DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **21**, Number **3**, July **2008**

Website: http://aimSciences.org

pp. **883–906**

GLOBAL WEAK SOLUTIONS TO THE CAMASSA-HOLM EQUATION

Zhenhua Guo

Center for Nonlinear Studies and Department of Mathematics
Northwest University, Xi'an 710069, China

Mina Jiang

Department of Mathematics, Huazhong Normal University
Wuhan, 430079, China

Zhian Wang

Department of Mathematical and Statistical Sciences
632CAB, University of Alberta, Edmonton
Alberta T6G 2G1, Canada

Gao-Feng Zheng

Department of Mathematics, Huazhong Normal University
Wuhan, 430079, China

Abstract. The existence of a global weak solution to the Cauchy problem for a one-dimensional Camassa-Holm equation is established. In this paper, we assume that the initial condition $u_0(x)$ has end states $u_\pm$, which has much weaker constraints than that $u_0(x) \in H^1(\mathbb{R})$ discussed in [30]. By perturbing the Cauchy problem around a rarefaction wave, we obtain a global weak solution as a limit of viscous approximation under the assumption $u_- < u_+$.

1. Introduction.

In this paper, we are concerned with the global existence of weak solutions to the Camassa-Holm equation

$$\partial_t u - \partial_x^2 \partial_t u + 3u\partial_x u = 2\partial_x u \partial_x^2 u + u\partial_x^3 u, \tag{1.1}$$

with initial data

$$u(0,x) = u_0(x) \to u_\pm \quad \text{as} \quad x \to \pm\infty, \tag{1.2}$$

which is formally equivalent to a dispersive shallow water equation [1]:

$$\begin{cases} \partial_t u + u\partial_x u + \partial_x P = 0, \qquad t>0, x \in \mathbb{R} \\ P(t,x) = \dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty} e^{-|x-y|}\left(u^2 + \frac{1}{2}(\partial_x u)^2\right)(t,y)dy, \end{cases} \tag{1.3}$$

where u is the fluid velocity in the x direction(or equivalently the height of water's free surface above a flat bottom). The equation (1.1) is completely integrable (see [15], [8] for the periodic case and [3], [10], [13] for the non-periodic case). A few non-linear dispersive and wave equations are lucky enough to be completely integrable

2000 *Mathematics Subject Classification.* Primary: 35D05, 35L60.

Key words and phrases. Camassa-Holm equation, global weak solution, vanishing viscosity method, Young measure.

in the sense that there exist a Lax pair formulation of the equation. This means in particular that they enjoy infinitely many conservation laws. In many cases these conservation laws provide control on high Sobolev norms which seems to be quite an exceptional event. For the general discussion on complete integrability in infinite dimensions, we refer to [19] for a detailed exposition. Quite a number of PDEs have been discovered to be completely integrable. However, most of them still remain obscure due to the lack of any physical significance. The equation

$$\partial_t u + 2K\partial_x u - \partial_x^2\partial_t u + 3u\partial_x u = 2\partial_x u\partial_x^2 u + u\partial_x^3 u, \tag{1.4}$$

which was discovered by Funchssteiner and Fokas [20], has enjoyed such obscurity. But a few years ago, this equation was rederived by Camassa and Holm [6] using an asymptotic expansion directly in the Hamiltanian for Euler's equations in the shallow water regime. They showed that (1.4) is Bi-Hamiltanian, i.e., it can be expressed in Hamiltanian form in two different ways. The novelty of Camassa and Holm's work was that they gave a physical derivation for (1.4) and showed that, for the special case $K = 0$, (1.4) possessed a solitary waves of the form $c\exp(-|x-ct|)$ with discontinuous first derivatives, which they named "peakon"(travelling wave solutions with a corner at their peak). More importantly, the peakons are orbitally stable (cf. [17])which means that the shape of the peakons is stable so that these wave patterns are physically recognizable, moreover, these peakons are solitons (cf. [16], [4]). Another feature of the Camassa-Holm equation is that it can be treated as a generalization of the Benjamin-Bona-Mahoney(BBM) equation or the Korteweg-de Vries(KdV) equation in some sense(See [6]). These three various equations all gave the good and consistent approximation for the full inviscid water wave equation in the small-amplitude and shallow-water regime. However, the Camassa-Holm equation has several important features that distinguish it from the BBM and KdV equations. Namely, while all solutions to BBM and KdV are global, the Camassa-Holm equation has global smooth solutions as well as smooth solutions that blow up (cf. [12]). Moreover, the only way singularities can develop in a solution corresponding to a smooth initial data decaying at infinity is in the form wave breaking: the slope u_x becomes unbounded while u stays bounded (cf. [9]) (this elaborate statement can be also found in [2] and [30]). Since the physical significance that (1.4) exhibits, which was discovered by Camassa and Holm, (1.4) has attracted a broad interest from researchers(see [1, 3, 4, 11, 15]). Cooper and Shepard [18] derived a variational approximation to the solitary waves of (1.4) for general K. In [7], the numerical solutions of time-dependent form and a discussion of the Camassa-Holm equation as a Hamiltanian system was presented. Boyd [2] derived a perturbation series for general K which converges even at the peak limit and gave three analytical representations for the spatially periodic generalization of the peakon called "Coshoidal wave". In [26], zero curvature formulation are given for the "dual hierarchies" of standard soliton equation hierarchies including the Camassa-Holm equation hierarchy. As pointed out exactly in [14], (1.1) represents the equation for geodesics on the diffeomorphism group.

In the paper [30], Xin and Zhang obtained the global-in-time existence of weak solutions to Camassa-Holm equations for the special case $K = 0$ with initial data $u(0,x) = u_0(x) \in H^1(\mathbb{R})$. Recently, Bressan and Constantin in [5] obtained the unique global conservative solutions of the Camassa-Holm equation with initial data in H^1. We observe that the assumption $u_0(x) \in H^1(\mathbb{R})$ implies $u_0(x) \to 0$ as $x \to \pm\infty$, which is a rigorous constraint in applications. The aim of this paper is to establish the global existence of the weak solution to problem (1.1)-(1.2) with initial data $u_0(x) \to u_\pm$ as $x \to \pm\infty$, where the initial data with end states $u_\pm$ has much weaker constraints than that $u_0(x) \in H^1(\mathbb{R})$. Toward this end, we assume that the limits $u_\pm$ of the initial data $u_0(x)$ at $x = \pm\infty$ satisfy $u_- < u_+$, i.e., a hyperbolic wave is a rarefaction wave. Under this circumstance, we perturb the Cauchy problem (1.1), (1.2) around the rarefaction wave $w^R(x/t)$ which satisfies the Riemann problem (2.1). Then we reformulate the Cauchy problem (1.1), (1.2) to a new Cauchy problem (3.1) and prove the global existence of a weak solution to this new problem, and thus a global weak solution of original problem (1.1), (1.2) follows. Moreover, we study the asymptotic behavior of the solution of problem (1.1), (1.2) and show that the solution tends to a rarefaction wave as $t \to \infty$.

Before giving the precise statements of the main results, we introduce the definition of a weak solution to the Cauchy problem (1.1)-(1.2) similarly as in [30]:

Definition 1.1. A continuous function $u = u(t,x)$ is said to be a global weak solution to the Cauchy problem (1.1)-(1.2) if

(1) $u(t,x) - \phi(t,x) \in C([0,\infty) \times \mathbb{R}) \cap L^\infty([0,\infty); H^1(\mathbb{R}))$ and

$$\|u-\phi\|_{H^1(\mathbb{R})} \le C(\|u_0-\phi_0\|_{H^1(\mathbb{R})}+1), \quad \forall\, t>0,$$
$$u_0(x) \to u_\pm \text{ as } x \to \pm\infty,$$

where C is a positive constant depending only on u_+, u_- and $\phi(t,x)$ satisfies

$$\begin{cases} \partial_t\phi + 3\phi\partial_x\phi = 0, \\ \phi(0,x) = \phi_0(x) = \dfrac{u_+ + u_-}{2} + \dfrac{u_+ - u_-}{2} K_q \displaystyle\int_0^{\sigma x} (1+y^2)^{-q} dy. \end{cases} \tag{1.5}$$

Here $\sigma > 0$ is an arbitrary constant and K_q is chosen such that $K_q \int_0^\infty (1+y^2)^{-q}\, dy = 1$ for each $q > 1/2$.

(2) $u(t,x)$ satisfies equation (1.1) in the sense of distributions and takes on the initial data pointwise.

The main result of this paper is as follows.

Theorem 1.1. *Suppose $u_- < u_+$, $u_0 - \phi_0 \in H^1(\mathbb{R})$ and ϕ_0 given in (1.5). Then the Cauchy problem (1.1)-(1.2) has a global weak solution. Furthermore, the global weak solution $u = u(t,x)$ satisfies*

$$\lim_{t\to+\infty} |u(t,x) - \phi(t,x)| = 0 \tag{1.6}$$

for all $x \in \mathbb{R}$.

Theorem 1.1 can be regarded as the extension of Theorem 1.2 in [30] as $u_- = u_+ = 0$. Our main goal is to study the stability of the simple wave to the Camassa-Holm Equations. That is, one shall be interested in the following initial-boundary problem

$$\begin{cases} \partial_t u - \partial_x^2\partial_t u + 3u\partial_x u = 2\partial_x u\partial_x^2 u + u\partial_x^3 u, & x \in (0,\infty) \\ u(t,0) = u_-, t \ge 0, & \\ u(0,x) = u_0(x) \to u_+, & \text{as } \ x \to +\infty. \end{cases} \tag{1.7}$$

Since the solution of (1.7) has a boundary at $x = 0$, the signs of the characteristic speeds $u_\pm$ divide the asymptotic state into five cases (c.f. [23]):

$$\begin{cases} (1) \;\; u_- < u_+ < 0, \\ (2) \;\; u_- < u_+ = 0, \\ (3) \;\; u_- < 0 < u_+, \\ (4) \;\; 0 = u_- < u_+, \\ (5) \;\; 0 < u_- < u_+. \end{cases} \tag{1.8}$$

When $u_- < 0$ and $u_+ = 0$, we observe that $\phi(x) = u_- e^{-x}$ is a stationary solution to (1.7). The interesting problem is to study the asymptotic behavior of solutions to the system (1.7), i.e., one can show that the initial-boundary problem (1.7) admits a unique global solution $u(t, x)$ which converges, as $t \to +\infty$, i) to the stationary solution or ii) to the rarefaction wave of the Burgers equation. However, the key point is to obtain the existence of the weak solutions. This is the main purpose of this paper. The stability of the simple wave to the Camassa-Holm Equations will be discussed in further work.

The main idea of proving Theorem 1.1 closely follows the method developed by Xin-Zhang in [30]. The major difference is that our assumption on initial datum is not in $H^1(\mathbb{R})$, not even in $L^2(\mathbb{R})$. One key observation is that we can perturb the initial datum with a rarefaction wave define by (2.1) because of $u_- < u_+$. The difference quantity, v, between the solution u to the original problem and the rarefaction wave ϕ satisfies another new "shallow-water-like" equation (see (3.1)). The initial datum of this difference converges to zero as $|x| \to +\infty$. The crucial element is to show that this new problem has a global weak solution. Note that the rarefaction wave ϕ is not in $L^1(\mathbb{R})$ for any $t > 0$, although it is smooth enough. Some extra efforts have to be made to deal with the complexity of the appearance of ϕ (when $\phi \equiv 0$ this new problem is nothing but the problem in [30]). We obtain the global existence result by using the vanishing viscosity method as performed in [30]. It turns out that the Young measures play a role in passaging limits of the approximate viscous solutions v^ε. A further question can be addressed: is this global weak solution u close to the rarefaction wave ϕ in some sense? Indeed, the positivity of $\partial_x \phi$ and the estimates for the approximation v^ε gives us the integrability of $v(\cdot, x)$ for a.e. x. This, combined with the control of $\partial_t v$, in turn shows that $v(t, x)$, i.e., $u(t, x) - \phi(t, x)$ converges to zero as $t \to +\infty$ for all x.

The rest of this paper is organized as follows. In section 2 we establish some preliminary estimates for the smooth rarefaction wave ϕ. In section 3, we reformulate the original problem to a new equivalent Cauchy problem and establish the global existence of this new problem with viscosity. In section 4, we show the existence of a global existence of problem (1.1) and (1.2) and examine the asymptotic behavior of solutions.

