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MATHEMATICS OF TRAVELING WAVES IN CHEMOTAXIS –REVIEW PAPER–

Zhi-An Wang

Department of Applied Mathematics
The Hong Kong Polytechnic University
Hung Hom, Hong Kong, China

Abstract. This article surveys the mathematical aspects of traveling waves of a class of chemotaxis models with logarithmic sensitivity, which describe a variety of biological or medical phenomena including bacterial chemotactic motion, initiation of angiogenesis and reinforced random walks. The survey is focused on the existence, wave speed, asymptotic decay rates, stability and chemical diffusion limits of traveling wave solutions. The main approaches are reviewed and related analytical results are given with sketchy proofs. We also develop some new results with detailed proofs to fill the gap existing in the literature. The numerical simulations of steadily propagating waves will be presented along the study. Open problems are proposed for interested readers to pursue.

Contents

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1. **Introduction.**

In the development of living system, there is continual interchanges or relaying of information between members of species at both the intercellular and intracellular levels. Such information is often conveyed by a coherent pattern or waveform that moves in space from a key element to a sequential development. Some practical examples include the calcium waves propagating on the surface of the egg of the fish *Medaka*, the propagation of an advantageous gene in a population, the multicellular propagating waves of myxobacteria in the early stages of fruiting body development, the depolarisation waves propagating along nerve axons, coherent swarms of motile micro-organisms advancing steadily through their environment toward a fresh supply of diffusing nutrient which they consume and seek chemotactically, and so on. These examples have been already transposed into theoretical mathematical models, cf. [58, Chapter 1]. The investigation of traveling waves always plays an crucial role in understanding the mechanisms behind various propagating wave patterns.

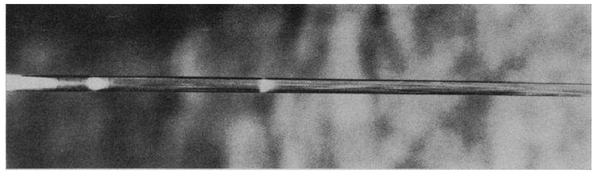
The prototypical reaction-diffusion model of admitting traveling wavefront solutions is the Fisher equation [20, 38], which reveals that the combination of reaction and diffusion greatly enhances the efficiency of information transferral via traveling waves of concentration changes in contrast to the diffusion alone. Although the Fisher equation was originally derived as a deterministic version of stochastic model for the propagation of a favored gene in a population, it has vigorous applications in physics and biology ranging from bacteria growth to combustion as well as animal dispersal. It is also a fundamental model from which numerous standard techniques were developed to analyze single species model with diffusive dispersal.

In spite of wide applicability of the Fisher equation, there are various wave propagating phenomena which can not be explained by simple reaction-diffusion mechanism. Instead they can be interpreted by chemotaxis, which, unlike diffusion, directs the motion of species up or down a chemical concentration gradient. There is a wealth of examples of traveling wave patterns driven by chemotaxis, such as the propagation of traveling band of bacterial toward the oxygen [1, 2], the outward propagation of concentric ring waves by *E. coli* [11, 12, 10], the spiral wave patterns during the aggregation of *Dictyostelium discoideum* [43, 23] and the migration of *Myxococcus xanthus* as traveling waves in the early stage of starvation-induced fruiting body development [85].

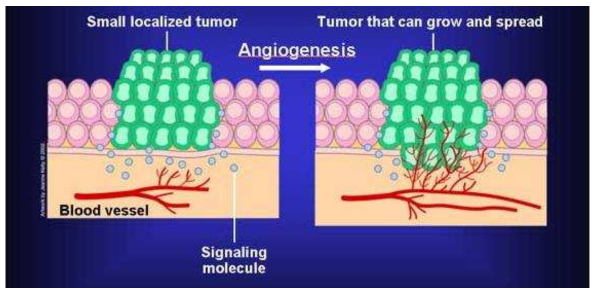
The prototypical chemotaxis model was proposed by Keller and Segel in the 1970s [33, 34] to describe the aggregation of cellular slime molds *Dictyostelium discoideum* in response to the chemical cyclic adenosine monophosphate (cAMP). In its general form, Keller-Segel model reads

$$\begin{cases} u_t = \nabla \cdot (d\nabla u - \chi u \nabla \phi(v)), \\ v_t = \varepsilon \Delta v + f(u,v) \end{cases} \tag{1.1}$$

where u and v denote the cell density and chemical concentration, respectively. $d > 0$ and $\varepsilon \geq 0$ are cell and chemical diffusion coefficients, respectively. $\chi > 0$ is called the chemotactic coefficient measuring the strength of the chemical signal.



(a)



(b)

FIGURE 1. (a) Photograph showing banded solitary waves of motile E. coli propagating in a capillary tube containing the oxygen and energy source. The figure is taken from article [1] for illustration. (b) An illustration of tumor angiogenesis adopted from the website of National Cancer Institute. The initiation of angiogenesis involves the secretion of the signalling molecule (i.e. endothelial angiogenesis growth factor) by tumor cells, which diffuse and induce neighboring endothelial cells of the blood vessels to migrate toward the tumor in order to build its own capillary network as shown in the picture.

Here $\phi(v)$ is referred to as the chemosensitivity function describing the signal detection mechanism and $f(u,v)$ is a function characterizing the chemical growth and degradation.

The typical examples of chemosensitivity function ϕ includes $\phi(v) = kv$ (linear law), $\phi(v) = k\log v$ (logarithmic law), or $\phi(v) = kv^m/(1+v^m)$ (receptor law), where $k > 0$ and $m \in \mathbb{N}$. The system with linear law $\phi(v) = kv$ and $f(u,v) = u - v$ was called the minimum chemotaxis model following the nomenclature of [13], see a review article [28] for the mathematical results of the minimum model. The logarithmic sensitivity $\phi(v) = k\log v$ follows from the Weber-Fechner law by which cells response to the chemical, and has prominent specific applications [35, 17, 6, 4]. The steady states for the logarithmic sensitivity with $f(u,v) = u - v$ was studied in [52, 61] and existence of global solutions was recently obtained in [86]. The receptor sensitivity law has been derived and applied in numerous models for chemotaxis, e.g. see [77, 78, 80] and references therein.

In this survey, we shall review the main mathematical approaches and results on the traveling wave solutions of chemotaxis model (1.1) with logarithmic chemosensitivity, and sketch necessary proofs. Some new results with proofs will be given in the paper to fill the gap remaining in the current literature. Solving the traveling wave problem of chemotaxis model (1.1) using the conventional approaches such as shooting method (i.e. phase plane analysis), confronts great challenges in general due to the high dimensionality of wave equations. Moreover the singularity caused by the logarithmic law makes the analysis even worse. Therefore some unconventional methods are demanded to overcome these difficulties. One of the major approaches in the literature was to employ various clever transformations to convert the traveling wave problem of chemotaxis models into equations/systems that can be solved, such as the Fisher equation, system of conservation laws. This article will review these transformation techniques and their consequences. Other methods will be only mentioned along the study without giving details due to the scope limit of present paper. We underline that there was another review paper [29] on the existence of traveling wave solutions of chemotaxis models. But we focus on different models and approaches. More importantly, we incorporate recent results and discuss many other aspects of traveling wave solutions beside the existence, such as wave speed, asymptotic decay rates, linear and nonlinear stability and chemical diffusion limits.

The chemotaxis model to be discussed in the paper is of the following form

$$\begin{cases} u_t = (du_x - \chi u \frac{v_x}{v})_x, \\ v_t = \varepsilon v_{xx} + \beta v - ug(v) \end{cases} \tag{1.2}$$

with $(x,t) \in \mathbb{R}\times[0,\infty)$, where $u(x,t)$ denotes the cell density and $v(x,t)$ the chemical (e.g. oxygen) concentration. The model (1.2) is a special case of model (1.1) with $\phi(v) = \log v, f(u,v) = \beta v - ug(v)$, which describes the chemotactic dynamics where cells move up the chemical concentration gradient and consume (or degrade) the chemical along the path. The function $g(v)$ is called the consumption rate function in the form

$$g(v) = v^m = \begin{cases} \text{constant rate}, & m = 0, \\ \text{sublinear rate}, & 0 < m < 1, \\ \text{linear rate}, & m = 1, \\ \text{superlinear rate}, & m > 1. \end{cases} \tag{1.3}$$

The constant β is to be determined to admit traveling wave solutions. A heuristic interpretation for β is the growth rate if $\beta > 0$ and degradation rate if $\beta < 0$. But we shall show that the traveling wave solutions do not exist for $\beta < 0$. The model (1.2)-(1.3) can describe generic biological processes depending upon the parameter values. When $0 \le m < 1$ and $\beta = 0$, (1.2) becomes the exact model proposed by Keller-Segel [35] to interpret the propagating traveling bands of bacterial chemotaxis experimentally observed in [1, 2], see Fig. 1 (a). When $m \ge 1$ and $\beta = 0$, the model (1.2) becomes a simplified version of models describing the initiation of angiogenesis, see Fig. 1 (b) for an illustration, whose mathematical analysis was intensively performed, see, for example, [21, 16, 15, 81]. Particularly when $m = 1$ and $\beta = 0$, (1.2) was also the exact system considered in [62, 72] to model the boundary motion of chemotactic bacteria. When $m = 1$ and $\beta \neq 0$, the model (1.2) was derived in [44, 65] as an example describing the reinforced random walks. Therefore the model (1.2)-(1.3) integrates numerous biological processes. The present article will be concentrated on the studies of traveling wave solutions. The research on other perspectives, such as the incorporation of cell kinetics [36, 42, 22, 8, 40, 66, 5, 59] and computations of traveling band formation [76, 41], will not be discussed though they are interesting as well. The alternative modeling approaches for traveling waves of chemotaxis, such as the kinetic descriptions in [74, 89], is also beyond the scope of our interest.

We complete this section by imposing the following initial conditions for the model (1.2)-(1.3)

$$(u,v)(x,0)=(u_0,v_0)(x)\to\begin{cases}(u_-,v_-) & \text{as } x\to-\infty,\\ (u_+,v_+) & \text{as } x\to+\infty.\end{cases} \tag{1.4}$$

Since u and v in (1.2) represent the biological particle densities, our attention will be restricted to the biologically relevant regime in which $u_\pm, v_\pm \geq 0$.

2. **Overview of the main approaches.**

2.1. **Preliminary.**

First of all, we note that the cell mass is conserved, which is obtained by integrating the first equation of (1.2)

$$\int_{\mathbb{R}} u(x,t)dx=\int_{\mathbb{R}} u_0(x)dx=:N$$

where N denote the cell mass which can be either a finite or infinity number. A traveling wave solution of (1.2) in $(x,t)\in\mathbb{R}\times[0,\infty)$ is a particular non-constant solution in the form

$$u(x,t)=U(z),\ v(x,t)=V(z),\ z=x-ct \tag{2.1}$$

with $U,V\in C^\infty(\mathbb{R})$ satisfying boundary conditions

$$U(\pm\infty)=u_\pm,\ V(\pm\infty)=v_\pm,\ U'(\pm\infty)=V'(\pm\infty)=0, \tag{2.2}$$

where c is the wave speed assumed to be non-negative ($c\geq 0$) without loss of generality, z is called the wave variable, the prime $'$ means the differentiation in z, u_-/u_+ and v_-/v_+ are called left/right end states of u and v, respectively, describing the asymptotic behavior of traveling wave solutions as $t\to+\infty/-\infty$.

To proceed, we give a definition below.

Definition 2.1. The traveling wave profile U is said to be a pulse if $u_-=u_+$ and a front if $u_-\neq u_+$.

Hereafter we shall adopt the following convention: we define the type of a traveling wave solution to (1.2) with a two word definition, e.g., a (pulse, front) solution means that U is a pulse and V is a front, and so on (the order of the terms is important and distinguished). In principle, there are four families of traveling waves solution as follows

$$(U,V)=\begin{cases}(pulse,\ pulse),\\ (pulse, front),\\ (front, front),\\ (front, pulse).\end{cases} \tag{2.3}$$

In the paper, we shall show that the system (1.2) may possess each of these four families of traveling wave solutions under suitable conditions on parameters.

Substituting the ansatz (2.1) into (1.2) and integrating the result yield

$$\begin{cases} dU'+cU-\chi V^{-1}V'U=C_0,\\ \varepsilon V''+cV'-UV^m+\beta V=0\end{cases} \tag{2.4}$$

where C_0 is an integration constant that relies on the boundary conditions of U and V such that

$$C_0 = u_\pm \Big[c - \chi \frac{V'(\pm\infty)}{V(\pm\infty)} \Big]. \tag{2.5}$$

Furthermore the evaluation of the second equation of (2.4) at $z = \pm\infty$ yields

$$u_\pm v_\pm^m = \beta v_\pm. \tag{2.6}$$

It is of importance to remark that the second term $\frac{V'(\pm\infty)}{V(\pm\infty)}$ in the square bracket of (2.5) is not necessarily zero since the traveling wave solution may exponentially decay to zero at infinity. If the cell mass $\int_{-\infty}^{\infty} U(z)dz (= N)$ is finite, then $C_0 = 0$. However conversely, $C_0 = 0$ does not exclude the possibility that cell mass is infinite if the traveling wave profile U is a front as shown later (see Remark 1).

The conventional method of solving (2.4) with (2.2) was to convert (2.4) into a system of first order ordinary differential equations (ODEs) and then perform the phase plane analysis. However the analysis of the corresponding first order ODE system of (2.4) faces two considerable challenges. The first one stems from $\varepsilon > 0$ since then one needs to set $W = V'$ and (2.4) becomes a three dimensional nonlinear ODE system (namely, there are three variable U, V and W in the system) for which there is no general mathematical theory available. The second challenge lies in the singularity of V^{-1} as $V \to 0$, which makes classical approaches not applicable, such as phase plane analysis. Hence the most fruitful approaches sought to over these challenges should involve deriving a rigorous reduction to a low-dimensional system and removing the singularity concurrently. The approaches discussed below are distinguished between $C_0 = 0$ and $C_0 \neq 0$.

2.2. **Case of zero integration constant ($C_0 = 0$).** If $C_0 = 0$, one can explicitly solve U in terms of V from the first equation of (2.4)

$$U(z) = Ce^{-\frac{c}{d}z} V^{\frac{\chi}{d}}(z), \tag{2.7}$$

where C is a translation constant of traveling wave solutions. Substituting (2.7) into the second equation of (2.4) leads to

$$\begin{cases} \varepsilon V'' + cV' - Ce^{-kz}V^r + \beta V = 0, \\ V(-\infty) = v_-, \ V(\infty) = v_+ \end{cases} \tag{2.8}$$

with

$$k = c/d, \ r = \chi/d + m. \tag{2.9}$$

Let $V(z)$ be a solution of (2.8) and τ satisfies $e^{\frac{c}{d}\tau} = C$. Then $V(z+\tau)$ is also a solution of (2.8) corresponding to $C = 1$. Without loss of generality, we let $C = 1$ in (2.8) hereafter and the corresponding traveling wave is called the normalized traveling wave. First we note that (2.8) does not have solutions for $c = 0$, namely, there is no standing wave.

Proposition 1. *If $c = 0$, then there is no traveling wave solution to (2.8) for any $v_\pm \geq 0$ and $\varepsilon \geq 0$.*

Proof. If $c = 0$, then $k = 0$ and the first equation of (2.8) is reduced to

$$\varepsilon V'' - V^r + \beta V = 0.$$

Clearly there is no nontrivial solution if $\varepsilon = 0$. Assuming $\varepsilon > 0$, then V is convex on $\mathbb{R}$ if $\beta \leq 0$ and can not satisfy the boundary condition in (2.8). Hence we consider $\beta > 0$. If $r = 1$, we can explicitly solve above equation and show that there is no nonnegative solution. If $r \neq 1$, we then write above equation as a system first-order of ordinary differential equations

$$\begin{cases} V' = W, \\ W' = \frac{1}{\varepsilon}(V^r - \beta V) \end{cases}$$

which has two equilibria $(0,0)$ and $(V_*,0)$ where $V_* = \beta^{1/(r-1)}$. If $r < 1$, the right hand side of above system is not differentiable at origin $(0,0)$ and no bounded solution exists. If $r > 1$, then the eigenvalue of the linearized matrices at $(0,0)$ and $(V_*,0)$ are $\pm\sqrt{-\beta/\varepsilon}$ and $\pm\sqrt{\beta(r-1)/\varepsilon}$, respectively. It can be readily found that $(0,0)$ is a center and $(V_*,0)$ is a saddle. Simple phase plane analysis will show there is no trajectory connecting $(V_*,0)$ to $(0,0)$ or itself that lies entirely in the region $V > 0$. The proof is then complete. □

To obtain the traveling wave solution, hereafter we assume that $c > 0$. Then evaluating the first equation of (2.8) at $z = \pm\infty$ yields

$$v_- = 0 \text{ for } \beta \in \mathbb{R} \text{ and } v_+ = 0 \text{ if } \beta \neq 0. \tag{2.10}$$

Then we immediately have the following result.

Theorem 2.1. *If $\beta < 0$, then (2.8) does not have a solution. In other words, there is no traveling wave solution to the system (1.2)-(1.3) if $\beta < 0$.*

Proof. Due to (2.10), $v_+ = 0$ if $\beta < 0$. Assume (2.8) has a solution V. Then it must be a pulse. But the equation (2.8) is not satisfied at the maximum point of $V(z)$, which is a contradiction. The proof is thus complete. □

Therefore we assume $\beta \geq 0$ throughout the paper in the case of $C_0 = 0$ in order to obtain a traveling wave solution.

2.2.1. *Zero chemical diffusion $\varepsilon = 0$.* Most of early studies, e.g. see [35, 62, 68, 32, 73, 70, 64, 41, 63, 25], considered only the case $\varepsilon = 0$ which reduced equation (2.8) to a first order ODE which can be explicitly solved. In such circumstance, the basic knowledge of ordinary differential equations is adequate. The results will be given in section 3.1.

