

Contents lists available at ScienceDirect

# Mathematical Biosciences



journal homepage: www.elsevier.com/locate/mbs

# **Original Research Article**

# A spatiotemporal model for the effects of toxicants on the competitive dynamics of aquatic species

# Xiumei Deng<sup>a</sup>, Qihua Huang<sup>a,\*</sup>, Zhi-An Wang<sup>b</sup>

<sup>a</sup> School of Mathematics and Statistics, Southwest University, Chongqing 400715, China
<sup>b</sup> Department of Applied Mathematics, Hong Kong Polytechnic University, Hong Kong, China

# ARTICLE INFO

Keywords: Competing species toxicant–taxis Population persistence Species diversity

# A B S T R A C T

In this paper, we develop a reaction-diffusion model with negative toxicant-taxis that incorporates spatiotemporally inhomogeneous toxicant input to investigate the impact of toxicants on the competitive dynamics of two species in a polluted aquatic environment. Here the negative toxicant-taxis models the evasive movement of avoiding toxicants by species. We establish the global well-posedness of the model, analyze the existence and stability of spatially homogeneous steady states, and derive sufficient conditions for species extinction and coexistence. Through linear stability analysis, we identify sufficient conditions on model parameters that destabilize spatially homogeneous steady states under spatiotemporally uniform toxicant input. Numerical experiments reveal the influence of key toxicant-related factors (input rate, taxis intensity, and diffusivity) on competition outcomes and species distributions. Notably, strong negative toxicant-taxis can induce spatial aggregation and segregation patterns between the species and the toxicant under uniform toxicant input. Our findings suggest that toxicant-taxis may promote population persistence and coexistence, particularly when the toxicant input is not uniform in space and time and the toxicant does not diffuse fast (i.e. weak diffusivity). However, strong toxicant diffusion can diminish the impact of taxis, adversely affecting population persistence and species coexistence.

### 1. Introduction

Over the last few decades, aquatic ecosystems have increasingly been burdened by a variety of toxic substances. These include heavy metals, pesticides, microplastics, and radioactive elements, which originate from a range of sources such as industrial waste, agricultural runoff, and natural resource extraction. These toxic substances can negatively impact the health and survival of individual organisms, disrupt breeding patterns and growth rates within populations, and ultimately alter the structure and function of complex aquatic communities. Monitoring and regulating these pollutants is essential for protecting aquatic environments. Several studies have proposed guidelines for water quality and innovative strategies to mitigate pollution in aquatic settings [1–7]. Additionally, assessing the impacts of toxicants in contaminated water bodies and deciphering the dynamics that influence species survival and biodiversity maintenance has become a paramount focus.

The ecological risks associated with toxicants have predominantly been studied in terms of their effects on individual organisms, including their reproductive capacity, survival rates, and development, within controlled laboratory settings and over short periods of time. However, in order to develop successful strategies for environmental management, it is imperative to comprehend the long-term effects of toxicants on the health of whole ecosystems. In this context, the use of mathematical models is crucial for converting the observed effects on individuals into an understanding of their implications at the population level. Numerous researchers have developed discrete-time difference equation models [8-11] and continuous-time ordinary differential equation (ODE) models [12-19] to explore the intricate dynamics between species and toxicants in contaminated aquatic ecosystems. Nevertheless, these models do not account for the spatial dispersal of populations and toxicants, even though both populations and toxicants have the potential to spread across space as a result of factors including active movement of organisms and passive diffusion driven by turbulent water. In a recent study [20], we introduced a diffusive population-toxicant model with negative toxicant-taxis to describe the dynamics between a population and a toxicant in contaminated lake environments. This model accounts for the spatial dispersal of both populations and toxicants and integrates the adaptive behavioral responses of the population to toxicant exposure (i.e evasive movement of avoiding toxicants). Specifically, it describes how individuals move

\* Corresponding author. E-mail addresses: dengxiumei95@163.com (X. Deng), qihua@swu.edu.cn (Q. Huang), mawza@polyu.edu.hk (Z.-A. Wang).

https://doi.org/10.1016/j.mbs.2024.109341

Received 1 October 2024; Received in revised form 22 October 2024; Accepted 12 November 2024 Available online 23 November 2024 0025-5564/© 2024 Published by Elsevier Inc. away from high-toxicant zones to regions with lower concentrations, aiming to enhance their survival, growth, and reproduction. The model allows us to explore the effects of various factors on the population's persistence and spatial distribution. These factors include the input of toxicants, their negative impact on population growth, and the dispersal behaviors of both the population and the toxicant.

It is important to note that the aforementioned models only describe the effects of toxicants on individual species. These models fall short of predicting the impact of environmental toxicants on species interactions. Species interactions, including competition, predation, and mutualism, are ubiquitous and play essential roles in maintaining ecosystem balance, promoting biodiversity, and ensuring the long-term health and stability of ecosystems. When two competing species coexist in a contaminated aquatic ecosystem, it is possible for two competing species to have different levels of vulnerability to toxicants. In this study, we aim to explore the impact of toxicants on the interspecific competition within a polluted aquatic environment. To achieve this, we extend the diffusive population-toxicant model presented in [20] to develop a model that describes the interacting dynamics between two competing species and a toxicant. This novel model comprises three reaction-diffusion equations. The first two equations govern the population dynamics of two competing species, incorporating the effects of toxicants, interspecific competition, and spatial dispersal. The third equation describes the dynamics of the toxicant, encompassing its diffusion, introduction into the environment, and degradation. This model can be used to elucidate the influence of the interplay between various factors, including the rate of toxicant input, the intensity of toxicant-taxis, and the diffusion capacity of toxicant, on competition outcomes and species distributions.

This paper is organized as follows. Section 2 introduces a spatiotemporal model that describes the interacting dynamics between two competing species and a toxicant within a polluted aquatic environment. Section 3 outlines the main analytical results for our model, including the global existence of classical solutions and their asymptotic behavior under specific conditions. The global existence of these classical solutions is proven in Section 4, with detailed proofs of their asymptotic behavior provided in Section 5. In Section 6, we conduct a linear stability analysis to identify parameter conditions leading to the instability of homogeneous steady states, suggesting the possibility of pattern formation. Section 7 presents numerical simulations demonstrating the emergence of spatial aggregation and segregation patterns from our model. Furthermore, we numerically investigate how the toxicant input pattern, the strength of toxicant-taxis, and the dispersal characteristics of the toxicant impact the competitive outcomes and spatial distributions of the two competing species. Finally, Section 8 concludes with a summary of our findings and proposes potential avenues for future research.

#### 2. Model formulation

We consider two species inhabiting the same polluted aquatic environment, where they compete for shared resources. Let  $\Omega$  represent the habitat of these species, defined as a bounded domain in  $\mathbb{R}^n$  with a smooth boundary, denoted by  $\partial\Omega$ . We denote the densities of the two species at location *x* and time *t* by u(x,t) and v(x,t), respectively. Additionally, let w(x,t) represent the concentration of the toxicant at location *x* and time *t*. The interaction between these two species and the toxicant can be described by the following mathematical model:

$u_t = d_1 \Delta u + \chi_1 \nabla \cdot (u \nabla w) + u(a - b_1 u - c_1 v - k_1 w),$	$x\in \varOmega, \ t>0,$
$v_t = d_1 \Delta v + \chi_2 \nabla \cdot (v \nabla w) + v(a - b_2 u - c_2 v - k_2 w),$	$x\in \varOmega,\ t>0,$
$w_t = d_2 \Delta w + H(x,t) - qw - p_1 uw - p_2 vw,$	$x\in \varOmega,\ t>0,$
$\nabla u \cdot \boldsymbol{n} = \nabla v \cdot \boldsymbol{n} = \nabla w \cdot \boldsymbol{n} = 0,$	$x\in\partial\Omega,\ t>0,$
$(u, v, w)(x, 0) = (u_0, v_0, w_0)(x).$	$x \in \Omega$ .

In the model, the parameters  $b_i$ ,  $c_i$ ,  $d_i$ ,  $k_i$ ,  $p_i$  (for i = 1, 2) and a, q are all positive constants, with the exception that  $\chi_1$  and  $\chi_2$  are non-negative.

The first and second equations of (2.1) describe the growth and spatial dispersal of species u and species v, respectively, under the influence of toxicant and competition. The terms  $d_1 \Delta u$  and  $d_1 \Delta v$  represent the random diffusion of the populations with a common diffusion coefficient  $d_1$ . The toxicant-taxis terms  $\chi_1 \nabla \cdot (u \nabla w)$  and  $\chi_2 \nabla \cdot (v \nabla w)$ signify the evasive movement of individuals from areas with high toxicant concentrations to areas with low toxicant concentrations. This movement is directed opposite to the toxicant gradient with  $\gamma_i$  (i = 1, 2) being the corresponding taxis coefficients. The parameter a represents the intrinsic growth rate of the species. The parameters  $b_1$  and  $c_2$  denote the intra-specific competition coefficients, which describe the effects of competition within each species. In contrast, the parameters  $b_2$  and  $c_1$  represent the inter-specific competition coefficients, which capture the effects of competition between the two species. The terms  $k_1 w$  and  $k_2 w$  quantify the impact of the toxicant on the population growth rates, where  $k_1$  and  $k_2$  are the corresponding effect coefficients that measure the sensitivity of each species to the toxicant.

The third equation of (2.1) represents a balance equation for the concentration of toxicant in the environment. The parameter  $d_2$  represents the diffusion coefficient of the toxicant. The function H(x,t) describes the location- and time-dependent input rate at which the exogenous toxicant enters the habitat. The parameter q denotes the per unit output rate of toxicant due to various factors, such as microbial degradation, environmental detoxification, and so on. The toxicant uptake rates by the populations from the environment, given by  $p_1uw$  and  $p_2vw$ , are modeled according to the law of mass action and are therefore proportional to both the concentration of contaminant and the population density, where  $p_1$  and  $p_2$  are the respective uptake coefficients.

The fourth line of (2.1) represents zero-flux boundary conditions corresponding to the first three equations, indicating that no individuals or toxicant enter or leave the habitat  $\Omega$  across its boundary, where *n* is the outward unit vector of  $\partial \Omega$ . The last line of (2.1) specifies the initial spatial distributions of the two species and the toxicant.

Model (2.1) extends not only the toxicant-dependent competition model proposed in [21], which is a system of ordinary differential equations, but also the toxicant-mediated reaction-diffusion equation model presented in [22], where the two species do not interact with each other directly; instead, their interactions are mediated through the effects of the toxicant.

In the absence of toxicant, model (2.1) simplifies to the following classical reaction–diffusion Lotka–Volterra competition model:

$$\begin{cases} u_t = d_1 \Delta u + u(a - b_1 u - c_1 v), & x \in \Omega, \ t > 0, \\ v_t = d_1 \Delta v + v(a - b_2 u - c_2 v), & x \in \Omega, \ t > 0, \\ \nabla u \cdot \mathbf{n} = \nabla v \cdot \mathbf{n} = 0, & x \in \partial\Omega, \ t > 0, \\ (u, v)(x, 0) = (u_0, v_0)(x), & x \in \Omega, \end{cases}$$

$$(2.2)$$

which has been extensively studied in the literature (see [23–26] and references therein). It has been shown that the global dynamics of solutions for model (2.2) is determined by the reaction coefficients  $a, b_i, c_i$  (i = 1, 2) (cf. [23,26]). Specifically (2.2) has four equilibria  $(0,0), (u_A,0), (0, v_B)$  and  $(u_*, v_*)$ , where  $u_A = \frac{a}{b_1}, v_B = \frac{a}{c_2}, u_* = \frac{a(c_2-c_1)}{b_1c_2-b_2c_1}, v_* = \frac{a(b_1-b_2)}{b_1c_2-b_2c_1}$ . Let  $\tilde{b} := b_1/b_2, \tilde{c} := c_1/c_2$ . Then the positive coexistence equilibrium  $(u_*, v_*)$  is globally asymptotically stable if  $\tilde{c} < 1 < \tilde{b}$  (weak competition), and the competitive exclusion equilibrium  $(u_A, 0)$  (resp.  $(0, v_B)$ ) is globally asymptotically stable if  $1 > \max{\tilde{b}, \tilde{c}}$  (resp.  $1 < \min{\tilde{b}, \tilde{c}}$ ). If  $\tilde{b} < 1 < \tilde{c}$  (strong competition), the coexistence steady state  $(u_*, v_*)$  is unstable, while both  $(u_A, 0)$  and  $(0, v_B)$  are locally stable depending on the initial data.

(2.1)

For the simplicity of analysis, we nondimensionalize model (2.1) by the rescalings

$$\begin{split} \hat{u} &= \frac{b_1 u}{a}, \quad \hat{v} = \frac{c_2 v}{a}, \quad \hat{w} = \frac{k_2 w}{a}, \quad \hat{t} = at, \quad \hat{x} = \sqrt{\frac{a}{d_1}}x, \quad \hat{\chi}_1 = \frac{a\chi_1}{d_1k_2}, \\ \hat{\chi}_2 &= \frac{a\chi_2}{d_1k_2}, \quad \hat{H}(\hat{x}, \hat{t}) = \frac{k_2 H(x, t)}{a^2}, \quad \hat{q} = \frac{q}{a}, \quad \hat{p}_1 = \frac{p_1}{b_1}, \quad \hat{p}_2 = \frac{p_2}{c_2}, \\ \hat{b} &= \frac{b_2}{b_1}, \quad \hat{c} = \frac{c_1}{c_2}, \quad \hat{d} = \frac{d_2}{d_1}, \quad k = \frac{k_1}{k_2}. \end{split}$$

Dropping the hats for notational convenience, system (2.1) reduces to

$$\begin{split} u_t &= \Delta u + \chi_1 \nabla \cdot (u \nabla w) + u(1 - u - cv - kw), & x \in \Omega, \ t > 0, \\ v_t &= \Delta v + \chi_2 \nabla \cdot (v \nabla w) + v(1 - bu - v - w), & x \in \Omega, \ t > 0, \\ w_t &= d\Delta w + H(x, t) - qw - p_1 uw - p_2 vw, & x \in \Omega, \ t > 0, \\ \nabla u \cdot \mathbf{n} &= \nabla v \cdot \mathbf{n} = \nabla w \cdot \mathbf{n} = 0, & x \in \partial\Omega, \ t > 0, \\ (u, v, w)(x, 0) &= (u_0, v_0, w_0)(x), & x \in \Omega. \end{split}$$

$$(2.3)$$

For the toxicant input rate H(x, t), we make the following basic assumption:

(*H*)  $H(x,t) \in C(\overline{\Omega} \times [0,\infty))$  is a nonnegative and bounded function.