Notation: Hereafter, we use C to denote generic constants without any confusion, which may change from line to line. When the dependence of the constant on some index or a function is important, we highlight it in the notation. $L^p = L^p(\mathbb{R}) (1 \le p \le \infty)$ denotes usual Lebesgue space with the norm

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}} |f(x)|^p \, dx \right)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

$$\|f\|_{L^\infty} = \operatorname*{ess\,sup}_{x \in \mathbb{R}} |f(x)|,$$

and the integral region $\mathbb{R}$ will be omitted if it does not cause any confusion. In the double integral, the differential $dxdt$ will be omitted often for the simplicity of the presentation, i.e., the integral $\int_0^t \int_{\mathbb{R}} f(t,x)dxdt$ is briefly denoted by $\int_0^t \int f(t,x)$.

2. **Preliminaries.**

To investigate the Cauchy problem (1.1) and (1.2), we first consider the following Riemann problem for non-viscous Burgers equation

$$\begin{cases} \partial_t w^R + 3w^R \partial_x w^R = 0 \ , \\ w^R(0,x) = w_0^R(x) = \begin{cases} u_- \ , & x < 0 \ , \\ u_+ \ , & x > 0 \ . \end{cases} \end{cases} \tag{2.1}$$

It is well known(for more details, see Smoller [27]), when $u_- < u_+$, that the solution of the Riemann problem (2.1) is the center rarefaction wave $w^R(t,x) = w^R(x/t)$ which reads

$$w^R(x/t) = \begin{cases} u_-, & x \le 3u_- t, \\ x/3t, & 3u_- t < x < 3u_+ t, \\ u_+, & x \ge 3u_+ t. \end{cases}$$

It is clear that the solution $w^R(x/t)$ is discontinuous. Using a similar approach applied in [24] and [28], the smooth solution of the Riemann solution $w^R(t,x)$ can be approximated by $\phi(t,x)$ which satisfies

$$\begin{cases} \partial_t \phi + \partial_x \left(\dfrac{3}{2}\phi^2 \right) = 0, \\ \phi(0,x) = \phi_0(x) = \dfrac{u_+ + u_-}{2} + \dfrac{u_+ - u_-}{2} K_q \displaystyle\int_0^{\sigma x} (1+y^2)^{-q} dy, \end{cases} \tag{2.2}$$

with $\sigma > 0$ an arbitrary constant and K_q chosen such that for each $q > \dfrac{1}{2}$ it holds that $K_q \displaystyle\int_0^{\infty} (1+y^2)^{-q} \ dy = 1$. It is straightforward to check that $\phi_0(x) \to u_\pm$ as $x \to \pm\infty$.

Since $\phi_0(x)$ is monotonically increasing, the method of the characteristic curve allows a unique smooth solution in all time. Then we have the following lemma.

Lemma 2.1. *There exists a unique smooth solution $\phi(t,x)$ to problem (2.2) which has the following properties by setting $\tilde{u} = \dfrac{1}{2}(u_+ - u_-) > 0$:*
(i) $u_- < \phi(t,x) < u_+, \partial_x \phi(t,x) > 0$ *for all* $(t,x) \in [0,\infty) \times \mathbb{R}$;

(ii) For any p with $1 \le p \le \infty$, there exists a constant $C_{p,q}$ depending on p,q such that

$$\begin{aligned} \|\partial_x \phi(t)\|_{L^p} &\le C_{p,q} \ \min(\sigma^{1-\frac{1}{p}} \tilde{u}, \tilde{u}^{\frac{1}{p}} t^{-1+\frac{1}{p}}), \\ \|\partial_x^2 \phi(t)\|_{L^p} &\le C_{p,q} \ \min\left(\sigma^{2-\frac{1}{p}} \tilde{u}, \sigma^{(1-\frac{1}{p})(1-\frac{1}{2q})} \tilde{u}^{-\frac{p-1}{2pq}} t^{-1-\frac{p-1}{2pq}} \right), \\ \|\partial_x^3 \phi(t)\|_{L^p} &\le C_{p,q} \ \min\left(\sigma^{3-\frac{1}{p}} \tilde{u}^{\frac{1}{p}}, a(\sigma, \tilde{u}, t) \right), \\ \|\partial_x^4 \phi(t)\|_{L^p} &\le C_{p,q} \ \min\left(\sigma^{4-\frac{1}{p}} \tilde{u}, b(\sigma, \tilde{w}, t) \right), \end{aligned}$$

where

$$\begin{aligned} a(\sigma,\tilde{u},t) &= \sigma^3\tilde{u}(1+\sigma\tilde{u}t)^{\frac{1}{p}-4}+\sigma^{2(1-\frac{1}{p})(1-\frac{1}{2q})}\tilde{u}^{-\frac{p-1}{pq}}t^{-\frac{1}{p}-(1-\frac{1}{p})(1+\frac{1}{q})} \\ &\quad +\sigma^{(2-\frac{1}{p})(1-\frac{1}{2q})}\tilde{u}^{-\frac{2p-1}{2pq}}t^{-1-\frac{2p-1}{2pq}}, \\ b(\sigma,\tilde{u},t) &= \sigma^3\tilde{u}(1+\sigma\tilde{u}t)^{\frac{1}{p}-5}+\sigma^{(3-\frac{2}{p})(1-\frac{1}{2q})}\tilde{u}^{-\frac{3p-2}{2pq}}t^{-(1+\frac{3}{2q})+\frac{1}{pq}} \\ &\quad +\sigma^{(3-\frac{1}{p})(1-\frac{1}{2q})}\tilde{u}^{-\frac{3p-1}{2pq}}t^{-1-\frac{3p-1}{2pq}}; \end{aligned}$$

(*iii*) *There exists a constant* C_q *depending on* q *such that*

$$\begin{aligned} &\int\left|\frac{(\partial_x^2\phi)^2}{\partial_x\phi}(t,x)\right|\,dx=\left\|\frac{(\partial_x^2\phi)^2}{\partial_x\phi}(t)\right\|_{L^1}\le C_q\ \min(\sigma^2\tilde{u},\sigma^{1-\frac{1}{2q}}\tilde{u}^{-\frac{1}{2q}}t^{-1-\frac{1}{2q}}), \\ &\int\left|\frac{(\partial_x^3\phi)^2}{\partial_x\phi}(t,x)\right|\,dx=\left\|\frac{(\partial_x^3\phi)^2}{\partial_x\phi}(t)\right\|_{L^1}\le C_q\ \min\left(\tilde{u}(\sigma^2+\sigma^4),\beta(q,\sigma,\tilde{u})\right), \\ &\int\left|\frac{(\partial_x^2\partial_t\phi)^2}{\partial_x\phi}(t,x)\right|\,dx=\left\|\frac{(\partial_x^2\partial_t\phi)^2}{\partial_x\phi}(t)\right\|_{L^1}\le C_q\ \min\left((\tilde{u}+\tilde{u}^3)\sigma^4,\gamma(q,\sigma,\tilde{u})\right), \end{aligned}$$

where

$$\begin{aligned} &\beta(q,\sigma,\tilde{u})=\sigma^{2-\frac{1}{q}}\tilde{u}^{-\frac{1}{q}}t^{-1-\frac{1}{q}}+\sigma^4\tilde{u}(1+\tilde{u}\sigma t)^{-6}+\ \sigma^{3-\frac{3}{2q}}\tilde{u}^{-\frac{3}{2q}}t^{-1-\frac{3}{2q}}; \\ &\gamma(q,\sigma,\tilde{u})=\sigma^{2-\frac{1}{q}}\tilde{u}^{-\frac{1}{q}}t^{-1-\frac{1}{q}}+\sigma^4\tilde{u}(1+\tilde{u}\sigma t)^{-6}+\sigma^2\tilde{u}t^{-2}+\sigma^{5-\frac{3}{2q}}\tilde{u}^{2-\frac{3}{2q}}t^{-1-\frac{3}{2q}}; \end{aligned}$$

(*iv*) $\|\partial_t^l\partial_x^k\phi\|_{L^\infty}\le C|w_+-w_-|^{l+k+1},\ l,k\ge 0,\ l+k\le 4$*;*

(*v*) $\sup_R|\phi(t,x)-w^R(x/t)|\to 0$ *as* $t\to\infty$*.*

Proof. The proof of the global existence of solutions to the Cauchy problem of conservation law (2.2) is very routine, which follows from the method of characteristics directly. It only remains to prove the last inequality in (*iii*) since the rest of estimates have been proved in [24, 25, 28, 32].

Indeed, the method of characteristic curve yields for all time t,

$$\phi(t,x)=\phi_0(x_0(t,x)), \tag{2.3}$$

where $x_0(t,x)$ is given by the relation

$$x=x_0(t,x)+3\phi_0(x_0(t,x))t. \tag{2.4}$$

Noting that

$$\frac{\partial x_0(t,x)}{\partial x}=\frac{1}{1+3\phi_0'(x_0)t},\quad \frac{\partial x_0(t,x)}{\partial t}=-\frac{3\phi_0(x_0)}{1+3\phi_0'(x_0)t}, \tag{2.5}$$

we have from (2.3),(2.4) and (2.5),

$$\partial_x\phi(t,x)=\frac{\phi_0'(x_0)}{1+3\phi_0'(x_0)t},\quad \partial_x^2\phi(t,x)=\frac{\phi_0''(x_0)}{(1+3\phi_0'(x_0)t)^3}, \tag{2.6}$$

and

$$\partial_x^2\partial_t\phi(t,x)=-\frac{3\phi_0(x_0)\phi_0'''(x_0)}{(1+3\phi_0'(x_0)t)^4}-\frac{9\phi_0'(x_0)\phi_0''(x_0)}{(1+3\phi_0'(x_0)t)^4}+\frac{27\phi_0(x_0)(\phi_0''(x_0))^2t}{(1+3\phi_0'(x_0)t)^5}, \tag{2.7}$$

where the prime means the derivative with respect to x_0.

From (2.2), one has

$$\phi_0'(x_0) = K_q\tilde{u}(1+(\sigma x_0)^2)^{-q}\sigma \le K_q\tilde{u}\sigma, \tag{2.8}$$

$$|\phi_0''(x_0)| \le 2q\sigma(K_q\tilde{u}\sigma)^{-\frac{1}{2q}}|\phi_0'(x_0)|^{1+\frac{1}{2q}}, \tag{2.9}$$

$$|\phi_0'''(x_0)| \le 2q(2q+3)\sigma^2(\sigma K_q\tilde{u})^{-\frac{1}{q}}|\phi_0'(x_0)|^{1+\frac{1}{q}}. \tag{2.10}$$

In addition, when $x_0 \ge 1$, we have

$$|\phi_0''(x_0)| \ge 2q\sigma^2(K_q\tilde{u}\sigma)^{-\frac{1}{q}}|\phi_0'(x_0)|^{1+\frac{1}{q}}. \tag{2.11}$$

Thus the Cauchy-Schwartz inequality gives the following estimate

$$\begin{aligned}\int\left|\frac{(\partial_x^2\partial_t\phi)^2}{\partial_x\phi}\right| dx &= \int\left|\frac{1+3\phi_0'(x_0)t}{\phi_0'(x_0)}\right|\Bigg(-\frac{3\phi_0(x_0)\phi_0'''(x_0)}{(1+3\phi_0'(x_0)t)^4}-\frac{9\phi_0'(x_0)\phi_0''(x_0)}{(1+3\phi_0'(x_0)t)^4}\\ &\quad+\frac{27\phi_0(x_0)(\phi_0''(x_0))^2t}{(1+3\phi_0'(x_0)t)^5}\Bigg)^2 \left(\frac{\partial x_0}{\partial x}\right)^{-1} dx_0\\ &\le C\int\frac{|\phi_0(x_0)\phi_0'''(x_0)|^2}{\phi_0'(x_0)(1+3\phi_0'(x_0)t)^6}\,dx_0 + C\int\frac{\phi_0'(x_0)\phi_0''(x_0)^2}{(1+3\phi_0'(x_0)t)^6}dx_0\\ &\quad+C\int\frac{|\phi_0(x_0)\phi_0''(x_0)^2|^2t^2}{\phi_0'(x_0)(1+3\phi_0'(x_0)t)^8}\,dx_0,\end{aligned} \tag{2.12}$$

where we used the variable transformation $dx_0 = \frac{\partial x_0}{\partial x}dx$.