2.2.2. *Nonzero chemical diffusion $\varepsilon > 0$.* The biological significance of $\varepsilon > 0$ was discussed in [27] and its analytical approximate solutions of the Cauchy problem was investigated in [69], and the steady-state problem was examined in [71], where $\beta = 0, m = 1$. When $\varepsilon > 0$, solving equation (2.8) is a little challenging due to the presence of variable coefficient e^{-kz} and analysis will be more involved than that for $\varepsilon = 0$. The first rigorous result on traveling wave solutions for $\varepsilon > 0$ was derived in [60] for $\beta = m = 0, r > 1$ by a change of dependent variable, and then in [18] for $\beta = m = 0, r = 1$ by a change of independent variable. The similar result was subsequently extended to $\beta = 0, 0 \leq m < 1, r > 1$ in [51] by the same method of [60]. The same idea was applied in [54] further to study the traveling wave solutions of a microscopic chemotactic random walk model. The paper [75] announced some existence results without proof for $m \geq 0, \beta \geq 0$ and $r \geq 1$. In this paper, we shall further develop the idea of [60, 18] to establish numerous results of traveling wave solutions for $\varepsilon > 0$, which not only supply the rigorous proofs to [75], but also derive many new results not obtained before, such as wave speed, asymptotic behavior and chemical diffusion limits. We shall outline the main ideas below and present the related results in section 3.

Change of dependent variable: With $r \neq 1$, we define

$$\mu = -\frac{k}{r-1} = -\frac{c}{\chi + d(m-1)} \tag{2.11}$$

and introduce a new variable $W(z)$ such that

$$V(z) = W(z)e^{-\mu z}. \tag{2.12}$$

Then substituting (2.12) into (2.8) and canceling $e^{-\mu z}$ yield that

$$\varepsilon W'' + sW' + f(W) = 0 \tag{2.13}$$

where

$$f(W) = \eta W - W^r \tag{2.14}$$

and

$$\begin{aligned} s &= c - 2\varepsilon\mu = c\Big[1 + \frac{2\varepsilon}{\chi + d(m-1)}\Big], \\ \eta &= \varepsilon\mu^2 - c\mu + \beta = \frac{c^2}{[\chi + d(m-1)]^2}\Big(\varepsilon + \chi + d(m-1)\Big) + \beta. \end{aligned} \tag{2.15}$$

It is easy to see that equation (2.13) resembles the traveling wave equation (9.3) of the Fisher equation. Hence the approaches for the Fisher wave equation can be applied to obtain W first, and then V and U by (2.12) and (2.7). The idea of introducing the change of dependent variable (2.12) was developed first in [60] for $\beta = 0, m = 0$ and then in [51] for $\beta = 0, 0 \leq m < 1$. Recently this approach was employed in [82] for $m = 1, \beta = 0$and in [54] for a microscopic chemotaxis model. In this paper, we shall employ this approach to explore the case $\beta \geq 0$. It turns out that the idea of transformation (2.12) produces very fruitful results which will be given in section 3.

Change of independent variable: If $\beta = 0$, we define a change of independent variable [18]

$$\tau = e^{-\frac{c}{\varepsilon}z}, \tag{2.16}$$

namely $\frac{d}{dz} = -\frac{c}{\varepsilon}e^{-\frac{c}{\varepsilon}z}\frac{d}{d\tau} = -\frac{c}{\varepsilon}\tau\frac{d}{d\tau}$, and $\tau \in [1, \infty)$ with

$$\tau = \begin{cases} 1, & \text{if } z = 0, \\ \infty, & \text{if } z = -\infty. \end{cases} \tag{2.17}$$

By defining $w(\tau) = V(-\varepsilon \ln \tau / c)$, we obtain from (2.8) that

$$w''(\tau) - \frac{\varepsilon}{c^2}e^{(\frac{2c}{\varepsilon} - k)z}w(\tau)^r = 0$$

which is

$$w''(\tau) = \alpha\tau^\theta w(\tau)^r \tag{2.18}$$

where

$$\alpha = \frac{\varepsilon}{c^2} > 0, \ \theta = \frac{k\varepsilon}{c} - 2 = \frac{\varepsilon}{d} - 2. \tag{2.19}$$

Now $\tau \in [0, \infty)$. Furthermore $z = -\infty$ corresponds to $\tau = \infty$ and $z = \infty$ corresponds to $\tau = 0$. Hence the boundary conditions for $w(\tau)$ is

$$w(0) = v_+, \ w(\infty) = 0. \tag{2.20}$$

Equation (2.18) is a type of linear Emden-Fowler equation [7]. The existence of solutions to equations (2.18)-(2.20) is guaranteed by [37]. The idea of using (2.16) to transform (2.8)-(2.10) into (2.18)-(2.20) was first given in [18] for the case $m = 0, \beta = 0$.

Now we explore the advantage and disadvantage of aforementioned two ideas. In relation to traveling wave solutions, the Fisher equation was studied more extensively than the Emden-Fowler equation. Hence the idea of making the change of dependent variable (2.12) may provide us more direct results such as the wave speed, stability and asymptotics. The defect of transformation (2.12) lies in the assumption $r \neq 1$, which is, however, remedied by making the change of independent variable (2.16). Unfortunately (2.16) only works for $\beta = 0$ that is not needed for (2.12). In this sense, the above two approaches are supplementary and one can not entirely cover the other. The scenario that both approaches can not solve is when $r = 1, \beta > 0$. The traveling wave problem for such case still remains open.

2.3. **Case of non-zero integration constant ($C_0 \neq 0$).** If $C_0 \neq 0$, U can be solved as an integral of V

$$U(z) = \left(C + \frac{C_0}{d}\int e^{\frac{c}{d}z}V^{-\frac{\chi}{d}}(z)dz\right)e^{-\frac{c}{d}z}V^{\frac{\chi}{d}}$$

which removes the singularity but introduces an integral into the problem. The resulting integral-differential equation is even more difficult to solve. The result for $C_0 \neq 0$ still largely remains open except for the case $m = 1$ which was solved in [83, 47, 48, 49] for $\varepsilon \geq 0$ and in [9] for $\varepsilon = 0$ by different methods, which will be sketched below.

2.3.1. *Change of dependent variables.* The idea of [83, 47, 48, 49] was a Hopf-Cole type transformation

$$h = -\frac{v_x}{v} = -\frac{\partial}{\partial x}\ln v \tag{2.21}$$

which upon the substitution into (1.2)-(1.4), after some algebra, yields a system of conservation laws

$$\begin{cases} u_t - \chi(uh)_x = du_{xx}, \\ h_t + (\varepsilon h^2 - u)_x = \varepsilon h_{xx} \end{cases} \tag{2.22}$$

for $x \in \mathbb{R}$ and $t \geq 0$ with the initial data

$$(u,h)(x,0) = (u_0,h_0)(x) \to \begin{cases} (u_-,h_-) & \text{as} \quad x \to -\infty, \\ (u_+,h_+) & \text{as} \quad x \to +\infty \end{cases} \tag{2.23}$$

where $h_\pm = -\frac{v_{0x}(\pm\infty)}{v_0(\pm\infty)}$. The existence and nonlinear stability of traveling wave solutions of the transformed system can be established by the theory of conservation laws and method of energy estimates. Then passing the results backward, we may obtain the existence and nonlinear stability of traveling wave solutions for the original system. This approach was very fruitful and particularly enabled us to obtain the nonlinear stability of traveling wave solutions. The detailed results will be presented in section 4.

2.3.2. *Change of independent variables.* When $m = 1$, we can write the chemical kinetics $g(u,v) = \beta v - uv = (\beta - u)v$ which is a separable function of u and v. Then the idea of [9] for separable models can be applied when $\varepsilon = 0$. Specifically, by a change of independent variable

$$dy = \frac{dz}{cV} \tag{2.24}$$

system (2.4) with $\varepsilon = 0$ becomes

$$\begin{cases} \frac{dU}{dy} = \frac{c}{d}V(C_0 - cU) + \frac{\chi}{d}(U-\beta)UV, \\ \frac{dV}{dy} = V^2(U-\beta) \end{cases} \tag{2.25}$$

which can be solved with classical method of phase plane analysis. The transformation (2.24) removes the singularity and obtain a solvable system simultaneously. The limitation is that it works only for $\varepsilon = 0$. In addition, the transformation (2.24) does not connect the solution of (2.25) to the original system (2.4) unless V can be explicitly found. When $\varepsilon > 0$, a system of three ODEs will result from (2.24) and the phase plane analysis is in general rather difficult. In contrast, the Hopf-Cole transformation (2.21) works for $\varepsilon \geq 0$, and more importantly it enables us to derive the nonlinear stability of traveling wave solutions besides the existence, see details in section 4. Hence for the particular case $m = 1$, the transformation (2.21) is more powerful and accessible than (2.24) and we shall not discuss the details of [9]. Of course, the transformation (2.24) can treat some other separable chemotaxis systems different from (1.2)-(1.3), see [9] for more examples.

We would like to conclude this section by mentioning a constructive approach introduced in [30] which investigated what kinds of chemotactic sensitivity, chemical production and decay functions may collectively produce traveling wave solutions for the chemotaxis models. Since the constructive approach in [30] did not address other issues of traveling wave solutions except existence, such as wave speed and stability, we shall not discuss this approach in detail in the paper.

3. Traveling wave solutions for $C_0 = 0$.

When $C_0 = 0$, the problem (2.4) subject to (2.2) becomes (2.8), see section 2.2. There are two separate cases to consider: $\varepsilon = 0$ and $\varepsilon > 0$.

3.1. Zero chemical diffusion $\varepsilon = 0$.

The existence of traveling wave solutions for $\varepsilon = 0$ and $\beta = 0$ has been studied very early in a series of papers [35, 62, 68, 32, 73, 70, 64, 41, 63, 25] where the traveling wave solutions were explicitly found for $0 \leq m < 1$. When $m = 1$, the existence and asymptotic behavior of traveling wave solutions were established in [83, 47, 82]. When $m > 1$, the non-existence of traveling wave solutions was shown in [75]. The case of $\beta > 0$ with $0 \leq m < 1$ will be investigated in the present paper. The theorem below will lump all these results with a complete proof.

Theorem 3.1. *Let $\varepsilon = 0$, $c > 0$ and $\beta \geq 0$ in (2.8). Then*

(i) if $\chi/d + m < 1$, there is no traveling wave solution.

(ii) if $\chi/d + m = 1$, traveling wave solutions exist, where the traveling wave profile (U, V) is a (pulse, pulse) if $\beta > 0$ and a (pulse, front) if $\beta = 0$.

(iii) if $\chi/d + m > 1$, traveling wave solutions exist if and only if $0 \leq m \leq 1$, and the traveling wave profile (U, V) is a

(1) (pulse, front) if $\beta = 0, 0 \leq m < 1$;

(2) (front, front) if $\beta = 0, m = 1$;

(3) (pulse, pulse) if $\beta > 0, 0 \leq m < 1$;

(4) (front, pulse) if $\beta > 0, m = 1$.

Proof. When $\varepsilon = 0$, the equation (2.8) with (2.10) reduces to

$$cV' - e^{-kz}V^r + \beta V = 0, \;\; V(-\infty) = 0, V(\infty) = v_+ \geq 0. \tag{3.1}$$

We have the following scenarios to consider.

(1) If $r = 1$, (3.1) is linear and can be explicitly solved to yield that

$$V(z) = C_* \exp\Big(-\frac{\beta}{c} z - \frac{1}{kc} e^{-kz}\Big)$$

and hence

$$U(z) = C_*^{\frac{\chi}{d}} \exp\bigg(-\Big(\frac{c}{d} + \frac{\chi\beta}{cd}\Big) z - \frac{\chi}{dkc} e^{-kz}\bigg)$$

where $C_* > 0$ is a constant. When $\beta \geq 0$, one can check that $U(\pm\infty) = 0, V(-\infty) = 0$. Furthermore $V(\infty) = 0$ if $\beta > 0$ and $V(\infty) = C_*$ if $\beta = 0$. This shows (ii).

(2) $r \neq 1$. Let $b(z) = \exp(-kz)/c$. Then equation (3.1) becomes a Bernoulli differential equation

$$V' + \frac{\beta}{c} V - b(z) V^r = 0. \tag{3.2}$$

With a change of variable, we can solve the above equation and obtain

$$V(z) = \Big[C_1 e^{-(1-r)\beta z/c} + C_2 e^{-kz} \Big]^{\frac{1}{1-r}} \tag{3.3}$$

where $C_1 > 0$ is a constant and

$$C_2 = \frac{r-1}{kc + \beta(r-1)}. \tag{3.4}$$

Then we have two cases to proceed.

Case 1: $r < 1$. Then $V(-\infty) \to \infty$ due to $\beta \geq 0$. Hence there is no traveling wave solution and the assertion (i) is proved.

Case 2: $r > 1$. Then $\frac{1}{1-r} < 0$. For $m \geq 0$, we have two cases to consider.

(a) If $\beta = 0$, then $C_2 = \frac{r-1}{kc}$. From (3.3), it has that

$$V(z) = \Big(C_1 + C_2 e^{-kz} \Big)^{\frac{1}{1-r}} \to \begin{cases} C_1^{\frac{1}{1-r}}, & \text{as } z \to \infty, \\ 0, & \text{as } z \to -\infty. \end{cases}$$

By the boundary condition (2.2), one can determine that $C_1 = v_+^{1-r} > 0$. Hence V is a traveling wavefront. By (2.7), one obtains

$$U(z) = \Big[v_+^{1-r} e^{(r-1)cz/\chi} + \frac{r-1}{kc} e^{-(k + (1-r)c/\chi)z} \Big]^{\frac{\chi}{d(1-r)}}. \tag{3.5}$$

Note that $k + (1-r)c/\chi > (<, =) 0$ is equivalent to $m < (>, =) 1$. It is easy to check that $U(\infty) = 0$ for all $m > 0$ and

$$U(-\infty) = \begin{cases} 0, & m < 1, \\ \frac{c^2}{\chi}, & m = 1, \\ \infty, & m > 1. \end{cases}$$

Therefore we conclude that if $\beta = 0$, traveling (pulse, front) solutions exist if and only if $0 \leq m < 1$ and traveling (front, front) solutions exist if and only if $m = 1$, and no bounded traveling solutions exist if $m > 1$.

(b) If $\beta > 0$, it follows that $-(1-r)\beta > 0$. Then from (3.3), one has $V(z) \to 0$ as $z \to \pm\infty$ since $C_2 > 0$. Furthermore, (2.7) yields that

$$U(z) = \Big[C_1 e^{(r-1)(c/\chi + \beta/c)z} + C_2 e^{-(k + (1-r)c/\chi)z} \Big]^{\frac{\chi}{d(1-r)}}. \tag{3.6}$$

Similar reasoning to case (a) asserts that if $\beta > 0$, traveling (pulse, pulse) solutions exist if and only if $0 \leq m < 1$ and traveling (front, pulse) solutions exist if and only if $m = 1$, bounded traveling solutions do not exist if $m > 1$.

Then the proof is complete. □

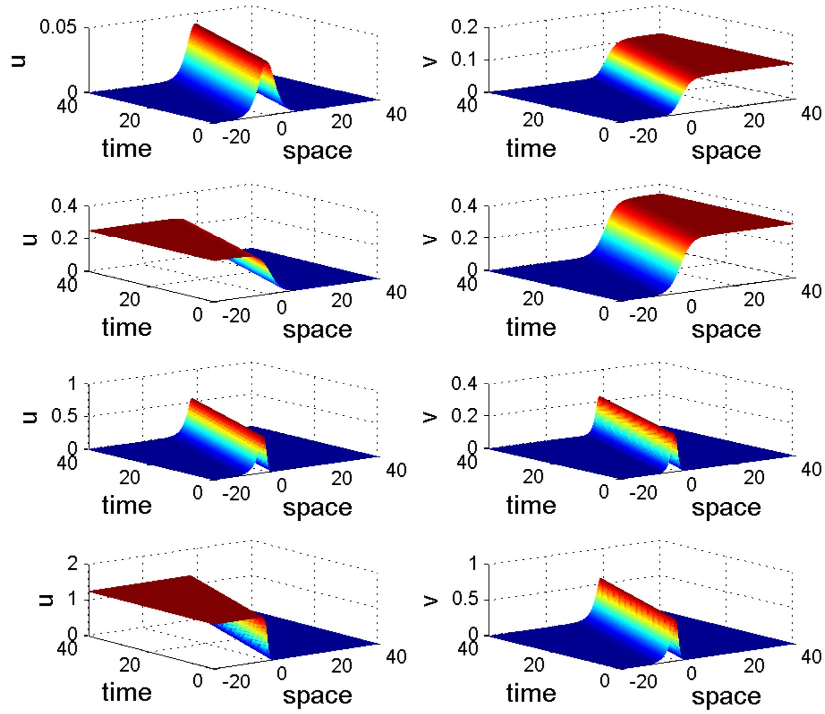


FIGURE 2. An illustration of four distinct families of traveling waves (U, V) shown in Theorem 3.1. First panel: $\beta = 0, m = 1/2$; second panel: $\beta = 0, m = 1$; third panel: $\beta = 1, m = 1/2$; fourth panel: $\beta = 1, m = 1$. Other parameter values are all the same: $C_1 = 3, = 1, d = 1, \chi = 1, c = 1$ where k and r are determined by $k = c/d, r = \chi/d + m$.

3.2. **Non-zero chemical diffusion** $\varepsilon > 0$**.** The main approaches $\varepsilon > 0$ have already been introduced in section 2.2.2. In this section, we shall present the relevant consequences of these approaches. We divide our analysis into cases $r \neq 1$ and $r = 1$.

3.2.1. *Case of* $r \neq 1$. For this case, the problem has been converted to a Fisher wave like equation (2.13) with (2.14) by the transformation (2.12), as shown in section 2.2.2. However to use the results for the Fisher equation that are cited in Theorem 9.1 (see Appendix in section 9), we need to perform some preliminary analysis to determine the parameter regime in which the conditions of Fisher problem are fulfilled. Here we present the results and proofs, which not only cover those of [60, 51, 54] for $\beta = 0$ but also give some new results such as minimum wave speed and asymptotic decay rates.