We shall explore the local and global dynamics of (2.3) with constant and non-constant input rate function H(x, t) in the subsequent sections.

#### 3. Main results

The first issue concerned is the global well-posedness of solutions to (2.3). For n = 2, the global existence and boundedness of solutions of (2.3) can be established for any  $\chi_i \ge 0$  (i = 1, 2) by slightly modifying the arguments of [20]. Hence below we state the global existence and boundedness of solutions of (2.3) for  $n \ge 3$  only.

**Theorem 3.1.** Let  $\Omega \subset \mathbb{R}^n$   $(n \geq 3)$  be a bounded domain with smooth boundary and the hypothesis (H) hold. Assume that  $(u_0, v_0, w_0) \in [W^{1,p}(\Omega)]^3$  with p > n and  $u_0, v_0, w_0 \geq 0 (\not\equiv 0)$ , and  $\chi_i$  (i = 1, 2) satisfy  $0 \leq \chi_i \leq \frac{2d}{3(n+2)(1+d)\tilde{w}}$  (3.1)

with

$$\bar{w} := \max\{\|w_0\|_{L^{\infty}(\Omega)}, \bar{w}\}, \ \bar{w} := \frac{\bar{H}}{q} \text{ and } \bar{H} := \max_{(x,t) \in (\bar{\Omega} \times [0,\infty))} H(x,t).$$
(3.2)

Then system (2.3) has a unique global classical solution  $(u, v, w) \in [C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))]^3$  satisfying u, v, w > 0 for all t > 0 and

 $\|u(\cdot,t)\|_{L^\infty(\varOmega)}+\|v(\cdot,t)\|_{L^\infty(\varOmega)}+\|w(\cdot,t)\|_{W^{1,\infty}(\varOmega)}\leq C,$ 

where C > 0 is a constant independent of t. Furthermore,  $0 < w \le \tilde{w}$  for all t > 0.

The second main issue to be addressed is how the discharged toxicant will affect a balanced aquatic ecosystem where interacting species coexist. To this end, we assume 0 < b, c < 1 for which two competing species coexist in the absence of toxicant as mentioned in Section 2. Moreover, for the sake of analytical convenience, we will assume  $p_1 = p_2 = q = 1$  since they are not within the focus of this paper. Consequently, system (2.3) becomes

$$\begin{split} u_t &= \Delta u + \chi_1 \nabla \cdot (u \nabla w) + u(1 - u - cv - kw), & x \in \Omega, \ t > 0, \\ v_t &= \Delta v + \chi_2 \nabla \cdot (v \nabla w) + v(1 - bu - v - w), & x \in \Omega, \ t > 0, \\ w_t &= d\Delta w + H(x, t) - w - uw - vw, & x \in \Omega, \ t > 0, \\ \nabla u \cdot \mathbf{n} &= \nabla v \cdot \mathbf{n} = \nabla w \cdot \mathbf{n} = 0, & x \in \partial\Omega, \ t > 0, \\ (u, v, w)(x, 0) &= (u_0, v_0, w_0)(x), & x \in \Omega. \end{split}$$
(3.3)

Next, we investigate the large-time behavior of solutions to system (3.3). For the asymptotic dynamics, in addition to the hypothesis (H), we require another assumption on the function H(x, t):

 $(\mathcal{H}_1) \lim_{t \to \infty} H(x,t) = h_0$  uniform in  $x \in \overline{\Omega}$  and  $\int_0^\infty \int_{\Omega} (H(x,t) - h_0)^2 dx dt < \infty$ , where  $h_0$  is a nonnegative constant.

We shall demonstrate that the asymptotic behavior of the solution to system (3.3) can be approximated by the asymptotic behavior of the solution to the following system:

$$\begin{aligned} u_t &= \Delta u + \chi_1 \nabla \cdot (u \nabla w) + u(1 - u - cv - kw), & x \in \Omega, \ t > 0, \\ v_t &= \Delta v + \chi_2 \nabla \cdot (v \nabla w) + v(1 - bu - v - w), & x \in \Omega, \ t > 0, \\ w_t &= d\Delta w + h_0 - w - uw - vw, & x \in \Omega, \ t > 0, \\ \nabla u \cdot \mathbf{n} &= \nabla v \cdot \mathbf{n} = \nabla w \cdot \mathbf{n} = 0, & x \in \partial\Omega, \ t > 0, \\ (u, v, w)(x, 0) &= (u_0, v_0, w_0)(x), & x \in \Omega. \end{aligned}$$

To achieve this, we first examine the existence and stability of spatially homogeneous steady states of system (3.4). For the sake of convenience, we will frequently employ the following notations:

$$\begin{aligned} \alpha &:= 3 - b - c - bc, \ \beta &:= 1 - c + k(1 - b), \\ h_1 &:= \frac{(1 - b)(2 - bk - k)}{(1 - bk)^2}, \ h_2 &:= \frac{(1 - c)(2k - 1 - c)}{(k - c)^2}, \\ h_3 &:= \frac{\alpha^2}{4(1 - bc)\beta}, \ \hbar_1 &:= \frac{3(k + 1) + \sqrt{9(1 - k)^2 + 4k}}{2k}, \\ \hbar_2 &:= \frac{1}{k + 1}, \ \hbar_3 &= \frac{[4 - (b + c)^2]\{\alpha[4 + (k + 1)^2] - \beta[4 - (b + c)^2]\}}{(1 - bc)[4 + (k + 1)^2]^2}, \\ \kappa_1 &:= \frac{1 + b + c - 3bc}{(1 - b)^2 + (1 + b)(1 - bc)}, \ \kappa_2 &:= \frac{(1 - c)^2 + (1 + c)(1 - bc)}{1 + b + c - 3bc}. \end{aligned}$$
(3.5)

**Remark 3.1.** One can easily check that  $h_1 \le h_3, h_2 \le h_3, h_1 > \max\{1, 1/k, h_3\}$  and  $\kappa_1 < 1 < \kappa_2$ . In particular,  $h_1 = h_2 = 1$  if k = 1.

A spatially homogeneous steady state of system (3.4) is an equilibrium of the following ODE system:

1 \

11

$$u_{t} = u(1 - u - cv - kw),$$
  

$$v_{t} = v(1 - bu - v - w),$$
  

$$w_{t} = h_{0} - w - uw - vw.$$
  
(3.6)

With straightforward calculations, we find that system (3.6) has five possible equilibria, which are given by

$$\begin{split} E_1 &:= (0, 0, w_1) = (0, 0, h_0), \\ E_2 &:= (0, v_2, w_2) = (0, \sqrt{1 - h_0}, 1 - \sqrt{1 - h_0}), \\ E_3 &:= (u_3, 0, w_3) = \left(\sqrt{1 - kh_0}, 0, \frac{1 - \sqrt{1 - kh_0}}{k}\right), \\ E_4 &:= (u_4, v_4, w_4) \\ &= \left(\frac{1 - c + (c - k)w_4}{1 - bc}, \frac{1 - b + (bk - 1)w_4}{1 - bc}, \frac{\alpha + \sqrt{\alpha^2 - 4h_0(1 - bc)\beta}}{2\beta}\right), \\ E_5 &:= (u_5, v_5, w_5) \\ &= \left(\frac{1 - c + (c - k)w_5}{1 - bc}, \frac{1 - b + (bk - 1)w_5}{1 - bc}, \frac{\alpha - \sqrt{\alpha^2 - 4h_0(1 - bc)\beta}}{2\beta}\right). \end{split}$$

We summarize the conditions for the existence and stability of these equilibria in Table 1, deferring a detailed discussion to Appendix. Furthermore, we visualize the regions of local stability of equilibria in the  $(k, h_0)$  plane, as depicted in Fig. 1.

The following theorem provides sufficient conditions under which the solutions of system (3.3) converge to either extinction or coexistence steady states.

**Theorem 3.2** (Global Stability). Suppose that the conditions in Theorem 3.1 and the additional hypothesis  $(\mathcal{H}_1)$  of H(x,t) hold, and (u, v, w) is the solution of system (3.3) obtained in Theorem 3.1 with 0 < b, c < 1. Then the following results hold:

Table 1

The conditions for the existence and stability of the equilibria of the ODE system (3.6) with $0 < b, c < 1$ .				
Equilibria	Existence	Local stability	Instability	
$E_1$	Always	$h_0 > \max\{1/k, 1\}$	$h_0 < \max\{1/k,1\}$	
<i>E</i> <sub>2</sub>	$h_0 < 1$	$k > 1, h_2 < h_0 < 1$	$\left\{ \begin{array}{l} k \leq 1, \ h_0 < 1 \ {\rm or} \\ k > 1, \ h_0 < h_2 \end{array} \right.$	
<i>E</i> <sub>3</sub>	$h_0 < 1/k$	$k < 1, \ h_1 < h_0 < 1/k$	$\left\{ \begin{array}{l} k \geq 1, \ h_0 < 1/k \ {\rm or} \\ k < 1, \ h_0 < h_1 \end{array} \right.$	
$E_4$	$\left\{ \begin{array}{l} \kappa_1 < k < 1, \ h_1 < h_0 < h_3 \ \text{or} \\ k = 1, \ 1 < h_0 < h_3 \ \text{or} \\ 1 < k < \kappa_2, \ h_2 < h_0 < h_3 \end{array} \right.$	/	Always unstable	
<i>E</i> <sub>5</sub>	$\begin{cases} k < 1, \ h_0 < h_1 \text{ or} \\ \kappa_1 < k < 1, \ h_1 \le h_0 < h_3 \text{ or} \\ k = 1, \ h_0 < h_3 \text{ or} \\ k > 1, \ h_0 < h_2 \text{ or} \\ 1 < k < \kappa_2, \ h_2 \le h_0 < h_3 \end{cases}$	Stable as long as it exists	/	



**Fig. 1.** A schematic of local stability regions for the ODE system (3.6) with 0 < b, c < 1, where LAS stands for "locally asymptotically stable".

(i) Assume that 
$$h_0 > h_1$$
, then  
 $\|(u, v, w) - (0, 0, w_1)\|_{L^{\infty}(\Omega)} \to 0 \text{ as } t \to \infty.$  (3.8)

(ii) Assume that  $\frac{w_5}{4d}(\chi_1^2 u_5 + \chi_2^2 v_5) \le 1$  and one of the following conditions holds:

• 
$$k \le 1$$
,  $h_0 < \min\{h_1, \hbar_3\}$ .

$$k > 1, h_0 < \min\{h_2, \hbar_3\}.$$

Then

$$\|(u, v, w) - (u_5, v_5, w_5)\|_{L^{\infty}(\Omega)} \to 0 \text{ as } t \to \infty.$$
(3.9)

The result in Theorem 3.2(i) asserts that the species will go extinction if the toxicant input rate is large irrespective of the dispersal strategies (both random diffusion and negative toxicant–taxis). However, the result in Theorem 3.2(ii) asserts that if the toxicant input rate is suitably small, then the two species with weak competition can coexist with the toxicant uniformly in space and time.

#### 4. Boundedness of solutions (Proof of Theorem 3.1)

The proof of Theorem 3.1 consists of two parts: local existence and a priori estimates. In the sequel, we denote  $\int_{\Omega} f(\cdot, t) dx$  and  $\int_0^t \int_{\Omega} f(\cdot, s) dx ds$  by  $\int_{\Omega} f$  and  $\int_0^t \int_{\Omega} f$ , respectively. In addition, we write  $\|\cdot\|_{L^p(\Omega)} =$ 

 $\|\cdot\|_{L^p}$  for short, and use *C* or  $C_i$  (i = 1, 2, ...) to denote generic constants which may vary in the context.

#### 4.1. Preliminary results

First, the existence of local-in-time classical solutions of system (2.3) can be readily shown by the Amann theorem (see [27,28]) and the positivity of solutions can be proved by the strong maximum principle.

**Lemma 4.1** (Local Existence). Let the assumptions in Theorem 3.1 hold. Then there exists a constant  $T_{\max} > 0$  such that system (2.3) has a unique classical solution  $(u, v, w) \in [C(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))]^3$ satisfying u, v, w > 0 for all  $t \in (0, T_{\max})$ . Moreover,

if 
$$T_{\max} < \infty$$
, then  $\lim_{t \neq T_{\max}} (\|u(\cdot, t)\|_{L^{\infty}} + \|v(\cdot, t)\|_{L^{\infty}} + \|w(\cdot, t)\|_{L^{\infty}}) = \infty$ .

**Lemma 4.2.** Let the conditions in Lemma 4.1 hold. Then the solution of (2.3) satisfies

$$w(x,t) \le \tilde{w} \quad \text{for all } t > 0, \tag{4.1}$$

where  $\tilde{w}$  is defined by Theorem 3.1.