Next we are going to estimate the three integrals on the right hand side of (2.12) respectively. To estimate the first integral, we break the integral domain $\mathbb{R}$ into two parts in order to use the inequality (2.11). That is

$$\int\frac{|\phi_0(x_0)\phi_0'''(x_0)|^2}{\phi_0'(x_0)(1+3\phi_0'(x_0)t)^6}\,dx_0 = \int_{|x_0|<1}+\int_{|x_0|\ge 1}. \tag{2.13}$$

Then we proceed to estimate the integrals on the right hand side of (2.13). Indeed, noting that $\phi_0(x_0)$ is bounded, we get from (2.8) and (2.10)

$$\begin{aligned}&\int_{|x_0|<1}\frac{|\phi_0(x_0)\phi_0'''(x_0)|^2}{\phi_0'(x_0)(1+3\phi_0'(x_0)t)^6}\,dx_0\\ \le\;& C_q\sigma^{4-\frac{2}{q}}\tilde{u}^{-\frac{2}{q}}\int_{|x_0|<1}\frac{|\phi_0'(x_0)|^{1+\frac{2}{q}}}{(1+3\phi_0'(x_0)t)^6}\,dx_0\\ \le\;& C_q\sigma^{4-\frac{2}{q}}\tilde{u}^{-\frac{2}{q}}(1+\tilde{u}K_q\sigma t)^{-6}\int_{|x_0|<1}|\phi_0'(x_0)|^{1+\frac{2}{q}}\,dx_0\\ \le\;& C_q\sigma^4\tilde{u}(1+\tilde{u}\sigma)^{-6},\end{aligned} \tag{2.14}$$

the last inequality resulting from the fact that $\int_{\mathbb{R}}(\phi_0'(x_0))^r\,dx_0 \le C_{r,q}\sigma^{r-1}\tilde{u}^r$, which is clear from (2.8).

Now we introduce the variable transformation $y = \phi_0'(x_0)t$ and deduce from (2.10) and (2.11) that

$$\begin{aligned}
&\int_{|x_0|<1} \frac{|\phi_0'(x_0)\phi_0'''(x_0)|^2}{\phi_0'(x_0)(1+3\phi_0'(x_0)t)^6}\ dx_0 \\
\leq\ & C\int_0^{\phi_0'(1)t} \frac{|\phi_0'''(x_0)|^2}{y(1+3y)^6|\phi_0''(x_0)|}\ dy \\
\leq\ & C\left(2q\sigma^2(K_q\tilde{u}\sigma)^{-\frac{1}{q}}\right)^{-1}\left(2q(2q+3)\sigma^2(K_q\tilde{u}\sigma)^{-\frac{1}{q}}\right)^2\int_0^\infty \frac{|\phi_0'(x_0)|^{1+\frac{1}{q}}}{y(1+3y)^6}\ dy \\
\leq\ & C_q\sigma^{2-\frac{1}{q}}\tilde{u}^{-\frac{1}{q}}t^{-1-\frac{1}{q}}\int_0^\infty (1+3y)^{-6+\frac{1}{q}}\ dy \\
\leq\ & C_q\sigma^{2-\frac{1}{q}}\tilde{u}^{-\frac{1}{q}}t^{-1-\frac{1}{q}}.
\end{aligned} \tag{2.15}$$

Thus the combination of (2.14) and (2.15) completes the estimate for the first integral on the right hand side of (2.12). Next we estimate the second integral on the right hand side of (2.12). In fact, it follows from (2.9) that

$$\begin{aligned}
&\int \frac{\phi_0'(x_0)\phi_0''(x_0)^2}{(1+3\phi_0'(x_0)t)^6}dx_0 \\
\leq\ & Ct^{-2}\int_0^{\phi_0'(0)} \frac{y|\phi_0''(x_0)|}{(1+3y)^6}dy \\
\leq\ & 2Cq\sigma(K_q\tilde{u}\sigma)^{-\frac{1}{2q}}t^{-2}\int_0^\infty \frac{y|\phi_0'(x_0)|^{1+\frac{1}{2q}}}{(1+3y)^6}dy \\
\leq\ & C_q\sigma^2\tilde{u}t^{-2}\int_0^\infty \frac{y}{(1+3y)^{-6}}dy \\
\leq\ & C_q\sigma^2\tilde{u}t^{-2}.
\end{aligned} \tag{2.16}$$

Furthermore, we estimate the third integral on the right hand side of (2.12) by

$$\int \frac{|\phi_0(x_0)\phi_0''(x_0)^2|^2t^2}{\phi_0'(x_0)(1+3\phi_0'(x_0)t)^8}\ dx_0 \tag{2.17}$$

$$\begin{aligned}
\leq\ & (K_q\sigma\tilde{u})^2t^2\int_0^\infty \frac{|\phi_0''(x_0)|^3}{y(1+3y)^8}dy \\
\leq\ & (K_q\sigma\tilde{u})^2t^2\int_0^\infty \frac{|\phi_0''(x_0)|^3}{y^{3+\frac{3}{2q}}(1+3y)^{6-\frac{3}{2q}}}dy \\
\leq\ & C_q\sigma^{5-\frac{3}{2q}}\tilde{u}^{2-\frac{3}{2q}}t^{-1-\frac{3}{2q}}\int_0^\infty (1+3y)^{-6+\frac{3}{2q}}dy \\
\leq\ & C_q\sigma^{5-\frac{3}{2q}}\tilde{u}^{2-\frac{3}{2q}}t^{-1-\frac{3}{2q}}.
\end{aligned} \tag{2.18}$$

Substitution of (2.14),(2.15),(2.16) and (2.18) into (2.12) yields

$$\int \left|\frac{(\partial_x^2\partial_t\phi)^2}{\partial_x\phi}\right|\ dx \leq \gamma(q,\sigma,\tilde{u}), \tag{2.19}$$

where

$$\gamma(q,\sigma,\tilde{u}) = \sigma^{2-\frac{1}{q}}\tilde{u}^{-\frac{1}{q}}t^{-1-\frac{1}{q}} + \sigma^4\tilde{u}(1+\tilde{u}\sigma t)^{-6} + \sigma^2\tilde{u}t^{-2} + \sigma^{5-\frac{3}{2q}}\tilde{u}^{2-\frac{3}{2q}}t^{-1-\frac{3}{2q}}.$$

On the other hand, since $\phi_0'(x_0) > 0$, it follows from (2.12) that

$$\begin{aligned}
&\int \left| \frac{(\partial_x^2 \partial_t \phi)^2}{\partial_x \phi} \right| \, dx \\
\leq \quad & C \int \frac{|\phi_0'''(x_0)|^2}{\phi_0'(x_0)} dx_0 + C \int |\phi_0'(x_0)||\phi_0''(x_0)|^2 dx_0 + C \int \frac{|\phi_0''(x_0)|^4}{\phi_0'(x_0)^3} dx_0 \\
\leq \quad & C_q \sigma^{4-\frac{2}{q}} \tilde{u}^{-\frac{2}{q}} \int (\phi_0'(x_0))^{1+\frac{2}{q}} dx_0 + C_q \sigma^{3-\frac{1}{q}} \tilde{u}^{1-\frac{1}{q}} \int (\phi_0'(x_0))^{2+\frac{1}{q}} dx_0 \\
& + C_q \sigma^{4-\frac{2}{q}} \tilde{u}^{-\frac{2}{q}} \int (\phi_0'(x_0))^{1+\frac{2}{q}} dx_0 \\
\leq \quad & C_q \sigma^4 \tilde{u} + C_q \sigma^4 \tilde{u}^3. \qquad (2.20)
\end{aligned}$$

Putting (2.19) and (2.20) together completes the proof for the last inequality in (iii) of Lemma 2.1. ☐

Corollary 2.2. *Let $\sigma = \tilde{u}, q = 1$ in the Lemma 2.1, then the solution $\phi(t,x)$ to (2.2) satisfies the following properties:*

(i) $u_- < \phi(t,x) < u_+,\ \partial_x \phi(t,x) > 0$ *for all* $(t,x) \in [0,+\infty) \times \mathbb{R}$*;*

(ii) *For any p with $1 \leq p \leq \infty$, there exists a constant C_p depending on p such that*

$$\begin{aligned}
\|\partial_x \phi(t)\|_{L^p} &\leq h(\tilde{u}) C_p (1+t)^{-1+\frac{1}{p}}, \\
\|\partial_x^2 \phi(t)\|_{L^p} &\leq h(\tilde{u}) C_p (1+t)^{-\frac{3p-1}{2p}}, \\
\|\partial_x^3 \phi(t)\|_{L^p} &\leq h(\tilde{u}) C_p (1+t)^{-2+\frac{1}{p}}, \\
\left\| \frac{(\partial_x^2 \phi)^2}{\partial_x \phi}(t) \right\|_{L^1} &\leq C h(\tilde{u}) (1+t)^{-\frac{3}{2}}, \\
\left\| \frac{(\partial_x^3 \phi)^2}{\partial_x \phi}(t) \right\|_{L^1} &\leq C h(\tilde{u}) (1+t)^{-2}, \\
\left\| \frac{(\partial_x^2 \partial_t \phi)^2}{\partial_x \phi}(t) \right\|_{L^1} &\leq C h(\tilde{u}) (1+t)^{-2},
\end{aligned}$$

where $h(\tilde{u})$ is a function of $\tilde{u}$ and satisfies $\lim_{\tilde{u} \to 0} h(\tilde{u}) = 0$;

(iii) $\|\partial_t^l \partial_x^k \phi\|_{L^\infty} \leq C|u_+ - u_-|^{l+k+1},\ l,k \geq 0,\ l+k \leq 4$*;*

(iv) $\sup_{\mathbb{R}} |\phi(t,x) - w^R(x/t)| \to 0$ *as* $t \to \infty$ *.*

3. **Global existence for the normalized problem with viscosity.** Let $v = u - \phi$, we can recast the Cauchy problem (1.1) to the following reformulated Cauchy problem

$$\left\{ \begin{array}{l}
\partial_t v - \partial_x^2 \partial_t v + \frac{3}{2} \partial_x \left((v+\phi)^2 - \phi^2 \right) \\
= 2(v+\phi) \partial_x^2 (v+\phi) + (v+\phi) \partial_x^3 (v+\phi) + \partial_x^2 \partial_t \phi, \\
v|_{t=0} = v_0(x) = u_0(x) - \phi_0(x) \to 0, \quad x \to \pm\infty.
\end{array} \right. \qquad (3.1)$$

For the convenience of presentation, we define $f(v) = \frac{3}{2}\left((v+\phi)^2 - \phi^2\right)$ and

$$g(v, \partial_x v, \partial_x^2 v, \partial_x^3 v, \partial_x^2 \partial_t v) = 2\partial_x (v+\phi) \partial_x^2 (v+\phi) + (v+\phi) \partial_x^3 (v+\phi) + \partial_x^2 \partial_t \phi + \partial_x^2 \partial_t v.$$

Then (3.1) takes the following form

$$\begin{cases} \partial_t v + \partial_x(f(v)) = g(v, \partial_x v, \partial_x^2 v, \partial_x^3 v, \partial_x^2\partial_t v), \\ v|_{t=0} = v_0(x) \to 0, \quad x \to \pm\infty. \end{cases} \tag{3.2}$$

It is formally equivalent to the following problem

$$\begin{cases} \partial_t v + v\partial_x v + \partial_x(P + \phi v - \phi^2) = 0, \\ \quad (t,x) \in [0,\infty) \times \mathbb{R}, \\ (-\partial_x^2 + I)P = \frac{1}{2}(\partial_x v + \partial_x \phi)^2 + (v+\phi)^2, \\ v(0,x) = v_0(x), \end{cases} \tag{3.2$'$}$$

where I denotes the identity operator. We plan to obtain a global solution of (3.2) as a weak limit of a viscosity solution approximation, which solves the following viscous problem

$$\begin{cases} \partial_t v^\varepsilon + \partial_x(f(v^\varepsilon)) = \varepsilon\partial_x^2 v^\varepsilon + g(v^\varepsilon, \partial_x v^\varepsilon, \partial_x^2 v^\varepsilon, \partial_x^3 v^\varepsilon, \partial_x^2\partial_t v^\varepsilon), \\ \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ v^\varepsilon(0,x) = v_0^\varepsilon(x) \to 0, \quad x \to \pm\infty, \end{cases} \tag{3.3}$$

where $0 < \varepsilon \leq 1, v_0^\varepsilon(x) = \eta_\varepsilon * v_0(x)$ and η_ε denotes the standard mollifier.

Next we are devoted to proving the global existence of solutions to problem (3.3), which consists of a local existence and the *a priori* estimates. For the sake of simplification, we will drop the superscript ε in $v^\varepsilon(t,x)$ to denote the solution of (3.3) in the rest of this section if there is no any ambiguity.