Lemma 3.1. *Let* $r \neq 1$ *and* $\beta \geq 0$. *Then*

(i) If $0 < r < 1$, *there is no traveling wave solution to (2.13).*

(ii) If $r > 1$, the nonnegative traveling wave solution $W(z)$ of (2.13) exists if and only if

$$s \geq s_* = 2\sqrt{\varepsilon\eta} \tag{3.7}$$

under which $W(z)$ is a wavefront with $W' < 0$ and satisfies the boundary conditions

$$W(-\infty) = \eta^{1/(r-1)},\ W(\infty) = 0,$$

as well as the following asymptotic behaviors:

(a) If $s > s_* = 2\sqrt{\varepsilon\eta}$, then

$$\begin{aligned} &W(z) \sim \eta^{1/(r-1)} - Ce^{\lambda_2^+ z},\ \text{as } z \to -\infty, \\ &W(z) \sim Ce^{\lambda_1^+ z},\ \text{as } z \to \infty \end{aligned} \tag{3.8}$$

where

$$\lambda_1^{\pm} = \frac{-s \pm \sqrt{s^2 - 4\varepsilon\eta}}{2\varepsilon} = \frac{(2\varepsilon\mu - c) \pm \sqrt{c^2 - 4\varepsilon\beta}}{2\varepsilon} \tag{3.9}$$

and

$$\lambda_2^{\pm} = \frac{-s \pm \sqrt{s^2 + 4\varepsilon\eta(r-1)}}{2\varepsilon} = \frac{(2\varepsilon\mu - c) \pm \sqrt{c^2 + 4\varepsilon(\eta r - \beta)}}{2\varepsilon}. \tag{3.10}$$

(b) If $s = s_* = 2\sqrt{\varepsilon\eta}$, then

$$\begin{aligned} &W(z) \sim \eta^{1/(r-1)} - Ce^{-\sigma z} + O(e^{-2\sigma z}),\ \text{as } z \to -\infty, \\ &W(z) \sim (A - Bz)e^{-\frac{s_*}{2\varepsilon}z} + O(z^2 e^{-\frac{s_*}{\varepsilon}z}),\ \text{as } z\ \to \infty \end{aligned} \tag{3.11}$$

where

$$\sigma = \frac{s_* - \sqrt{s_*^2 + 4\varepsilon\eta(r-1)}}{2\varepsilon} = (1 - \sqrt{r})\sqrt{\frac{\eta}{\varepsilon}}(< 0).$$

Proof. First note that (2.13) has two equilibria: $(W_1, \xi_1) = (0, 0)$ which always exists, and $(W_2, \xi_2) = (\eta^{1/(r-1)}, 0)$ which exists if $\eta > 0$. It can be shown that if (W_2, ξ_2) does not exist, then there does not exist a homoclinic trajectory connecting the unstable manifold to the stable manifold of the equilibrium (W_1, ξ_1). If (W_2, ξ_2) exists and $0 < r < 1$, then the kinetic function $f(W)$ defined in (2.14) is not differentiable at the origin, which becomes a singular Fisher equation that does not admit traveling wave solutions [56]. So we assume that $r > 1$ which implies that $\mu < 0, \eta > 0$ and hence $s > 0$ and (W_2, ξ_2) exists. For the sake of convenience, we rewrite (2.13) as a system of ordinary differential equations

$$\begin{cases} W' = \xi, \\ \xi' = -\frac{s}{\varepsilon}\xi - \frac{1}{\varepsilon}f(W). \end{cases} \tag{3.12}$$

Let J_1 and J_2 be the linearized matrices of system (3.12) at the equilibria (W_1, ξ_1) and (W_2, ξ_2), respectively. Then the eigenvalues of matrix J_1 and J_2 can be readily computed, as given by (3.9) and (3.10).

It is straightforward to check that $\lambda_1^- \leq \lambda_1^+ < 0$ if $s^2 \geq 4\varepsilon\eta$. On the other hand, if $r > 1, \beta \geq 0$, the equation (2.13) is a Fisher wave equation (9.3) satisfying condition (9.2) with $\varrho = \eta^{1/(r-1)}$ and $\lambda_2^- < 0 < \lambda_2^+$. Then the results of Lemma 3.1 are direct consequences of Theorem 9.1 and Theorem 9.2, see section 9. The proof is completed. □

Using (2.7) and (2.12), we can derive the non-existence and existence results for the traveling wave solutions of the original chemotaxis model (1.2) based on the results of $W(z)$. First noticing that $r = \frac{\chi}{d} + m$, the following nonexistence of traveling wave solutions for $\frac{\chi}{d} + m < 1$ follows from Lemma 3.1(i) directly.

Theorem 3.2. *Let $\beta \geq 0$. If $\frac{\chi}{d} + m < 1$, there is no traveling wave solution to (1.2)-(1.4).*

If $r = \frac{\chi}{d} + m > 1$, there are two cases ($\beta > 0$ and $\beta = 0$) to consider the existence of traveling wave solutions. Employing Lemma 3.1 together with (2.7), (2.11) and (2.12), we first obtain the following existence results for $\beta > 0$.

Theorem 3.3. *Let $\beta > 0$ and $\frac{\chi}{d} + m > 1$ with $0 \leq m \leq 1$. Then the traveling wave solution (U, V) of (1.2)-(1.4) exists if and only if*

$$c \geq c_* = 2\sqrt{\varepsilon\beta}. \tag{3.13}$$

Moreover the solution (U, V) has the following asymptotic behavior as $|z| \to \infty$:

(a) If $c > c_* = 2\sqrt{\varepsilon\beta}$, then

$$U(z) \sim \begin{cases} \left(\eta^{1/(r-1)} + Ce^{\lambda_2^+ z}\right)^{\frac{\chi}{d}} e^{\delta_1 z}, \text{ as } z \to -\infty, \\ Ce^{\delta_2 z}, \text{ as } z \to \infty \end{cases} \tag{3.14}$$

and

$$V(z) \sim \begin{cases} \left(\eta^{1/(r-1)} + Ce^{\lambda_2^+ z}\right) e^{-\mu z}, \text{ as } z \to -\infty, \\ Ce^{(\lambda_1^+ - \mu) z}, \text{ as } z \to \infty \end{cases} \tag{3.15}$$

where

$$\delta_1 = -\frac{\chi\mu + c}{d} = \frac{c(1-m)}{\chi + d(m-1)}, \quad \delta_2 = \frac{\chi(\lambda_1^+ - \mu) - c}{d} < 0. \tag{3.16}$$

(b) If $c = c_* = 2\sqrt{\varepsilon\beta}$, it has that

$$U(z) \sim \begin{cases} \left(\eta^{1/(r-1)} - Ce^{-\sigma_* z} + O(e^{-2\sigma_* z})\right)^{\frac{\chi}{d}} e^{\delta_1^* z}, \text{ as } z \to -\infty \\ \left((A - Bz)e^{-\tau_1 z} + O(z^2 e^{-\tau_2 z})\right)^{\frac{\chi}{d}} e^{-\frac{c_*}{d} z}, \text{ as } z \to \infty \end{cases} \tag{3.17}$$

and

$$V(z) \sim \begin{cases} \left(\eta^{1/(r-1)} - Ce^{-\sigma_* z} + O(e^{-2\sigma_* z})\right) e^{-\mu_* z}, \text{ as } z \to -\infty \\ (A - Bz)e^{-\tau_1 z} + O(z^2 e^{-\tau_2 z}), \text{ as } z \to \infty \end{cases} \tag{3.18}$$

where

$$\begin{aligned} &\sigma_* = (1 - \sqrt{r})\sqrt{\frac{\eta_*}{\varepsilon}}, \ \eta_* = \frac{4\varepsilon\beta}{[\chi + d(m-1)]^2}\Big(\varepsilon + \chi + d(m-1)\Big) + \beta, \\ &\mu_* = -\frac{c_*}{\chi + d(1-m)}, \ \delta_1^* = -\frac{\chi\mu_* + c_*}{d} = \frac{2\sqrt{\varepsilon\beta}(1-m)}{\chi + d(m-1)}, \\ &\tau_1 = \frac{s_*}{2\varepsilon} + \mu_* = \frac{c_*}{2\varepsilon} = \sqrt{\frac{\beta}{\varepsilon}} \ (> 0), \\ &\tau_2 = \frac{s_*}{\varepsilon} + \mu_* = \frac{c_*}{\varepsilon} - \mu_* = 2\sqrt{\frac{\beta}{\varepsilon}}\left(1 + \frac{\varepsilon}{\chi + d(m-1)}\right) \ (> 0). \end{aligned} \tag{3.19}$$

Proof. By Theorem 9.1, the wave speed s for (2.13) is $s \geq s_* = 2\sqrt{\varepsilon f'(0)}$ which is equivalent to $c \geq c_* = 2\sqrt{\varepsilon\beta}$ after simple calculation by using (2.15). Using (2.12), we can translate the decay rates of $W(z)$ obtained in Theorem 3.1 to $V(z)$ directly with trivial calculations, and hence derive the asymptotic decay rates of V as announced in the Theorem. Furthermore combining (2.7) and (2.12), one has

$$U(z) = W^{\frac{\chi}{d}} e^{-\frac{\chi\mu + c}{d} z}.$$

Then applying (3.8) and (3.11) into above equation and (2.12), we can readily derive the asymptotic decay rates for U as given in the Theorem. Now we proceed to determine under what condition for m, the limits of (U, V) as $z \to \pm\infty$ exist. First from (3.16), we have

$$\begin{cases} \delta_1 > 0, & 1 - \frac{\chi}{d} \leq m < 1, \\ \delta_1 = 0, & m = 1, \\ \delta_1 < 0, & m > 1. \end{cases}$$

Hence as $z \to -\infty$, it has that

$$U(z) \to \begin{cases} 0, & 1 - \frac{\chi}{d} \leq m < 1, \\ \eta, & m = 1, \\ \infty, & m > 1 \end{cases} \tag{3.20}$$

which immediately shows that there does not exist a bounded traveling wave solution if $m > 1$. Hence it is required that $0 \leq m \leq 1$ to guarantee the existence of U. The proof is complete. □

We should point out that the asymptotics (3.14)-(3.18) are also true for $\beta = 0$. But when $\beta = 0$, we have $c_* = 0$. If $c = c_* = 0$ which implies $\sigma_* = \mu_* = \delta_1^* = \tau_1 = \tau_2 = 0$. Then it is straightforward to check from (3.17)-(3.18) that $(U(z), V(z)) \to (\infty, \infty)$ as $z \to \infty$. So the traveling wave solution does not exist when $c = c_* = 0$ and we have the following results for $\beta = 0$ by using Lemma 3.1.

Theorem 3.4. *Let $\beta = 0$ and $\frac{\chi}{d} + m > 1$ with $0 \leq m \leq 1$. Then the traveling wave solution (U, V) of (1.2)-(1.4) exists if and only if*

$$c > c_* = 0$$

and the solution (U, V) has the following asymptotic behavior as $|z| \to \infty$:

$$U(z) \sim \begin{cases} \left(\eta^{1/(r-1)} + Ce^{\lambda_2^+ z}\right)^{\frac{\chi}{d}} e^{\delta_1 z}, \text{ as } z \to -\infty, \\ Ce^{-\frac{c}{d} z}, \text{ as } z \to \infty \end{cases} \tag{3.21}$$

and

$$V(z) \sim \begin{cases} \left(\eta^{1/(r-1)} + Ce^{\lambda_2^+ z}\right) e^{-\mu z}, \text{ as } z \to -\infty, \\ v_+ - Ce^{-\frac{c}{\varepsilon} z}, \text{ as } z \to \infty. \end{cases} \tag{3.22}$$

Proof. Note that when $\beta = 0$, $\lambda_1^+ - \mu = 0$ and hence (3.21) is a direct consequence of (3.21). In addition, the asymptotics of V as $z \to \infty$ given in (3.22) can be obtained from the first equation of (2.8) directly, see also [82]. □

The following Theorem addresses the type of traveling wave profiles and monotonicity of traveling wave solutions.

Theorem 3.5. *Let $\beta \geq 0$ and $\frac{\chi}{d} + m > 1$ with $0 \leq m \leq 1$. Let (3.13) hold. Then*

(i) If $0 \leq m < 1$, the traveling wave solution profile (U, V) is a (pulse, pulse) if $\beta > 0$ and a (pulse, front) with $V' > 0$ if $\beta = 0$.

(ii) If $m = 1$, the traveling wave solution profile (U, V) is a (front, pulse) with $U' < 0$ if $\beta > 0$ and a (front, front) with $U' < 0, V' > 0$ if $\beta = 0$.

Proof. We first derive the types of traveling wave profiles for the case $c > c_* = 2\sqrt{\varepsilon\beta}, \beta \geq 0$. Using the definition of λ_1^+ in (3.9), we have

$$\lambda_1^+ - \mu = \frac{-c + \sqrt{c^2 - 4\varepsilon\beta}}{2\varepsilon} \begin{cases} < 0, & \beta > 0 \\ = 0, & \beta = 0 \end{cases} \tag{3.23}$$

and hence

$$\delta_2 = \frac{\chi}{d}(\lambda_1^+ - \mu) - \frac{c}{d} < 0.$$

Therefore for $1 - \frac{\chi}{d} \leq m \leq 1$, it follows from (3.17) and (3.21) that

$$U(z) \to 0 \text{ as } z \to \infty \text{ for all } \beta \geq 0$$

which, along with (3.20), shows that $U(\infty)$ is bounded and $U(z)$ is a pulse for $0 \leq m < 1$ and a front for $m = 1$.

Noticing that $\mu < 0$, we have from (3.15) that

$$V(z) \to 0 \text{ as } z \to -\infty$$

and

$$V(z) \to \begin{cases} 0, & \beta > 0, \\ C, & \beta = 0 \end{cases} \quad \text{as z} \to \infty. \tag{3.24}$$

So $V(z)$ is a pulse for $\beta > 0$ and front for $\beta = 0$. We are left to derive the types of traveling wave profiles for the case $c = c_* = 2\sqrt{\varepsilon\beta}, \beta > 0$. In this case, we have $\mu_* < 0, \tau_2 > 0$ when $1 - \frac{\chi}{d} < m \leq 1$, as well as $\delta_1^* > 0$ if $1 - \frac{\chi}{d} < m < 1$ and $\delta_1^* = 0$ if $m = 1$. Hence $V(z) \to 0$ as $z \to \pm\infty$, $U(z) \to 0$ as $z \to +\infty$, $U(z) \to 0$ as $z \to -\infty$ if $1 - \frac{\chi}{d} < m < 1$, and $U(z) \to \eta^{1/(r-1)} + \text{constant}$ as $z \to -\infty$ if $m = 1$.

Next we show the wave monotonicity. We first consider the case $\beta = 0$ where V is a wavefront for both $0 \leq m < 1$ and $m = 1$. Indeed by the second equation of (2.4) with $\beta = 0$ and (2.2), we can easily derive that

$$V' = \frac{1}{\varepsilon} e^{-\frac{c}{\varepsilon} z} \int_{-\infty}^{z} e^{\frac{c}{\varepsilon}\xi} U V^m d\xi.$$

It immediately follows that $V' > 0$ for any $m \geq 0$ due to $U, V > 0$ for all $z \in (-\infty, \infty)$.

Next we proceed to consider the case $\beta > 0, m = 1$ where U is a front. Since $W' < 0$, we have from (2.12) that $W' = e^{\mu z}(V' + \mu V) < 0$. Hence $V' + \mu V < 0$. Note that when $m = 1$, $\mu = -\frac{k}{\chi/d} = -\frac{c}{\chi}$. Therefore $V' - \frac{c}{\chi}V < 0$, namely $c - \chi V' V^{-1} > 0$. Then by the first equation of (2.4), one has

$$U' = \frac{U}{d}(c - \chi V' V^{-1}) > 0.$$

The proof is finished. □

Remark 1. From Theorem 3.5 we see that the wave profile U is a front when $m = 1$, which corresponds to infinite cell mass. This shows the fact that cell mass may still be infinite even if $C_0 = 0$, as mentioned in section 2.1.

TABLE 1. A summary for the non-existence and existence of traveling wave solutions of model (1.2) in various parameter regimes, and the wave profile, wave speed and wave monotonicity if traveling wave solutions exist.

Parameter Regimes	Wave Profile (U, V)	Wave Speed	Wave Monotonicity
$\frac{\chi}{d}+m<1$ or $m>1$ or $\beta<0$	None	None	None
$\frac{\chi}{d}+m>1$, $0\le m<1$ $\beta>0$	(Pulse, Pulse)	$c\ge c_*=2\sqrt{\varepsilon\beta}$	None
$\frac{\chi}{d}+m\ge 1$, $0\le m<1$ $\beta=0$	(Pulse, Front)	$c>c_*=0$	$V'>0$
$\frac{\chi}{d}+m>1$, $m=1$ $\beta>0$	(Front, Pulse)	$c=\chi\left(\frac{u_--\beta}{\varepsilon+\chi}\right)^{1/2}$	$U'<0$
$\frac{\chi}{d}+m>1$, $m=1$ $\beta=0$	(Front, Front)	$c=\chi\left(\frac{u_-}{\varepsilon+\chi}\right)^{1/2}$	$U'<0, V'>0$

For convenience, we summarize the wave profiles, wave speed and the monotonicity of traveling wave solutions in Table 1.

3.2.2. *Case of* $r=1$. With $r=\chi/d+m=1$, the problem (2.8) becomes

$$\begin{cases} \varepsilon V''(z)+cV'(z)-e^{-kz}V(z)+\beta V(z)=0, \\ V(-\infty)=0,\ V(\infty)=v_+\ge 0. \end{cases} \tag{3.25}$$

Since the transformation (2.12) fails when $r=1$, we ought to employ idea of making the change of independent variable as shown in section 2.2.2. This method only works for $\beta=0$. Hence we assume $\beta=0$ and consider the following problem

$$\begin{cases} w''(\tau)=\alpha\tau^\theta w(\tau),\ 0<\tau<\infty \\ w(0)=v_+,\ w(\infty)=0 \end{cases} \tag{3.26}$$

where α,θ are given in (2.19).