**Proof.** Note that u, v, w > 0 for all t > 0 (see Lemma 4.1), we can show the above result by a comparison principle applied to the third equation of system (2.3) and omit the proof for brevity.

**Lemma 4.3.** Suppose the assumptions in Lemma 4.1 hold and let (u, v, w) be a solution of (2.3). Then one has

$$\|u(\cdot,t)\|_{L^1} + \|v(\cdot,t)\|_{L^1} \le M_0 \text{ for all } t \in (0,T_{\max}),$$
(4.2)

where  $M_0 > 0$  is a constant independent of t.

**Proof.** Integrating the first two equations of (2.3) along with the homogeneous Neumann boundary conditions, we have

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} (u+v) + \int_{\Omega} (u+v) \\ &= 2 \int_{\Omega} u - \int_{\Omega} u^2 - k \int_{\Omega} uw + 2 \int_{\Omega} v - \int_{\Omega} v^2 - \int_{\Omega} vw - (b+c) \int_{\Omega} uv. \end{aligned}$$
By the facts that  $u, v, w > 0$  and the Young inequality, we have
$$\begin{aligned} \frac{d}{dt} \int_{\Omega} (u+v) + \int_{\Omega} (u+v) &\leq 2 \int_{\Omega} u - \int_{\Omega} u^2 + 2 \int_{\Omega} v - \int_{\Omega} v^2 \\ &\leq \frac{1}{2} \int_{\Omega} u^2 + 2|\Omega| - \int_{\Omega} u^2 + \frac{1}{2} \int_{\Omega} v^2 + 2|\Omega| - \int_{\Omega} v^2, \end{aligned}$$

then  

$$\frac{d}{dt} \int_{\Omega} (u+v) + \int_{\Omega} (u+v) + \frac{1}{2} \int_{\Omega} (u^2+v^2) \le 4|\Omega|,$$

which together with the Gronwall inequality yields (4.2).  $\Box$ 

**Lemma 4.4.** Suppose the assumptions in Lemma 4.1 hold and let (u, v, w) be a solution of system (2.3). If there exists r > n/2 such that

$$\sup_{t \in (0, T_{max})} \|u(\cdot, t)\|_{L^r} + \sup_{t \in (0, T_{max})} \|v(\cdot, t)\|_{L^r} \le M_1$$
(4.3)

for some constant  $M_1 > 0$ . Then there is a constant C > 0 independent of t such that

 $\|u(\cdot,t)\|_{L^\infty}+\|v(\cdot,t)\|_{L^\infty}+\|w(\cdot,t)\|_{W^{1,\infty}}\leq C\ for\ all\ t\in(0,T_{\max}).$ 

**Proof.** The proof is the same as that of [29, Lemma 3.1]. We omit the details here.  $\Box$ 

#### 4.2. A priori estimates

Now we are devoted to proving Theorem 3.1 by deriving a uniform bound for u(x, t) and v(x, t) in  $L^r(\Omega)$  with  $r = \frac{n}{2} + 1$ . We first introduce a weight function to be used later.

**Lemma 4.5.** Let  $\tilde{w}$  be given by (3.2),  $\chi_i$  (i = 1, 2) satisfy (3.1) and  $r = \frac{n}{2} + 1$ . For all  $\zeta \in [0, \tilde{w}]$  and  $\sigma := \frac{d(r-1)}{6r|(1+d)\tilde{w}|^2}$ , we define a function  $\mu(\zeta) := e^{\sigma\zeta^2}$ . (4.4)

Then it follows that

$$\frac{(1+d)^2}{r-1}\frac{{\mu'}^2(\zeta)}{\mu(\zeta)} + \chi_i^2(r-1)\mu(\zeta) + \chi_i\mu'(\zeta) \le \frac{d}{r}\mu''(\zeta).$$
(4.5)

**Proof.** Clearly to prove, (4.5), it amounts to show the following inequality:

$$\frac{J_1(\zeta) + J_2(\zeta) + J_3(\zeta)}{J_4(\zeta)} \le 1,$$
(4.6)

where

- - 1

$$J_{1}(\zeta) := \frac{(1+d)^{2}}{r-1} \frac{\mu^{\prime 2}(\zeta)}{\mu(\zeta)}, \quad J_{2}(\zeta) := \chi_{i}^{2}(r-1)\mu(\zeta),$$
  
$$J_{3}(\zeta) := \chi_{i}\mu^{\prime}(\zeta), \quad J_{4}(\zeta) := \frac{d}{r}\mu^{\prime\prime}(\zeta).$$
(4.7)

By tedious calculations, we have that, for  $\zeta \in [0, \tilde{w}]$ ,

$$\frac{J_1(\zeta)}{J_4(\zeta)} = \frac{2r}{r-1} \frac{(1+d)^2 \sigma \zeta^2}{d+2d\sigma \zeta^2} \le \frac{2r}{r-1} \frac{(1+d)^2 \sigma \zeta^2}{d} \le \frac{2r}{r-1} \frac{(1+d)^2 \sigma \tilde{w}^2}{d} = 1/3,$$
(4.8)

$$\frac{J_2(\zeta)}{J_4(\zeta)} = \frac{r(r-1)\chi_i^2 \mu(\zeta)}{2d\sigma\mu(\zeta) + 4d\sigma^2 \zeta^2 \mu(\zeta)} \le \frac{r(r-1)\chi_i^2}{2d\sigma} = \frac{3r^2(1+d)^2 \chi_i^2 \tilde{w}^2}{d^2} \le 1/3,$$
(4.9)

$$\frac{J_{3}(\zeta)}{J_{4}(\zeta)} = \frac{2r\chi_{i}\sigma\zeta\mu(\zeta)}{2d\sigma\mu(\zeta) + 4d\sigma^{2}\zeta^{2}\mu(\zeta)} \le \frac{r\chi_{i}\zeta}{d} \le \frac{r\chi_{i}\tilde{w}}{d} \le \frac{r\tilde{w}}{d} \le \frac{r\tilde{$$

Combining (4.8), (4.9) and (4.10), we obtain (4.6), which gives (4.5) and completes the proof.  $\Box$ 

**Lemma 4.6.** Suppose the assumptions in Lemma 4.1 hold and let (u, v, w) be a solution of system (2.3). Let  $\chi_i$  (i = 1, 2) satisfy (3.1) and  $r = \frac{n}{2} + 1$ . Then

 $\|u(\cdot,t)\|_{L^r} + \|v(\cdot,t)\|_{L^r} \le C \text{ for all } t \in (0,T_{\max}),$ (4.11)

where C > 0 is a constant independent of t.

#### **Proof.** The definition of $\mu(\zeta)$ implies

$$\mu'(w(x,t)) = 2\sigma w(x,t)\mu(w(x,t)) > 0.$$
(4.12)

It follows from (4.1) that

$$1 \le \mu(w(x,t)) \le \mu(\tilde{w}) \quad \text{for all } (x,t) \in \Omega \times (0,T_{max}). \tag{4.13}$$

Using the equations in system (2.3), we have

$$\begin{aligned} \frac{1}{r} \frac{d}{dt} \int_{\Omega} u^r \mu(w) &= \int_{\Omega} u^{r-1} \mu(w) [\Delta u + \chi_1 \nabla \cdot (u \nabla w) + u(1 - u - cv - kw)] \\ &+ \frac{1}{r} \int_{\Omega} u^r \mu'(w) [d \Delta w + H(x, t) - qw - p_1 uw - p_2 vw] \\ &= -(r-1) \int_{\Omega} u^{r-2} \mu(w) |\nabla u|^2 - \int_{\Omega} u^{r-1} \mu'(w) \nabla u \cdot \nabla w \\ &- \chi_1(r-1) \int_{\Omega} u^{r-1} \mu(w) \nabla u \cdot \nabla w - \chi_1 \int_{\Omega} u^r \mu'(w) |\nabla w|^2 \\ &+ \int_{\Omega} u^r \mu(w)(1 - u - cv - kw) - d \int_{\Omega} u^{r-1} \mu'(w) \nabla u \cdot \nabla w \\ &- \frac{d}{r} \int_{\Omega} u^r \mu''(w) |\nabla w|^2 \\ &+ \frac{1}{r} \int_{\Omega} u^r \mu'(w) (H(x, t) - qw - p_1 uw - p_2 vw). \end{aligned}$$

$$(4.14)$$

Using the positivity of u, v, w for all t > 0,  $\mu'(w) > 0$ ,  $w \le \tilde{w}$  and (4.12)–(4.13), we find

$$\int_{\Omega} u^{r} \mu(w)(1 - u - cv - kw) + \frac{1}{r} \int_{\Omega} u^{r} \mu'(w)(H(x, t) - qw - p_{1}uw - p_{2}vw)$$

$$\leq \int_{\Omega} u^{r} \mu(w)(1 - u) + \frac{1}{r} \int_{\Omega} u^{r} \mu'(w)H(x, t)$$

$$\leq \int_{\Omega} u^{r} \mu(w)(1 + 2\sigma \tilde{w}\bar{H} - u).$$
(4.15)

Then we adding  $\frac{1}{r} \int_{\Omega} u^r \mu(w)$  on both sides of (4.14) and using (4.15), we get

$$\frac{1}{r}\frac{d}{dt}\int_{\Omega}u^{r}\mu(w) + \frac{1}{r}\int_{\Omega}u^{r}\mu(w) + (r-1)\int_{\Omega}u^{r-2}\mu(w)|\nabla u|^{2} \\
+ \frac{d}{r}\int_{\Omega}u^{r}\mu''(w)|\nabla w|^{2} \\
\leq -(1+d)\int_{\Omega}u^{r-1}\mu'(w)\nabla u\cdot\nabla w - \chi_{1}(r-1)\int_{\Omega}u^{r-1}\mu(w)\nabla u\cdot\nabla w \\
+ \chi_{1}\int_{\Omega}u^{r}\mu'(w)|\nabla w|^{2} + \int_{\Omega}u^{r}\mu(w)\Big(1 + \frac{1}{r} + 2\sigma\tilde{w}\tilde{H} - u\Big).$$
(4.16)

Using the Young inequality, we get

$$- (1+d) \int_{\Omega} u^{r-1} \mu'(w) \nabla u \cdot \nabla w \leq \frac{r-1}{4} \int_{\Omega} u^{r-2} \mu(w) |\nabla u|^2 + \frac{(1+d)^2}{r-1} \int_{\Omega} u^r \frac{\mu'^2(w)}{\mu(w)} |\nabla w|^2,$$
 (4.17)

and

$$-\chi_{1}(r-1)\int_{\Omega} u^{r-1}\mu(w)\nabla u \cdot \nabla w$$
  

$$\leq \frac{r-1}{4}\int_{\Omega} u^{r-2}\mu(w)|\nabla u|^{2} + \chi_{1}^{2}(r-1)\int_{\Omega} u^{r}\mu(w)|\nabla w|^{2}.$$
(4.18)

The combination (4.16), (4.17) and (4.18) yields

$$\frac{1}{r}\frac{d}{dt}\int_{\Omega}u^{r}\mu(w) + \frac{1}{r}\int_{\Omega}u^{r}\mu(w) + \frac{1}{r}\int_{\Omega}u^{r}\mu(w) + \frac{r-1}{2}\int_{\Omega}u^{r-2}\mu(w)|\nabla u|^{2} + \frac{d}{r}\int_{\Omega}u^{r}\mu''(w)|\nabla w|^{2} + \frac{(1+d)^{2}}{r-1}\int_{\Omega}u^{r}\frac{\mu'^{2}(w)}{\mu(w)}|\nabla w|^{2} + \chi_{1}^{2}(r-1)\int_{\Omega}u^{r}\mu(w)|\nabla w|^{2} + \chi_{1}\int_{\Omega}\mu(w)u^{r}\left(1 + \frac{1}{r} + 2\sigma\tilde{w}\bar{H} - u\right).$$
(4.19)

X. Deng et al.

It follows from Lemma 4.5 that

$$\frac{(1+d)^2}{r-1} \int_{\Omega} u^r \frac{\mu'^2(w)}{\mu(w)} |\nabla w|^2 + \chi_1^2(r-1) \int_{\Omega} u^r \mu(w) |\nabla w|^2 + \chi_1 \int_{\Omega} u^r \mu'(w) |\nabla w|^2 \le \frac{d}{r} \int_{\Omega} u^r \mu''(w) |\nabla w|^2.$$
(4.20)

Substituting (4.20) into (4.19), we can find a constant  $C_1 > 0$  such that

$$\begin{split} &\frac{1}{r}\frac{d}{dt}\int_{\Omega}u^{r}\mu(w)+\frac{1}{r}\int_{\Omega}u^{r}\mu(w)+\frac{r-1}{2}\int_{\Omega}u^{r-2}\mu(w)|\nabla u|^{2}\\ &\leq \mu(\tilde{w})\int_{\Omega}u^{r}\Big(1+\frac{1}{r}+2\sigma\tilde{w}\bar{H}-u\Big)\leq C_{1}, \end{split}$$

where we have used the fact that  $u^r(\theta - u)$  is bounded for any  $\theta > 0$ and  $u \ge 0$ . The using (4.13),  $u_0 \in W^{1,p}(\Omega)$  (p > n) and the Sobolev embedding  $W^{1,p}(\Omega) \hookrightarrow L^{\infty}(\Omega) \hookrightarrow L^{r}(\Omega)$ , we find that two constants  $C_2 > 0$  and  $C_3 > 0$  such that

$$\|u(\cdot,t)\|_{L^{r}}^{r} \leq \int_{\Omega} u^{r} \mu(w) \leq \max\{\mu(\tilde{w})\|u_{0}\|_{L^{r}}^{r}, rC_{1}\} \leq C_{2}(\|u_{0}\|_{W^{1,p}}^{r}+1) \leq C_{3}$$

for all  $t \in (0, T_{max})$ . Performing the same procedure to v, we can get a positive constant  $C_4$  such that  $||v(\cdot,t)||_{L^r} \leq C_4$  for all  $t \in (0, T_{\text{max}})$ . This completes the proof.  $\Box$ 

Proof of Theorem 3.1. The combination of Lemma 4.1, Lemmas 4.4 and 4.6 directly gives the global existence and boundedness of solutions for system (2.3), namely Theorem 3.1.