Since we can rewrite (3.3) as follows

$$\begin{cases} \partial_t v + v\partial_x v + \partial_x(P + \phi v - \phi^2) = \varepsilon\partial_x^2 v, \\ \quad (t,x) \in [0,\infty) \times \mathbb{R}, \\ (-\partial_x^2 + I)P = \frac{1}{2}(\partial_x v + \partial_x \phi)^2 + (v+\phi)^2, \\ v(0,x) = v_0(x), \end{cases} \tag{3.4}$$

by the standard argument for a nonlinear parabolic equation (cf.[30] also), one can obtain the local well-posedness result for $v_0(x) \in H^2(\mathbb{R})$. Precisely, we have the following local existence result.

Lemma 3.1. (Local existence). *Let $v_0 \in H^2(\mathbb{R})$. Then for each $\varepsilon > 0$, there exists a positive constant $T > 0$, such that the Cauchy problem (3.3) admits a unique smooth solution $v(t,x) \in C([0,T), H^2(\mathbb{R})) \cap L^2([0,T), H^3(\mathbb{R}))$.*

To obtain the global existence, it only remains to derive the *a priori* estimates, which is given in the following lemma.

Lemma 3.2. (A priori estimates). *Let $v_0 \in H^2(\mathbb{R})$ and $v(t,x)$ be a solution obtained in Lemma 3.1, then it holds that*

$$\|v(t)\|_{H^1}^2 + \int_0^t \int \partial_x\phi(v^2 + (\partial_x v)^2) + \varepsilon\int_0^t \int ((\partial_x v)^2 + (\partial_x^2 v)^2) \\ \leq C_1(\|v_0\|_{H^1}^2 + h(\tilde{u})), \tag{3.5}$$

$$\|\partial_x^2 v(t)\|_{L^2} + \varepsilon\int_0^t \int |\partial_x^3 v(s,\cdot)|^2 dxds \leq C_2(1 + \|v_0\|_{H^1}^2 + h(\tilde{u})) \tag{3.6}$$

where $C_1 > 0$ is a constant independent of ε and $C_2 > 0$ is a constant depending on ε.

Proof. We first prove inequality (3.5). To this end, we multiply (3.3) by v and integrate it with respect to x over $\mathbb{R}$ to get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int (v^2+(\partial_x v)^2)dx + \varepsilon\int(\partial_x v)^2dx + \int v\partial_x(f(v))dx \\ = \quad & 2\int v\partial_x(v+\phi)\partial_x^2(v+\phi)dx + \int v(v+\phi)\partial_x^3(v+\phi)dx + \int \partial_x^2\partial_t\phi v dx. \end{aligned} \tag{3.7}$$

The third term of the left hand side of (3.7) is estimated as

$$\int v\partial_x(f(v))dx = -\frac{3}{2}\int v^2\partial_x v dx - 3\int \phi v v_x dx = \frac{3}{2}\int \partial_x\phi v^2 dx. \tag{3.8}$$

Moreover, the second term of the right hand side of (3.7) can be rearranged as

$$\begin{aligned} &\int v(v+\phi)\partial_x^3(v+\phi)dx = -\int \partial_x(v(v+\phi))\partial_x^2(v+\phi)dx \\ = \quad & -2\int v\partial_x(v+\phi)\partial_x^2(v+\phi)dx - \int v\partial_x^2\phi\partial_x v dx - \int v\partial_x\phi\partial_x^2\phi dx \\ & + \int \phi\partial_x v\partial_x^2 v dx + \int \phi\partial_x\phi\partial_x^2 v dx \\ = \quad & -2\int v\partial_x(v+\phi)\partial_x^2(v+\phi)dx + \frac{1}{2}\int v^2\partial_x^3\phi dx - \int \partial_x\phi\partial_x^2\phi v dx \\ & -\frac{1}{2}\int \partial_x\phi(\partial_x v)^2dx + \int \partial_x^2(\phi\partial_x\phi)v dx. \end{aligned} \tag{3.9}$$

Substituting (3.8) and (3.9) into (3.7), we get the following identity

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int (v^2+(\partial_x v)^2)\ dx + \varepsilon\int(\partial_x v)^2dx + \frac{3}{2}\int \partial_x\phi v^2\ dx + \frac{1}{2}\int \partial_x\phi(\partial_x v)^2\ dx \\ = \quad & \frac{1}{2}\int \partial_x^3\phi v^2\ dx - \int \partial_x\phi\partial_x^2\phi v\ dx + \int \partial_x^2(\phi\partial_x\phi)v\ dx + \int \partial_x^2\partial_t\phi v\ dx. \end{aligned} \tag{3.10}$$

Applying Young inequality and Corollary 2.2 yields

$$\begin{aligned} & \frac{1}{2}\int \partial_x^3\phi v^2\ dx - \int \partial_x\phi\partial_x^2\phi v\ dx + \int \partial_x^2(\phi\partial_x\phi)v\ dx + \int \partial_x^2\partial_t\phi v\ dx \\ \le \quad & \frac{1}{2}\|\partial_x^3\phi\|_{L^\infty}\int v^2dx + \frac{1}{4}\int \partial_x\phi v^2dx + \int(\partial_x^2\phi)^2\partial_x\phi dx + \frac{1}{4}\int \partial_x\phi v^2dx \\ & + \int \frac{(\partial_x^2(\phi\partial_x\phi))^2}{\partial_x\phi}dx + \frac{1}{4}\int \partial_x\phi v^2dx + \int \frac{(\partial_x^2\partial_t\phi)^2}{\partial_x\phi}dx \\ \le \quad & Ch(\tilde{u})(1+t)^{-4}\int v^2dx + \frac{3}{4}\int v^2\partial_x\phi dx + \|\partial_x\phi\|_\infty\int(\partial_x^2\phi)^2dx \\ & + \int \frac{(\partial_x^2(\phi\partial_x\phi))^2}{\partial_x\phi}dx + \int \frac{(\partial_x^2\partial_t\phi)^2}{\partial_x\phi}dx \\ \le \quad & Ch(\tilde{u})(1+t)^{-4}\int v^2dx + \frac{3}{4}\int v^2\partial_x\phi dx + Ch(\tilde{u})(1+t)^{-\frac{5}{2}} \\ & + Ch(\tilde{u})(1+t)^{-2} + \int \frac{(\partial_x^2(\phi\partial_x\phi))^2}{\partial_x\phi}dx. \end{aligned} \tag{3.11}$$

In addition, noting that $\phi(t,x)$ is bounded, we have from Corollary 2.2 that

$$\begin{aligned}
&\int \frac{(\partial_x^2(\phi\partial_x\phi))^2}{\partial_x\phi}\,dx = \int \frac{(2\partial_x\phi\partial_x^2\phi + \phi\partial_x^3\phi + (\partial_x^2\phi)^2)^2}{\partial_x\phi}dx \\
\leq\ & C\int \partial_x\phi(\partial_x^2\phi)^2dx + C\int \frac{\phi^2(\partial_x^3\phi)^2}{\partial_x\phi}dx + C\int \frac{(\partial_x^2\phi)^4}{\partial_x\phi}dx \\
\leq\ & C\|\partial_x\phi\|_{L^\infty}\int(\partial_x^2\phi)^2dx + C\|\phi\|_{L^\infty}^2\int \frac{(\partial_x^3\phi)^2}{\partial_x\phi}dx + C\|\partial_x^2\phi\|_{L^\infty}^2\int \frac{(\partial_x^2\phi)^2}{\partial_x\phi}dx \\
\leq\ & Ch(\tilde u)(1+t)^{-2}. \qquad (3.12)
\end{aligned}$$

Therefore, substituting (3.12) into (3.11), one has that

$$\begin{aligned}
&\frac{1}{2}\int \partial_x^3\phi v^2\ dx - \int_{\mathbb{R}} \partial_x\phi\partial_x^2\phi v\ dx + \int \partial_x^2(\phi\partial_x\phi)v\ dx + \int \partial_x^2\partial_t\phi v\ dx \\
\leq\ & Ch(\tilde u)(1+t)^{-4}\int v^2\ dx + \frac{3}{4}\int \partial_x\phi v^2\ dx + Ch(\tilde u)(1+t)^{-2}. \qquad (3.13)
\end{aligned}$$

Hence, the substitution of (3.13) into (3.11) yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int(v^2+(\partial_x v)^2)\ dx + \frac{3}{4}\int \partial_x\phi v^2\ dx + \frac{1}{2}\int \partial_x\phi(\partial_x v)^2\ dx + \varepsilon\int(\partial_x v)^2dx \\
\leq\ & Ch(\tilde u)(1+t)^{-4}\int v^2\ dx + Ch(\tilde u)(1+t)^{-2}. \qquad (3.14)
\end{aligned}$$

Integrating (3.14) with respect to t over $[0,t]$, we arrive at

$$\begin{aligned}
&\int(v^2+(\partial_x v)^2)dx + \int_0^t\int \partial_x\phi(v^2+(\partial_x v)^2)dxd\tau + \varepsilon\int_0^t\int(\partial_x v)^2dxd\tau \\
\leq\ & Ch(\tilde u)\int_0^t(1+s)^{-4}\int v^2\ dxd\tau + \|v_0\|_{H^1}^2 + Ch(\tilde u), \qquad (3.15)
\end{aligned}$$

which implies

$$\int v^2\ dx \leq Ch(\tilde u)\int_0^t(1+s)^{-4}\int v^2\ dxd\tau + \|v_0\|_{H^1}^2 + Ch(\tilde u). \qquad (3.16)$$

Applying Gronwall's inequality to (3.16) gives us the inequality

$$\int v^2\ dx \leq C(h(\tilde u) + \|v_0\|_{H^1}^2). \qquad (3.17)$$

Substituting (3.17) into (3.15), we obtain that

$$\|v(t)\|_{H^1}^2 + \int_0^t\int \partial_x\phi(v^2+(\partial_x v)^2)\ dxd\tau + \varepsilon\int_0^t\int(\partial_x v)^2dxd\tau \leq C(\|v_0\|_{H^1}^2 + h(\tilde u)). \qquad (3.18)$$

To finish the proof of (3.5), it remains to estimate $\int_0^t\int|\partial_x^2 v|^2dxd\tau$. Toward this end, we differentiate the first equation of (3.4) with respect to x and get

$$\partial_t\partial_x v + (\partial_x v)^2 + v\partial_x^2 v + P - \left(\frac{1}{2}(\partial_x v+\partial_x\phi)^2 + (v+\phi)^2\right) + \partial_x^2(\phi v - \phi^2) = \varepsilon\partial_x^3 v.$$

We integrate the above equation to derive that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int(v^2+(\partial_x v)^2)dx+\varepsilon\int((\partial_x v)^2+(\partial_x^2 v)^2)dx\\
=\;&\frac{3}{2}\int\partial_x(v^2)\phi dx+\frac{5}{2}\int\partial_x v(\partial_x\phi)^2dx-\frac{1}{2}\int\partial_x^2\phi\partial_x(v^2)dx\\
&-\int\partial_x\phi(\partial_x v)^2dx+\int\partial_x\phi(\partial_x v)^2dx+2\int\phi\partial_x^2\phi\partial_x v dx\\
\le\;&\frac{5}{2}\int\partial_x v(\partial_x\phi)^2dx+2\int\phi\partial_x^2\phi\partial_x v dx+\int\partial_x\phi(\partial_x v)^2dx+\frac{1}{2}\int\partial_x^3\phi v^2dx\\
\le\;&5\left(\int\partial_x\phi(\partial_x v)^2dx+\int(\partial_x\phi)^3dx\right)+\int\partial_x\phi(\partial_x v)^2dx+\int\frac{\phi^2(\partial_x^2\phi)^2}{\partial_x\phi}dx\\
&+\int\partial_x\phi(\partial_x v)^2dx+\frac{1}{2}\|\partial_x^3\phi\|_{L^\infty}\|v\|_{L^2}^2\\
\le\;&7\int\partial_x\phi(\partial_x v)^2dx+5\int(\partial_x\phi)^3dx+\int\frac{\phi^2(\partial_x^2\phi)^2}{\partial_x\phi}dx+\frac{1}{2}\|\partial_x^3\phi\|_{L^\infty}\|v\|_{L^2}^2.
\end{aligned}$$

Integrating the above inequality, and applying Corollary 2.2 as well as (3.18), we get the boundedness of $\int_0^t\int|\partial_x^2 v|^2dxd\tau$. Together with (3.18), we obtain (3.5).