The the existence of unique solution to (3.26) with $w'(\tau)<0$ was guaranteed by [37], see also [18], since $\theta>-2$. Then using the results of [14, Chapter IV], we can readily derive the asymptotic behavior of the solution to (3.25), as follows.

Theorem 3.6. *Let $\beta=0$. Then for every $c>0$ there is a unique monotone solution (up to a translation) to problem (3.25) with $v_+>0$ and $V'>0$. Furthermore the solution has the following asymptotic behavior as $z\to\pm\infty$:*

$$V(z)\sim e^{\frac{c}{4\varepsilon}\left(\frac{\varepsilon}{d}-2\right)z-\frac{2d}{c\sqrt{\varepsilon}}e^{-\frac{c}{2d}z}},\ \text{as } z\to-\infty,$$
$$V(z)-v_+\sim e^{-\frac{c}{\varepsilon}z},\ \text{as } z\to\infty.$$

Proof. Since $V(z) = w(\tau)$, the existence of V follows from the existence of $w(\tau)$ which solves (3.26). Due to $V'(z) = -\frac{c}{\varepsilon}e^{-\frac{c}{\varepsilon}z}w'(\tau)$ and $w'(\tau) < 0$, it follows that $V' > 0$ for all $z \in \mathbb{R}$. Now we assume that $V(0) = \varrho > 0$ and study the asymptotics of V as $z \to \infty$ by considering the following problem

$$\begin{cases} \varepsilon V'' + cV' - e^{-kz}V = 0, z \in (0, \infty) \\ V(0) = \varrho,\ V(\infty) = v_+. \end{cases} \tag{3.27}$$

Let $V' = \rho$ and $X = \begin{bmatrix} V \\ \rho \end{bmatrix}$. Then we can write the equation (3.27) as

$$X' = (A + B(z))X \tag{3.28}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{c}{\varepsilon} \end{bmatrix},\ B(z) = \begin{bmatrix} 0 & 0 \\ \frac{e^{-kz}}{\varepsilon} & 0 \end{bmatrix}.$$

It is straightforward to obtain the eigenvalues of A as

$$\lambda_1 = -\frac{c}{\varepsilon},\ \lambda_2 = 0 \tag{3.29}$$

with corresponding eigenvectors

$$\mathbf{x}_i = \begin{bmatrix} 1 \\ \lambda_i \end{bmatrix}, i = 1, 2.$$

Considering the fact that

$$\int_0^\infty |B(z)|dz = \frac{1}{\varepsilon}\int_0^\infty e^{-kz}dz = \frac{1}{\varepsilon k} < \infty,$$

then by [14, Chapter IV (Theorem 2)], the solution of system (3.28) satisfies

$$X \sim c_1 e^{\lambda_1 z}\mathbf{x}_1 + c_2 e^{\lambda_2 z}\mathbf{x}_2,\ \text{as } z \to \infty$$

and hence $V(z) = c_1 e^{-\frac{c}{\varepsilon}z} + c_2$. Applying the boundary condition yields that $c_1 = \varrho - v_+$ and $c_2 = v_+$. Therefore it follows that

$$V - v_+ \sim e^{-\frac{c}{\varepsilon}z},\ \text{as } z \to \infty.$$

Next we proceed to consider the asymptotics as $z \to -\infty$ and consider the problem

$$\begin{cases} \varepsilon V'' + cV' - e^{-kz}V = 0, z \in (-\infty, 0) \\ V(-\infty) = 0,\ V(0) = \varrho. \end{cases} \tag{3.30}$$

Note that the coefficient e^{-kz} is singular at $z = -\infty$, it is very challenging to solve the problem (3.30) directly. Instead we shall transform the problem to a non-singular problem via the transformation (2.16). Hence we first consider the following transformed problem

$$\begin{cases} w''(\tau) = \frac{\varepsilon}{c^2}\tau^\theta w(\tau),\ \tau \in [1, \infty) \\ w(1) = \varrho,\ w(\infty) = 0 \end{cases} \tag{3.31}$$

where θ is define in (2.19). For convenience, we let $f(\tau) = \frac{\varepsilon}{c^2}\tau^\theta$ and apply the results in [14] to derive the asymptotical behavior of solutions to (3.31). To this end, we need verify $\int_1^\infty |f^{-\frac{3}{2}}f''|d\tau < \infty$. Indeed noticing that $f''(\tau) = \frac{\varepsilon}{c^2}\theta(\theta-1)\tau^{\theta-2}$ and $-\frac{\theta}{2} - 2 = -\frac{\varepsilon k}{2c} - 1 < -1$, we find

$$\begin{aligned}\int_1^\infty |f^{-\frac{3}{2}}f''|d\tau &= \frac{\varepsilon}{c^2}|\theta(\theta-1)|\int_1^\infty |f^{-\frac{3}{2}}\tau^{\theta-2}|d\tau \\ &\leq \frac{c}{\sqrt{\varepsilon}}|\theta(\theta-1)|\int_1^\infty \tau^{-\frac{\theta}{2}-2}d\tau < \infty.\end{aligned}$$

Then the result of [14, Chapter IV (Theorem 14)] asserts that the system (3.31) has a solution $w(\tau)$ for $\tau > 1$ with the following asymptotic behavior as $\tau \to \infty$:

$$\begin{aligned}w(\tau) \sim [f(\tau)]^{-\frac{1}{4}}e^{-\int_1^\tau [f(\xi)]^{\frac{1}{2}}d\xi} &= \left(\frac{c^2}{\varepsilon}\right)^{\frac{1}{4}}\tau^{-\frac{\theta}{4}}e^{-\frac{\sqrt{\varepsilon}}{c}\int_1^\tau \xi^{\frac{\theta}{2}}d\xi} \\ &= \left(\frac{c^2}{\varepsilon}\right)^{\frac{1}{4}}e^{\frac{2\sqrt{\varepsilon}}{c(\theta+2)}}\tau^{-\frac{\theta}{4}}e^{-\frac{2\sqrt{\varepsilon}}{c(\theta+2)}\tau^{\frac{\theta}{2}+1}}\end{aligned}$$

which along with (2.19) yields

$$w(\tau) \sim \tau^{-\frac{\varepsilon k-2c}{4c}}e^{-\frac{2}{k\sqrt{\varepsilon}}\tau^{\frac{\varepsilon k}{2c}}}.$$

Thanks to (2.16) and (2.17), we have

$$V(z) = w(e^{-\frac{c}{\varepsilon}z}) \sim e^{(\frac{k}{4}-\frac{c}{2\varepsilon})z-\frac{2}{k\sqrt{\varepsilon}}e^{-\frac{k}{2}z}}, \text{ as } z \to -\infty.$$

Hence we obtain the asymptotic behavior of solutions as $z \to \pm\infty$ by noticing that $k = c/d$. Next we show the monotonicity of V. To this end, we integrate the first equation of (3.30) over $(-\infty, z)$ and obtain

$$V' = e^{-\frac{c}{\varepsilon}z}\int_{-\infty}^z e^{(\frac{c}{\varepsilon}-k)\xi}V(\xi)d\xi > 0$$

for all $z \in (-\infty, \infty)$ since $V(\xi) \geq 0$ for all $\xi \in \mathbb{R}$. □

Remark 2. The existence of traveling wave solutions discussed above require $r \geq 1$. Indeed for $r < 1$, it was shown in [64, 18] that there was so-called singular traveling wave solution (U, V) to the system (1.2) with $m = 0$ and $\beta = 0$, namely $U(z) = V(z) = 0$ if $z \leq z_0$ and $U(z) > 0, V(z) > 0$ if $z > z_0$ for some z_0, where the case $\varepsilon = 0$ and $\varepsilon > 0$ are treated in [64] and [18], respectively.

4. **Traveling wave solutions for $C_0 \neq 0$.** When $C_0 \neq 0$, U can be solved as an integral of V which removes the singularity but concurrently introduces an integral and hence does not virtually reduce the difficulty. New ideas should be sought to attack such a problem. As far as we know, the only available result was developed in [83, 47, 48, 49] for the case $m = 1$, where a Hopf-Cole transformation was essentially employed to transform the original chemotaxis system (1.2) into a system of conservation laws, see section 2.2. In this section, we will present more details of this approach and corresponding results in [83, 47, 48, 49]. We first examine the transformed problem (2.22)-(2.23) and then pass the results back to v by (2.21). Here we only discuss the case $\varepsilon > 0$ and refer the readers to [47] for the case $\varepsilon = 0$.

Because the chemotaxis model (1.2) with $m = 1$ describes the directed movement of cells toward the chemical which is consumed by cells when they encounter, the wave is an "invasion" pattern. That is, the wave profile of u decreases from its tail to front and that of v increases from its tail to the front, which means $u_x < 0, v_x > 0$.

From the transformation (2.21), it follows that $h \le 0$. Hence the physical region of (u,h) is

$$\mathscr{X} = \{(u,h) \mid u \ge 0, h \le 0, u_\pm \ge 0, h_\pm \le 0 \}. \tag{4.1}$$

Later we shall apply the theory of conservation laws to show the system (2.22)-(2.23) admits viscous shock wave solutions which are nonlinearly stable if $u_+ > 0$.

4.1. **Traveling wave solutions of the transformed system.** We first show that the transformed system (2.22) without viscosity is a genuinely nonlinear hyperbolic system under some conditions for ε. In the absence of the viscous terms, (2.22) becomes

$$\begin{cases} u_t - \chi(uh)_x = 0, \\ h_t + (\varepsilon h^2 - u)_x = 0. \end{cases} \tag{4.2}$$

By the standard procedure, we can show that if $0 < \varepsilon < 1$, the hyperbolic system (4.2) is genuinely nonlinear (see [49]). Hence the viscous shock waves can be expected.

Now we substitute the traveling wave ansatz

$$(u,h)(x,t) = (U,H)(z), \; z = x - ct.$$

into (2.22), and obtain the traveling wave equations

$$\begin{cases} -cU' - \chi(UH)' = dU'', \\ -cH' + (\varepsilon H^2 - U)' = \varepsilon H'' \end{cases} \tag{4.3}$$

with boundary conditions

$$(U,H)(z) \;\to\; (u_\pm, h_\pm) \text{ as } z \to \pm\infty \tag{4.4}$$

where $u_\pm \ge 0$ and $h_\pm \le 0$.

Integrating (4.3) once yields that

$$\begin{cases} dU' = -cU - \chi UH + \varrho_1 =: F(U,H), \\ \varepsilon H' = -cH + \varepsilon H^2 - U + \varrho_2 =: G(U,H) \end{cases} \tag{4.5}$$

where ϱ_1 and ϱ_2 are constants satisfying

$$\begin{aligned} \varrho_1 &= cu_- + \chi u_- h_- = cu_+ + \chi u_+ h_+, \\ \varrho_2 &= ch_- - \varepsilon(h_-)^2 + u_- = ch_+ - \varepsilon(h_+)^2 + u_+ \end{aligned} \tag{4.6}$$

which gives

$$\begin{aligned} c(u_+ - u_-) &= \chi(u_- h_- - u_+ h_+), \\ c(h_+ - h_-) &= \varepsilon(h_+)^2 - \varepsilon(h_-)^2 + u_- - u_+. \end{aligned} \tag{4.7}$$

Eliminating c from (4.7) yields

$$\frac{u_+ - u_-}{h_+ - h_-} = \frac{\chi(u_- h_- - u_+ h_+)}{\varepsilon(h_+)^2 - \varepsilon(h_-)^2 + u_- - u_+}. \tag{4.8}$$

Now if $u_\pm$ and $h_\pm$ satisfy (4.8), we can drive an equation for c from (4.6)

$$c^2 + \chi h_- c + \chi u_+ \left(\frac{\varepsilon(h_+^2 - h_-^2)}{u_+ - u_-} - 1 \right) = 0. \tag{4.9}$$

Note that when ε is small such that $\frac{\varepsilon(h_+^2 - h_-^2)}{u_+ - u_-} - 1 < 0$, the discriminant of the quadratic (4.9) is positive and hence (4.9) gives two solutions with opposite signs, where the positive c corresponds to the wave speed of the second characteristic family of system (2.22) and the negative c corresponds to that of the first characteristic family. Hereafter we only consider the case of $c > 0$ and the analysis extends to the case $c < 0$. The positive wave speed c is given by

$$c = -\frac{\chi h_-}{2} + \frac{1}{2}\sqrt{\chi^2 h_-^2 + 4u_+\chi\left(1 - \varepsilon\frac{h_+^2 - h_-^2}{u_+ - u_-}\right)}. \tag{4.10}$$

Since $h_- < 0$ and $\frac{\varepsilon(h_+^2 - h_-^2)}{u_+ - u_-} - 1 < 0$, then $c \geq \chi|h_-| = -\chi h_-$, which is equivalent to

$$c + \chi h_- \geq 0. \tag{4.11}$$

We can calculate that entropy condition of the shock of second characteristic family (cf. [49]) gives

$$0 \leq u_+ < u_-, \quad h_- < h_+ \leq 0 \tag{4.12}$$

which will be used later to prove the existence of traveling waves.

The first result concerning the existence of traveling wave solutions of (2.22)-(2.23), namely, the existence of solutions to (4.3)-(4.4), is as follows.

Theorem 4.1 ([49])**.** *Let (4.12) hold. If ε is small, then the system (4.2) admits a monotone shock profile $(U, H)(x - ct)$, which is unique up to a translation and satisfies $U' < 0$, $H' > 0$, where the wave speed c is given by (4.10). Moreover the solution profile (U, H) decays exponentially as $z \to \pm\infty$*

$$\begin{pmatrix} U \\ H \end{pmatrix} \sim \begin{pmatrix} u_\pm \\ h_\pm \end{pmatrix} + \begin{pmatrix} c_{1\pm} \\ c_{2\pm} \end{pmatrix} e^{\lambda_\pm z}, \text{ as } z \to \pm\infty$$

where $c_{1\pm}$ and $c_{2\pm}$ are positive constants and

$$\lambda_\pm = -\Big(\frac{c + \chi h_\pm}{2d} + \frac{c - 2\varepsilon h_\pm}{2\varepsilon}\Big) + \sqrt{\Big(\frac{c + \chi h_\pm}{2d} - \frac{c - 2\varepsilon h_\pm}{2\varepsilon}\Big)^2 + \frac{\chi u_\pm}{\varepsilon d}}.$$

Proof. We only outline the main steps of the proof and details were given in [49].

Step 1. It is easy to see that ODE system (4.5) has two and only two equilibria (u_-, h_-) and (u_+, h_+). The Jacobian matrices of the linearized system of (4.5) about equilibria $(u_\pm, v_\pm)$ are

$$J(u_\pm, v_\pm) = \begin{bmatrix} \frac{-c - \chi h_\pm}{d} & -\frac{\chi u_\pm}{d} \\ -\frac{1}{\varepsilon} & \frac{-c + 2\varepsilon h_\pm}{\varepsilon} \end{bmatrix}. \tag{4.13}$$

By calculating the eigenvalue of $J(u_\pm, h_\pm)$, one can show that the equilibrium (u_-, h_-) is a saddle and (u_+, h_+) is a stable node if $\varepsilon > 0$ is suitably small.

Step 2. Denote the region enclosed by nullclines of system (4.5) in the fourth quadrant by

$$\mathscr{G} = \left\{(U, H) \;\middle|\; \frac{\varrho_1}{\chi H + c} \leq U \leq \varepsilon H^2 - cH + \varrho_2\right\} \tag{4.14}$$

whose edges are denoted by

$$\Gamma_1 = \{(U, H) \mid F(U, H) = -U(\chi H + c) + \varrho_1 = 0, u_+ < U < u_-, h_- < H < h_+\},$$

$$\Gamma_2 = \{(U, H) \mid G(U, H) = \varepsilon H^2 - cH - U + \varrho_2 = 0, u_+ < U < u_-, h_- < H < h_+\}.$$

Then we can show that $\mathscr{G}$ is an invariant set of system (4.5), see an illustration in Fig. 3.

Step 3. By computing the tangential direction of the edges Γ_1 and Γ_1 at (u_-, h_-), we can find the direction of unstable manifold of system (4.5)at (u_-, h_-) is between the tangent lines of Γ_1 and Γ_2 at (u_-, h_-) and points into the region $\mathscr{G}$. Since the manifold is trapped inside the invariant region $\mathscr{G}$, this unstable manifold has go to

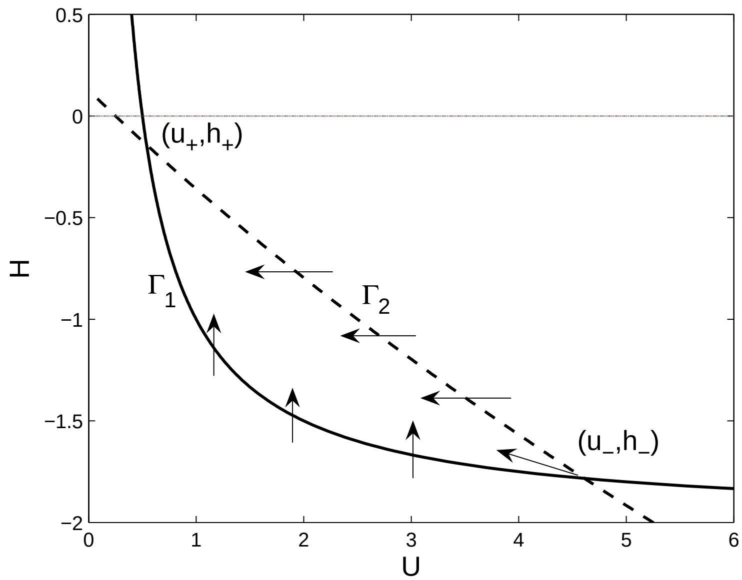


FIGURE 3. A numerical plot of the phase portrait of system (4.5), where the solid line (hyperbola) represents the nullcline $U(\chi H + c) = \varrho_1$ and dashed line (parabola) represents the nullcline $U = \varepsilon H^2 - cH + \varrho_2$. The parameter values are $\varrho_1 = 1, \varrho_2 = 1/4, c = 2, \varepsilon = 1/2, \chi = 1$.

the stable equilibrium (u_+, h_+) by the Poincaré-Bendixson theorem, and generates a trajectory connecting (u_-, h_-) and (u_+, h_+) which corresponds to a traveling wave solution of (4.2). From (4.5) and (4.14), we see that $U' < 0$ and $H' > 0$.

