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Global dynamics of an SIS epidemic model with cross-diffusion: applications to quarantine measures

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Abstract

This paper considers an SIS model with a cross-diffusion dispersal strategy for the infected individuals describing the public health intervention measures (like quarantine) during the outbreak of infectious diseases. The model adopts the frequency-dependent transmission mechanism and includes demographic changes (i.e. population recruitment and death) subject to homogeneous Neumann boundary conditions. We first establish the existence of global classical solutions with the uniform-in-time bound. Then, we define the basic reproduction number R_0 by a weighted variational form. Due to the presence of the cross-diffusion on infected individuals, we employ a change of variable and apply the index theory along with the principal eigenvalue theory to establish the threshold dynamics in terms of R_0 based on the fact that the sign of the principal eigenvalue of the weighted eigenvalue problem is the same as that of the corresponding unweighted eigenvalue problem. Furthermore, we obtain the global stability of the unique disease-free equilibrium and constant endemic equilibrium under some conditions. Finally, we discuss some open questions and use numerical simulation to demonstrate the applications of our analytical results, showing that the cross-diffusion dispersal strategy can reduce the value

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of R_0 and help eradicate the diseases even if the habitat is high-risk in contrast to the situation without cross-diffusion.

Keywords: SIS epidemic model, cross-diffusion, basic reproduction number, threshold dynamics

Mathematics Subject Classification numbers: 35A01, 35B40, 35B44, 35K57, 35Q92, 92C17

1. Introduction and main results

1.1. SIS models with cross-diffusion

Infectious diseases [7, 8, 22] have brought in tremendous impacts on public health and the global economy such as the unprecedented novel coronavirus disease 2019 (COVID-19). Mathematical modelings and analysis of infectious diseases have had a long history and numerous results are available (see [18, 21, 38]). In epidemiology, the basic reproduction number of an infection, denoted by R_0, is the expected number of cases directly generated by one case in a population where all individuals are susceptible to infection. This number is the threshold determining if an emerging infectious disease can spread in a population. Specifically, the infection persists if $R_0 > 1$ while becomes extinct in the long run if $R_0 < 1$. Generally, the larger the value of R_0, the harder it is to control the epidemic. It is therefore of mathematical and biological importance to properly define and give explicit estimates of R_0 (see [20, 50]). It is noteworthy that the value of R_0 can vary, even for the same disease strain, depending on external factors such as environmental conditions, public health policy governing the detection and movement pattern of the infected population, and so on. Among a large number of mathematical works based on reaction-diffusion (or with advection) models (see [2, 32, 39, 51] and reference therein), most (if not all) mathematical models have assumed that both susceptible and infected individuals employ homogeneous diffusive movements. However, this assumption leaves out the effects of human behaviors and public health quarantine measures on the mobility of individuals during the outbreak of disease such as COVID-19 [22, 28, 49]. To fill this gap, this paper aims to introduce the cross-diffusion for the infected individuals (i.e. the dispersal of the infected individuals depend on the density of the susceptible population) into the SIS model and explore the effect of the human intervention on the propagation of infectious diseases, particularly on the basic reproduction number R_0. There are many SIS epidemic models, we choose, among others, the SIS model with frequency-dependent transmission mechanism (see [16]) and demographic change (i.e. population growth/recruitment and death). That is, denoting the population density of the susceptible and infected individuals at position $x \in \Omega \subset \mathbb{R}^n$ and time $t > 0$ by $S(x,t)$ and $I(x,t)$, respectively, we consider the following SIS model with cross-diffusion on I:

$$\begin{cases} S_t = d_S \Delta S + \Lambda(x) - \theta S - \alpha(x)\frac{SI}{S+I} + \beta(x) I, & x \in \Omega, t > 0, \\ I_t = d_I \Delta[\gamma(S) I] + \alpha(x)\frac{SI}{S+I} - [\beta(x) + \eta(x)] I, & x \in \Omega, t > 0, \\ \partial_\nu S = \partial_\nu I = 0, & x \in \partial\Omega, t > 0, \\ (S(x,0), I(x,0)) = (S_0, I_0), & x \in \Omega, \end{cases} \tag{1.1}$$

where the homogeneous Neumann boundary condition means that no individuals cross the habitat boundary $\partial\Omega$; the susceptible individuals are assumed to move randomly with rate d_S

while infected individuals adopt a density-dependent type cross-diffusion with a positive rate function $\gamma(S) \in C^3([0,\infty))$ satisfying

$$\gamma'(S) > 0 \text{ for all } S \in [0,\infty). \tag{1.2}$$

Note that $\Delta[\gamma(S)I] = \nabla\cdot(\gamma(S)\nabla I) + \nabla\cdot(I\gamma'(S)\nabla S)$. The cross-diffusion along with the condition (1.2) indicates that the infected individuals will move away from the area with a higher density of susceptible individuals (like quarantine measure) while dispersing themselves at a rate increasing with respect to the density of susceptible individuals (crowd avoidance). The model (1.1) has included demography changes (recruitment and death of population), where the recruitment of the susceptible population is represented by $\Lambda(x) - \theta S$ with $\Lambda(x)$ being a non-negative Hölder continuous function on $\overline{\Omega}$ and $\theta \geqslant 0$ being a constant; $\alpha(x)$, $\beta(x)$ and $\eta(x)$ are non-negative Hölder continuous functions on $\overline{\Omega}$ accounting for the disease transmission rate, recovery rate, and death rate of the infected individuals, respectively.