Next, we derive (3.6). Indeed, it is straightforward to deduce from (3.4) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\partial_x^2 v(t,x)|^2dx+\varepsilon\int|\partial_x^3 v(t,x)|^2dx\\
=\;&2\int(v\partial_x v\partial_x^2 v)(t,x)dx+\int\left[\partial_x^3 v\left(v\partial_x^2 v+\frac{1}{2}(\partial_x v)^2+P+\phi v-\phi^2\right)\right](t,x)dx\\
\le\;&2\|v(t,\cdot)\|_{L^\infty}\|\partial_x v(t,\cdot)\|_{H^1}^2+\frac{\varepsilon}{4}\int|\partial_x^3 v(t,x)|^2dx+\frac{1}{\varepsilon}\|v(t,\cdot)\|_{L^\infty}^2\|\partial_x^2 v(t,\cdot)\|_{L^2}^2\\
&+\frac{1}{8\varepsilon}\int(\partial_x v)^4(t,x)dx+\frac{1}{2\varepsilon}\int|P_1(t,x)|^2dx+\frac{1}{2}\int|\partial_x v(t,x)|^2dx\\
&+\frac{1}{2}\int|\partial_x^2 P_2(t,x)|^2dx+\frac{\varepsilon}{4}\int|\partial_x^3 v(t,x)|^2dx+\frac{1}{\varepsilon}\|\phi(t,x)\|_{L^\infty}^2\|v(t,\cdot)\|_{L^2}^2\\
&+2\int|\partial_x v(t,x)|^2dx+\|\phi\|_{L^\infty}^2\|\partial_x^2\phi\|_{L^2}^2+\|\partial_x\phi\|_{L^4}^4, \qquad (3.19)
\end{aligned}$$

where

$$P_1=\frac{1}{4}\int e^{-|x-y|}(\partial_x\phi+\partial_x v)^2(t,y)dy \quad\text{and}\quad P_2=\frac{1}{2}\int e^{-|x-y|}(\phi+v)^2(t,y)dy.$$

Then we have

$$\|P_1(t,\cdot)\|_{L^2}^2\le C\|(\partial_x v+\partial_x\phi)(t,\cdot)\|_{L^4}^4\le C(\|\partial_x v(t,\cdot)\|_{L^4}^4+\|\partial_x\phi(t,\cdot)\|_{L^4}^4) \qquad (3.20)$$

and

$$\begin{aligned}
&\|\partial_x^2 P_2(t,\cdot)\|_{L^2}^2\\
\le\;&C\|(v+\phi)(t,\cdot)\|_{L^\infty}^2\|(\partial_x^2 v+\partial_x^2\phi)(t,\cdot)\|_{L^2}^2+C\|(\partial_x v+\partial_x\phi)(t,\cdot)\|_{L^4}^4\\
\le\;&C(\|v(t,\cdot)\|_{L^\infty}^2+\|\phi(t,\cdot)\|_{L^\infty}^2)(\|\partial_x^2 v(t,\cdot)\|_{L^2}^2+\|\partial_x^2\phi(t,\cdot)\|_{L^2}^2)\\
&+C\|\partial_x v(t,\cdot)\|_{L^4}^4+C\|\partial_x\phi(t,\cdot)\|_{L^4}^4. \qquad (3.21)
\end{aligned}$$

By the Gagliardo-Nirenberg inequality, it follows that

$$\|\partial_x v(t,\cdot)\|_{L^4}^4\le C\|v(t,\cdot)\|_{L^\infty}^2\|\partial_x^2 v(t,\cdot)\|_{L^2}^2. \qquad (3.22)$$

Substituting (3.22), (3.21) and (3.20) into (3.19), one has

$$\frac{1}{2}\frac{d}{dt}\int|\partial_x^2 v(t,\cdot)|^2dx+\frac{\varepsilon}{2}\int|\partial_x^3 v(t,\cdot)|^2dx\le\frac{C}{(1+t)^{3/2}}+C\|\partial_x v(t,\cdot)\|^2_{H^1(\mathbb{R})} \quad (3.23)$$

where $C=C(\varepsilon,\|v(t,\cdot)\|_{L^\infty},h(\widetilde{u}))$, which can be chosen independent of time due to

$$\|v(t,\cdot)\|_{L^\infty}\le\sqrt{2}\|v(t,\cdot)\|_{H^1}\le\sqrt{2}\|v_0\|_{H^1}+C,\forall\ t\in[0,T). \quad (3.24)$$

Now integrating (3.23) with respect to t over $[0,t]$ and applying inequality (3.5) establish inequality (3.6). Hence the proof of Lemma 3.2 is completed. □

The combination of Lemma 3.1 and Lemma 3.2 gives the global existence theorem.

Theorem 3.3. *Assume $v_0^\varepsilon(x)\in H^2(\mathbb{R})$ for each $\varepsilon>0$, then there exists a unique solution $v^\varepsilon=v^\varepsilon(t,x)\in C([0,\infty);H^2(\mathbb{R}))\cap L^2([0,\infty);H^3(\mathbb{R}))$ to the Cauchy problem (3.3). Furthermore, the solution $v^\varepsilon(t,x)$ satisfies (3.5) and (3.6).*

4. **Existence of a weak solution.**

In this section, we will prove the global existence of weak solution to problem (3.2) using the vanishing viscosity method. It turns out that the difficulty is how to pass limits of the viscous solution v^ε. A Young measure argument guarantees the weak limit v which corresponds to a global weak solution to (3.2). As a consequence, $u=v+\phi$ is a global weak solution to the Cauchy problem (1.1)-(1.2). Before giving the proof, we first establish some crucial estimates for $\partial_x v^\varepsilon$.

Lemma 4.1. *Suppose $(v^\varepsilon,P_\varepsilon)$ satisfies (3.4). Then there exists a positive constant C depending only on $\|v_0\|_{H^1},h(\tilde{u})$, $|u_-|,|u_+|$ such that*

$$\partial_x v^\varepsilon\le\frac{2}{t}+C,\ \forall\ t>0,\ x\in\mathbb{R}.$$

Proof. Let $q_\varepsilon=\partial_x v^\varepsilon$. Then it satisfies

$$\begin{cases}\partial_t q_\varepsilon+(v^\varepsilon+\phi)\partial_x q_\varepsilon+\frac{1}{2}q_\varepsilon^2+\partial_x\phi q_\varepsilon-\varepsilon\partial_x^2 q_\varepsilon=A_\varepsilon,\\ q_\varepsilon(0,x)=\partial_x v^\varepsilon(0,x)\equiv\partial_x v_0^\varepsilon(x),\end{cases} \quad (4.1)$$

where $A_\varepsilon=(v^\varepsilon+\phi)^2-P_\varepsilon+5/2(\partial_x\phi)^2-\partial_x^2\phi v^\varepsilon+2\phi\partial_x\phi$. It follows from the fact

$$P_\varepsilon(t,x)=\frac{1}{2}\int_{-\infty}^{+\infty}e^{-|x-y|}\left((v^\varepsilon+\phi)^2(t,y)+\frac{1}{2}(\partial_y v^\varepsilon+\partial_y\phi)^2(t,y)\right)\,dy$$

that for any $t>0$,

$$\begin{aligned}&\|P_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}\\ \le\ &\frac{1}{4}\int_{-\infty}^{+\infty}(\partial_y v^\varepsilon+\partial_y\phi)^2(t,y)\,dy+\frac{1}{2}\int_{-\infty}^{+\infty}e^{-|x-y|}\|(v^\varepsilon+\phi)^2\|_{L^\infty}\,dy\\ \le\ &\frac{1}{2}\left(\|\partial_x v^\varepsilon(t,\cdot)\|^2_{L^2}+\|\partial_x\phi(t,\cdot)\|^2_{L^2}\right)+2(\|v^\varepsilon(t,\cdot)\|^2_{L^\infty}+\|\phi(t,\cdot)\|^2_{L^\infty})\\ \le\ &5(\|v^\varepsilon(t,\cdot)\|^2_{H^1}+\|\partial_x\phi(t,\cdot)\|^2_{L^2}+\|\phi(t,\cdot)\|^2_{L^\infty})\\ \le\ &C(\|v_0\|_{H^1},h(\tilde{u}),|u_-|,|u_+|).\end{aligned} \quad (4.2)$$

Then we can deduce that

$$\begin{aligned}
&\|A_\varepsilon\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\\
\le\ & 5\sup_{0\le t<\infty}(\|v^\varepsilon(t,\cdot)\|^2_{L^\infty}+\|\phi(t,\cdot)\|^2_{L^\infty}+\|P_\varepsilon(t,\cdot)\|_{L^\infty}\\
&\qquad+\|\partial_x\phi(t,\cdot)\|^2_{L^\infty}+\|\partial_x^2\phi(t,\cdot)\|^2_{L^\infty}\|v_\varepsilon(t,\cdot)\|^2_{L^\infty})\\
\le\ & C(\|v_0\|_{H^1},h(\tilde{u}),|u_-|,|u_+|). \qquad (4.3)
\end{aligned}$$

Since $|\partial_x\phi q_\varepsilon|\le\frac14 q_\varepsilon^2+2\|\partial_x\phi(t,\cdot)\|^2_{L^\infty}$, it follows that

$$\partial_t q_\varepsilon+(v^\varepsilon+\phi)\partial_x q_\varepsilon+\frac14 q_\varepsilon^2-\varepsilon\partial_x^2 q_\varepsilon\le\frac14 R^2,$$

for some $R>0$. Let $Q_\varepsilon(t)$ solves the following ODE

$$\begin{cases}\dfrac{d}{dt}Q_\varepsilon+\dfrac14 Q_\varepsilon^2=R^2\\ Q_\varepsilon(0)=\max\{0,\partial_x v_0^\varepsilon\}.\end{cases}$$

Then the comparison principle gives us

$$\partial_x v^\varepsilon(t,x)\equiv q_\varepsilon(t,x)\le Q_\varepsilon(t).$$

Direct computation tells us that $Q_\varepsilon(t)\le\dfrac{2}{t}+4R$, which in turn implies our Lemma. □

Using the same technique as in [30] and the previous estimates for ϕ and v^ε, we get the following uniform local space-time higher integrability estimate for $\partial_x v^\varepsilon$.

Lemma 4.2. *Let $\alpha=2k/(2l+1)$ with $k,l\in\mathbb{N}$ and $l\ge k$. Assume $a>b$ and $T>0$. Then there exists a positive constant $C=C(a,b,T,\|v_0\|_{H^1},h(\tilde{u}),\alpha)$, but independent of ε, such that*

$$\int_0^T\int_a^b|\partial_x v^\varepsilon(t,x)|^{2+\alpha}dxdt\le C.$$

Proof. Let $0\le\chi\le1,\chi\in C_0^\infty((a-1,b+1))$, and $\chi\equiv1$ on $[a,b]$. Set $\theta(\xi)=(1+\alpha)\int_0^\xi\max\{1,s^\alpha\}ds$ for $\xi\in\mathbb{R}$. Multiplying the equation (4.1) by $\xi(x)\theta'(q_\varepsilon)$ and integrating the resultant equation on $[0,T]\times\mathbb{R}$, we get

$$\begin{aligned}
&\int_0^T\int\chi(x)\left(q_\varepsilon\theta(q_\varepsilon)-\frac12 q_\varepsilon^2\theta'(q_\varepsilon)\right)dxdt\\
=\ &\int\chi(x)\theta(q_\varepsilon)\Big|_0^T-\int_0^T\int(\partial_x\phi\chi+(v+\phi)\chi')\theta(q_\varepsilon)dxdt+\int_0^T\int\chi\partial_x\phi q_\varepsilon\theta'(q_\varepsilon)dxdt\\
&+\varepsilon\int_0^T\int\chi'\theta'(q_\varepsilon)\partial_x q_\varepsilon dxdt+(\partial_x q_\varepsilon)^2\chi\theta''(q_\varepsilon)-\int_0^T\int A_\varepsilon\chi\theta'(q_\varepsilon)dxdt. \qquad (4.4)
\end{aligned}$$

By the definition, we have

$$\begin{aligned}
&\int_0^T\int\chi(x)\left(q_\varepsilon\theta(q_\varepsilon)-\frac12 q_\varepsilon^2\theta'(q_\varepsilon)\right)dxdt\\
\ge\ &\int_0^T\int_{\{|q_\varepsilon|\ge1\}}\chi\left(\frac{1-\alpha}{2}|q_\varepsilon|^{2+\alpha}+\alpha|q_\varepsilon|\right)dxdt+\int_0^T\int_{\{|q_\varepsilon|<1\}}\chi\left(\alpha+\frac12\right)|q_\varepsilon|^2dxdt\\
\ge\ &\left(1-\frac{\alpha}{2}\right)\int_0^T\int\chi|q_\varepsilon|^{2+\alpha}dxdt.
\end{aligned}$$