#### 5. Asymptotic behavior of solutions

In this section, we shall prove the asymptotic behavior of solutions solving system (3.3) by constructing some proper Lyapunov functionals. Let us first recall a basic result.

**Lemma 5.1** ([30, Lemma 1.1]).Let  $\theta \ge 0$ , m > 0 be constants,  $\psi(t) \ge 0$ ,  $\int_{\theta}^{\infty} \omega(t) dt < \infty$ . Assume that  $\varphi \in C^{1}([\theta, \infty))$ ,  $\varphi$  is bounded from below and satisfies

 $\varphi'(t) \leq -m\psi(t) + \omega(t)$  in  $[\theta, \infty)$ .

If either  $\psi \in C^1([\theta, \infty))$  and  $\psi'(t) \leq \kappa$  in  $[\theta, \infty)$  for some constant  $\kappa > 0$ , or  $\psi \in C^{\varrho}([\theta, \infty))$  and  $\|\psi\|_{C^{\varrho}([\theta, \infty))} \leq \kappa$  for some constants  $0 < \varrho < 1$  and  $\kappa > 0$ , then  $\lim_{t\to\infty} \psi(t) = 0$ .

Moreover, we need higher regularity of the solution given in the following.

**Lemma 5.2.** Let (u, v, w) be the unique global bounded classical solution of system (3.3) given by Theorem 3.1. Then for any given  $0 < \rho < 1$ , there exists a constant C > 0 such that

$$\begin{split} & \|u(\cdot,t)\|_{C^{2+\varrho,1+\frac{\rho}{2}}(\bar{\Omega}\times[1,\infty))} + \|v(\cdot,t)\|_{C^{2+\varrho,1+\frac{\rho}{2}}(\bar{\Omega}\times[1,\infty))} \\ & + \|w(\cdot,t)\|_{C^{2+\varrho,1+\frac{\rho}{2}}(\bar{\Omega}\times[1,\infty))} \leq C. \end{split}$$

Proof. This proof is based on the standard parabolic regularity theory [31, Theorem 1.3] and standard parabolic Schauder theory [32]. The proof details can follow the same way as the proof [33, Lemma 3.4] and details are omitted for brevity.  $\Box$ 

**Proof of Theorem 3.2(i).** Define the following Lyapunov functional:  $\mathcal{E}_{1}(t) := \mathcal{E}_{1}[u(t), v(t), w(t)] = \alpha_{1} \int_{\Omega} u + \beta_{1} \int_{\Omega} v + \frac{1}{2} \int_{\Omega} (w - w_{1})^{2},$ 

where  $\alpha_1 = \frac{w_1^2}{kw_1-2} > 0$ ,  $\beta_1 = \frac{w_1^2}{w_1-2} > 0$  if  $h_0 > 2\max\{1/k, 1\}$ . First, we can check that  $\mathcal{E}_1(t) \ge 0$  for all t > 0 due to the positivity of u, v. Next,

we shall show that there exist two positive constants  $\eta_1$  and  $\xi_1$  such that

$$\frac{d}{dt}\mathcal{E}_1(t) \le -\eta_1 \mathcal{F}_1(t) + \xi_1 \int_{\Omega} (H(x,t) - h_0)^2$$
holds for all  $t > 0$ , where
$$(5.1)$$

 $\mathcal{F}_1(t) = \int [u^2 + v^2 + (w - w_1)^2].$ 

d

$$J_{\Omega}$$
 The time derivative of the functional  $\mathcal{E}(t)$  of

The time derivative of the functional  $\mathcal{E}_1(t)$  along trajectories of system (3.3) is given by

$$\begin{split} \frac{d}{dt} \mathcal{E}_{1}(t) &= \alpha_{1} \int_{\Omega} u_{t} + \beta_{1} \int_{\Omega} v_{t} + \int_{\Omega} (w - w_{1})w_{t} \\ &= \alpha_{1} \int_{\Omega} u(1 - u - cv - kw) + \beta_{1} \int_{\Omega} v(1 - bu - v - w) \\ &+ \int_{\Omega} (w - w_{1})(H(x, t) - w - uw - vw) - d \int_{\Omega} |\nabla w|^{2} \\ &= - \int_{\Omega} \left[ \alpha_{1}u^{2} + \beta_{1}v^{2} + (w - w_{1})^{2} + (\alpha_{1}k - w_{1})u(w - w_{1}) \\ &+ (\beta_{1} - w_{1})v(w - w_{1}) \right] \\ &- (\alpha_{1}kw_{1} - \alpha_{1} - w_{1}^{2}) \int_{\Omega} u - (\beta_{1}w_{1} - \beta_{1} - w_{1}^{2}) \int_{\Omega} v \\ &- (\alpha_{1}c + \beta_{1}b) \int_{\Omega} uv \\ &- \int_{\Omega} (u + v)w^{2} - d \int_{\Omega} |\nabla w|^{2} + \int_{\Omega} (w - w_{1})(H(x, t) - h_{0}) \\ &= - \int_{\Omega} \left[ \alpha_{1}u^{2} + \beta_{1}v^{2} + (w - w_{1})^{2} + \frac{2\alpha_{1}}{w_{1}}u(w - w_{1}) \\ &+ \frac{2\beta_{1}}{w_{1}}v(w - w_{1}) \right] - \alpha_{1} \int_{\Omega} u - \beta_{1} \int_{\Omega} v \\ &- (\alpha_{1}c + \beta_{1}b) \int_{\Omega} uv - \int_{\Omega} (u + v)w^{2} - d \int_{\Omega} |\nabla w|^{2} \\ &+ \int_{\Omega} (w - w_{1})(H(x, t) - h_{0}) \\ &= - \int_{\Omega} \Theta_{1}A_{1}\Theta_{1}^{T} - \alpha_{1} \int_{\Omega} u - \beta_{1} \int_{\Omega} v \\ &- (\alpha_{1}c + \beta_{1}b) \int_{\Omega} uv - \int_{\Omega} (u + v)w^{2} \\ &- d \int_{\Omega} |\nabla w|^{2} + \int_{\Omega} (w - w_{1})(H(x, t) - h_{0}), \end{split}$$

where

$$\Theta_1 = (u, v, w - w_1) \text{ and } A_1 = \begin{bmatrix} \alpha_1 & 0 & \frac{\alpha_1}{w_1} \\ 0 & \beta_1 & \frac{\beta_1}{w_1} \\ \frac{\alpha_1}{w_1} & \frac{\beta_1}{w_1} & 1 \end{bmatrix},$$

and  $\Theta_1^T$  denotes the transpose of  $\Theta_1$ . It follows from  $\alpha_1 > 0$  and  $\beta_1 > 0$ that

$$|A_{1,1}| := \begin{vmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{vmatrix} = \alpha_1 \beta_1 > 0.$$

By some calculations, one can check that

$$|A_1| = \alpha_1 \beta_1 - \frac{\alpha_1 \beta_1^2}{w_1^2} - \frac{\alpha_1^2 \beta_1}{w_1^2} > 0$$

is equivalent to

 $(w_1 - 2)(kw_1 - 2) > (k + 1)w_1 - 4,$ 

that is

$$kh_0^2 - 3(k+1)h_0 + 8 > 0.$$

Solving the above inequality, one has

$$h_0 < \hbar$$
 or  $h_0 > \hbar_1$ 

where

$$\hbar := \frac{3(k+1) - \sqrt{9(1-k)^2 + 4k}}{2k}$$

and  $h_1$  is given by (3.5). By a simple calculation, we obtain that h < 1 $2 \max\{1/k, 1\} < \hbar_1$ . Then  $A_1$  is a positive definite matrix if  $h_0 > \hbar_1$ . This means that there exists a positive constant  $\eta_1$  such that

$$\Theta_1 A_1 \Theta_1^T \ge 2\eta_1 |\Theta_1|^2$$

based on the Sylvester criterion. Under the condition  $h_0 > h_1$ , one has that

$$\frac{d}{dt}\mathcal{E}_1(t) \leq -2\eta_1 \mathcal{F}_1(t) + \int_{\Omega} (w - w_1)(H(x, t) - h_0) dt$$

Using the Young inequality and the definition of  $\mathcal{F}_1(t)$ , we get that

$$\begin{split} \frac{u}{dt} \mathcal{E}_1(t) &\leq -2\eta_1 \mathcal{F}_1(t) + \int_{\Omega} (w - w_1) (H(x, t) - h_0) \\ &\leq -2\eta_1 \mathcal{F}_1(t) + \eta_1 \int_{\Omega} (w - w_1)^2 + \frac{1}{4\eta_1} \int_{\Omega} (H(x, t) - h_0)^2 \\ &\leq -\eta_1 \mathcal{F}_1(t) + \xi_1 \int_{\Omega} (H(x, t) - h_0)^2, \end{split}$$

where  $\xi_1 := \frac{1}{4\eta_1} > 0$ , which gives (5.1). We proceed to show (3.8). From the definitions of  $\mathcal{E}_1(t)$  and  $\mathcal{F}_1(t)$ , we get  $\mathcal{E}_1(t) \in C^1([1,\infty))$  and  $\mathcal{F}_1(t) \ge 0$ . Using the regularity of u, v, wobtained in Lemma 5.2, it is easy to see that  $\mathcal{F}_1(t) \in C^{\varrho/2}([1,\infty))$  and  $\|\mathcal{F}_1\|_{C^{\varrho/2}([1,\infty))} \leq \kappa$  in  $[1,\infty)$  for some constant  $\kappa > 0$ . Then applying Lemma 5.1 and hypothesis ( $\mathcal{H}_1$ ), we obtain  $\lim_{t\to\infty} \mathcal{F}_1(t) = 0$ , that is

 $\lim \left( \|u\|_{L^2} + \|v\|_{L^2} + \|w - w_1\|_{L^2} \right) = 0.$ (5.2)

By Lemma 5.2, we get

 $\|u\|_{W^{1,\infty}}+\|v\|_{W^{1,\infty}}+\|w-w_1\|_{W^{1,\infty}}\leq C_1 \ \, \text{for all} \ t>1.$ 

Then using the Gagliardo-Nirenberg inequality, one has

 $\|u\|_{L^{\infty}} \leq C_2 \|u\|_{W^{1,\infty}}^{1/2} \|u\|_{L^2}^{1/2} \leq C_1^{1/2} C_2 \|u\|_{L^2}^{1/2},$ 

which combined with (5.2) implies

 $\lim_{t \to \infty} \|u\|_{L^{\infty}} = 0.$ 

Similar to the above we can get that

 $\lim_{t \to \infty} (\|v\|_{L^{\infty}} + \|w - w_1\|_{L^{\infty}}) = 0.$ 

Thus (3.8) holds, and the proof is completed.

Proof of Theorem 3.2(ii). Define the following Lyapunov functional:

 $\mathcal{E}_2(t) := \mathcal{E}_2[u(t), v(t), w(t)] = I_1(t) + I_2(t) + I_3(t),$ (5.3)

 $I_1(t) = \int_{\Omega} \left( u - u_5 - u_5 \ln \frac{u}{u_5} \right), \ I_2(t) = \int_{\Omega} \left( v - v_5 - v_5 \ln \frac{v}{v_5} \right),$  $I_3(t) = \frac{1}{2w_5} \int_O (w - w_5)^2.$ 

First, we show that  $\mathcal{E}_2(t) \ge 0$  for all t > 0. In fact, letting  $\phi(f) =$  $f - f_* \ln f$  and using the Taylor expansion, for all positive f and  $f_*$ , we have

$$f - f_* - f_* \ln \frac{f}{f_*} = \phi(f) - \phi(f_*) = \phi'(f_*)(f - f_*) + \frac{1}{2}\phi''(\delta)(f - f_*)^2$$
$$= \frac{f_*}{2\delta^2}(f - f_*)^2,$$
(5.4)

where  $\delta$  is between f and  $f_*$ . Then setting f = u and  $f_* = u_5$ , from (5.4) we can find  $\delta_1$  between *u* and  $u_5$  such that

$$u - u_5 - u_5 \ln \frac{u}{u_5} = \frac{u_5}{2\delta_1^2} (u - u_5)^2 \ge 0,$$

which implies  $I_1(t) \ge 0$ . Similarly we have that  $I_2(t) \ge 0$ . Then it follows that  $\mathcal{E}_2(t) \ge 0$  for all t > 0.