Step 4. Finally the asymptotic decay rates of (U, H) as $z \to \pm\infty$ can be derived by calculating the eigenvalues of Jacobian matrix (4.13).

The proof is complete. □

4.2. **Passing results to the original system.** In the preceding subsection, we establish the existence of traveling wave solutions to the transformed system (2.22)-(2.23). Now we are ready to transfer the results back to the original system (1.2)-(1.4).

Theorem 4.2 ([50]). *The model (1.2)-(1.4) with $m = 1$ has a unique (up to a translation) monotone bounded traveling wave solution $(U, V)(x - ct)$ with $U' < 0, V' > 0$ provided that $\varepsilon \geq 0$ is small, such that $v_+ > v_- = 0, u_- > u_+ = \beta \geq 0$, where the wave speed c is given by*

$$c = \chi\left[\frac{u_-}{\chi + \varepsilon(1 - u_+/u_-)}\right]^{1/2} = \begin{cases} \chi\sqrt{\frac{u_-}{\chi+\varepsilon}}, & \beta = 0, \\ \chi\sqrt{\frac{u_-}{\chi+\varepsilon(1-\beta/u_-)}}, & \beta > 0. \end{cases} \tag{4.15}$$

Moreover the traveling wave solution (U, V) has the following asymptotic behavior

$$U(z) \sim \begin{cases} u_- + Ce^{\lambda_- z}, & \text{as } z \to -\infty, \\ \beta + Ce^{\lambda_+ z}, & \text{as } z \to \infty, \end{cases}$$

and

$$V(z) \sim \begin{cases} Ce^{-h_- z - \frac{C}{\lambda_-} e^{\lambda_- z}}, \text{ as } z \to -\infty, \\ Ce^{-\frac{C}{\lambda_+} e^{\lambda_+ z}}, \text{ as } z \to \infty, \end{cases}$$

where C is a generic positive constant and

$$h_- = \frac{c(\beta - u_-)}{\chi u_-}.$$

Proof. Note that u in the system (1.2) remains the same as that in (2.22) and so does U. Hence we only need to translate the results from H to V. Recalling the transformation (2.21), we have

$$H(z) = -(\ln V)_z$$

which yields that

$$V(z) = Ce^{-\int_0^z H(y)dy}$$

where $C = V(0)$ is a positive constant. Then the results for V can be readily derived from H only if $h_+ = 0$, see more details in [50]. The assertion $u_+ = \beta$ is obtained from (2.6) by the fact $m = 1$ and $v_+ > 0$. Finally when $h_+ = 0$, we can solve from the first equation of (4.6) that $h_- = \frac{c(\beta - u_-)}{\chi u_-}$. Then the asymptotic decay rates of $(U, V)(z)$ can be readily derived from the decay results in Theorem 4.1. □

Remark 3. From (2.5), one has that

$$C_0 = c\beta$$

which validates the possibility $C_0 \neq 0$ when $\beta > 0$.

Remark 4. The smallness assumption of the chemical diffusion ε imposed in Theorem 4.2 can be removed for the case $u_+ = \beta = 0$ using a similar approach as in section 3.2.1, see [82]. However such an approach does not apply to the case $u_+ > 0$. Hence the existence/nonexistence of traveling wave solutions with $u_+ > 0$ for large $\varepsilon > 0$ still remains open.

Remark 5. For the same parameter values $m = 1, \beta = 0$, when $C_0 = 0$, the results in Theorem 4.2 are consistent with those in Theorem 3.5 (ii), while for $m = 1, \beta > 0$, the solution profile (front, front) for $C_0 = c\beta > 0$ (see Theorem 4.2) is different from the profile (front, pulse) for $C_0 = 0$ (see Theorem 3.5). This shows that $C_0 = 0$ and $C_0 \neq 0$ may generate different results and it is necessary to distinguish them.

If (1.2) is reviewed as a model to describe the initiation of angiogenesis as in [16, 82], it is of importance to incorporate the cell growth, since angiogenesis activator increases and prompts the division and growth of vascular endothelial cells in order to form new blood vessels during the initiation of angiogenesis. Hence in a recent paper [3], a logistic growth term is incorporated into the model (1.2) and the existence of traveling wave profile (front, front) was established.

4.3. **Numerical simulations of wave propagation.** Due to the singular term v^{-1}, it is unfeasible to obtain the numerical solutions of the model (1.2) without the approximation technique. The Hopf-Cole transformation (2.21) makes not only analytical study possible but also numerical exploration achievable with standard numerical schemes. In this section, we shall illustrate the numerical simulations of propagating traveling waves, and discuss the biological implications.

As the most interesting solution component, the cell density u in the model (1.2) remains the same as the one in the transformed system (2.22). Hence we can find the numerical solution u by numerically solving the transformed model (2.22) directly. The traveling wave propagation will be simulated in a finite spatial domain with Dirichlet conditions compatible with the initial data. The parameter values will be chosen to meet the requirement (4.8). In the simulation, the initial data are set as

$$\begin{aligned} u_0(x) &= \tilde{u}(x) = \bar{u} + 1/(1+\exp(2(x-20))), \\ h_0(x) &= \tilde{h}(x) = \bar{h} + 1/(1+\exp(-2(x-20))) \end{aligned} \tag{4.16}$$

with end states $\tilde{u}_- = \bar{u}+1, \tilde{u}_+ = \bar{u}, \tilde{h}_- = \bar{h}, \tilde{h}_+ = \bar{h}+1$. We consider two sets of parameter values: (1) $\varepsilon = 0$ and $\beta > 0$; (2) $\varepsilon > 0$ and $\beta = 0$. The former may describe the reinforced random walk and the latter may account for the directed movement of endothelial cells during the initiation of angiogenesis, see section 1 for details.

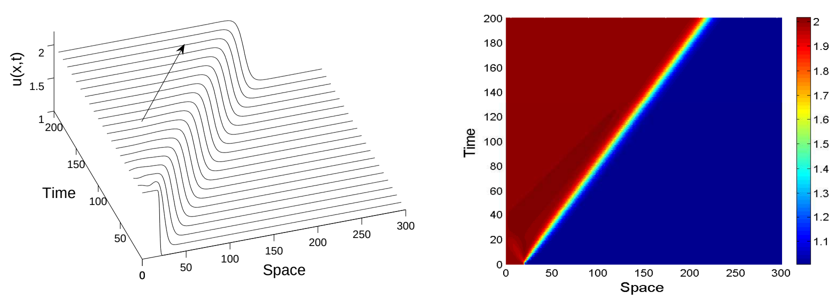
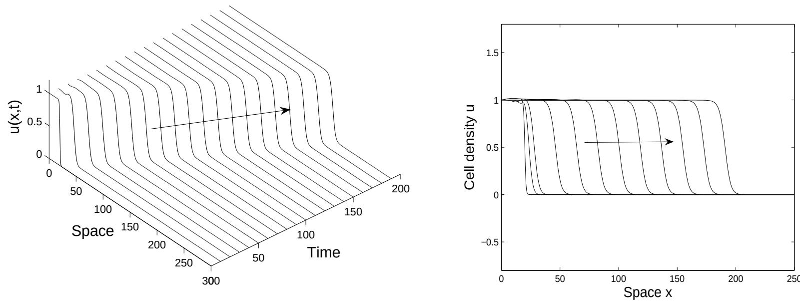


FIGURE 4. A numerical simulation of wave propagation of cell density u to the model (2.22) in the spatial domain where $\varepsilon = 0, d = 2, \chi = 0.5$ and initial data are given in (4.16). The arrow indicates the propagating direction of traveling waves.

In Fig. 4, we simulate the wave propagation of the model (2.22) with $\varepsilon = 0$ and $\beta > 0$, where the domain $\Omega = (0, 300)$ with mesh size 0.5. We choose $d = 2, \bar{u} = 1, \bar{h} = -1$, and hence $\tilde{u}_- = 2, \tilde{u}_+ = 1, \tilde{h}_- = -1, \tilde{h}_+ = 0$. Fig. 4 (a) is a three dimensional visualization of traveling waves propagating in the spatial field, and Fig. 4 (b) plots the temporal-spatial wave pattern formation. From Fig. 4 (a), we see that the solution from the initial data oscillates for a short time and then quickly form a stable propagating wave. In relation to the biological motivation of the model with $\varepsilon = 0$, $\beta > 0$, Fig. 4 illustrates a spatial movement pattern of random walkers in response to the chemical signal. Here $\beta > 0$ is reflected by the fact $\beta = u_+ = 1$.

Fig. 5 plots the propagating waves generated by the model (2.22) with $\varepsilon = 0.1 > 0$ and $\beta = 0$, where $\bar{u} = 0$ and other parameter values are the same as those in Fig. 4. Here Fig. 5 (a) is a three dimensional visualization of traveling waves propagating in the spatial field, and Fig. 5 (b) plots the evolution of solutions from the initial profile to a monotone wavefront which fits our theoretical results. In this parameter regime, the model (1.2) describes the directed migration of endothelial cells toward



(a) (b)

FIGURE 5. A numerical plot of traveling wavefronts of cell density u to the model (2.22) propagating in the space as time evolves, where $\varepsilon = 0.1, d = 2, \chi = 0.9$ and initial data are $u_0 = 1/(1 + \exp(2(x-20))), h_0 = -1 + 1/(1 + \exp(-2(x-20)))$. The arrow indicates the propagating direction of traveling waves.

the signal molecule VEGF. Such process is exactly shown by our simulations where $\beta = u_+ = 0$. One difference observed between Fig. 4 and Fig. 5 is that the transient time taken from initial data to a steady wave in Fig. 4 is longer than that in Fig. 5. This implies that the chemical diffusion ε may enhance the process of propagating wave formation, which is not found in the theoretical results.

5. **Stability of traveling wave solutions.** In contrast to the existence results, the stability result of traveling wave solutions was much less. The first stability result was obtained in [73] for the case $\varepsilon = \beta = 0, 0 \le m < 1$ by the spectrum analysis and followed in [25] for $\varepsilon = \beta = m = 0$. An essential progress on the stability of traveling wave solutions was made in [47, 49] where the nonlinear stability was proved for the case $m = 1, \beta \ge 0$ by the method of energy estimates. These results will be presented in this section.

5.1. **Linear stability/instability.** Let us first recall the stability results from [73, 60]. Let $(u, v)(x, t) = (u, v)(z, t)$, where $z = x - ct$. Then the system with $\beta = 0$ becomes

$$\begin{cases} u_t = (du_z - \chi u \frac{v_z}{v})_z + cu_z, \\ v_t = \varepsilon v_{zz} + cv_z - uv^m. \end{cases} \tag{5.17}$$

To derive the linearized stability of traveling wave solutions, we consider a small perturbation of the traveling wave solution $(U(z), V(z))$

$$\begin{bmatrix} u(z,t) \\ v(z,t) \end{bmatrix} = \begin{bmatrix} U(z) \\ V(z) \end{bmatrix} + \epsilon \begin{bmatrix} \tilde{u}(z,t) \\ \tilde{v}(z,t) \end{bmatrix}, \ 0 < \epsilon \ll 1.$$

Substituting this into (5.17) and keeping only the first-order terms in ϵ give the equations for $(\tilde{u}(z,t), \tilde{v}(z,t))$, which is also the linearized system of (5.17) about (U, V) and reads

$$\begin{cases} u_t = \Big(du_z + cu - \chi\Big(\frac{U}{V}v_z + U(\frac{1}{V})_z v + \frac{V'}{V}u\Big)\Big)_z, \\ v_t = \varepsilon v_{zz} + cv_z - (V^m u + mUV^{m-1}v) \end{cases} \tag{5.18}$$

where the tildes have been dropped for convenience.

Now we look for solutions to the linearized system (5.18) by setting

$$\begin{bmatrix} u(z,t) \\ v(z,t) \end{bmatrix} = \begin{bmatrix} \bar{u}(z) \\ \bar{v}(z) \end{bmatrix} e^{\lambda t}$$

which on substituting into (5.18) gives upon canceling the exponentials

$$\begin{cases} \left(d\bar{u}_z + c\bar{u} - \chi(\frac{U}{V}\bar{v}_z + U(\frac{1}{V})_z\bar{v} + \frac{V'}{V}\bar{u})\right)_z = \lambda\bar{u}, \\ \varepsilon\bar{v}_{zz} + c\bar{v}_z - V^m\bar{u} - mUV^{m-1}\bar{v} = \lambda\bar{v} \end{cases} \tag{5.19}$$

with $\bar{u}$ and $\bar{v}$ considered in the space

$$X = \left\{ \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \middle| \ \bar{u}, \bar{v} \in L^1(\mathbb{R}) \text{ and } \int_{\mathbb{R}} \bar{u}(z)dz = 0 \right\}.$$

It was first shown in [73] that $\text{Re}(\lambda) \leq 0$ when $\varepsilon = 0, 0 \leq m < 1$, and the eigenmode of the zero eigenvalue is a uniform small displacement of the steadily propagating waves, which implies the linearized stability of traveling wave solutions. Later the same linearized stability result was shown in [25] for $\varepsilon = m = 0$, which was indeed covered by the results of [73]. We summarize these results in the following theorem.

Theorem 5.1 ([73]). *If $\varepsilon = 0$ and $0 \leq m < 1$ in (5.19), then $\text{Re}(\lambda) \leq 0$, where the eigenmode $(\bar{u}, \bar{v})$ associated with $\lambda = 0$ describes a uniform small displacement of the steadily propagating waves. That is, the traveling wave solution to (1.2)-(1.4) with $\beta = \varepsilon = 0$ and $0 \leq m < 1$ is linearly stable.*

When $\varepsilon > 0$, Nagai and Ikeda [60] proved the instability of traveling wave solutions to (1.2)-(1.4) for $m = 0, \beta = 0$ in both space X and exponentially weighted space

$$X_\omega = \left\{ \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \in X \middle| \ \omega\bar{u}, \omega\bar{v} \in L^1(\mathbb{R}) \right\}$$

where

$$\omega(z) = \begin{cases} e^{-\rho_1 z} & \text{for } z < -1, \\ e^{\rho_2 z} & \text{for } z > 1 \end{cases}$$

with $\rho_1 \geq l_1, \rho_2 \geq l_2$. Here constants $l_1 = \frac{c}{\chi - d}, l_2 = -\frac{c}{d}$ are such that $e^{l_1 z}$ is the decay rate of traveling wave solutions $U(z)$ and $V(z)$ as $z \to -\infty$, and $e^{l_2 z}$ is the decay rate of $U(z)$ as $z \to \infty$. Then the instability result of [60] is as follows:

Theorem 5.2 ([60]). *Let $\varepsilon > 0, \beta = 0$ and $\chi > d$ in (5.19). Then*

$$\sigma(\mathscr{L}) \cap \{\lambda \in \mathbb{C} \mid \text{Re}(\lambda) > 0\} \neq \emptyset \text{ and } \sigma(\mathscr{L})_\omega \cap \{\lambda \in \mathbb{C} \mid \text{Re}(\lambda) > 0\} \neq \emptyset$$

where $\sigma(\mathscr{L})$ and $\sigma(\mathscr{L})_\omega$ denote the spectrum of $\mathscr{L}$ in X and in X_ω, respectively.

Theorem 5.2 immediately asserts that traveling wave solutions of the model (1.2)-(1.4) with $m = \beta = 0$ are unstable in both space X and exponentially weighted space X_ω. Namely the chemical diffusion destabilize the traveling wave solutions, which comes out counter-intuitively though the experiment favors the stability [35]. It is worthwhile to study the stability problem further in more appropriate solution spaces.

5.2. **Nonlinear stability.** The linearized stability/instability traveling wave solutions of (1.2) was derived for the case $0 \le m < 1$. When $m = 1$, the linearized system of (1.2) about the traveling wave solution contains eigenvalues with zero real part which cannot be removed with the exponential weight function as in [60]. Hence the stability of (1.2) with $m = 1$ becomes is a challenging problem. In [47, 49], the nonlinear stability of traveling wave solutions of the transformed model (2.22)-(2.23) with $m = 1, \beta > 0$ was established. The stability result for the original system (1.2)-(1.4) based on the results of [47, 49] was given in [50]. These results will be presented in this section.