Based on the estimates for ϕ and v^ε, all the terms on the right hand side of (4.4) can be controlled by some constant depending only on $a, b, T, \|v_0\|_{H^1}, h(\tilde{u})$, and α (c.f. [30]). According to the definition of χ, the Lemma is established. ☐

Lemma 4.3. *There exists a subsequence $\{v_{\varepsilon j}, P_{\varepsilon j}\}$ of the sequence $\{v_\varepsilon, P_\varepsilon\}$ and some functions (v, P) with $v \in L^\infty([0,\infty); H^1(\mathbb{R}))$ and $P \in L^\infty([0,\infty); W^{1,\infty}_{loc}(\mathbb{R}))$ such that $v_{\varepsilon j} \to v$ as $j \to +\infty$ uniformly on each compact subset of $\mathbb{R}^+ \times \mathbb{R}$ and $P_{\varepsilon j} \to P$ in $L^q_{loc}(\mathbb{R}^+ \times \mathbb{R})$ as $j \to +\infty$ for $1 < q < +\infty$.*

Proof. It is shown in Lemma 3.2 that $\{v_\varepsilon\}$ is bounded in $L^\infty(\mathbb{R}^+, H^1(\mathbb{R}))$. Let

$$(-\partial_x^2 + I)P_{1\varepsilon} = \frac{1}{2}(\partial_x v^\varepsilon + \partial_x \phi)^2, \quad (-\partial_x^2 + I)P_{2\varepsilon} = (v^\varepsilon + \phi)^2,$$

Then it is easy to see that $P_\varepsilon = P_{1\varepsilon} + P_{2\varepsilon}$ and

$$\|P_{1\varepsilon}(t,\cdot)\|^2_{H^1} = \frac{1}{2}\int P_{1\varepsilon}(\partial_x v^\varepsilon + \partial_x\phi)^2 \le \|P_\varepsilon(t,\cdot)\|_{L^\infty}(\|v^\varepsilon(t,\cdot)\|^2_{H^1} + \|\partial_x\phi(t,\cdot)\|^2_{L^2}),$$

as well as

$$\begin{aligned}
& \|\partial_x P_{2\varepsilon}(t,\cdot)\|^2_{L^2} \\
= \; & \int\left(\int e^{-|x-y|}(\partial_y v^\varepsilon + \partial_y\phi)(t,y)(v^\varepsilon+\phi)(t,y)dy\right)^2 dx \\
\le \; & \|(v^\varepsilon+\phi)(t,\cdot)\|^2_{L^\infty}\int\left(\int e^{-|x-y|}|(\partial_y v^\varepsilon + \partial_y\phi)(t,y)|dy\right)^2 dx \\
\le \; & \|(v^\varepsilon+\phi)(t,\cdot)\|^2_{L^\infty}\int\left(\int e^{-|x-y|}dy\int e^{-|x-y|}(\partial_y v^\varepsilon + \partial_y\phi)^2(t,y)dy\right) dx \\
\le \; & 2\|(v^\varepsilon+\phi)(t,\cdot)\|^2_{L^\infty}\int\left(\int e^{-|x-y|}(\partial_y v^\varepsilon + \partial_y\phi)^2(t,y)dy\right) dx \\
= \; & 4\|(v^\varepsilon+\phi)(t,\cdot)\|^2_{L^\infty}\|(\partial_y v^\varepsilon + \partial_y\phi)(t,\cdot)\|^2_{L^2} \\
\le \; & 8\|(v^\varepsilon+\phi)(t,\cdot)\|^2_{L^\infty}\left(\|\partial_y v^\varepsilon(t,\cdot)\|^2_{L^2} + \|\partial_y\phi(t,\cdot)\|^2_{L^2}\right).
\end{aligned}$$

Here we have used Hölder inequality and Fubini theorem. Now it is easy to deduce from (4.2), Corollary 2.2 and Lemma 3.2 that

$$\|\partial_x P_\varepsilon(t,\cdot)\|_{L^2} \le C$$

for all $t > 0$, where C is a constant independent of ε and t. Then it is clear that $\{\partial_t v^\varepsilon\}$ is bounded in $L^2([0,T]\times\mathbb{R})$ for any $T > 0$ from the above inequality, equation (3.4) and Lemma 3.2. Therefore there exist some function $v \in L^\infty([0,\infty); H^1(\mathbb{R}))$ such that

$$\begin{aligned}
& v^{\varepsilon j} \rightharpoonup v && \text{weakly in } L^\infty([0,\infty); H^1(\mathbb{R})), \\
& v^{\varepsilon j} \to v && \text{uniformly for any compact set in } \mathbb{R}^+ \times \mathbb{R}.
\end{aligned}$$

It can be also easily shown that $\{P_\varepsilon\}$ is uniformly bounded in $L^\infty([0,\infty); W^{1,r}_{\text{loc}}(\mathbb{R})) \cap L^\infty([0,\infty); W^{2,1}_{\text{loc}}(\mathbb{R}))$ for any $r \in [1, +\infty)$. Note that

$$(-\partial_x^2 + I)\frac{\partial P_\varepsilon}{\partial t} = 2(v^\varepsilon + \phi)\left(\frac{\partial v^\varepsilon}{\partial t} + \frac{\partial\phi}{\partial t}\right) + (q_\varepsilon + \partial_x\phi)\left(\frac{\partial q_\varepsilon}{\partial t} + \partial_x\partial_t\phi\right).$$

One can show that $(v^\varepsilon+\phi)\left(\frac{\partial v^\varepsilon}{\partial t}+\frac{\partial \phi}{\partial t}\right)$ is uniformly bounded in $L^2([0,T]\times\mathbb{R})$ for any $T>0$. Furthermore, it holds that

$$\begin{aligned}
&(q_\varepsilon+\partial_x\phi)\left(\frac{\partial q_\varepsilon}{\partial t}+\partial_x\partial_t\phi\right)\\
=\;& -(q_\varepsilon+\partial_x\phi)(v^\varepsilon+\phi)\partial_x q_\varepsilon-\frac{1}{2}q_\varepsilon^2(q_\varepsilon+\partial_x\phi)-\partial_x\phi(q_\varepsilon+\partial_x\phi)\\
&+A_\varepsilon(q_\varepsilon+\partial_x\phi)+\varepsilon\partial_x^2 q_\varepsilon(q_\varepsilon+\partial_x\phi)+\partial_t\partial_x\phi(q_\varepsilon+\partial_x\phi)\\
=\;& -\frac{1}{2}\partial_x\left((v^\varepsilon+\phi)q_\varepsilon^2\right)-\partial_x\phi(v^\varepsilon+\phi)\partial_x q_\varepsilon-\partial_x\phi(q_\varepsilon+\partial_x\phi)q_\varepsilon\\
&+\varepsilon\partial_x\left(q_\varepsilon\partial_x q_\varepsilon\right)-\varepsilon(\partial_x q_\varepsilon)^2+(q_\varepsilon+\partial_x\phi)(A_\varepsilon+\partial_t\partial_x\phi),
\end{aligned}$$

which is uniformly bounded in $L^1([0,T);W^{-1,1}_{\text{loc}}(\mathbb{R}))$. By the standard elliptic regularity theory and the fact that $W^{-1,1}_{\text{loc}}(\mathbb{R})\hookrightarrow W^{-(1+\delta),r}_{\text{loc}}(\mathbb{R})$ for any $\delta>0$ and $r>1, r$ close to 1, we obtain that $\{\partial_t P_\varepsilon\}$ is uniformly bounded in $L^1([0,T);W^{1-\delta,r}_{\text{loc}}(\mathbb{R}))$ for some $\delta>0, r>1$. So $P_\varepsilon\in W^{1,1}_{\text{loc}}(\mathbb{R}^+\times\mathbb{R}))\cap L^\infty([0,\infty);W^{1,\infty}_{\text{loc}}(\mathbb{R}))$ and there exists $P\in L^\infty([0,\infty);W^{1,\infty}_{\text{loc}}(\mathbb{R}))$ such that

$$P_{\varepsilon j}\to P \quad \text{in} \;\; L^q_{\text{loc}}(\mathbb{R}^+\times\mathbb{R})).$$

This completes the proof of Lemma 4.3. ∎

Now let $\mu_{t,x}(\lambda)$ be the Young measure associated with $\{q_\varepsilon\}\equiv\{\partial_x v^\varepsilon\}$, see [30]. Then for any continuous function $f=f(\lambda)$ with $f(\lambda)=o(|\lambda|^r)$ and $\partial_\lambda f(\lambda)=o(|\lambda|^{r-1})$ as $|\lambda|\to\infty$ and $r<2$, and $\forall\,\psi\in L^s_c(\mathbb{R})$ with $1/s+r/2=1$, there holds

$$\lim_{\varepsilon\to 0^+}\int f(q_\varepsilon(t,x))\psi(x)dx=\int\overline{f(q)}\psi(x)dx$$

uniformly in each compact subset of $\mathbb{R}^+$. Here

$$\overline{f(q)}=\int f(\lambda)d\mu_{t,x}(\lambda)\in C([0,\infty);L^{r'/r}_{\text{loc}}(\mathbb{R}))$$

with $r'\in(r,2)$. Moreover, for all $T>0$, we have

$$\lim_{\varepsilon\to 0^+}\int_0^T\!\!\int g(q_\varepsilon)\varphi dxdt=\int_0^T\!\!\int\overline{g(q)}\varphi dxdt$$

and

$$\lambda\in L^l_{\text{loc}}(\mathbb{R}^+\times\mathbb{R}\times\mathbb{R},dt\otimes dx\otimes d\mu_{t,x}(\lambda))\quad \text{for all}\;\; l<3,$$

where $g=g(x,t,\lambda)$ is a continuous function satisfying $g=o(|\lambda|^l)$ as $|\lambda|\to\infty$ for some $l<3$, and $\varphi=\varphi(t,x)\in L^m([0,T]\times\mathbb{R})$ with $l/3+1/m<1$. And also $\lambda\in L^\infty([0,\infty);L^2(\mathbb{R}\times\mathbb{R},dx\otimes d\mu_{x,t}(\lambda)))$, $\overline{q}(t,x)=\partial_x v(t,x)$.

We furthermore give the following Lemma.

Lemma 4.4. $\mu_{t,x}(\lambda)=\delta_{\overline{q}(t,x)}(\lambda)$ *for a.e.* $(t,x)\in\mathbb{R}^+\times\mathbb{R}$.

Proof. We sketch the proof which is comparable to the proof in [30]. The main difference is that the appearance of nontrivial ϕ in the present case. The proof is divided into six steps.

Step 1.

Let $E = E(\lambda) \in W^{2,\infty}(\mathbb{R})$ be a given convex function with $E(\lambda) = O(|\lambda|)$ and $DE(\lambda) = O(1)$ as $|\lambda| \to \infty$. Then it follows that

$$\begin{aligned} & \partial_t \left(E(q_\varepsilon)\right) + \partial_x \left((v^\varepsilon + \phi)E(q_\varepsilon)\right) \\ = \;& q_\varepsilon E(q_\varepsilon) + \partial_x \phi E(q_\varepsilon) - \frac{1}{2} q_\varepsilon^2 DE(q_\varepsilon) - \partial_x \phi q_\varepsilon DE(q_\varepsilon) \\ & + DE(q_\varepsilon) A_\varepsilon + \varepsilon \partial_x \left(DE(q_\varepsilon)\partial_x q_\varepsilon\right) - \varepsilon D^2 E(q_\varepsilon)(\partial_x q_\varepsilon)^2. \end{aligned}$$

Since $\{\sqrt{\varepsilon}\partial_x q_\varepsilon\}$ is uniformly bounded in $L^2(\mathbb{R}^+ \times \mathbb{R})$, we obtain that

$$\begin{aligned} & \partial_t \overline{E(q)} + \partial_x((v+\phi)\overline{E(q)}) \\ \le \;& \overline{qE(q) - \frac{1}{2} q^2 DE(q) + \partial_x \phi E(q) - \partial_x \phi q DE(q)} \\ & + \overline{DE(q)} \left((v+\phi)^2 - P + \frac{5}{2}(\partial_x \phi)^2 - \partial_x^2 \phi v + 2\phi \partial_x \phi \right). \end{aligned}$$

In the following we will denote $A = (v+\phi)^2 - P + \frac{5}{2}(\partial_x\phi)^2 - \partial_x^2\phi v + 2\phi\partial_x\phi$.