Relevant results on (1.1) with $\gamma(S) = 1$. To put our research into perspective, we recall some related results developed for the SIS model (1.1) with $\gamma(S) = 1$. When the demographic changes are not considered (i.e. $\Lambda(x) = \theta = \eta(x) = 0$), by integrating the sum of the two equations of (1.1), one immediately finds that the total population is conserved, namely

$$\int_\Omega [S(x,t) + I(x,t)]\,\mathrm{d}x = \int_\Omega (S_0 + I_0)\,\mathrm{d}x =: N, \quad \forall\, t > 0,$$

where the constant $N > 0$ denotes the number of total population. For this case, Allen *et al* [2] first introduced the basic reproduction number R_0 via a variational formula and established the existence, uniqueness and global stability of the *disease-free equilibrium* (DFE) if $R_0 < 1$. When $R_0 > 1$, they proved the existence and uniqueness of the *endemic equilibrium* (EE), and explored the asymptotic behavior of the unique EE as $d_S \to 0$. Particularly, they conjectured that this unique EE is globally stable, which was later confirmed in [41] for the cases of $d_I = d_S$ or $\alpha(x) = r\beta(x)$ with constant $r > 1$. The results in [41] imply that the disease will persist in the high-risk domain Ω (namely $\int_\Omega \alpha(x)\mathrm{d}x > \int_\Omega \beta(x)\mathrm{d}x$). When α and β are temporally and spatially inhomogeneous, Peng and Zhao [43] showed that the disease will persist in the high-risk domain Ω, and the joint effect of spatial heterogeneity and temporal periodicity may enhance the persistence of the disease. In addition, [19] explored the existence of traveling wave solutions. When the demographic changes are included (i.e. $\Lambda(x)$, θ, $\eta(x) > 0$), the total population is no longer conserved and the analysis will be more involved. The first result seemed to be obtained by Li *et al* in [31] where the global existence and boundedness of classical solutions as well as the threshold dynamics in terms of the basic reproduction number R_0 were studied. By the uniform persistence theory, they showed that the disease will persist uniformly and hence at least one EE exists in the high-risk domain. The asymptotic profiles of EE for large and/or small of d_S or d_I were further obtained in [31]. These findings imply that a varying total population may enhance disease persistence, thereby posing greater challenges to the disease control. In addition, Li *et al* [33] introduced an infectious population oriented taxis advection term for S (i.e. the susceptible moves away from the density gradient of the infected individuals) with varying/conserved total population and showed that such a cross-diffusion does not contribute to eradication of the disease. Last but not least, we refer readers to [34, 47, 48] for some results on SIS models with taxis movement in the S-equation, and [11, 17, 32, 42, 52, 53] and the references therein for more results on various SIS epidemic models with random diffusion.

This paper aims to study the SIS epidemic model (1.1) with cross-diffusion for I and explore how the cross-diffusion dispersal strategy can play positive roles in controlling the spread of disease. Since this cross-diffusion describes the outcome of quarantine measures to the population mobility during the outbreak of infectious disease, our results will elucidate whether the quarantine measures help to control the disease spread from a theoretical perspective. As we know, this is the first work on the SIS epidemic model (1.1) with cross-diffusion (i.e. $\gamma(S)$ is non-constant) and there are no results available for such kind of models. Our main goals include the following:

- Establish the global well-posedness of solutions (global existence and stability) to (1.1) under suitable conditions;
- Investigate the effects of cross-diffusion on the persistence and extinction of the infectious disease.

The main challenge in the analyses arises from the cross-diffusion structure in the I-equation. The SIS model with taxis-like advection in the S-equation considered in [33] is significantly different from (1.1) with the cross-diffusion in the I-equation. For the model of [33], the L^∞ boundedness of I can be directly obtained from the I-equation based on the boundedness of L^1 by using the result 'L^1-boundedness implies L^∞-boundedness' for classical reaction-diffusion equations proved in [1]. But for the cross-diffusion SIS system (1.1), the boundedness of I can not be obtained directly from the I-equation alone. This needs more complicated coupling estimates to establish the global boundedness of solutions under the structural hypothesis (H2) below, as shown in section 2.