5.2.1. *Nonlinear stability of traveling wave solutions to the transformed system.* To study the nonlinear stability of traveling wave solutions for the transformed system (2.22)-(2.23), the technique of taking anti-derivative will be used to define the perturbations as

$$(u, h)(x, t) = (U, H)(x - ct + x_0) + (\phi_x, \psi_x)(x, t) \tag{5.20}$$

where x_0 is a translation constant and assumed to be zero without loss of generality. Hence

$$(\phi(x, t), \psi(x, t)) = \int_{-\infty}^{x} (u(y, t) - U(y - ct), h(y, t) - H(y - ct))dy \tag{5.21}$$

for all $x \in \mathbb{R}$ and $t \ge 0$, and

$$\phi(\pm\infty, t) = 0, \ \psi(\pm\infty, t) = 0 \text{ for all } t > 0.$$

We assume that the initial perturbation is made along the vector $(u_+ - u_-, h_+ - h_-)$ and hence

$$\int_{-\infty}^{+\infty} (u_0 - U, h_0 - H)(x)dx = (0, 0). \tag{5.22}$$

The initial condition of the perturbation (ϕ, ψ) is thus given by

$$(\phi_0, \psi_0)(x) = \int_{-\infty}^{x} (u_0 - U, h_0 - H)(y)dy. \tag{5.23}$$

The asymptotic stability of traveling wave solutions of (2.22) means that $(u - U, v - H)(x, t) = (\phi_x, \psi_x)(x, t) \to 0$ as $t \to \infty$, which is given in [47] for $\varepsilon = 0$ and in [49] for $\varepsilon > 0$, as follows:

Theorem 5.3 ([47, 49])**.** *Let $(U, V)(x - ct)$ be a viscous shock profile of (2.22) obtained in Theorem 4.1. If $\varepsilon \ge 0$ is small and $u_+ > 0$, then there exists a constant $\varepsilon_0 > 0$ such that if $\|u_0 - U\|_{W^{1,2}} + \|h_0 - H\|_{W^{1,2}} + \|(\phi_0, \psi_0)\|_{L^2} \le \varepsilon_0$, the Cauchy problem (2.22)-(2.23) has a unique global solution $(u, h)(x, t)$ such that $u(x, t) \ge \delta_0 > 0$ for some $\delta_0 > 0$ for all $x \in \mathbb{R}$, $t \ge 0$, and*

$$(u - U, h - H) \in (C([0, \infty); W^{1,2}) \cap L^2([0, \infty); W^{1,2}))^2$$

with

$$\sup_{x \in \mathbb{R}} |(u, h)(x, t) - (U, H)(x - ct)| \to 0 \quad as \quad t \to \infty. \tag{5.24}$$

Proof. The proof of Theorem 5.3 is based on iterative L^2 energy estimates. The method of energy estimates for the nonlinear stability of viscous shock profiles of conservation laws was first introduced independently by Matsumura and Nishihara in [55] and by Goodman in [24] with further developments over the years, see [53]. Here we establish the stability results of traveling wave solutions without the smallness assumption on the wave strength. Since the proof is nontrivial and involves many technical treatments, we omit the details and refer the interested readers to [47, 49]. □

5.2.2. *Nonlinear stability of traveling wave solutions to the original system.* Passing the stability results of the transformed system (2.22)-(2.23) to the original system (1.2)-(1.4), we have the following asymptotic nonlinear stability:

Theorem 5.4 ([50]). *Let $m = 1, \beta > 0$ and $(U, V)(x - ct)$ be a traveling wave solution of (1.2)-(1.4) with $u_+ = \beta > 0$ obtained in Theorem 4.2. Then there exists a constant $\varepsilon_0 > 0$ such that if $\|u_0 - U\|_{W^{1,2}} + \|(\ln v_0)_x - (\ln V)_x\|_{W^{1,2}} + \|(\phi_0, \psi_0)\|_{L^2} \le \varepsilon_0$, where*

$$\phi_0(x) = \int_{-\infty}^{x} (u_0 - U)(y)dy, \ \ \psi_0(x) = -\ln v_0(x) + \ln V(x),$$

then the Cauchy problem (1.2)-(1.4) has a unique global solution $(u, v)(x, t)$ satisfying $u(x, t) > \delta_0$ for all $x \in \mathbb{R}$, $t \ge 0$ for some $\delta_0 > 0$, such that

$$(u - U, v_x/v - V_x/V) \in C([0, \infty); W^{1,2}) \cap L^2([0, \infty); W^{1,2}).$$

Moreover the solution (u, v) has the following asymptotic nonlinear stability

$$\sup_{x\in\mathbb{R}} |(u, v)(x, t) - (U, V)(x - ct)| \to 0, \quad \text{as} \quad t \to \infty.$$

Proof. The result for u has been given in Theorem 5.4. It remains to translate the results from h to v. By the transformation (2.21) and (5.21), one has that

$$\frac{v(x,t)}{V(x-ct)} = e^{\int_{-\infty}^{x}(H(\xi-ct)-h(\xi,t))d\xi} = e^{\psi(x,t)}.$$

Next we show that $\psi(x, t) \to 0$ as $t \to \infty$. Indeed it has been shown in [49] (see the Proposition 4.3 and the proof of Theorem 4.1 in [49]) that $\|\psi(\cdot, t)\|_{L^2} < \infty$ and $\|\psi_x(\cdot, t)\|_{W^{1,2}} \to 0$ as $t \to \infty$. Then

$$\begin{aligned}\psi^2(x,t) &= 2\int_{-\infty}^{x} \psi\psi_y(y,t)dy \\ &\le 2\Big(\int_{\mathbb{R}} \psi^2 dx\Big)^{1/2}\Big(\int_{\mathbb{R}} \psi_x^2 dx\Big)^{1/2} \to 0 \text{ as } t \to \infty\end{aligned}$$

which implies $\psi(x, t) \to 0$ as $t \to \infty$. Note that $V(x-ct)$ is a traveling wave solution which is uniformly bounded in x. Then

$$|v(x,t) - V(x - ct)| = V(x - ct)|1 - e^{\psi(x,t)}| \to 0 \text{ as } t \to \infty$$

for all $x \in \mathbb{R}$. The proof is complete. □

5.3. **Numerical simulations of stability of traveling wave solutions.** As mentioned in section 4.3, the Hopf-Cole transformation (2.21) enables us to compute the transformed model (2.22) numerically. In this section, we continue to simulate the stability of traveling wave solution component u. Since the numerical scheme is restricted to a finite domain, we consider the perturbations that vanish outside the finite domain to approximate the real situation. To this end, we set the initial data (u_0, h_0) as

$$\begin{aligned} u_0(x) &= 1 + \frac{1}{1+\exp(2(x-100))} + \frac{0.5\sin x}{((x-100)/10)^2+1}, \\ h_0(x) &= -1 + \frac{1}{1+\exp(-2(x-100))} + \frac{0.5\sin x}{((x-100)/10)^2+1}, \end{aligned} \tag{5.25}$$

so that the initial perturbation belong to $W^{1,2}(\mathbb{R})$ as required by Theorem 5.3. Then the evolution of the numerical solution u for the case $\varepsilon > 0$ is plotted in Fig. 6 where we see that the solution gradually stabilizes to a traveling wavefront, as proved by our theoretical results.

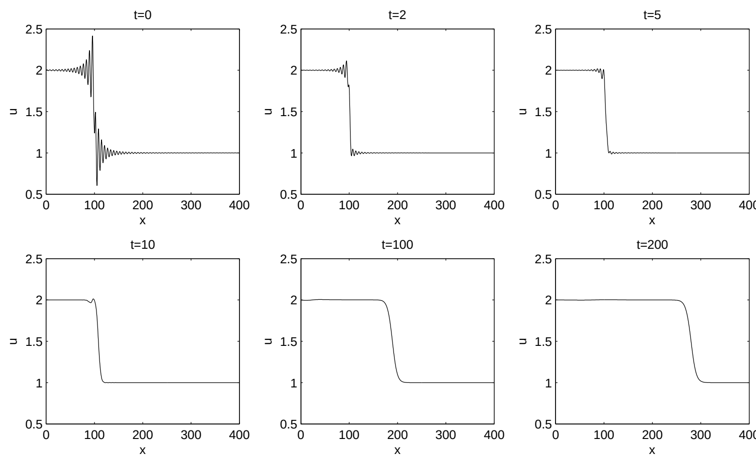


FIGURE 6. Numerical illustration of the stability of traveling wavefront u for $\varepsilon = 0.1 > 0$, where $d = 2, \chi = 0.45$ and initial data are given by (5.25).

Before concluding this section, it is worth mentioning a work [57] where the existence and instability of traveling wave solutions to the same model (1.2) with $m = 1, \beta > 0$ was established based on the spectral analysis. There are two principle differences between [57] and the results in the present paper. First the paper [57] proves the instability of traveling wave profile (front, pulse), and here the nonlinear stability of different traveling wave profile (front, front) in Theorem 5.4 is established. This difference is caused by the assumption that the integration constant C_0 in the first equation of (2.4) is zero in [57], and nonzero in the present paper. When $C_0 = 0$, the existence of traveling wave profile (front, pulse) was also given in Theorem 3.5. The instability result of [57] indeed fill the gap of stability problem for this case. Second, in the present paper, the unique wave speed is identified, however in [57] only the minimum wave speed is found.

6. **Wave speed.** Wave speed is an important quantity in traveling wave solutions, since it describes how fast one species spreads/disperses from one location to another. For example, it may describe how fast a disease is transmitted or a tumor spreads in the tissue. The wave speed of system (1.2)-(1.4) can be obtained either explicitly or implicitly. For the sake of presentation, we denote the wave speed by c_0 for $\varepsilon = 0$ and by c_ε for $\varepsilon > 0$, and denote the traveling wave solutions by (U_0, V_0) for $\varepsilon = 0$ and by $(U_\varepsilon, V_\varepsilon)$ for $\varepsilon > 0$. The following results are distinguished by linear consumption rate ($m = 1$) and nonlinear consumption rate ($m \neq 1$).

6.1. **Wave speed for $m = 1$.** When $m = 1$ and $C_0 \neq 0$, the wave speed has been given in (4.15). When $m = 1$ and $C_0 = 0$ the traveling wave solution (U, V) is (front, front) if $\beta = 0$ and (front, pulse) if $\beta > 0$. The wave speed can be explicitly found in terms of system parameters by matching the asymptotics of solutions at $z = \pm\infty$. The result is:

Theorem 6.1. *Let $m = 1$ and $C_0 = 0$. Assume that $u_- > \beta$, then*

(i) If $\varepsilon = 0$, then the wave speed is unique and given by

$$c_0 = \sqrt{\chi(u_- - \beta)}.$$

(ii) If $\varepsilon > 0$, then the wave speed is uniquely given by

$$c_\varepsilon = \chi\sqrt{\frac{u_- - \beta}{\varepsilon + \chi}}.$$

(iii) $|c_\varepsilon - c_0| = \mathcal{O}(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof. (i) First notice that when $\varepsilon = 0, \beta \geq 0$, the solution U is given by (3.6). Moreover when $m = 1$, $\mu = -\frac{c_\varepsilon}{\chi}$, $\frac{\chi}{(r-1)d} = 1$, and the constant C_2 can be calculated as

$$C_2 = \frac{r-1}{kc + \beta(r-1)} = \frac{1}{c^2/\chi + \beta}.$$

Then the traveling wave solution $U(z)$ is further expressed by

$$U(z) = \left(C_1 e^{(r-1)(c/\chi+\beta/c)z} + \frac{1}{c^2/\chi + \beta}\right)^{-1}$$

Hence

$$u_- = U(-\infty) = \left(\frac{1}{c^2/\chi + \beta}\right)^{-1}$$

which yields the wave speed

$$c = \sqrt{\chi(u_- - \beta)}.$$

(ii) When $m = 1$, $\eta = \varepsilon\frac{c_\varepsilon^2}{\chi^2} + \frac{c_\varepsilon^2}{\chi} + \beta$. Then from (3.14), we have

$$U(-\infty) = u_- = \eta^{\frac{\chi}{(r-1)d}} = \eta$$

which gives

$$c_\varepsilon^2\left(\frac{\varepsilon}{\chi^2} + \frac{1}{\chi}\right) = u_- - \beta.$$

If $u_- \geq \beta$, the wave speed is given by $c_\varepsilon = \chi\sqrt{\frac{u_- - \beta}{\varepsilon + \chi}}$, as required.

(iii) From (i) and (ii), we can easily find

$$|c_\varepsilon - c_0| \leq \sqrt{\frac{u_- - \beta}{\chi}}\varepsilon$$

which yields the assertion. ◻

From the above results we see that the wave speed decreases with respect to the chemical growth rate β, which seems counter-intuitive. The reason is the chemotactic saturation effect caused by the singular term v^{-1} which indicates that cells indeed might become less sensitive for higher concentration v. Hence fast growth of v will decease the chemotactic speed due to this saturation effect. Another expected observation from our results is that the wave speed is a decreasing function of the chemical diffusion coefficient ε because the fast dispersal of the chemical may diminish the gradient of the chemical concentration. On the other hand, intuitively the wave speed should be a increasing function of chemosensitivity coefficient χ, which is exactly indicated by the formulas in Theorem 6.1. In other words, the wave speed formulas given in the Theorem 6.1 is consistent with all rational perceptions and connected to all necessary parameters in the model.

6.2. **Wave speed for $0 \le m < 1$.** For the case of $0 \le m < 1$, the existence of traveling wave solutions was established only for $C_0 = 0$. Since traveling wave solutions do not exist when $m > 1$, we consider the case $0 \le m < 1$ under which the traveling wave solution (U, V) is (pulse, front) if $\beta = 0$ and (pulse, pulse) if $\beta > 0$. In this case, U is always a pulse.

Theorem 6.2. *Let $0 \le m < 1$ such that $\chi/d + m > 1$. Then*

(i) If $\varepsilon = 0$, then the wave speed is given by

$$c_0 = (1-m)v_+^{m-1}\left[N - \beta\int_{-\infty}^{\infty} V_0^{1-m}dz\right].$$

(ii) If $\varepsilon > 0$, the wave speed is

$$c_\varepsilon = (1-m)v_+^{m-1}\left[N + \beta\int_{-\infty}^{\infty} V_\varepsilon^{1-m}dz\right] - (1-m)v_+^{m-1}\varepsilon\int_{-\infty}^{\infty}\frac{V_\varepsilon''}{V_\varepsilon^m}dz.$$

Proof. (i) From the second equation of (2.4), we obtain for $\varepsilon = 0$

$$\frac{cV'}{V^m} = U - \beta V^{1-m}.$$

The integration of above equation gives the wave speed c_0.

(ii) The proof is similar to the proof of (i). So we omit the details. ☐

Corollary 1. *(i) If $\beta = 0$ and $0 < m < 1$, then the wave speed c_0 is given by*

$$c_0 = (1-m)v_+^{m-1}N.$$

(ii) If $m = \beta = 0$, then the wave speeds c_ε and c_0 are equal and given by

$$c_\varepsilon = c_0 = \frac{N}{v_+}.$$

The wave speeds in Corollary 1 were initially obtained in [68, 73], which can be viewed as a special case of Theorem 6.2. It is worthwhile to note that the wave speed for $m = 1$ is unique and that for $0 \le m < 1$ is not except for $m = \beta = 0$ as given in Corollary 1. Indeed, for $0 < m < 1$, we have shown that the wave speed has a minimum speed $c_* = 2\sqrt{\varepsilon\beta}$, and the traveling wave solution exists for any wave speed greater than the minimum speed. This implies that there are infinite many wave speeds for each $\varepsilon > 0$.

Remark 6. Except the special cases in Corollary 1, Theorem 6.2 does not give a precise wave speed due to the unknown integrals which are however finite. But it provides a structure of wave speed which allows us to speculate the impact of parameters on the wave speed.

7. **Chemical diffusion limits.** The parameter ε represents the chemical diffusion, which is larger than the cell diffusion d in general. However in applications, ε might be small compared to cell diffusion. For example, in bacterial chemotaxis shown in Fig. 1 (a), the chemical (i.e. oxygen) is carried by large water molecules which diffuse slowly. In angiogenesis, the chemical is often the substrate or network tissue which is almost static. Most of past studies considered $\varepsilon = 0$ and $\varepsilon > 0$ separately.

In contrast the chemical diffusion limit as $\varepsilon \to 0$ was much less studied, and was first studied in [60] for the case $m = \beta = 0$ with an extension in [82] to $\beta = 0, m = 1$.

When ε varies, both the traveling wave speed and traveling wave solution will change. Hence to prove the convergence of traveling wave solutions as $\varepsilon \to 0$, the convergence of wave speed needs to be established first. As shown in the previous section, $c_\varepsilon \to c_0$ as $\varepsilon \to 0$ when $m = 1$ (see Theorem 6.1) or $m = \beta = 0$ (see Corollary 1 (ii)). Hereafter we denote the traveling wave solutions of (1.2) for $\varepsilon > 0$ by $(U_\varepsilon, V_\varepsilon)$, and for $\varepsilon = 0$ by (U_0, V_0). We have the following results.

Theorem 7.1 ([60]). *Let $m = \beta = 0$ and $\chi/d > 1$. Then*

$$\|U_\varepsilon - U_0\|_{L^\infty} = \mathcal{O}(\sqrt{\varepsilon}),\ \|V_\varepsilon - V_0\|_{L^\infty} = \mathcal{O}(\varepsilon).$$

The similar result can be obtained to the case of $\beta = 0$ and $0 \le m < 1$.