Next, we show that under certain conditions, there exist two constants  $\eta_2 > 0$  and  $\xi_2 > 0$  such that

$$\frac{d}{dt}\mathcal{E}_{2}(t) \leq -\eta_{2}\mathcal{F}_{2}(t) + \xi_{2} \int_{\Omega} (H(x,t) - h_{0})^{2}$$
(5.5)

holds for all t > 0, where

$$F_2(t) = \int_{\Omega} [(u - u_5)^2 + (v - v_5)^2 + (w - w_5)^2].$$

For this purpose, we use the first equation of (3.3) and the fact that  $1 - u_5 - cv_5 - kw_5 = 0$  to estimate  $I_1(t)$  as follows

$$\frac{d}{dt}I_{1}(t) = \int_{\Omega} \left(1 - \frac{u_{5}}{u}\right)u_{t}$$

$$= -u_{5} \int_{\Omega} \left|\frac{\nabla u}{u}\right|^{2} - \chi_{1}u_{5} \int_{\Omega} \frac{\nabla u \cdot \nabla w}{u} + \int_{\Omega} (u - u_{5})(1 - u - cv - kw)$$

$$= -u_{5} \int_{\Omega} \left|\frac{\nabla u}{u}\right|^{2} - \chi_{1}u_{5} \int_{\Omega} \frac{\nabla u \cdot \nabla w}{u}$$

$$- \int_{\Omega} (u - u_{5})^{2} - c \int_{\Omega} (u - u_{5})(v - v_{5})$$

$$- k \int_{\Omega} (u - u_{5})(w - w_{5}).$$
(5.6)

Using the second equation of (3.3) and the fact that  $1-bu_5-v_5-w_5=0$ , we obtain

$$\frac{d}{dt}I_{2}(t) = \int_{\Omega} \left(1 - \frac{v_{5}}{v}\right)u_{t}$$

$$= -v_{5}\int_{\Omega} \left|\frac{\nabla v}{v}\right|^{2} - \chi_{2}v_{5}\int_{\Omega} \frac{\nabla v \cdot \nabla w}{v} + \int_{\Omega} (v - v_{5})(1 - bu - v - w)$$

$$= -v_{5}\int_{\Omega} \left|\frac{\nabla v}{v}\right|^{2} - \chi_{2}v_{5}\int_{\Omega} \frac{\nabla v \cdot \nabla w}{v}$$

$$-\int_{\Omega} (v - v_{5})^{2} - b\int_{\Omega} (u - u_{5})(v - v_{5})$$

$$-\int_{\Omega} (v - v_{5})(w - w_{5}).$$
(5.7)

Furthermore with the fact  $h_0 - w_5 - u_5 w_5 - v_5 w_5 = 0$  and using the third equation of (3.3), we get

$$\begin{aligned} \frac{d}{dt}I_{3}(t) &= \frac{1}{w_{5}} \int_{\Omega} (w - w_{5})w_{t} \\ &= -\frac{d}{w_{5}} \int_{\Omega} |\nabla w|^{2} + \frac{1}{w_{5}} \int_{\Omega} (w - w_{5})(H(x, t) - w - uw - vw) \\ &= -\frac{d}{w_{5}} \int_{\Omega} |\nabla w|^{2} + \frac{1}{w_{5}} \int_{\Omega} (w - w_{5})(H(x, t) - h_{0}) \\ &- \frac{1}{w_{5}} \int_{\Omega} (w - w_{5})^{2} - \int_{\Omega} (u - u_{5})(w - w_{5}) \\ &- \int_{\Omega} (v - v_{5})(w - w_{5}) - \frac{1}{w_{5}} \int_{\Omega} (u + v)(w - w_{5})^{2}. \end{aligned}$$
(5.8)

Combining (5.3), (5.6), (5.7) with (5.8) gives that

$$\frac{d}{dt}\mathcal{E}_2(t) = -\int_{\Omega}\Theta_2 A_2 \Theta_2^T - \int_{\Omega} \Lambda B \Lambda^T + \frac{1}{w_5} \int_{\Omega} (w - w_5)(H(x, t) - h_0) - \frac{1}{w_5} \int_{\Omega} (u + v)(w - w_5)^2,$$
(5.9)

where  $\Theta_2 = (u - u_5, v - v_5, w - w_5), \Lambda = (\frac{\nabla u}{u}, \frac{\nabla v}{v}, \nabla w)$  and  $A_2, B$  are symmetric matrices denoted by

$$A_2 := \begin{bmatrix} 1 & \frac{b+c}{2} & \frac{k+1}{2} \\ \frac{b+c}{2} & 1 & 1 \\ \frac{k+1}{2} & 1 & \frac{1}{w_5} \end{bmatrix}, B := \begin{bmatrix} u_5 & 0 & \frac{\chi_1 u_5}{2} \\ 0 & v_5 & \frac{\chi_2 v_5}{2} \\ \frac{\chi_1 u_5}{2} & \frac{\chi_2 v_5}{2} & \frac{\lambda_2 v_5}{2} \end{bmatrix}$$

To proceed, we claim that  $A_2$  and B are positive definite and positive semi-definite respectively. Notice b < 1 and c < 1 imply b + c < 2. Then

$$|A_{2,1}| := \left| \begin{array}{c} 1 & \frac{b+c}{2} \\ \frac{b+c}{2} & 1 \end{array} \right| = 1 - \frac{(b+c)^2}{4} > 0,$$

and

$$\begin{split} |A_2| &= \frac{1}{w_5} |A_{21}| + \frac{1}{4} [2(b+c)(k+1) - 4 - (k+1)^2] \\ &> |A_{2,1}| \Big[ \frac{1}{w_5} - \frac{4 + (k+1)^2}{4 - (b+c)^2} \Big] > 0 \\ \text{if} \\ w_5 &< \frac{4 - (b+c)^2}{4 + (k+1)^2}. \end{split}$$
(5.10)

With (5.10) in hand and combining the conditions of existence of  $E_5$ (see Table 1), we can directly calculate that  $|A_2| > 0$  if either  $k \leq 1$ 1,  $h_0 < \min\{h_1, \hbar_3\}$  or k > 1,  $h_0 < \min\{h_2, \hbar_3\}$ . Based on the Sylvester criterion,  $A_2$  is a positive definite matrix and we can find a constant  $\eta_2 > 0$  such that

$$\Theta_2 A_2 \Theta_2^T \ge 2\eta_2 |\Theta_2|^2. \tag{5.11}$$

For B, one can deduce that

$$|B_1| := \begin{vmatrix} u_5 & 0 \\ 0 & v_5 \end{vmatrix} = u_5 v_5 > 0,$$
  
and

$$|B| = \frac{u_5 v_5}{4} \left[ \frac{4d}{w_5} - (u_5 \chi_1^2 + v_5 \chi_2^2) \right] \ge 0$$

if and only if  $\frac{w_5}{4d}(\chi_1^2 u_5 + \chi_2^2 v_5) \le 1$ . Then *B* is a positive semi-definite matrix, and hence

$$ABA^{T} \ge 0. \tag{5.12}$$

Substituting (5.11) and (5.12) into (5.9), we have

$$\frac{d}{dt}\mathcal{E}_{2}(t) \leq -2\eta_{2}\mathcal{F}_{2}(t) + \frac{1}{w_{5}}\int_{\Omega} (w - w_{5})(H(x, t) - h_{0})$$

Then we can use the similar arguments as for (5.1), we obtain (5.5)with  $\xi_2 := \frac{1}{4\eta_2 w_s^2} > 0$ . Using Lemmas 5.1 and 5.2 again as for (3.8), we obtain (3.9) and hence complete the proof of Theorem 3.2.

Remark 5.1. From the proof of Theorem 3.2, the convergence rate of the solution is elusive, as the rate at which H(x,t) converges to  $h_0$ when t approaches infinity is unknown (see  $(\mathcal{H}_1)$ ). In addition to the hypotheses of Theorem 3.2, if we further assume that

 $||H(x,t) - h_0||_{L^2} \le C_1 e^{-C_2 t},$ 

where  $C_1, C_2 > 0$  are constant. Then under the conditions of Theorem 3.2, we have

 $\|(u, v, w) - (0, 0, w_1)\|_{L^{\infty}(\Omega)} \to 0$  exponentially as  $t \to \infty$ ,

or

 $\|(u, v, w) - (u_5, v_5, w_5)\|_{L^{\infty}(\Omega)} \to 0$  exponentially as  $t \to \infty$ .

#### 6. Linear instability analysis

The main purpose of this section is to investigate whether the PDE system (3.4) can generate spatially heterogeneous solutions in certain parameter regimes. From Table 1, we see that the equilibria of the ODE system (3.6)  $E_1$ ,  $E_2$ ,  $E_3$ ,  $E_5$  are locally asymptotically stable under certain conditions, while  $E_4$  is unstable. In what follows, we will derive conditions under which the equilibria  $E_1$ ,  $E_2$ ,  $E_3$ ,  $E_5$  become unstable in the presence of spatial structure. To facilitate this analysis, we linearize system (3.4) about a constant steady state  $(u_s, v_s, w_s)$  to get

$$\begin{cases} \Phi_t = \mathbf{A}\Delta\Phi + J\Phi, & x \in \Omega, \ t > 0, \\ \nabla\Phi \cdot \mathbf{v} = 0, & x \in \partial\Omega, \ t > 0, \\ \Phi(\cdot, 0) = (u_0 - u_s, v_0 - v_s, w_0 - w_s)^{\mathsf{T}}, & x \in \Omega, \end{cases}$$
(6.1)

where T denotes the transpose and

0.

(6.3)

$$\boldsymbol{\Phi} = \begin{bmatrix} u - u_s \\ v - v_s \\ w - w_s \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 1 & 0 & \chi_1 u_s \\ 0 & 1 & \chi_2 v_s \\ 0 & 0 & d \end{bmatrix},$$

$$\boldsymbol{J} = \begin{bmatrix} 1 - 2u_s - cv_s - kw_s & -cu_s & -ku_s \\ -bv_s & 1 - bu_s - 2v_s - w_s & -v_s \\ -w_s & -w_s & -1 - u_s - v_s \end{bmatrix}.$$

Let  $W_{a}(x)$  represent the eigenfunction of the following eigenvalue problem:

$$\Delta W_{\rho}(x) + \rho^2 W_{\rho}(x) = 0 \text{ in } \Omega, \text{ and } \frac{\partial W_{\rho}(x)}{\partial \nu} = 0 \text{ on } \partial \Omega,$$

where  $\rho$  is the wave number. Since (6.1) is linear, the solution  $\Phi(x,t)$ has the form of

$$\Phi(x,t) = \sum_{\rho \ge 0} C_{\rho} e^{\lambda t} W_{\rho}(x), \tag{6.2}$$

where the constants  $C_{\rho}$  are determined by Fourier expansion of the initial conditions in terms of  $W_{\rho}(x)$ , and  $\lambda$  is the temporal eigenvalue. Inserting (6.2) into (6.1) yields

$$\mathcal{A}W_{\rho}(x) = -\rho^2 \mathbf{A}W_{\rho}(x) + JW_{\rho}(x),$$

which means that  $\lambda$  is the eigenvalue of the matrix given in Box I. By calculating the eigenvalue of matrix  $\mathcal{J}_{\rho}$ , we obtain the eigenvalues  $\lambda(\rho^2)$ as the roots of the characteristic equation given in Box II.

For the toxicant-only equilibrium  $E_1 = (0, 0, w_1)$ , we get

$$\begin{split} P_{1}(\lambda) &:= \operatorname{Det}(\lambda I - \mathcal{J}_{\rho}(E_{1})) \\ &= \begin{vmatrix} \lambda + \rho^{2} - (1 - kw_{1}) & 0 & 0 \\ 0 & \lambda + \rho^{2} - (1 - w_{1}) & 0 \\ w_{1} & w_{1} & \lambda + d\rho^{2} + 1 \\ &= [\lambda + \rho^{2} - (1 - kw_{1})][\lambda + \rho^{2} - (1 - w_{1})](\lambda + d\rho^{2} + 1) = 0 \end{split}$$

Then the eigenvalues of  $E_1$  are

$$\lambda_1 = -\rho^2 + 1 - kw_1, \lambda_2 = -\rho^2 + 1 - w_1, \lambda_3 = -d\rho^2 - 1$$

It is straightforward to verify that when  $h_0 > \max\{1/k, 1\}$ , we have  $\lambda_1 < 0, \lambda_2 < 0$ , and  $\lambda_3 < 0$ . Therefore, the toxicant-only equilibrium  $E_1$  is linearly stable, and no bifurcation will arise from  $E_1$  if  $h_0 > 1$  $\max\{1/k, 1\}.$ 

For the semi-trivial equilibrium  $E_2 = (0, v_2, w_2)$ , we have

$$\begin{split} P_{2}(\lambda) &:= \operatorname{Det}(\lambda I - \mathcal{J}_{\rho}(E_{2})) \\ &= \begin{vmatrix} \lambda + \rho^{2} - (1 - cv_{2} - kw_{2}) & 0 & 0 \\ bv_{2} & \lambda + \rho^{2} + v_{2} & \chi_{2}v_{2}\rho^{2} + v_{2} \\ w_{2} & w_{2} & \lambda + d\rho^{2} + 1 + v_{2} \end{vmatrix} \\ &= [\lambda + \rho^{2} - (1 - cv_{2} - kw_{2})][\lambda^{2} + \mathcal{A}_{1}\lambda + \mathcal{B}_{1}] = 0, \end{split}$$

$$\begin{aligned} \mathcal{A}_1 &= (1+d)\rho^2 + 2v_2 + 1 > 0, \\ \mathcal{B}_1 &= d\rho^4 + (1+v_2 + dv_2 - v_2w_2\chi_2)\rho^2 + v_2(1+v_2 - w_2) \end{aligned}$$

