrightarrow C^0(0,1)$ asserts that $\|w\|_{L^\infty(0,1)} \le c_{13}$ for some constant $c_{13} > 0$. This, together with Lemma 3.3 and extensibility criterion in Lemma 2.2, indicates that $T_{max} = +\infty$. Hence the existence of global classical solutions is proved. Next we are devoted to proving the exponential convergence of the global solution by borrowing an idea from [22]. To this end, we first derive from the first equation of (1.2) that the quantity $u - \bar{u}$ satisfies

$$\begin{cases} (u-\bar{u})_t = (u-\bar{u})_{xx} - (uw)_x, & x \in (0,1), t > 0, \\ (u-\bar{u})(x,0) = u_0(x) - \bar{u}, & x \in [0,1], \\ (u-\bar{u})_x|_{x=0,1} = 0, & t > 0. \end{cases} \tag{3.15}$$

Then multiplying the first equation of (3.15) by $-(u-\bar{u})_{xx}$ and using the Cauchy–Schwarz inequality, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^1 (u-\bar{u})_x^2 dx + \int_0^1 (u-\bar{u})_{xx}^2 dx &= \int_0^1 (u-\bar{u})_{xx}(uw)_x dx \\ &\le \frac{1}{2}\int_0^1 (u-\bar{u})_{xx}^2 dx + \frac{1}{2}\int_0^1 (uw)_x^2 dx. \end{aligned}$$

This, along with the inequality $(uw)_x^2 \le 2(w^2u_x^2 + u^2w_x^2)$ and the boundedness of u and w, gives a constant $c_{14} > 0$ such that

$$\frac{d}{dt}\int_0^1 (u-\bar{u})_x^2 dx + \int_0^1 (u-\bar{u})_{xx}^2 dx \le \int_0^1 (uw)_x^2 dx \le c_{14}\int_0^1 \left(u_x^2 + w_x^2\right) dx. \tag{3.16}$$

Furthermore the second inequality of Lemma 3.2 with Lemma 3.3 yields a constant c_{15} such that

$$\int_0^t\int_0^1 u_x^2 dx ds \le \|u\|_{L^\infty}\int_0^t\int_0^1 \frac{u_x^2}{u} dx ds \le c_{15} \quad \text{for all } t > 0.$$

Then using the above inequality and integrating (3.16) with respect to t, one has

$$\int_0^1 (u-\bar{u})_x^2 dx \le \int_0^1 (u_0-\bar{u})_x^2 dx + c_{14}(c_{15}+1)\int_0^t\int_0^1 w_x^2 dx \le c_{16} \tag{3.17}$$

for some constant $c_{16} > 0$, where the second inequality of Lemma 3.2 has been used.

Next we employ the Csiszár–Kullback–Pinsker inequality (cf. [4]) with Lemma 3.5 to obtain that

$$\|u-\bar{u}\|_{L^1}^2 \le 2\bar{u}\int_0^1 u \ln\frac{u}{\bar{u}} dx \le 2\bar{u}G(u_0,w_0)e^{-\alpha t}. \tag{3.18}$$

Notice that the Gagliardo–Nirenberg inequality yields a constant $c_{17} > 0$ such that

$$\|u-\bar{u}\|_{L^\infty} \le c_{17}\left(\left\|(u-\bar{u})_x\right\|_{L^2}^{\frac{2}{3}} \cdot \|u-\bar{u}\|_{L^1}^{\frac{1}{3}} + \|u-\bar{u}\|_{L^1}\right).$$

Then the application of (3.17) and (3.18) to above inequality asserts that

$$\|u-\bar{u}\|_{L^\infty} \le c_{18} e^{-\frac{\alpha}{6}t} \quad \text{for all } t>0$$

for some $c_{18}>0$. Finally we prove that w converges to zero exponentially. This is obvious. Indeed from Lemma 3.5, we obtain a constant c_{19} such that

$$\|w\|_{L^2}^2 + \|w_x\|_{L^2}^2 \le c_{19} G(u_0, w_0) e^{-\alpha t}.$$

Note that if $f|_{x=0,1}=0$, then by the Hölder inequality one has

$$f^2(x) = \int_0^x \left[f^2(\xi)\right]' d\xi = 2\int_0^x f(\xi) f'(\xi) d\xi \le 2\|f\|_{L^2}\|f_x\|_{L^2} \le \|f\|_{L^2}^2 + \|f_x\|_{L^2}^2.$$

Therefore the application of above inequality gives

$$\|w\|_{L^\infty} \le \left(\|w\|_{L^2}^2 + \|w_x\|_{L^2}^2\right)^{\frac{1}{2}} \le \sqrt{c_{19} G(u_0, w_0)} e^{-\frac{\alpha}{2}t},$$

which completes the proof of Theorem 1.1 by choosing $\beta = -\frac{\alpha}{6}$. □

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