Theorem 7.2 ([82]). *Let $\beta = 0$ and $0 \le m \le 1$ such that $r = \chi/d + m \ge 1$. Then*

$$\|U_\varepsilon - U_0\|_{L^\infty} = \mathcal{O}(\varepsilon) + \mathcal{O}(|c_\varepsilon - c_0|),\ \|V_\varepsilon - V_0\|_{L^\infty} = \mathcal{O}(\varepsilon) + \mathcal{O}(|c_\varepsilon - c_0|).$$

Particularly when $m = 1$, then

$$\|U_\varepsilon - U_0\|_{L^\infty} = \mathcal{O}(\varepsilon),\ \|V_\varepsilon - V_0\|_{L^\infty} = \mathcal{O}(\varepsilon).$$

Proof. The proof is a generalization of [82]. When $\beta = 0$, (2.8) becomes

$$\varepsilon V_\varepsilon'' - c_\varepsilon V_\varepsilon' - e^{-kz} V_\varepsilon = 0.$$

Define $\mathbf{V} = V_\varepsilon - V_0$. Then

$$\varepsilon \mathbf{V}'' + c_0 \mathbf{V}' + (c_\varepsilon - c_0) V_\varepsilon' + \varepsilon V_0'' = g(z, V_\varepsilon) - g(z, V_0) \tag{7.1}$$

where $g(z, V) = e^{-kz} V^r$. Then multiplying (7.1) by $\mathbf{V}$ and using the mean value theorem, we have

$$\varepsilon \mathbf{V}\mathbf{V}'' + c_0 \mathbf{V}\mathbf{V}' + (c_\varepsilon - c_0) V_\varepsilon' \mathbf{V} + \varepsilon V_0'' \mathbf{V} = r e^{-kz} \xi^{r-1} \mathbf{V}^2 \ge 0 \tag{7.2}$$

since $r e^{-kz} \xi^{r-1} \ge 0$, where ξ is between V_0 and V_ε. Then integrating (7.2) on both sides and using the fact $\mathbf{V}(\infty) = \mathbf{V}'(\infty) = 0$, we derive that

$$\frac{\varepsilon}{2} (\mathbf{V}^2)' + \frac{c_0}{2} \mathbf{V}^2 \le \varepsilon \int_z^\infty \mathbf{V} V_0'' dz + (c_\varepsilon - c_0) \int_z^\infty V_\varepsilon' \mathbf{V} dy. \tag{7.3}$$

Let z_0 be a point at which $|\mathbf{V}(z)|$ attains its maximum on $\mathbb{R}$. Then $(\mathbf{V}^2)'(z_0) = 2\mathbf{V}'(z_0)\mathbf{V}(z_0) = 0$ and hence

$$\mathbf{V}(z) \le \mathbf{V}(z_0) \le \frac{2}{c_0} \Big(\varepsilon \|V_0''\|_{L^1(\mathbb{R})} + |c_\varepsilon - c_0| \cdot \|V_\varepsilon'\|_{L^1(\mathbb{R})} \Big). \tag{7.4}$$

From the proof of Theorem 3.1, we know that when $\varepsilon = 0, \beta = 0$ and $r = \chi/d + m > 1$, the traveling wave solution component V_0 is

$$V_0 = \left(v_+^{1-r} + \frac{r-1}{kc} e^{-kz} \right)^{\frac{1}{1-r}},$$

which yields that

$$V_0'' \sim \left(v_+^{1-r} e^{\frac{k(r-1)}{r} z} + \frac{r-1}{kc} e^{-\frac{k}{r} z} \right)^{\frac{r}{1-r}}.$$

This implies that V_0'' decays exponentially as $|z| \to \infty$ and hence $\|V_0''\|_{L^1(\mathbb{R})} < \infty$. Moreover since $V_\varepsilon' > 0$ when $\beta = 0$ (see Theorem 3.5), we have $\|V_\varepsilon'\|_{L^1(\mathbb{R})} = v_+ - v_-$.

Therefore (7.4) yields the estimate for $\|V_\varepsilon - V_0\|_{L^\infty}$ ($=|\mathbf{V}(z)|$) as announced in the theorem. With (2.7) and the mean value theorem, we deduce that

$$\begin{aligned} U_\varepsilon - U_0 &= e^{-\frac{c_\varepsilon}{d}z} V_\varepsilon^{\frac{\chi}{d}} - e^{-\frac{c_0}{d}z} V_0^{\frac{\chi}{d}} \\ &= e^{-\frac{c_\varepsilon}{d}z}(V_\varepsilon^{\frac{\chi}{d}} - V_0^{\frac{\chi}{d}}) + V_0^{\frac{\chi}{d}}(e^{-\frac{c_\varepsilon}{d}z} - e^{-\frac{c_0}{d}z}) \\ &= \frac{\chi}{d} e^{-\frac{c_\varepsilon}{d}z} \zeta_1^{\frac{\chi}{d}-1}(V_\varepsilon - V_0) - \frac{z}{d} e^{-\frac{z}{d}\zeta_2}(c_\varepsilon - c_0) \end{aligned}$$

where ζ_1 is between V_0 and V_ε and ζ_2 is between c_ε and c_0. This implies the announced estimate for $\|U_\varepsilon - U_0\|_{L^\infty}$. Noticing that when $m = 1$, $|c_\varepsilon - c_0| = \mathcal{O}(\varepsilon)$ from Theorem 6.1(iii). Then the proof is complete. □

Remark 7. The above chemical diffusion limits were derived for the case $C_0 = 0$. For the case $C_0 \neq 0$, the analogous result was derived in [50] for the case $m = 1$ by the method of geometric singular perturbation.

In the remaining part of this section, we shall prove the ε-convergence of traveling wave solutions by the geometric singular perturbation approach which does not require the priori convergence of wave speed. This result applies to all parameter regimes where $C_0 = 0$ and $r = \chi/d + m \neq 1$.

Theorem 7.3. *Let $(U_\varepsilon, V_\varepsilon)$ be the traveling wave solution of system (1.2) with $\varepsilon \geq 0$. Then it holds that*

$$|(U_\varepsilon, V_\varepsilon) - (U_0, V_0)| \to 0, \text{ as } \varepsilon \to 0.$$

Proof. We rewrite (2.13) as a form of slow system

$$\begin{cases} W_z = \rho = \phi(W, \rho), \\ \varepsilon \rho_z = -(c - 2\varepsilon\mu)\rho - (\varepsilon\mu^2 - c\mu + \beta)W + W^r =: \varphi(W, \rho, \varepsilon). \end{cases} \tag{7.5}$$

By introducing a fast variable $\tau = z/\varepsilon$, we convert the slow system (7.5) into a fast system as follows

$$\begin{cases} W_\tau = \varepsilon\phi(W, \rho), \\ \rho_\tau = \varphi(W, \rho, \varepsilon). \end{cases} \tag{7.6}$$

Setting $\varepsilon = 0$ in slow system (7.5), we obtain an invariant manifold $\mathcal{M}_0$ defined by

$$\mathcal{M}_0 = \left\{ (W, \rho) \mid \rho = h_0(W) = \frac{1}{c}[(c\mu - \beta)W + W^r] \right\}$$

where $W(z)$ satisfies

$$W_z = \frac{1}{c}[(c\mu - \beta)W + W^r]. \tag{7.7}$$

Now we verify that the solution $V(z) = W(z)e^{-\mu z}$ is the solution of system (1.2) with $\varepsilon = 0$. Indeed substituting this into (7.7) yields the equation $V' + \frac{\beta}{c}V - \frac{1}{c}e^{(r-1)\mu z}V^r = 0$ which is the same as equation (3.2) by noticing that $k = -(r-1)\mu$. We can solve (7.7) and obtain a solution

$$W(z) = \left[C_1 e^{(1-r)(\beta/c - \mu)} z + C_2 \right]^{\frac{1}{1-r}} \tag{7.8}$$

where C_1, C_2 are the same as those in (3.3). If $r > 1$, then $W(z)$ is a traveling wave solution, denoted by W_0, of (7.7) satisfying $W_0(z) \to C_2^{\frac{1}{1-r}}$ as $z \to -\infty$ and $W_0(z) \to 0$ as $z \to +\infty$.

Note that at any point of $\mathcal{M}_0$, $\frac{\partial\varphi(W,\rho,0)}{\partial\rho} = -c \neq 0$. Hence $\mathcal{M}_0$ is normally hyperbolic for fast system (7.6) with $\varepsilon = 0$. By Fenichel's invariant manifold theorem [31], for $\varepsilon > 0$ sufficiently small, there is a slow manifold $\mathcal{M}_\varepsilon$ that lies within $O(\varepsilon)$ neighborhood of $\mathcal{M}_0$ and is diffeomorphic to $\mathcal{M}_0$. Moreover it is locally invariant under the flow of (7.6) and can be written as

$$\mathcal{M}_\varepsilon = \{(W,\rho) \mid \rho = h_\varepsilon(W) = h_0(W) + O(\varepsilon)\}. \tag{7.9}$$

Therefore the slow system (7.5) on the manifold $\mathcal{M}_0$ can be written as

$$W_z = \phi(W, h_\varepsilon(W)) = \frac{1}{c}[(c\mu - \beta)W + W^r] + O(\varepsilon)$$

which is indeed a regular perturbation of (7.7). It is straightforward to check that the differential equation (7.8) has an one-dimensional unstable manifold near w_- and an one-dimensional stable manifold near $w_+ = 0$. The transversal intersection of these two manifolds gives the heteroclinic orbit $W_0(z)$. By the geometric singular perturbation theory (see [19] or [31]), for sufficiently small $\varepsilon > 0$, there is a heteroclinic orbit $W_\varepsilon(z)$ of (7.5) on $\mathcal{M}_\varepsilon$, which is a small perturbation of $W_0(z)$ with

$$|W_\varepsilon(z) - W_0(z)| = O(\varepsilon) \to 0 \text{ as } \varepsilon \to 0. \tag{7.10}$$

By the transformation (2.12), we have for each $z \in \mathbb{R}$

$$|V_\varepsilon(z) - V_0(z)| = |W_\varepsilon(z) - W_0(z)|e^{-\mu z} = O(\varepsilon)e^{-\mu z} \to 0 \text{ as } \varepsilon \to 0. \tag{7.11}$$

Furthermore by the transformation (2.7), it follows that for $z \in \mathbb{R}$

$$|U_\varepsilon(z) - U_0(z)| = |V_\varepsilon(z) - V_0(z)|^{\frac{\chi}{d}} e^{-\frac{c}{d}z} = O(\varepsilon^{\frac{\chi}{d}})e^{-(\mu + c/d)z} \to 0 \text{ as } \varepsilon \to 0. \tag{7.12}$$

The proof is thus complete. □

8. **Open problem.** The chemotaxis model (1.2) has very prominent applications in biomedical or biological sciences, and its mathematical richness is partially demonstrated in this paper via traveling wave solutions whose studies involve numerous mathematical methods and techniques. The peculiar feature of model (1.2) different from other chemotaxis models lies in the logarithmic chemotactic sensitivity, i.e. $\phi(v) = \ln v$, which has been successfully applied to interpret the propagation of traveling bands of bacterial chemotaxis [1, 2]. It is also mathematically necessary to reproduce the experimentally observed traveling bands [54]. Hence the study of chemotaxis models with logarithmic chemotactic sensitivity has its genuine application-driven motivation. Of course the mathematical studies of such model should not be restricted to the traveling wave solutions. When $m = 1$, there have been a few studies for the model (1.2) on other aspects, such as self-similar solutions and collapse behavior [67], the global existence and large time behavior of solutions [90, 26, 46, 84, 79], and Cauchy problem [45].

Though the study of traveling wave solutions to (1.2) has received extensive attentions since it was born, there are still numerous interesting questions remaining open. We outline below and hope to arouse readers's interests.

The first question is the stability/instability of traveling wave solutions to the model (1.2). Although there is some stability theory for the cross diffusion models (e.g., see [87]), the logarithmic sensitivity in the chemotaxis model (1.2) brings the fundamental difficulties to the study of stability of traveling wave solutions. The stability and instability results shown in section 5 are limited to particular cases. For example, the linear stability in [73] was derived for $\varepsilon = \beta = 0, 0 \leq m < 1$, and the instability of [60] was shown only for $m = \beta = 0, \varepsilon > 0$, where traveling wave

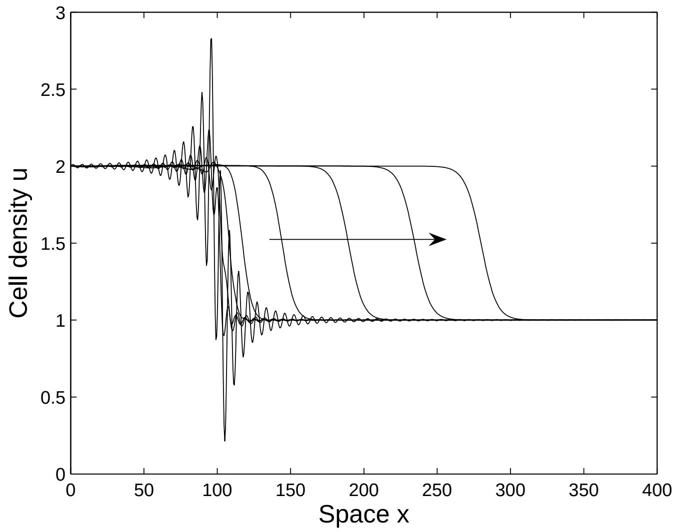


FIGURE 7. Numerical simulations of the stability of traveling wavefronts u for $u_+ > 0$ with a large initial perturbation, where $\varepsilon = 0.1, d = 2$. Other parameter and initial data are: $u_0 = 1 + \frac{1}{1+\exp(2(x-100))} + \frac{\sin x}{((x-100)/10)^2+1}, v_0 = -1 + \frac{1}{1+\exp(-2(x-100))} + \frac{\sin x}{((x-100)/10)^2+1}$, $\chi = 0.45$. The arrow indicates the propagating direction of traveling waves.

solutions (U, V) are (pulse, front). The nonlinear stability of [47, 49] was proved only for $m = 1$ and $u_+ = \beta > 0$, where traveling wave solutions (U, V) are (front, front). The paper [57] shows the instability of traveling wave profile (front, pulse) for the case $m = 1, \beta > 0$. The model (1.2) admits four families of traveling wave solutions (see Theorem 3.1 and Theorem 3.5). Basically the stability problem of traveling wave profiles (pulse, pulse) entirely remains open. Hence we could address many questions concerning the stability of traveling wave solutions, as follows:

1. Though the paper [60] has shown the linear instability of traveling wave solutions for $m = \beta = 0$, this result does not favor the experimental observation shown in Fig. 1 where the stability of traveling bands was observed. Note that when $\varepsilon = \beta = 0, 0 \leq m < 1$, it was shown in [73] that the traveling profile (pulse, front) of (1.2) is linearly stable. Hence the incorporation of the chemical diffusion (i.e. $\varepsilon > 0$) is not expected to destabilize the system. It is worthwhile to reexamine the stability of traveling wave solutions of (1.2) for $\varepsilon > 0, 0 < m < 1$ to support the experiment.
2. The nonlinear stability of wave profile (front, front) for $m = 1$ was established in [47, 49] for $u_+ = \beta > 0$. But the stability for $u_+ = 0$ remains open. Noting that approaches used in [47, 49] fail when $u_+ = 0$, new ideas are needed to tackle this case. Moreover the numerical simulation Fig. 7 shows the traveling wave solution is also stable for a large perturbation, which however leaves another open problem for the future.

3. Nonlinear stability of traveling wave solutions was established for $m = 1, \beta > 0$. A nature question is how about the nonlinear stability for $m \neq 1$. Particularly when $0 \leq m < 1$ and $\beta > 0$, the wave profile (U, V) is a (pulse, pulse). The stability of such a wave profile has not been studied previously and leads to a very interesting question to be explored.
4. When $C_0 = 0$, the parameter β plays a crucial role in determining the wave profile V, where V is a front if $\beta = 0$ and a pulse if $\beta > 0$, see Theorem 3.5. The stability results mentioned above were only for $\beta > 0$. The stability of traveling wave solutions for $\beta = 0$ entirely remains open.

The last but not least question we want to stress is the study of traveling wave solutions in two dimensions. As we show, all available results of traveling wave solutions of the model (1.2) are restricted to one dimension. The study of two-dimensional traveling waves (e.g., planar waves) largely remains open and is very deserved to be attempted in the future.

9. **Appendix: Revisit of Fisher wave problem.**

One of the cornerstones of modern mathematical biology is the Fisher equation [20] whose traveling wave solution was first extensively discussed in the pioneer work by Kolmogorov-Petrovskii-Piskunov [38]. In its general form, the Fisher equation considered in [20, 38] for a scalar function $u(x, t)$ is as follows:

$$u_t = \varepsilon u_{xx} + f(u), \ x \in \mathbb{R}, \ t \geq 0 \tag{9.1}$$

where $f(u)$ is a nonlinear function accounting for the kinetics of $u(x, t)$. Of particular interest was the study of the existence and stability of traveling wave of (9.1) where $f(u)$ satisfies the following conditions

$$\begin{aligned}
&(1)\ f(u), f'(u) \in C[0, \infty),\\
&(2)\ f(0) = f(\varrho) = 0 \text{ for some } \varrho > 0,\\
&(3)\ f(u) > 0 \text{ for all } u \in (0, \varrho) \text{ and } f(u) < 0 \text{ for } u \in (\varrho, \infty),\\
&(4)\ f'(0) > 0 \text{ and } f'(\varrho) < 0.
\end{aligned} \tag{9.2}$$

A prototypical form of $f(u)$ satisfying condition (9.2) is $f(u) = u(1-u/\varrho)$ where ϱ is called the carrying capacity. Due to the physical/biological interest (for example u denotes the population density in animal dispersal), u is assumed to be nonnegative usually.

The traveling wave solutions of (9.1)-(9.2) is a particular solution $u(x, t) = U(x-ct) = U(z)$ which satisfies a second order ordinary differential equation

$$\varepsilon U'' + cU' + f(U) = 0 \tag{9.3}$$

with boundary condition

$$U(-\infty) = \varrho, \ U(\infty) = 0 \tag{9.4}$$

where $z = x - ct$ is referred to as the wave variable and c is called the wave speed. Multiplying both sides of (9.3) by U' and integrating the result over $z \in \mathbb{R}$ yields

$$c = \frac{\int_0^\varrho f(U)dU}{\int_{\mathbb{R}} U'^2 dz} > 0. \tag{9.5}$$

Writing (9.3) as a system of first order ordinary differential equations

$$\begin{cases} U' = V, \\ V' = -\frac{c}{\varepsilon}V - \frac{1}{\varepsilon}f(U) \end{cases} \tag{9.6}$$

then the existence of solutions to (9.3)-(9.4) is equivalent to finding a trajectory for (9.6) in the $U-V$ phase plane that emanates from the unstable equilibrium $(\varrho, 0)$ to the stable equilibrium $(0,0)$, cf. [58]. The existence and asymptotic decay of solutions to (9.6) are well-known and summarized in the following theorem (cf. [58, 88, 39, 38]).