Note that the condition for the local stability of  $E_2$  in the ODE system (3.6) (see Appendix) is  $1 - cv_2 - kw_2 < 0$ . This implies that  $\rho^2 - (1 - cv_2) - kw_2 < 0$ .  $cv_2 - kw_2 > 0$  for all  $\rho \ge 0$ . Since  $A_1 > 0$  for all  $\rho \ge 0$ , Eq. (6.3) has no purely imaginary roots. This indicates that Hopf bifurcation emerging from the equilibrium  $E_2$  is impossible By a simple calculation, we find that  $B_1 > 0$  if and only if

$$\chi_2 < \bar{\chi}(\rho^2) := \frac{d\rho^4 + (1 + v_2 + dv_2)\rho^2 + v_2(1 + v_2 - w_2)}{v_2 w_2 \rho^2} \text{ for } \rho > 0,$$

where  $1+v_2-w_2 > 0$  and then  $\bar{\chi}(\rho^2) > 0$ . The above analysis reveals that all eigenvalues of Eq. (6.3) have negative real parts, indicating that no spatially inhomogeneous patterns emerge when  $\chi_2 < \bar{\chi}(\rho^2)$ . When  $\chi_2 =$  $\bar{\chi}(\rho^2)$ , Eq. (6.3) has one zero eigenvalue and two negative eigenvalues. While, when  $\chi_2 > \bar{\chi}(\rho^2)$ , Eq. (6.3) possesses a positive eigenvalue,

$$\begin{aligned} \mathcal{J}_{\rho} &:= -\rho^{2}\mathbf{A} + J \\ &= \begin{bmatrix} -\rho^{2} + 1 - 2u_{s} - cv_{s} - kw_{s} & -cu_{s} & -\chi_{1}u_{s}\rho^{2} - ku_{s} \\ -bv_{s} & -\rho^{2} + 1 - bu_{s} - 2v_{s} - w_{s} & -\chi_{2}v_{s}\rho^{2} - v_{s} \\ -w_{s} & -w_{s} & -d\rho^{2} - 1 - u_{s} - v_{s} \end{bmatrix}. \end{aligned}$$

$$\begin{aligned} & \text{Det}(\lambda I - \mathcal{J}_{\rho}) \\ & = \begin{vmatrix} \lambda + \rho^2 - (1 - 2u_s - cv_s - kw_s) & cu_s & \chi_1 u_s \rho^2 + ku_s \\ bv_s & \lambda + \rho^2 - (1 - bu_s - 2v_s - w_s) & \chi_2 v_s \rho^2 + v_s \\ w_s & w_s & \lambda + d\rho^2 + 1 + u_s + v_s \end{vmatrix} = 0. \end{aligned}$$

#### Box II.

Box I.

suggesting that the equilibrium  $E_2$  is linearly unstable for the PDE system, and a steady-state bifurcation may occur in this scenario.

Next, we consider the semi-trivial equilibrium  $E_3 = (u_3, 0, w_3)$ . Following a similar analysis, we obtain

$$P_{2}(\lambda) := \text{Det}(\lambda I - \mathcal{J}_{\rho}(E_{3})) = [\lambda + \rho^{2} - (1 - bu_{3} - w_{3})][\lambda^{2} + \mathcal{A}_{2}\lambda + \mathcal{B}_{2}]$$
  
= 0,  
(6.4)

where

$$\begin{split} \mathcal{A}_2 &= (1+d)\rho^2 + 2u_3 + 1 > 0, \\ \mathcal{B}_2 &= d\rho^4 + [1+u_3 + du_3 - u_3w_3\chi_1]\rho^2 + u_3(1+u_3 - kw_3). \end{split}$$

Under the conditions that ensure the local stability of  $E_3$  for the ODE system (3.6), we have  $1 - bu_3 - w_3 < 0$ , which implies that  $\rho^2 - (1 - bu_3 - w_3) > 0$  for  $\rho \ge 0$ . It can be easily verified that Eq. (6.4) has no pure imaginary roots, and thus the Hopf bifurcation arising from  $E_3$  is impossible due to  $A_2 > 0$ . Furthermore, it is easy to check that  $B_2 > 0$  if and only if  $\chi_1 < \hat{\chi}(\rho^2)$ , where

$$\hat{\chi}(\rho^2) := \frac{d\rho^4 + [1 + u_3 + du_3]\rho^2 + u_3(1 + u_3 - kw_3)}{u_3 w_3 \rho^2} > 0.$$

Consequently, when  $\chi_1 < \hat{\chi}(\rho^2)$ , all the eigenvalues of (6.4) have negative real parts, indicating that  $E_3$  is linearly stable for the PDE system (3.4). When  $\chi_1 = \hat{\chi}(\rho^2)$ , Eq. (6.4) has one zero eigenvalue and two negative eigenvalues. However, when  $\chi_1 > \hat{\chi}(\rho^2)$ , one of the eigenvalues of (6.4) becomes a positive real number, leading to the instability of  $E_3$  and hence the steady-state bifurcation may arise.

In a similar way, for the coexistence equilibrium  $E_5 = (u_5, v_5, w_5)$ , using the fact that

$$\begin{cases} 1 - u_5 - cv_5 - kw_5 = 0, \\ 1 - bu_5 - v_5 - w_5 = 0, \\ h_0 - w_5 - u_5w_5 - v_5w_5 = 0, \end{cases}$$

we get that

$$P_4(\lambda) := \lambda^3 + A_3 \lambda^2 + B_3 \lambda + C_3 = 0,$$
(6.5)

where

$$\begin{aligned} \mathcal{A}_{3} &= (2+d)\rho^{2} + u_{5} + v_{5} + \frac{h_{0}}{w_{5}} > 0, \\ \mathcal{B}_{3} &= (\rho^{2} + u_{5}) \Big[ (1+d)\rho^{2} + v_{5} + \frac{h_{0}}{w_{5}} \Big] + (\rho^{2} + v_{5}) \Big( d\rho^{2} + \frac{h_{0}}{w_{5}} \Big) \\ &- v_{5} w_{5} (\chi_{2} \rho^{2} + 1) - u_{5} w_{5} (\chi_{1} \rho^{2} + k) - b c u_{5} v_{5}, \\ \mathcal{C}_{3} &= (\rho^{2} + u_{5}) (\rho^{2} + v_{5}) \Big( d\rho^{2} + \frac{h_{0}}{w_{5}} \Big) - u_{5} w_{5} (\chi_{1} \rho^{2} + k) (\rho^{2} + v_{5}) \\ &- v_{5} w_{5} (\chi_{2} \rho^{2} + 1) (\rho^{2} + u_{5}) - b c u_{5} v_{5} \Big( d\rho^{2} + \frac{h_{0}}{w_{5}} \Big) \\ &+ u_{5} v_{5} w_{5} [c (\chi_{2} \rho^{2} + 1) + b (\chi_{1} \rho^{2} + k)]. \end{aligned}$$

$$(6.6)$$

Simple calculation yields

$$\mathcal{A}_{3}\mathcal{B}_{3} - \mathcal{C}_{3} = \left[ (1+d)\rho^{2} + u_{5} + \frac{h_{0}}{w_{5}} \right] \left[ (1+d)\rho^{2} + v_{5} + \frac{h_{0}}{w_{5}} \right] \left( 2\rho^{2} + u_{5} + v_{5} \right) - bcu_{5}v_{5}(2\rho^{2} + u_{5} + v_{5}) - u_{5}v_{5}w_{5}[c(\chi_{2}\rho^{2} + 1) + b(\chi_{1}\rho^{2} + k)] - u_{5}w_{5}(\chi_{1}\rho^{2} + k) \left[ (1+d)\rho^{2} + u_{5} + \frac{h_{0}}{w_{5}} \right] - v_{5}w_{5}(\chi_{2}\rho^{2} + 1) \left[ (1+d)\rho^{2} + v_{5} + \frac{h_{0}}{w_{5}} \right].$$
(6.7)

Since the coefficients of (6.5) are real numbers, the equation has three roots, one of which must be real, while the other two can be either real or complex conjugates. According to the Routh–Hurwitz criterion [34, Appendix B] for third-order polynomials, all the eigenvalues are negative or have negative real parts if and only if

$$A_3 > 0, C_3 > 0 \text{ and } A_3B_3 - C_3 > 0.$$

Given  $A_3 > 0$ , the coexistence equilibrium  $E_5$  is unstable if and only if  $C_3 < 0$  or  $A_3B_3 - C_3 < 0$ . Since at least one eigenvalue has negative real part if  $A_3 > 0$ , from the Appendix of [35], we see that one eigenvalue of (6.4) is a positive real number, and the other two eigenvalues are either negative real numbers or complex conjugates with negative real parts if  $C_3 < 0$ . Therefore, Eq. (6.4) has complex roots with nonnegative real parts only if  $C_3 \ge 0$ . In particular, there is a zero root when  $C_3 = 0$ . If  $C_3 > 0$ , again by the Appendix of [35], the equation has two eigenvalues (either real or complex) with positive real parts if  $A_3B_3 - C_3 < 0$ . Summarizing the above discussion, we have the following results.

**Lemma 6.1.** Suppose that the equilibria  $E_1$ ,  $E_2$ ,  $E_3$ ,  $E_5$  are locally asymptotically stable with respect to the ODE system (3.6) (see Table 1 or Fig. 1). Then the following hold for the system (3.4):

- (i)  $E_1$  is linearly stable for any  $\chi_1$ ,  $\chi_2 \ge 0$ .
- (ii)  $E_2$  is linearly stable if  $\chi_2 < \bar{\chi}(\rho^2)$  for any  $\rho > 0$ , while unstable and a steady-state bifurcation may occur if  $\chi_2 > \bar{\chi}(\rho^2)$  for some  $\rho > 0$ .
- (iii) E<sub>3</sub> is linearly stable if χ<sub>1</sub> < χ̂(ρ<sup>2</sup>) for any ρ > 0, while unstable and a steady-state bifurcation may occur if χ<sub>1</sub> > χ̂(ρ<sup>2</sup>) for some ρ > 0.
- (iv)  $E_5$  is linearly stable if  $C_3 > 0$  and  $A_3B_3 C_3 > 0$ , while unstable if  $\chi_1$  or  $\chi_2$  is large such that  $C_3 < 0$  or  $A_3B_3 C_3 < 0$ , where the Hopf bifurcation is impossible when  $C_3 < 0$ .

#### 7. Numerical simulations and applications

In this section, we first demonstrate that spatially and temporally homogeneous (i.e., constant) toxicant input rates can generate spatially inhomogeneous patterns from the PDE system (3.4). We then explore how spatially and temporally varying toxicant input, negative toxicanttaxis and toxicant diffusivity influence the competition outcomes by resorting to numerical studies.

#### 7.1. Homogeneous toxicant input

In this subsection, we assume that the toxicant input rate  $H(x,t) \equiv h_0$  with a positive constant  $h_0$ . From Lemma 6.1, we see that when the toxicant–taxis coefficients  $\chi_1, \chi_2$  are sufficiently large, the spatially homogeneous steady states  $E_2$ ,  $E_3$ , and  $E_5$ , which are stable in ODE system, become unstable in the PDE system (3.4). Below we aim to numerically illustrate that this taxis-driven instability can give rise to spatially inhomogeneous patterns.

We first demonstrate that spatial patterns may emerge from the coexistence steady state  $E_5 = (u_5, v_5, w_5)$ . To this end, we let  $b = 0.6, c = 0.4, k = 1, h_0 = 0.9$ . Note that the allowable wave numbers  $\rho$  are discrete in a bound domain, for instance, for a one-dimensional domain  $\Omega = (0, l), \rho = n\pi/l$  (n = 1, 2, ...). For numerical simulations, we choose  $l = 12\pi, n = 12$  such that  $\rho = 1$ . Then from Eqs. (6.6) and (6.7), we get

$$C_3 = 3.7463 - 0.2094\chi_1 - 0.1538\chi_2 < 0 \Leftrightarrow \chi_2 > \frac{3.7463 - 0.2094\chi_1}{0.1538}$$
(7.1)

and

$$\mathcal{A}_3 \mathcal{B}_3 - \mathcal{C}_3 = 36.1350 - 0.7731\chi_1 - 0.5012\chi_2 < 0 \Leftrightarrow \chi_2 > \frac{36.1350 - 0.7731\chi_1}{0.5012}.$$
  
Since  $\chi_1 > 0$ , it clearly holds that

$$\frac{3.7463 - 0.2094\chi_1}{0.1538} < \frac{36.1350 - 0.7731\chi_1}{0.5012}.$$

Then according to Lemma 6.1(iv), the coexistence equilibrium  $E_5$  is unstable if (7.1) holds. We choose  $\chi_1 = 10, \chi_2 = 11$  to satisfy (7.1). Fig. 2 shows the long-time solution profiles of system (3.4) with the initial data set as a small random perturbation of  $E_5$ . The figure reveals the emergence of spatial aggregation and segregation patterns between the populations and the toxicant, with higher (lower) population densities observed in regions of lower (higher) toxicant concentrations. For Fig. 2, we calculate the long-time average density of species *u* (approximating  $\frac{1}{|\Omega|} \int_{\Omega} u(x, 600) dx := \bar{u}(600)$ ). The results show that  $\bar{u}(600) = 0.2602$ , which is less than the equilibrium density  $u_5 = 0.3321$ . Similarly, for species *v*,  $\bar{v}(600) = 0.1639$ , which is also below its equilibrium value  $v_5 = 0.2214$ . This is because the aggregation of individuals in low toxicant regions leads to stronger intra-specific competition, resulting in a decline in the overall population density.

Next, we present an example for which spatial patterns can arise from the steady state  $E_3 = (u_3, 0, w_3)$ , where species *u* wipes out species *v*, based on Lemma 6.1(iii). We choose  $b = 0.6, c = 0.4, k = 0.5, h_0 = 1$ . Then according to Lemma 6.1(iii), the constant semi-trivial steady state  $E_3$  is linearly stable if

$$\chi_1 < \hat{\chi}(\rho^2) = \frac{\rho^4 + 2.4142\rho^2 + 1}{0.4142\rho^2}$$

for all  $\rho > 0$ , but  $E_3$  becomes unstable if  $\chi_1 > \hat{\chi}(\rho^2)$  for some  $\rho > 0$ . Similar to Fig. 2, we choose  $l = 12\pi$ , n = 12 such that  $\rho = 1$ , then  $\hat{\chi}(1) = 10.6572$ . Fig. 3 shows long-time solution profiles of (3.4) with the initial data set as a small random perturbation from the semi-trivial equilibrium  $E_3$ . When  $\chi_1 < \hat{\chi}(1)$ , the solution of system (3.4) converges to the constant steady state  $E_3$  (see Fig. 3(a)). However, as the taxis coefficient  $\chi_1$  exceeds  $\hat{\chi}(1)$ , spatially inhomogeneous patterns emerge from  $E_3$ . Importantly, this pattern demonstrates the coexistence of both species, which implies that high toxicant–taxis can shift the competitive outcome from exclusion to coexistence, potentially promoting biodiversity. This phenomenon occurs because one species concentrates in areas with low toxicant levels, which leaves more space and resources for the other species to share.