Step 2.

A similar argument as above applied to $E(\lambda) = \lambda$ give us that

$$\partial_t \overline{q} + \partial_x((v+\phi)\overline{q}) = \frac{1}{2}\overline{q^2} + A.$$

So $\partial_t \overline{q} + (v+\phi)\partial_x \overline{q} = \frac{1}{2}\overline{q^2} - \overline{q}^2 - \partial_x\phi\overline{q} + A$. It can be shown that

$$\partial_t E(\overline{q}) + \partial_x((v+\phi)E(\overline{q})) = (\partial_x\phi + \overline{q})E(\overline{q}) + DE(\overline{q}) \left(\frac{1}{2}\overline{q^2} - \overline{q}^2 - \partial_x\phi\overline{q} + A \right).$$

Hence we get

$$\begin{aligned} & \partial_t(\overline{E(q)} - E(\overline{q})) + \partial_x((v+\phi)(\overline{E(q)} - E(\overline{q}))) \\ \le \;& \int \left(\lambda E(\lambda) - \frac{1}{2}\lambda^2 DE(\lambda) + \partial_x\phi(E(\lambda) - \lambda DE(\lambda)) \right) d\mu_{t,x}(\lambda) \\ & - \frac{1}{2} DE(\overline{q})(\overline{q^2} - \overline{q}^2) + \left(\overline{DE(q)} - DE(\overline{q}) \right) A \\ & + \frac{1}{2} DE(\overline{q})\overline{q}^2 - \overline{q}E(\overline{q}) + \partial_x\phi(\overline{q}DE(\overline{q}) - E(\overline{q})). \end{aligned} \tag{4.5}$$

Step 3.

Define

$$Q_R(\lambda) = \begin{cases} \frac{1}{2}\lambda^2, & \text{if } |\lambda| \le R, \\ R|\lambda| - \frac{1}{2}R^2, & \text{if } |\lambda| > R, \end{cases}$$

and $Q_R^+(\lambda) = \chi_{\{\lambda \ge 0\}} Q_R(\lambda), Q_R^-(\lambda) = \chi_{\{\lambda < 0\}} Q_R(\lambda)$, where $R > 0$ and χ_A denotes the characteristic function of the set A. Since $Q_R(\lambda)$ is convex, we have

$$\begin{aligned} & 0 \le \overline{Q_R(q)} - Q_R(\overline{q}) \\ & = \frac{1}{2}(\overline{q^2} - \overline{q}^2) - \frac{1}{2}\left(\int (|\lambda| - R)^2 \chi_{\{|\lambda| \ge R\}} d\mu_{t,x}(\lambda) - (|\overline{q}| - R)^2 \chi_{\{|\overline{q}| \ge R\}} \right). \end{aligned}$$

It can be shown that $\overline{q}(t,x) \rightharpoonup q_0(x) = \partial_x v_0(x)$ as $t \to 0^+$ in $L^2(\mathbb{R})$. Then

$$\lim_{t \to 0^+} \int (\overline{q}(t,x))^2 dx \ge \int (q_0(x))^2 dx.$$

However, the energy estimate gives us

$$\lim_{t \to 0^+} \int (\overline{q}(t,x))^2 dx \le \int \overline{q^2}(t,x) dx \le \int (q_0(x))^2 dx.$$

Therefore, $\lim_{t\to 0^+}\int(\overline{q}(t,x))^2dx=\int(q_0(x))^2dx$ and

$$\lim_{t\to 0}\int\left(\overline{Q_R^{\pm}(q)}-Q_R^{\pm}(\overline{q})\right)(t,x)dx=0. \tag{4.6}$$

Step 4.
Since $q_\varepsilon(t,x)$ and $\overline{q}(t,x)$ are bounded from above by $2/t+C$, $\operatorname{supp}\mu_{t,x}(\cdot)\subset(-\infty,2/t+C)$. Applying (4.5) to $E(\lambda)=Q_R^+(\lambda)$, we obtain that for $R>2/t+C$

$$\begin{aligned}
&\partial_t(\overline{Q_R^+(q)}-Q_R^+(\overline{q}))+\partial_x((v+\phi)(\overline{Q_R^+(q)}-Q_R^+(\overline{q})))\\
\le\ & -\partial_x\phi\int\frac{\lambda^2}{2}\chi_{\{\lambda\ge 0\}}d\mu_{t,x}(\lambda)+\partial_x\phi\frac{\overline{q}^2}{2}\chi_{\{\overline{q}\ge 0\}}+\left(\overline{DQ_R^+(q)}-DQ_R^+(\overline{q})\right)A\\
\le\ & \left(\overline{DQ_R^+(q)}-DQ_R^+(\overline{q})\right)A,
\end{aligned}$$

where we have used the fact that $\frac{\lambda^2}{2}\chi_{\{\lambda\ge 0\}}$ is a convex function and $\partial_x\phi\ge 0$. Then for $t>\frac{2}{R-C}$, we derive that

$$\begin{aligned}
\int(\overline{Q_R^+(q)}-Q_R^+(\overline{q}))(t,x)dx\ \le\ & \int(\overline{Q_R^+(q)}-Q_R^+(\overline{q}))(\frac{2}{R-C},x)dx\\
&+\int_{\frac{2}{R-C}}^t\int A\left(\overline{DQ_R^+(q)}-DQ_R^+(\overline{q})\right)dxds,
\end{aligned}$$

which is

$$\frac{1}{2}\int(\overline{q_+^2}-\overline{q}_+^2)(t,x)dx\le\frac{1}{2}\int(\overline{q_+^2}-\overline{q}_+^2)(\frac{2}{R-C},x)dx+\int_{\frac{2}{R-C}}^t\int A(\overline{q_+}-\overline{q}_+)dxds,$$

where $h_+=\max\{0,h\},h_-=\min\{0,h\}$. Letting $R\to+\infty$, we have

$$\int(\overline{q_+^2}-\overline{q}_+^2)(t,x)dx\le 2\int_0^t\int A(\overline{q_+}-\overline{q}_+)dxds. \tag{4.7}$$

Step 5.
Applying (4.5) to $E(\lambda)=Q_R^+(\lambda)$, we get

$$\begin{aligned}
&\partial_t(\overline{Q_R^-(q)}-Q_R^-(\overline{q}))+\partial_x((v+\phi)(\overline{Q_R^+(q)}-Q_R^+(\overline{q})))\\
\le\ & -\frac{R}{2}\left(\int\lambda(\lambda+R)\chi_{\{\lambda\le -R\}}d\mu_{t,x}(\lambda)-\overline{q}(\overline{q}+R)\chi_{\{\overline{q}\le -R\}}\right)\\
&-\frac{1}{2}DQ_R^-(\overline{q})(\overline{q^2}-\overline{q}^2)+A\left(\overline{DQ_R^-(q)}-DQ_R^-(\overline{q})\right)\\
&+\partial_x\phi\left(\int(Q_R^-(\lambda)-\lambda DQ_R^-(\lambda))d\mu_{t,x}(\lambda)-(Q_R^-(\overline{q})-\overline{q}DQ_R^-(\overline{q}))\right).
\end{aligned} \tag{4.8}$$

Note that

$$
\begin{aligned}
&\int (Q_R^-(\lambda) - \lambda DQ_R^-(\lambda))d\mu_{t,x}(\lambda) - (Q_R^-(\overline{q}) - \overline{q}DQ_R^-(\overline{q}))\\
= & -\int_{-R}^0 \left(\frac{\lambda^2}{2}\chi_{\{-R<\lambda\le 0\}} + \frac{R^2}{2}\chi_{\{\lambda\le -R\}}\right) d\mu_{t,x}(\lambda) + \frac{\overline{q}^2}{2}\chi_{\{-R<\overline{q}\le 0\}}\\
&+\frac{R^2}{2}\chi_{\{\overline{q}\le -R\}}\\
= & -\int Q_R^-(\lambda)d\mu_{t,x}(\lambda) + Q_R^-(\overline{q}) - \int R(\lambda+R)\chi_{\{\lambda\le -R\}}d\mu_{t,x}(\lambda)\\
&+R(\overline{q}+R)\chi_{\{\overline{q}\le -R\}}\\
\le & -\int R(\lambda+R)\chi_{\{\lambda\le -R\}}d\mu_{t,x}(\lambda) + R(\overline{q}+R)\chi_{\{\overline{q}\le -R\}}.
\end{aligned}
$$

Then integrating (4.8) over $[0,t)\times\mathbb{R}$, and using (4.6), we have

$$
\begin{aligned}
&\int \left(\overline{Q_R^-(q)} - Q_R^-(\overline{q})\right)(t,x)dx\\
\le & -\frac{R}{2}\int_0^t\int \left(\int \lambda(\lambda+R)\chi_{\{\lambda\le -R\}}d\mu_{s,x}(\lambda) - \overline{q}(\overline{q}+R)\chi_{\{\overline{q}\le -R\}}\right) dxds\\
&+\frac{R}{2}\int_0^t\int (\overline{q^2} - \overline{q}^2)(s,x)dsdx + \int_0^t\int A\left(\overline{DQ_R^-(q)} - DQ_R^-(\overline{q})\right) dxds\\
&+\int_0^t\int \partial_x\phi\left(-\int R(\lambda+R)\chi_{\{\lambda\le -R\}}d\mu_{s,x}(\lambda) + R(\overline{q}+R)\chi_{\{\overline{q}\le -R\}}\right) dxds.
\end{aligned}
$$

Step 6.

From the identity

$$
\begin{aligned}
&\overline{Q_R^-(q)} - Q_R^-(\overline{q})\\
&= \frac{1}{2}\left(\overline{q_-^2} - \overline{q}_-^2\right) - \frac{1}{2}\left(\int (\lambda+R)^2\chi_{\{\lambda\le -R\}}d\mu_{t,x}(\lambda) - (\overline{q}+R)^2\chi_{\{\overline{q}\le -R\}}\right)
\end{aligned}
$$

and (4.7), (4.8), we get

$$
\begin{aligned}
&\int \left(\frac{1}{2}(\overline{q_+^2} - \overline{q}_+^2) + \overline{Q_R^-(q)} - Q_R^-(\overline{q})\right)(t,x)dx\\
\le\ & R\int_0^t\int \left(\frac{1}{2}(\overline{q_+^2} - \overline{q}_+^2) + \overline{Q_R^-(q)} - Q_R^-(\overline{q})\right)(s,x)dxds\\
&+\frac{R}{2}\int_0^t\int \left(\int R(\lambda+R)\chi_{\{\lambda\le -R\}}d\mu_{s,x}(\lambda) - R(\overline{q}+R)\chi_{\{\overline{q}\le -R\}}\right) dxds\\
&+\int_0^t\int A\left(\overline{DQ_R^-(q)} - DQ_R^-(\overline{q}) + \overline{q_+} - \overline{q}_+\right) dxds\\
&+C\int_0^t\int \left(-\int R(\lambda+R)\chi_{\{\lambda\le -R\}}d\mu_{s,x}(\lambda) + R(\overline{q}+R)\chi_{\{\overline{q}\le -R\}}\right) dxds,
\end{aligned}
$$

where $C>0$ is a constant such that $\|\partial_x\phi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\le C$.