Theorem 9.1. *Let (9.2) hold. Then (9.1) has a unique (up to a translation) bounded nonnegative traveling wave solution $U(z) = U(x-ct)$ with $U' < 0$ for all $z \in \mathbb{R}$ if and only if*

$$c \geq \bar{c} = 2\sqrt{\varepsilon f'(0)}. \tag{9.7}$$

Under the condition (9.7), if we define

$$\lambda_{\pm}(\xi) = \frac{-c \pm \sqrt{c^2 - 4\varepsilon f'(\xi)}}{2\varepsilon}, \tag{9.8}$$

then $U(z) = U(x-ct)$ has the following asymptotic behavior as $|z| \to \infty$:

(1) If $c > \bar{c} = 2\sqrt{\varepsilon f'(0)}$, then

$$U(z) = \begin{cases} \varrho - Ce^{\lambda_+(\varrho)z}, & z \to -\infty \\ Ce^{\lambda_+(0)z}, & z \to \infty. \end{cases}$$

(2) If $c = \bar{c} = 2\sqrt{\varepsilon f'(0)}$, then

$$U(z) = \begin{cases} \varrho - Ce^{-\beta z} + O(e^{-2\beta z}), & z \to -\infty \\ (A - Bz)e^{-\frac{\bar{c}}{2\varepsilon}z} + O(z^2 e^{-\frac{\bar{c}}{\varepsilon}z}), & z \to \infty \end{cases}$$

where A, B, C are positive constants and

$$\beta = \frac{\bar{c} - \sqrt{\bar{c}^2 - 4\varepsilon f'(\varrho)}}{2\varepsilon}.$$

The Fisher equation (9.1) satisfying condition (9.2) is often referred to as the classical (or regular) Fisher problem, which is mostly frequently studied. But there is a scenario where the kinetic function $f(u)$ is singular at $u = 0$ satisfying

$$\begin{aligned} &(1)\ f(u) \in C[0,\infty), f'(u) \in C(0,\infty), \\ &(2)\ f(0) = f(\varrho) = 0 \text{ for some } \varrho > 0, \\ &(3)\ f(u) > 0 \text{ for all } u \in (0,\varrho) \text{ and } f(u) < 0 \text{ for } u \in (\varrho, \infty), \\ &(4)\ \lim_{u\to 0} f'(u) = \infty \text{ and } f(u) = O(u^\sigma) \text{ as } u \to 0 \text{ with } 0 < \sigma < 1, \\ &(5)\ f'(\varrho) < 0. \end{aligned} \tag{9.9}$$

Equation (9.1) subject to (9.9) is called the singular Fisher equation. The typical example of $f(u)$ satisfying (9.9) is $f(u) = u^\sigma(1-u)$ with $0 < \sigma < 1$. In this case, we have the following results [56].

Theorem 9.2. *The Fisher equation (9.1) with condition (9.9) does not have a traveling wave solution satisfying boundary condition (9.4).*

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REFERENCES

[1] J. Adler, *Chemotaxis in bacteria*, Science, **153** (1966), 708–716.

[2] J. Adler, *Chemoreceptors in bacteria*, Science, **166** (1969), 1588–1597.

[3] S. B. Ai, W. Z. Huang and Z. A. Wang, *Traveling wave solutions to a chemotaxis system with logistic growth*, preprint, 2012.

[4] W. Alt and D. A. Lauffenburger, *Transient behavior of a chemotaxis system modelling certain types of tissue inflammation*, J. Math. Biol., **24** (1987), 691–722.

[5] S. Balasuriya and G. A. Gottwald, *Wavespeed in reaction diffusion systems, with applications to chemotaxis and population pressure*, J. Math. Biol., **61** (2010), 377–399.

[6] D. Balding and D. L. McElwain, *A mathematical model of tumour-induced capillary growth*, J. Theor. Biol., **114** (1985), 53–73.

[7] R. Bellman, "Stability Theory of Differential Equations," McGraw-Hill Book Company, New York-Toronto-London, 1953.

[8] R. D. Benguria, M. C. Depassier and V. Mendez, *Minimum speed of fronts of reaction-convenction-diffusion equations*, Phys. Rev. E (3), **69** (2004), 031106, 7 pp.

[9] F. S. Berezovskaya, A. S. Novozhilov and G. P. Karev, *Families of traveling impulses and fronts in some models with cross-diffusion*, Nonlinear Analysis: Real World Applications, **9** (2008), 1866–1881.

[10] M. P. Brenner, L. S. Levitor and E. O. Budrene, *Physical mechanisms for chemotactic pattern formation by bacterial*, Biophys. J., **74** (1988), 1677–1693.

[11] E. Budrene and H. Berg, *Complex patterns formed by motile cells of Escherichia coli*, Nature, **349** (1991), 630–633.

[12] E. Budrene and H. Berg, *Dynamics of formation of symmetrical patterns by chemotactic bacteria*, Nature, **376** (1995), 49–53.

[13] S. Childress, *Chemotactic collapse in two dimensions*, in "Modelling of Patterns in Space and Time" (Heidelberg, 1983), Lect. Notes in Biomath., **55**, Springer, Berlin, (1984), 61–66.

[14] W. A. Coppel, "Stability and Asymptotic Behavior of Differential Equations," D. C. Heath and Co., Boston, 1965.

[15] L. Corrias, B. Perthame and H. Zaag, *A chemotaxis model motivated by angiogenesis*, C. R. Acad. Sci. Paris. Ser. I., **336** (2003), 141–146.

[16] L. Corrias, B. Perthame and H. Zaag, *Global solutions of some chemotaxis and angiogenesis system in high space dimensions*, Milan j. Math., **72** (2004), 1–28.

[17] F. W. Dahlquist, P. Lovely and D. E. Jr Koshland, *Qualitative analysis of bacterial migration in chemotaxis*, Nature, New Biol., **236** (1972), 120–123.

[18] Y. Ebihara, Y. Furusho and T. Nagai, *Singular solution of traveling waves in a chemotactic model*, Bull. Kyushu Inst. Tech. Math. Natur. Sci., **39** (1992), 29–38,.

[19] N. Fenichel, *Geometric singular perturbation theory of ordinary differential equations*, J. Differential Equations, **31** (1979), 53–98.

[20] R. A. Fisher, *The advance of advantageous genes*, Ann. Eugenics, **7** (1937), 355–369.

[21] M. A. Fontelos, A. Friedman and B. Hu, *Mathematical analysis of a model for the initiation of angiogenesis*, SIAM J. Math. Anal., **33** (2002), 1330–1355.

[22] M. Funaki, M. Mimura and T. Tsujikawa, *Travelling front solutions arising in the chemotaxis-growth model*, Interfaces Free Bound., **8** (2006), 223–245.

[23] R. E. Goldstein, *Traveling-wave chemotaxis*, Phys. Rev. Lett., **77** (1996), 775–778.

[24] J. Goodman, *Nonlinear asymptotic stability of viscous shock profiles for conservation laws*, Arch. Rat. Mech. Anal., **95** (1986), 325–344.

[25] S. Gueron and N. Liron, *A model of herd grazing as a traveling wave: Chemotaxis and stability*, J. Math. Biol., **27** (1989), 595–608.

[26] J. Guo, J. X. Xiao, H. J. Zhao and C. J. Zhu, *Global solutions to a hyperbolic-parabolic coupled system with large initial data*, Acta Math. Sci. Ser. B Engl. Ed, **29** (2009), 629–641.

[27] M. Holz and S. H. Chen, *Spatio-temporal structure of migrating chemotactic band of escherichia coli*, Biophys. J., **26** (1979), 243–262.

[28] D. Horstmann, *From 1970 until present: The Keller-Segel model in chemotaxis and its consequences. I.* Jahresber. Deutsch. Math.-Verein., **105** (2003), 103–165.

[29] D. Horstmann, *From 1970 until present: The Keller-Segel model in chemotaxis and its consequences. II.* Jahresber. Deutsch. Math.-Verein., **106** (2004), 51–69.

[30] D. Horstmann and A. Stevens, *A constructive approach to traveling waves in chemotaxis*, J. Nonlin. Sci., **14** (2004), 1–25.

[31] C. K. R. T. Jones, *Geometric singular perturbation theory*, in "Dynamical Systems" (ed. J. Russell) (Montecatini Terme, 1994), Lecture Notes in Math., **1609**, Springer, Berlin, 1995.

[32] E. F. Keller and G. M. Odell, *Necessary and sufficient conditions for chemotactic bands*, Math. Biosci., **27** (1975), 309–317.

[33] E. F. Keller and L. A. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theor. Biol., **26** (1970), 399–415.

[34] E. F. Keller and L. A. Segel, *Model for chemotaxis*, J. Theor. Biol., **30** (1971), 225–234.

[35] E. F. Keller and L. A. Segel, *Traveling bands of chemotactic bacteria: A theorectical analysis*, J. Theor. Biol., **26** (1971), 235–248.

[36] C. R. Kennedy and R. Aris, *Traveling waves in a simple population model involving growth and death*, Bull. Math. Biol., **42** (1980), 397–429.

[37] I. T. Kiguradse, *On the non-negative non-increasing solutions of non-linear second order differential equations*, Ann. Mat. Pura ed Appl. (4), **81** (1969), 169–192.

[38] A. N. Kolmogorov, I. G. Petrovskii and N. S. Piskunov, *A study of the equation of diffusion with increase in the quantity of matter, and its application to a biological problem*, Byul. Moskovskogo Gos. Univ., **1** (1937), 1–25.

[39] M. Kot, "Elementals of Mathematical Biology," Cambridge University Press, Cambridge, 2001.

[40] K. A. Landman, M. J. Simpson, J. L Slater and D. F. Newgreen, *Diffusive and chemotactic cellular migration: Smooth and disconcinuous traveling wave solutions*, SIAM J. Appl. Math., **65** (2005), 1420–1442.

[41] I. R. Lapidus and R. Schiller, *A model for traveling bands of chemotactic bacteria*, Biophy. J., **22** (1978), 1–13.

[42] D. Lauffenburger, C. R. Kennedy and R. Aris, *Traveling bands of chemotactic bacteria in the context of population growth*, Bull. Math. Biol., **46** (1984), 19–40.

[43] K. J. Lee, E. C. Cox and R. E. Goldstein, *Competing patterns of signaling activity in dictyostelium discoideum*, Phys. Rev. Lett., **76** (1996), 1174–1177.

[44] H. A. Levine and B. D. Sleeman, *A system of reaction diffusion equations arising in the theory of reinforced random walks*, SIAM J. Appl. Math., **57** (1997), 683–730.

[45] D. Li, T. Li and K. Zhao, *On a hyperbolic-parabolic system modeling chemotaxis*, Math. Models Methods Appl. Sci., **21** (2011), 1631–1650.

[46] T. Li, R. H. Pan and K. Zhao, *Global dynamics of a hyperbolic-parabolic model arising from chemotaxis*, SIAM. J. Appl. Math., **72** (2012), 417–443.

[47] T. Li and Z.A. Wang, *Nonlinear stability of traveling waves to a hyperbolic-parabolic system modeling chemotaxis*, SIAM J. Appl. Math., **70** (2009/10), 1522–1541.

[48] T. Li and Z.A. Wang, *Nonlinear stability of large amplitude viscous shock waves of a hyperbolic-parabolic system arising in chemotaxis*, Math. Models Methods Appl. Sci., **20** (2010), 1967–1998.

[49] T. Li and Z.A. Wang, *Asymptotic nonlinear stability of traveling waves to conservation laws arising from chemotaxis*, J. Differential Equations, **250** (2011), 1310–1333.

[50] T. Li and Z.A. Wang, *Steadily propogating waves of a chemotaxis model*, Math. Biosci., **240** (2012), 161–168.

[51] Y. Li, *The existence of traveling waves in a biological model for chemotaxis*, Acta Mathematicae Applicatae Sinica (in Chinese), **27** (2004), 123–131.

[52] C.-S. Lin, W.-M. Ni and I. Takagi, *Large amplitude stationary solutions to a chemotaxis system*, J. Differential Equations, **72** (1988), 1–27.

[53] T.-P. Liu and Y. Zeng, *Time-asymptotic behavior of wave propagation around a viscous shock profile*, Comm. Math. Phy., **290** (2009), 23–82.

[54] R. Lui and Z. A. Wang, *Traveling wave solutions from microscopic to macroscopic chemotaxis models*, J. Math. Biol., **61** (2010), 739–761.

[55] A. Matsumura and K. Nishihara, *On the stability of travelling wave solutions of a one-dimensional model system for compressible viscous gas,* Japan J. Appl. Math., **2** (1985), 17–25.

[56] P. M. McCabe, J. A. Leach and D. J. Needham, *A note on the non-existence of permanent form traveling wave solutions in a class of singular reaction-diffusion problems*, Dynamical Systems, **17** (2002), 131–135.

[57] M. Meyries, *Local well posedness and instability of travelling waves in a chemotaxis model*, Adv. Differential Equations, **16** (2011), 31–60.

[58] J. D. Murray, "Mathematical Biology. I. An Introduction," Third edition, Interdisciplinary Applied Mathematics, **17**, Springer-Verlag, New York, 2002.

[59] G. Nadin, B. Perthame and L. Ryzhik, *Traveling waves for the Keller-Segel system with fisher birth terms*, Interfaces Free Bound., **10** (2008), 517–538.

[60] T. Nagai and T. Ikeda, *Traveling waves in a chemotaxis model,* J. Math. Biol., **30** (1991), 169–184.

[61] W.-M. Ni, *Diffusion, cross-diffusion, and their spike-layer steady states*, Notices Amer. Math. Soc., **1** (1998), 9–18.

[62] R. Nossal, *Boundary movement of chemotactic bacterial populations*, Math. Biosci., **13** (1972), 397–406.

[63] A. Novick-Cohen and L. A. Segel, *A gradually slowing traveling band of chemotactic bacteria*, J. Math. Biol., **19** (1984), 125–132.

[64] G. M. Odell and E. F. Keller, *Traveling bands of chemotactic bacteria revisited*, J. Theor. Biol., **56** (1976), 243–247.

[65] H. Othmer and A. Stevens, *Aggregation, blowup and collapse: The ABC's of taxis in reinforced random walks*, SIAM J. Appl. Math., **57** (1997), 1044–1081.

[66] C. H. Ou and W. Yuan, *Traveling wavefronts in a volume-filling chemotaxis model*, SIAM Appl. Dyn. Sys., **8** (2009), 390–416.

[67] M. Rascle and C. Ziti, *Finite time blow-up in some models of chemotaxis*, J. Math. Biol., **33** (1995), 388–414.

[68] G. Rosen, *On the propogation theory for bands of chemotactic bacteria*, Math. Biosci., **20** (1974), 185–189.

[69] G. Rosen, *Analytical solution to the initial value problem for traveling bands of chemotactic bacteria*, J. Theor. Biol., **49** (1975), 311–321.

[70] G. Rosen, *Existence and nature of band solutions to generic chemotactic transport equations*, J. Theor. Biol., **59** (1976), 243–246.

[71] G. Rosen, *Steady-state distribution of bacteria chemotactic toward oxygen*, Bull. Math. Biol., **40** (1978), 671–674.

[72] G. Rosen, *Theoretical significance of the condition $\delta = 2\mu$ in bacterial chemotaxis*, Bull. Math. Biol., **45** (1983), 151–153.

[73] G. Rosen and S. Baloga, *On the stability of steadily propogating bands of chemotactic bacteria*, Math. Biosci., **24** (1975), 273–279.

[74] J. Saragosti, V. Calvez, N. Bournaveas, A. Buguin, P. Silberzan and B. Perthame, *Mathematical description of bacterial traveling pulses*, PLoS Computational Biology, **6** (2010), e1000890, 12 pp.

[75] H. Schwetlick, *Traveling waves for chemotaxis systems*, Proc. Appl. Math. Mech., **3** (2003), 476–478.

[76] T. L. Scribner, L. A. Segel and E. H. Rogers, *A numerical study of the formation and propogation of traveling bands of chemotactic bacteria*, J. Theor. Biol., **46** (1974), 189–219.

[77] L. A. Segel, *A theoretical study of receptor mechanisms in bacterial chemotaxis*, SIAM J. Appl. Math., **32** (1977), 653–665.

[78] J. A. Sherratt, E. H. Sage and J. D. Murray, *Chemical control of eukaryotic cell movement: A new model*, J. Theor. Biol., **162** (1993), 23–40.

[79] Y. S. Tao, L. H. Wang and Z. A. Wang, *Large-time behavior of a parabolic-parabolic chemotaxis model with logarithmic sensitivity in one dimension*, Discrete Contin. Dyn. Syst.-Series B, doi:10.3934/dcdsb.2013.18.821, 2013.

[80] M. J. Tindall, P. K. Maini, S. L. Porter and J. P. Armitage, *Overview of mathematical approaches used to model bacterial chemotaxis. II. Bacterial populations*, Bull. Math. Biol., **70** (2008), 1570–1607.

[81] C. Walker and G. F. Webb, *Global existence of classical solutions for a haptoaxis model*, SIAM J. Math. Anal., **38** (2006/07), 1694–1713.

[82] Z.A. Wang, *Wavefront of an angiogenesis model*, Discrete Contin. Dyn. Syst.-Series B, **17** (2012), 2849–2860.

[83] Z.A. Wang and T. Hillen, *Shock formation in a chemotaxis model*, Math. Methods. Appl. Sci., **31** (2008), 45–70.

[84] Z.A. Wang and K. Zhao, *Global dynamics and diffusion limit of a one-dimensional repulsive chemotaxis model*, Comm. Pure Appl. Anal., accpted, 2012.

[85] R. Welch and D. Kaiser, *Cell behavior in traveling wave patterns of myxobacteria*, Proc. Natl. Acad. Sci., **98** (2001), 14907–14912.

[86] M. Winkler, *Global solutions in a fully parabolic chemotaxis system with singular sensitivity*, Math. Methods Appl. Sci., **34** (2011), 176–190.

[87] Y. P. Wu, *Stability of travelling waves for a cross-diffusion model*, J. Math. Anal. Appl, **215** (1997), 388–414.

[88] J. Xin, *Front propagation in heterogeneous media*, SIAM Review, **42** (2000), 161–230.

[89] C. Xue, H. J. Hwang, K. J. Painter and R. Erban, *Travelling waves in hyperbolic chemotaxis equations*, Bull. Math. Biol., **73** (2011), 1695–1733.

[90] M. Zhang and C. J. Zhu, *Global existence of solutions to a hyperbolic-parabolic system*, Proc. Amer. Math. Soc., **135** (2006), 1017–1027.

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E-mail address: mawza@polyu.edu.hk