#### 7.2. Inhomogeneous toxicant input

This subsection investigates the influence of time- and locationdependent toxicant input rate H(x,t), the strengths of toxicant-taxis  $\chi_1$  and  $\chi_2$ , as well as the toxicant diffusivity d, on the competitive outcomes and spatial distributions of the two competing species. In addition to the case assumed in the hypothesis ( $\mathcal{H}_1$ ) (see the end of Section 2), we shall consider two additional cases for the function H(x,t):

 $(\mathcal{H}_2) \lim_{t \to \infty} H(x,t) = h(x)$ , where h(x) is a nonnegative function about *x*.

 $(\mathcal{H}_3)$  H(x,t) is a time-periodic function.

In the following, we shall numerically explore these three cases separately.

**Case 1:** H(x,t) **satisfies**  $(\mathcal{H}_1)$ . Recall that Theorem 3.2 establishes sufficient conditions under which the solutions of system (3.3) converge to the spatially homogeneous coexistence steady state  $E_5$  with the hypothesis  $(\mathcal{H}_1)$ . These conditions include the toxicant input function H(x,t) satisfying hypothesis  $(\mathcal{H}_1)$  and the toxicant–taxis coefficients  $\chi_1$  and  $\chi_2$  satisfying  $\frac{w_5}{4d}(\chi_1^2 u_5 + \chi_2^2 v_5) \leq 1$ . For simulations, we choose three different representations for the input rate function H(x,t) that satisfy hypothesis  $(\mathcal{H}_1)$ :

$$H_1(x,t) = h_0, \ H_2(x,t) = h_0 + \frac{1}{1+t}, \ H_3(x,t) = h_0 + \frac{\sin(\pi x)}{1+t}.$$
 (7.2)

We also choose the following model parameters:

 $b = 0.6, c = 0.4, d = 1, k = 1.3, h_0 = 0.5,$ 

such that system (3.4) (with constant toxicant input rate  $h_0$ ) has a unique coexistence steady state  $E_5 = (u_5, v_5, w_5) = (0.4835, 0.4515,$ 0.2584). According to Theorem 3.2, when  $\chi_1$  and  $\chi_2$  satisfy the condition  $\frac{w_5}{4d}(\chi_1^2 u_5 + \chi_2^2 v_5) \le 1$ , the solution of system (3.3) converges to  $E_5$ . Numerical results confirm this convergence for all toxicant input functions  $H_1, H_2, H_3$ . However, when we choose sufficiently large  $\chi_1$ and  $\chi_2$  such that  $\frac{w_5}{4d}(\chi_1^2 u_5 + \chi_2^2 v_5) > 1$ , we observe that the solution of system (3.3) no longer approaches the spatially homogeneous steady state  $E_5$ . Instead, it converges to a spatially inhomogeneous coexistence solution, as illustrated in Fig. 4. Moreover, the spatial distributions of the two species and the toxicant significantly depend on the specific form of the input function H(x, t).

**Case 2:** H(x,t) **satisfies**  $(\mathcal{H}_2)$ . Theoretically analyzing the asymptotic dynamics of system (3.3) with H(x,t) satisfying assumption  $(\mathcal{H}_2)$  is challenging. Therefore, we investigate the impact of toxicant–taxis strengths  $\chi_1$  and  $\chi_2$  on the competition outcomes and spatial distributions of the two species through numerical solutions of system (3.3). To this end, we let

$$H(x,t) = 1.4(1+e^{-t})(1+\cos(\pi x))$$

such that assumption ( $\mathcal{H}_2$ ) holds. We also choose the following parameter values: b = 0.1, c = 0.2, d = 0.01, k = 1. Fig. 5 illustrates the asymptotic behavior of system (3.3) for different toxicant–taxis coefficients. As shown in Fig. 5(a), both species go extinct in the absence of negative toxicant–taxis effects (i.e.,  $\chi_1 = \chi_2 = 0$ ). However, increasing the negative toxicant–taxis strength for one species may lead to the extinction of the other species (Figs. 5(b) and 5(c)). Furthermore, increasing both coefficients  $\chi_1$  and  $\chi_2$  results in the coexistence of the two species. These findings suggest that negative toxicant–taxis (i.e. avoidance of toxicants) is beneficial for population persistence and species diversity, as expected. This is because most individuals aggregate in regions with low toxicant concentrations due to the avoidance of toxicants, thereby improving their survival chances.

**Case 3:** H(x,t) **satisfies** ( $H_3$ ). We choose the following timeperiodic toxicant input function:



**Fig. 2.** Numerical simulations of large-time solution profiles of (3.4) in the interval  $\Omega = (0, 12\pi)$ , where  $b = 0.6, c = 0.4, d = 1, h_0 = 0.9, k = 1, \chi_1 = 10, \chi_2 = 11, u_5 = 0.3321, v_5 = 0.2214, w_5 = 0.5793$ . The initial data  $(u_0, v_0, w_0) = (u_5 + 0.01 \cdot \text{rand}, v_5 + 0.01 \cdot \text{rand}, w_5 + 0.01 \cdot \text{rand})$ .



**Fig. 3.** Numerical simulations of large-time solution profiles of (3.4) in the interval  $\Omega = (0, 12\pi)$ , where  $b = 0.6, c = 0.4, d = 1, k = 0.5, h_0 = 1, u_3 = 0.7071, w_3 = 0.5858$ . The initial data  $(u_0, v_0, w_0) = (u_3 + 0.01 \cdot \text{rand}, 0 + 0.01 \cdot \text{rand})$ .



Fig. 4. Numerical simulations of large-time solution profiles of (3.3) with (7.2), where  $b = 0.6, c = 0.4, d = 1, k = 1.3, h_0 = 0.5$  and  $\chi_1 = \chi_2 = 40$ . The initial value  $u_0 = u_5 + 0.01 \cos(\pi x), v_0 = v_5 + 0.01 \cos(\pi x), w_0 = w_5 + 0.01 \cos(\pi x)$ , where  $u_5 = 0.4835, v_5 = 0.4515, w_5 = 0.2584$ .



**Fig. 5.** A comparison of the effects of toxicant-taxis on the large-time solution profiles of (3.3) with the hypothesis ( $\mathcal{H}_2$ ), where b = 0.1, c = 0.2, d = 0.01, k = 1,  $H(x,t) = 1.4(1 + e^{-t})(1 + \cos(\pi x))$  and initial data ( $u_0, v_0, w_0$ ) = (1 + 0.01 · rand, 1 + 0.01 · rand).

#### $H(x,t) = 0.5 + 0.5\sin(t/6),$

and the same initial conditions and other model parameters as in Fig. 5. The numerical patterns of system (2.1) for different values of  $\chi_1$  and  $\chi_2$  are shown in Fig. 6. It is observed that the time-periodic toxicant input function H(x, t) leads to time-periodic solutions. When both toxicant–taxis coefficients  $\chi_1$  and  $\chi_2$  are small, the solution of system (3.3) exhibits spatial homogeneity (see Fig. 6(a)). Conversely, spatially inhomogeneous patterns emerge when  $\chi_1$  or  $\chi_2$  becomes large (see Fig. 6(b)–(d)).

Finally, we shall explore the impact of toxicant diffusivity on the population dynamics. Note that in Fig. 5, we have assumed  $(\mathcal{H}_2)$  and a very weak diffusivity of the toxicant (i.e., d = 0.01). Under these assumptions, we observed that the negative toxicant–taxis may promote population persistence and species diversity. A natural question arises: what will happen if the toxicant diffusion is not small? To explore this, we keep the same initial conditions and parameters as in Fig. 5, but vary the toxicant diffusion coefficient *d*. The long-time solution behaviors for different values of *d* are illustrated in Fig. 7 where we observe that increasing the toxicant diffusion rate *d* is detrimental to population persistence. This is because fast diffusion of the toxicant reduces the heterogeneity of its spatial distribution, thereby diminishing the effectiveness of the toxicant–taxis for species *u* and *v*. Similar behaviors are observed when the toxicant input function satisfies condition ( $\mathcal{H}_3$ ) (not shown here for brevity).

#### 8. Discussion

The pollution of aquatic environments is a major global concern, with toxic substances posing serious risks to the health of aquatic ecosystems. It has become increasingly vital to assess and predict the impacts of environmental toxicants, as well as to identify the factors that affect population persistence and species diversity. Mathematical models play a critical role in this process by translating the effects observed at the individual level into broader population-level outcomes. In this paper, we explore the effects of toxicants on the competitive interactions between two species in a polluted environment using a reaction-diffusion equation model that incorporates negative toxicanttaxis. This model extends existing population-toxicant ODE frameworks and toxicant-mediated single-species reaction-diffusion models. Our enhanced model facilitates the analysis of how different factors, such as toxicant input, taxis intensities, and toxicant dispersal, affect the competitive outcomes and spatial distributions of the two species.

We analyzed the existence and stability of the space-absent ODE system (3.6) under the assumption of weak interspecific competition. Under this assumption, the two species coexist in the absence of toxicants. However, with the introduction of toxicant, our findings indicate that while high toxicant levels lead to the extinction of both species, relatively low toxicant levels result in complex dynamics, including several types of bistability (see Table 1 and Fig. 1). In these bistable scenarios, the competition outcomes depend on the initial values. Note that the equilibria of the ODE system (3.6) correspond to the spatially homogeneous steady states of the PDE system (3.4). We established sufficient conditions under which the asymptotic behavior of the solution to system (3.4) (see Theorem 3.2).

We then investigated the possibility that the PDE system (3.4) admits spatially heterogeneous steady-state solutions. Specifically, we analyzed whether the stable equilibria of the ODE system (3.6) become unstable in the presence of spatial components (diffusion and/or toxicant–taxis) through linear stability analysis. Our findings reveal that random diffusion does not destabilize the steady states, but strong negative toxicant–taxis does (see Lemma 6.1). This assertion was verified through numerical simulation of system (3.4) under certain conditions (see Figs. 2 and 3).



**Fig. 6.** A comparison of the effects of toxicant-taxis on the large-time solutions of (3.3) when the hypothesis ( $\mathcal{H}_3$ ) holds, where b = 0.1, c = 0.2, d = 0.01, k = 1,  $H(x, t) = 0.5 + 0.5 \sin(t/6)$  and initial data ( $u_0, v_0, w_0$ ) = (1 + 0.01 · rand, 1 + 0.01 · rand).



Fig. 7. A comparison of the effects of toxicant diffusion on the large-time solutions of (3.3) with  $b = 0.1, c = 0.1, k = 1.1, \chi_1 = \chi_2 = 1$  and  $H(x,t) = 1.4(1 + e^{-t})(1 + \cos(\pi x))$ . The initial data  $(u_0, v_0, w_0) = (1 + 0.01 \cdot \operatorname{rand}, 1 + 0.01 \cdot \operatorname{rand}, 1 + 0.01 \cdot \operatorname{rand})$ .

We numerically analyzed the effects of the toxicant input rate H(x,t), toxicant-taxis coefficients  $\chi_i$  (i = 1, 2) and the toxicant diffusivity d on the long-term dynamics of (3.3). We found that, under spatiotemporally inhomogeneous toxicant inputs, toxicant-taxis promotes species diversity when the toxicant diffusion coefficient is small (see Figs. 5 and 6). This advantage stems from the fact that spatiotemporal toxicant inputs, coupled with low toxicant diffusion, create regions with relatively low toxicant levels. This motivates individuals to actively move towards these regions for survival. However, we observed that a strong diffusion capacity of the toxicant can diminish the impact of taxis, which is detrimental to population persistence and species diversity (see Fig. 7).

Several avenues for future research remain to be explored. Firstly, in Theorem 3.2, we only proved the large-time behavior of the solution to the reaction-diffusion system (3.3) for very high or very low toxicant input rates. Investigating the long-term dynamics of system (3.3) with intermediate toxicant input rates presents a significant challenge and is left for future work. Secondly, our numerical simulations in Figs. 2 and 3 suggest that spatially inhomogeneous stationary solutions with segregation structures exist for system (3.4). However, rigorously proving the existence of nonconstant stationary solutions with segregation structures for system (3.4) remains an interesting open problem. Thirdly, for analytical simplicity, the spatio-temporal model describing the impact of toxicants on the competition dynamics of two species, which involves three reaction-diffusion equations, assumes weak interspecific competition, equal random diffusion for both species, and equal decay and uptake rates ( $q = p_1 = p_2 = 1$ ). Relaxing these assumptions would lead to a more realistic but analytically more complicated model, which we leave for future investigation. Lastly, many species inhabit advective environments with unidirectional flow, such as rivers and streams. To investigate the impact of toxicants on the competitive dynamics between two species in such an environment, a reaction-diffusion-advective model can be derived by incorporating advective terms into model (3.3). This resulting reaction-diffusionadvective model also extends the spatiotemporal models studied in [36, 37], which focus on the effects of toxicants on a single species in contaminated rivers.