Note that

$$
0\le \overline{DQ_R^-(q)} - DQ_R^-(\overline{q}) + \overline{q_+} - \overline{q}_+ = -\int (\lambda+R)\chi_{\{\lambda\le -R\}}d\mu_{t,x}(\lambda) + (\overline{q}+R)\chi_{\{\overline{q}\le -R\}}.
$$

Let $L > 0$ be a constant such that $\|A_\varepsilon\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})} \le L/2$ (see (4.3)). Then

$$\begin{aligned}
&\int_0^t\int A\left(\overline{DQ_R^-(q)} - DQ_R^-(\overline{q}) + \overline{q_+} - \overline{q}_+\right)dxds\\
\le\ & \frac{R}{2}\int_0^t\int\left(\int \frac{L}{R}(R+\lambda)\chi_{\{\lambda\le -R\}}d\mu_{t,x}(\lambda) - \frac{L}{R}(R+\overline{q})\chi_{\{\overline{q}\le -R\}}\right)dxds.
\end{aligned}$$

So for $R \ge 2\sqrt{L} + 4C$, we have

$$\begin{aligned}
&\frac{R}{2}\int_0^t\int\left(\int R(\lambda+R)\chi_{\{\lambda\le -R\}}d\mu_{s,x}(\lambda) - R(\overline{q}+R)\chi_{\{\overline{q}\le -R\}}\right)dxds\\
&+\int_0^t\int A\left(\overline{DQ_R^-(q)} - DQ_R^-(\overline{q}) + \overline{q_+} - \overline{q}_+\right)dxds\\
&+C\int_0^t\int\left(-\int R(\lambda+R)\chi_{\{\lambda\le -R\}}d\mu_{s,x}(\lambda) + R(\overline{q}+R)\chi_{\{\overline{q}\le -R\}}\right)dxds\\
\le\ & \frac{R}{2}\int_0^t\int\left\{\int\left(R\left(1-\frac{L}{R^2}\right)-2C\right)(\lambda+R)\chi_{\{\lambda\le -R\}}d\mu_{s,x}(\lambda)\right.\\
&\qquad\left. -\left(R\left(1-\frac{L}{R^2}\right)-2C\right)(\overline{q}+R)\chi_{\{\overline{q}\le -R\}}\right\}dxds\\
\le\ & 0.
\end{aligned}$$

Therefore for $R \ge 2\sqrt{L} + 4C$, we get

$$\begin{aligned}
&\int\left(\frac{1}{2}(\overline{q_+^2} - \overline{q}_+^2) + \overline{Q_R^-(q)} - Q_R^-(\overline{q})\right)(t,x)dx\\
\le\ & R\int_0^t\int\left(\frac{1}{2}(\overline{q_+^2} - \overline{q}_+^2) + \overline{Q_R^-(q)} - Q_R^-(\overline{q})\right)(s,x)dxds.
\end{aligned}$$

Gronwall's inequality implies that for $R \ge 2\sqrt{L} + 4C$,

$$\int\left(\frac{1}{2}(\overline{q_+^2} - \overline{q}_+^2) + \overline{Q_R^-(q)} - Q_R^-(\overline{q})\right)(t,x)dx = 0, \quad \forall\ t \ge 0.$$

Letting $R \to +\infty$, one obtains that

$$\int\left(\overline{q^2} - \overline{q}^2\right) \le 0, \quad \forall\ t \ge 0.$$

So $\int \overline{q^2} = \int \overline{q}^2$ for all $t \ge 0$. Consequently, one has that

$$\mu_{t,x}(\lambda) = \delta_{\overline{q}(t,x)}(\lambda), \quad \text{a.e. } (t,x) \in \mathbb{R}^+\times\mathbb{R}.$$

The proof of Lemma 4.4. is finished. □

Now we are in a position to prove the main results, i.e., Theorem 1.1.

From Lemma 4.2 and Lemma 4.4, we deduce that $\partial_x v^\varepsilon \to \partial_x v$ as $\varepsilon \to 0^+$ in $L^2_{\rm loc}(\mathbb{R}^+\times\mathbb{R})$, which implies $u(t,x) = v(t,x) + \phi(t,x)$ is the desired global weak solution to the Cauchy problem (1.1)-(1.2). The proof is standard, see, e.g., [30]. So we omit the details. To finish the proof, it remains to investigate the asymptotic behavior of the solution $v(t,x)$. Recalling (3.5), we have

$$\int_0^\infty\int \partial_x\phi v^2 dxdt \le C.$$

Fubini's Theorem gives us that

$$\int \left(\partial_x \phi \int_0^\infty v^2(t,x)\ dt \right)\ dx \le C.$$

Hence there is a Lebesgue measure zero set $N_1 \subset \mathbb{R}$ such that

$$v(\cdot, x) \in L^2(\mathbb{R}) \quad \text{for all} \ \ x \in \mathbb{R} \setminus N_1. \tag{4.9}$$

On the other hand, it follows from the proof of Lemma 4.3 that $\|\partial_x P(t,\cdot)\|_{L^2} \le C$ for all $t > 0$. This, (3.2′) and the estimates for ϕ, v gives us that $\partial_t v \in L^\infty([0,\infty); L^2(\mathbb{R}))$. We may assume that

$$\sup_{t\in\mathbb{R}^+} \int (\partial_t v(t,x))^2 dx < +\infty.$$

Denote

$$E_n = \{x \in \mathbb{R} | (\partial_t v(t,x))^2 \ge n \quad \text{for some} \ \ t > 0\}$$

and $N_2 = \bigcap_{n=1}^{\infty} E_n$. It follows that $n|E_n| \le \sup_{t\in\mathbb{R}^+} \int (\partial_t v(t,x))^2 dx$. Then $|N_2| = 0$, and $\forall\ x \in \mathbb{R}\setminus N_2$, there exists some $M \in \mathbb{N}$, such that

$$(\partial_t v(t,x))^2 \le M \quad \text{for all} \ \ t > 0. \tag{4.10}$$

Let $N = N_1 \cup N_2$. Then $|N| = 0$ and (4.8) (4.9) gives

$$\lim_{t\to+\infty} |v(t,x)| = 0 \quad \text{for} \ \ x \in \ \mathbb{R}\setminus N. \tag{4.11}$$

For any $x \in N$, there is a sequence $\{x_j\} \subset \mathbb{R}\setminus N$, such that $x_j \to x$ as $j \to +\infty$. Since

$$\begin{aligned} |v(t,x)| &\le |v(t,x_j)| + |v(t,x) - v(t,x_j)| \\ &\le |v(t,x_j)| + |x - x_j|^{\frac{1}{2}} \|v(t,\cdot)\|_{H^1} \\ &\le |v(t,x_j)| + C|x - x_j|^{\frac{1}{2}}, \end{aligned}$$

we conclude that

$$\lim_{t\to+\infty} |v(t,x)| = 0, \quad \text{for} \ \ x \in N. \tag{4.12}$$

Then (1.6) follows immediately from (4.10), (4.11). This completes the proof of the Theorem 1.1.

With Theorem 1.1 and Lemma 2.1 (v), it is easy to observe that the weak solution $u(t,x)$ of Cauchy problem (1.1) and (1.2) approaches, as $t \to \infty$, the rarefaction wave $w^R(x/t)$ determined by (2.1). We summarize this observation in the following Theorem.

Theorem 4.5. *Let $u_- < u_+$ and $u_0 - w_0^R \in H^1(\mathbb{R})$ and w_0^R given in (2.1). Then the Cauchy problem (1.1)-(1.2) has a global weak solution $u = u(t,x)$ satisfying*

$$\lim_{t\to+\infty} |u(t,x) - w^R(x/t)| = 0, \quad \text{for all} \ \ x \in \mathbb{R}.$$

Acknowledgements. We are very grateful to Professor Changjiang Zhu for bringing this problem to our attention and also for enormous invaluable discussion. We also thank one of the referees who gives useful comments and suggests many updated references. Guo is supported in part by the NSFC (Grant No.10401012) and a research grant at the Northwest University. Jiang is supported by the NSFC (Grant No. 10171037). Wang is supported by the F. S. China Scholarship of the University of Alberta and Zheng is supported in part by the NSFC (Grant No.10626023 and 10571069).

REFERENCES

[1] M. S. Alber, R. Camassa, D. D. Holm and J. E. Marsden, *The geometry of peaked solitons and billiard solutions o f a class of integrable PDEs*, Lett. Math. Phys., **32** (1994), 137–151.

[2] J. P. Boyd, *Peakons and coshodal waves: travelling wave solutions of the Camassa-Holm equation*, Appl. Math. Comput., **81** (1997), 173–181.

[3] R. Beals, D. Sattinger and J. Szmigielski, *Acoustic scattering and the extended korteweg-de vris hierarchy*, Adv. Math., **140** (1998), 190–206.

[4] R. Beals, D. Sattinger and J. Szmigielski, *Multi-peakons and a theorem of Stieltjes*, Inverse Problem, **15** (1999), 1–4.

[5] A. Bressan and A. Constantin, *Global conservative solutions of the Camassa-Holm equation*, Arch. Rat. Mech. Anal., **183** (2007), 215–239.

[6] R. Camassa and D. D. Holm, *An integrable shollow water equation with peaked solitons*, Phys. Rev. Letters, **71** (1993),1661–1664.

[7] R. Camassa, D. D. Holm and J. M. Hyman, *A new integrable shallow water equation*, Advance in Applied Mechanics, **31** (Y.-Y. Wu and J. W. Hutchinson, Eds), Academic, (1994) 1–33.

[8] A. Constantin, *On the inverse spectral problem for the Camassa-Holm equation*, J. Funct. Anal., **155** (1998), 352–363.

[9] A. Constantin, *Existence of permanent and breaking waves for a shallow water equation: a geometric approach*, Ann. Inst. Fourier(Grenoble), **50** (2000), 321–362.

[10] A. Constantin, *On the scattering problem for the Camassa-Holm equation*, Proc. Roy. Soc. London Ser. A, **457** (2001), 953–970.

[11] A. Constantin and J. Escher, *Wave breaking for nonlinear nonlocal shallow water equations*, Acta Mathematica, **181** (1998), 229–243.

[12] A. Constantin and J. Escher, *On the Cauchy problem for a family of quasilinear hyperbolic equations*, Commun. Partial Diff. Equations, **23** (1998), 1449–1458.

[13] A. Constantin, V. Gerdjikov and R.Ivanov, *Inverse scattering transform for the Camassa-Holm equation*, Inverse Problems, **22** (2006), 2197–2207.

[14] A. Constantin and B. Kolev, *Geodesic flow on the diffeomorphism group of the circle*, Comment. Math. Helv., **78** (2003), 787–804.

[15] A. Constantin and H. P. Mckean, *A shallow water equation on the circle*, Comm. Pure Appl. Math., **52** (1999), 949–982.

[16] A. Constantin and L. Molinet, *Orbital stability of solitary waves for a shallow water equation*, Physica D., **157** (2001), 75–89.

[17] A.Constantin and W. Strauss, *Stability of peakons*, Comm. Pure Appl. Math., **53** (2000), 603–610.

[18] F. Cooper and H. Shepard, *Solitons in the Camassa-Holm shallow water equation*, Phys. Lett., A, **194** (1994), 246–250.

[19] P. G. Drazin and R. S. Johnson, “Solitons: an Introduction,” Cambridge University Press, Cambridge, 1989.

[20] B. Fuchssteiner and A. S. Fokas, *Symplectic structures, their Bäcklund transformations and hereditary symmetries*, Phys. D, **4** (1981/82), 47–66.

[21] A. Jeffrey, “Nonlinear Wave Motion,” Pitman Monographs and Surveys in Pure and Applied Mathematics, 43.

[22] A. Jeffrey, *Decay estimates for parabolic conservation laws*, Proc. Estonian Acad. Sci. Phys. Math., **48** (1999), 265–267.

[23] T. P. Liu, A. Matsumura and K. Nishihara, *Behavior of solutions for the Burgerd equation with boundary corresponding to rarefaction waves*, SIAM J. Math. Anal., **29** (1998), 293–308.

[24] A. Matsumura and K. Nishihara, *Asymptotics toward the rarefaction waves of the solutions of a one-dimentional model system for compressible viscous gas*, Japan J. Appl. Math., **3** (1986), 1–13.

[25] K. Nishihara, *A note on the stability of travelling wave solution of Burgers equation*, Japan J. Appl. Math., **21** (1985), 27–35.

[26] J. Schiff, *Zero curvature formulations of dual hierarchies*, J. Math. Phys., **37** (1996), 1928–1938.

[27] J. Smoller, "Shock Waves and Reaction-Diffusion Equations," Springer-Verlag, New York, 1983.

[28] Z. A. Wang and C. J. Zhu, *Stability of the rarefaction wave for the generalized KdV-Burgers equation*, Acta. Math. Sci., **22B** (2002), 319–328.

[29] G. B. Whitham, "Linear and Nonlinear Waves, Pure and Applied Mathematics," Wiley-Intersciences, New York-London-Sydney, 1974.

[30] Z.-P. Xin and P. Zhang, *On the weak solutions to a shallow water equation*, Commun. Pure Appl. Math., **53** (2000), 1411–1433.

[31] T. Yang, H. J. Zhao and C. J. Zhu, *Asymptotic behavior of solution to a hyperbolic system with relaxition and boundary effect*, J. Differential Equations, **163** (2000), 348–380.

[32] C. J. Zhu, *Asymptotic behavior of solution for P-system with relaxation*, J. Differential Equations, **180** (2002), 273–306.

Received April 2007; revised October 2007.

E-mail address: gfzheng@mail.ccnu.edu.cn
E-mail address: zhenhua.guo.math@gmail.com
E-mail address: zhian@ualberta.ca
E-mail address: jmn3911@sina.com