#### CRediT authorship contribution statement

Xiumei Deng: Writing – original draft, Validation, Methodology, Formal analysis, Conceptualization. Qihua Huang: Writing – review & editing, Supervision, Methodology, Conceptualization. Zhi-An Wang: Writing – review & editing, Supervision, Methodology, Formal analysis, Conceptualization.

# Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

#### Acknowledgements

The research of Q. Huang is supported by the National Natural Science Foundation of China (No.12271445). The research of Z.-A. Wang is partially supported by the NSFC/RGC Joint Research Scheme sponsored by the Research Grants Council of Hong Kong and the National Natural Science Foundation of China (Project No. N\_PolyU509/22) and Hong Kong RGC GRF grant No. PolyU 15305824.

#### Appendix. Local stability of equilibria of the ODE system (3.6)

We derive conditions for local stability of equilibria  $(u_s, v_s, w_s)$  by using the Jacobian matrix for (3.6), as long as real parts of eigenvalues

of the Jacobian, evaluated at the equilibria, are nonzero. The Jacobian matrix for (3.6) is

$$J(u_s, v_s, w_s) = \begin{bmatrix} 1 - 2u_s - cv_s - kw_s & -cu_s & -ku_s \\ -bv_s & 1 - bu_s - 2v_s - w_s & -v_s \\ -w_s & -w_s & -1 - u_s - v_s \end{bmatrix}$$

It follows from the expression of toxicant-only equilibrium  $E_1$  that the eigenvalues of the Jacobian matrix at  $E_1$  are

$$\lambda_{1,1} = 1 - kw_1, \ \lambda_{1,2} = 1 - w_1, \ \lambda_{1,3} = -1.$$

We find that  $\lambda_{1,1} < 0$  and  $\lambda_{1,2} < 0$  if and only if  $h_0 > \max\{1/k, 1\}$ , which implies that the equilibrium  $E_1$  is locally asymptotically stable. Moreover,  $E_1$  is unstable if  $h_0 < \max\{1/k, 1\}$ . For the semi-trivial equilibrium  $E_2$ , the eigenvalues of the Jacobian matrix are

$$\begin{split} \lambda_{2,1} &= 1 - cv_2 - kw_2, \ \lambda_{2,2} = \frac{-(1 + 2v_2) + \sqrt{(1 + 2v_2)^2 - 4v_2(1 + v_2 - w_2)}}{2} \\ \lambda_{2,3} &= \frac{-(1 + 2v_2) - \sqrt{(1 + 2v_2)^2 - 4v_2(1 + v_2 - w_2)}}{2}. \end{split}$$

Recalling the expression and the conditions of existence of  $E_2$  (see (3.7) and Table 1), we obtain that

$$1 + v_2 - w_2 = 2\sqrt{1 - h_0} > 0,$$

and

 $(1+2v_2)^2 - 4v_2(1+v_2-w_2) = 1 + 4v_2w_2 > 0,$ 

which imply  $\lambda_{2,2} < 0$  and  $\lambda_{2,3} < 0$ . Using the fact that  $1 - v_2 - w_2 = 0$ , we have

$$\lambda_{2,1} = 1 - cv_2 - kw_2 = 1 - c + (c - k)w_2 < 0$$

if k > 1 and  $h_2 < h_0 < 1$ . Hence,  $E_2$  is locally asymptotically stable if k > 1 and  $h_2 < h_0 < 1$ . Furthermore, we obtain that  $\lambda_{2,1} > 0$  if

$$\begin{cases} k \le 1, h_0 < 1 \text{ or} \\ k > 1, h_0 < h_2. \end{cases}$$
(A.1)

Then  $E_2$  is unstable if (A.1) holds.

Similarly, the eigenvalues of the Jacobian matrix of the semi-trivial equilibrium  $E_3$  are

$$\lambda_{3,1} = 1 - bu_3 - w_3, \ \lambda_{3,2} = \frac{-(1 + 2u_3) + \sqrt{(1 + 2u_3)^2 - 4u_3(1 + u_3 - kw_3)}}{2},$$
  
$$\lambda_{3,3} = \frac{-(1 + 2u_3) - \sqrt{(1 + 2u_3)^2 - 4u_3(1 + u_4 - kw_3)}}{2}.$$

From the expression and the existence conditions of  $E_3$ , we get

$$1 + u_3 - kw_3 = 2\sqrt{1 - kh_0} > 0$$

and

$$(1+2u_3)^2 - 4u_3(1+u_3-kw_3) = 1 + 4ku_3w_3 > 0,$$

which mean that  $\lambda_{3,2}$ ,  $\lambda_{3,3}$  are negative. Moreover, it follows from  $1 - u_3 - kw_3 = 0$  that

$$\lambda_{3,1} = 1 - bu_3 - w_3 = 1 - b + (bk - 1)w_3 < 0$$

if k < 1 and  $h_1 < h_0 < 1/k$ . Then the semi-trivial equilibrium  $E_3$  is locally asymptotically stable if k < 1 and  $h_1 < h_0 < 1/k$ . Moreover, we get that  $\lambda_{3,1} > 0$  if

$$\begin{cases} k \ge 1, \ h_0 < 1/k \text{ or} \\ k < 1, \ h_0 < h_1. \end{cases}$$
(A.2)

That is,  $E_2$  is unstable if (A.2) is satisfied.

Furthermore, the characteristic equation for the Jacobian evaluated at the coexistence equilibrium  $E_5$  is given by

$$|\lambda I - J(u_5, v_5, w_5)| = \lambda^3 + S_2 \lambda^2 + S_1 \lambda + S_0 = 0,$$

where

$$S_0 := u_5 v_5 \Big[ (1 - bc) \frac{h_0}{w_5} - \beta w_5 \Big],$$
  

$$S_1 := (1 - bc) u_5 v_5 + \frac{(u_5 + v_5)h_0}{w_5} - (ku_5 + v_5)w_5,$$
  

$$S_2 := u_5 + v_5 + \frac{h_0}{w_5}.$$

Straightforward calculations give us that  $S_0 > 0$  and  $S_2 > 0$ . Then, by the Routh–Hurwitz criterion [34, Appendix B], we know that  $E_5$ is locally asymptotically stable if  $S_1S_2 - S_0 > 0$ . It follows from the expression and the existence conditions of  $E_5$  that  $S_1S_2 - S_0 > 0$  if

$$k < 1, h_0 < h_1 \text{ or}$$
  

$$\kappa_1 < k < 1, h_1 \le h_0 < h_3 \text{ or}$$
  

$$k = 1, h_0 < h_3 \text{ or}$$
  

$$k > 1, h_0 < h_2 \text{ or}$$
  

$$1 < k < \kappa_2, h_2 \le h_0 < h_3.$$
  
(A.3)

Then the coexistence equilibrium  $E_5$  is locally asymptotically stable if (A.3) holds. In a similar way, we obtain that  $E_4$  is unstable.

#### Data availability

No data was used for the research described in the article.

#### References

- J.A. Camargo, Á. Alonso, Ecological and toxicological effects of inorganic nitrogen pollution in aquatic ecosystems: a global assessment, Environ. Int. 32 (2006) 831–849.
- [2] W.H. Clements, C. Kotalik, Effects of major ions on natural benthic communities: an experimental assessment of the US Environmental Protection Agency aquatic life benchmark for conductivity, Freshw. Sci. 35 (2016) 126–138.
- [3] J.W. Fleeger, K.R. Carman, R.M. Nisbet, Indirect effects of contaminants in aquatic ecosystems, Sci. Total Environ. 317 (2003) 207–233.
- [4] T. Hanazato, Pesticide effects on freshwater zooplankton: an ecological perspective, Environ. Pollut. 112 (2001) 1–10.
- [5] A.J. Smith, C.P. Tran, A weight-of-evidence approach to define nutrient criteria protective of aquatic life in large rivers, J. N. Am. Benthol. Soc. 29 (2010) 875–891.
- [6] S. Yang, F. Xu, F. Wu, S. Wang, B. Zheng, Development of PFOS and PFOA criteria for the protection of freshwater aquatic life in China, Sci. Total Environ. 470 (2014) 677–683.
- [7] T.F. Zabel, S. Cole, The derivation of environmental quality standards for the protection of aquatic life in the UK, Water Environ. J. 13 (1999) 436–440.
- [8] R.A. Erickson, S.B. Cox, J.L. Oates, T. Anderson, C.J. Salice, K.R. Long, A Daphnia population model that considers pesticide exposure and demographic stochasticity, Ecol. Model. 275 (2014) 37–47.
- [9] T.I. Hayashi, M. Kamo, Y. Tanaka, Population-level ecological effect assessment: estimating the effect of toxic chemicals on density-dependent populations, Ecol. Res. 24 (2009) 945–954.
- [10] J.A. Spromberg, W.J. Birge, Modeling the effects of chronic toxicity on fish populations: The influence of life-history strategies, Environ. Toxicol. Chem. 24 (2005) 1532–1540.
- [11] J.A. Spromberg, J.P. Meador, Relating chronic toxicity responses to populationlevel effects: A comparison of population-level parameters for three salmon species as a function of low-level toxicity, Ecol. Model. 199 (2006) 240–252.

- [12] T.G. Hallam, C.E. Clark, G.S. Jordan, Effects of toxicants on populations: a qualitative approach II. First order kinetics, J. Math. Biol. 18 (1983) 25–37.
- [13] T.G. Hallam, C.E. Clark, R.R. Lassiter, Effects of toxicants on populations: a qualitative approach I. Equilibrium environmental exposure, Ecol. Model. 18 (1983) 291–304.
- [14] T.G. Hallam, J.T. De Luna, Effects of toxicants on populations: a qualitative: approach III. Environmental and food chain pathways, J. Theoret. Biol. 109 (1984) 411–429.
- [15] Q. Huang, L. Parshotam, H. Wang, C. Bampfylde, M.A. Lewis, A model for the impact of contaminants on fish population dynamics, J. Theoret. Biol. 334 (2013) 71–79.
- [16] Q. Huang, G. Seo, C. Shan, Bifurcations and global dynamics in a toxin-dependent aquatic population model, Math. Biosci. 296 (2018) 26–35.
- [17] G. Lan, C. Wei, S. Zhang, Long time behaviors of single-species population models with psychological effect and impulsive toxicant in polluted environments, Phys. A 521 (2019) 828–842.
- [18] Z. Ma, G. Cui, W. Wang, Persistence and extinction of a population in a polluted environment, Math. Biosci. 101 (1990) 75–97.
- [19] D.M. Thomas, T.W. Snell, S.M. Jaffar, A control problem in a polluted environment, Math. Biosci. 133 (1996) 139–163.
- [20] X. Deng, Q. Huang, Z.-A. Wang, Global dynamics and pattern formation in a diffusive population-toxicant model with negative toxicant-taxis, SIAM J. Appl. Math. 83 (2023) 2212–2236.
- [21] C. Shan, Q. Huang, Direct and indirect effects of toxins on competition dynamics of species in an aquatic environment, J. Math. Biol. 78 (2019) 739–766.
- [22] X. Deng, Q. Huang, A toxicant-taxis reaction-diffusion model for the interaction between two species and a toxicant in an aquatic ecosystem, Chaos Solitons Fractals 183 (2024) 114930.
- [23] P. Brown, Decay to uniform states in ecological interactions, SIAM J. Appl. Math. 38 (1980) 22–37.
- [24] A. Jüngel, Diffusive and nondiffusive population models, in: Mathematical Modeling of Collective Behavior in Socio-Economic and Life Sciences, Springer, 2010, pp. 397–425.
- [25] Y. Lou, W.-M. Ni, Diffusion, self-diffusion and cross-diffusion, J. Differential Equations 131 (1996) 79–131.
- [26] D. Mottoni, Qualitative analysis for some quasilinear parabolic systems, Inst. Math. Polish Acad. Sci. Zam 190 (1979) 11–79.
- [27] H. Amann, Dynamic theory of quasilinear parabolic equations. II. Reactiondiffusion systems, Differential Integral Equations 3 (1990) 13–75.
- [28] H. Amann, Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems, in: Function Spaces, Differential Operators and Nonlinear Analysis, Springer, 1993, pp. 9–126.
- [29] H. Jin, Z.-A. Wang, Global stability of prey-taxis systems, J. Differential Equations 262 (2017) 1257–1290.
- [30] M.X. Wang, Note on the Lyapunov functional method, Appl. Math. Lett. 75 (2018) 102–107.
- [31] M.M. Porzio, V. Vespri, Hölder estimates for local solutions of some doubly nonlinear degenerate parabolic equations, J. Differential Equations 103 (1993) 146–178.
- [32] O.A. Ladyzhenskaia, V.A. Solonnikov, N.N. Ural'tseva, Linear and Quasi-linear Equations of Parabolic Type, vol. 23, American Mathematical Soc., 1968.
- [33] Z.-A. Wang, L. Wu, Lotka-Volterra diffusion-advection competition system with dynamical resources, Discrete Contin. Dyn. Syst. Ser. B 28 (2023) 3322–3348.
- [34] J.D. Murray, Mathematical Biology: I. An Introduction. Interdisciplinary Applied Mathematics, vol. 17, Springer, 2002.
- [35] P. Liu, J.P. Shi, Z.-A. Wang, Pattern formation of the attraction-repulsion Keller-Segel system, Discrete Contin. Dyn. Syst. Ser. B 18 (2013) 2597–2625.
- [36] P. Zhou, Q. Huang, A spatiotemporal model for the effects of toxicants on populations in a polluted river, SIAM J. Appl. Math. 82 (2022) 95–118.
- [37] J. Xing, Q. Huang, H. Nie, Dynamical analysis of a diffusive population-toxicant model with toxicant-taxis in polluted aquatic environments, Math. Biosci. 372 (2024) 109193.