1.2. Main results

Throughout this paper, we suppose that the initial value (S_0, I_0) satisfies

$$0 \leqslant S_0 \in W^{1,\infty}(\Omega),\ I_0 \in C(\overline{\Omega}) \text{ with } I_0 \geqslant 0 \text{ and } \int_\Omega I_0(x)\,\mathrm{d}x > 0, \tag{1.3}$$

and the following conditions hold:

(H0) The functions $\Lambda(x),\ \alpha(x),\ \beta(x), \eta(x)$ are positive and Hölder continuous on $\overline{\Omega}$, and θ is a positive constant.

Moreover, $\gamma(S)$ is assumed to fulfill the following conditions:

(H1) $\gamma(S) \in C^3([0,\infty))$, $\gamma'(S) > 0$ and $\gamma(0) = 1$;
(H2) There exist some positive constants $\mathcal{K}_0$ and $\mathcal{K}_1$ such that $\gamma(S) \leqslant \mathcal{K}_0$ and $\gamma'(S) \leqslant \mathcal{K}_1$.

Note that in the hypothesis (H1), $\gamma(0)$ can be any positive constant, which however can be absorbed into d_I. Hence, we simply assume $\gamma(0) = 1$ without loss of generality.

Our first main result concerning the global boundedness of solutions is given below.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and hypotheses (H0)-(H2) hold. Then (1.1) with (1.3) admits a unique classical solution $(S,I) \in [C(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty))]^2$ satisfying $S, I > 0$ on $\overline{\Omega} \times (0,\infty)$. Moreover, there exists a constant $C > 0$ independent of $\mathcal{K}_1$ such that*

$$\|S\|_{W^{1,\infty}} + \|I\|_{L^\infty} \leqslant C\left(1 + \mathcal{K}_1^{12}\right)\mathrm{e}^{C\left(1+\mathcal{K}_1^4\right)} =: M(\mathcal{K}_1) \text{ for all } t > 0. \tag{1.4}$$

Remark 1.1. When considering the mass action infection mechanism (see [26]), namely $\frac{SI}{S+I}$ is replaced by SI in (1.1), theorem 1.1 can hold without the assumption (H2) since the boundedness of S can follow directly from the comparison principle.

Next we shall explore how the cross-diffusion affects the basic reproduction number R_0. To this end, we consider the stationary problem

$$\begin{cases} d_S\Delta S + \Lambda(x) - \theta S - \alpha(x)\frac{SI}{S+I} + \beta(x) I = 0, & x\in\Omega,\\ d_I\Delta[\gamma(S)I] + \alpha(x)\frac{SI}{S+I} - [\beta(x)+\eta(x)]I = 0, & x\in\Omega,\\ \partial_\nu S = \partial_\nu I = 0, & x\in\partial\Omega. \end{cases} \tag{1.5}$$

It is easy to check that (1.5) has a unique semi-trivial solution $(\widetilde{S}(x),0) =: (\widetilde{S},0)$ satisfying $0<\widetilde{S}\leqslant \frac{1}{\theta}\max_{x\in\overline{\Omega}}\Lambda(x)$ and

$$d_S\Delta S + \Lambda(x) - \theta S = 0 \text{ in } \Omega;\ \partial_\nu S = 0 \text{ on } \partial\Omega.$$

$(\widetilde{S},0)$ is called the DFE. An EE, denoted by $(\hat{S}(x),\hat{I}(x))$, is a solution of (1.5) satisfying $\hat{I}(x)\geqslant 0$ and $\hat{I}(x)\not\equiv 0$ on Ω . In fact, if EE exists, then the maximum principle and the Hopf boundary lemma for elliptic equations assert that $\hat{S}(x)>0, \hat{I}(x)>0$ in $\overline{\Omega}$. By the nomenclature from [2], we define the low-risk site Ω^- and the high-risk site Ω^+ as:

$$\Omega^- = \{x\in\Omega:\ \alpha(x)<\beta(x)+\eta(x)\},\quad \Omega^+ = \{x\in\Omega:\ \alpha(x)>\beta(x)+\eta(x)\}.$$

The domain Ω is called a low-risk domain if $\int_\Omega \alpha(x)\mathrm{d}x < \int_\Omega[\beta(x)+\eta(x)]\mathrm{d}x$ and a high-risk domain if $\int_\Omega\alpha(x)\mathrm{d}x \geqslant \int_\Omega[\beta(x)+\eta(x)]\mathrm{d}x$.

Now we define the basic reproduction number R_0 of (1.1) by the following variational form (see the motivation detailed in section 3.1):

$$R_0 := R_0\big(d_I,\gamma(\widetilde{S})\big) = \sup_{0\neq w\in H^1(\Omega)} \frac{\int_\Omega \alpha(x) w^2\mathrm{d}x}{\int_\Omega\left\{d_I|\nabla(\gamma^{\frac{1}{2}}(\widetilde{S})w)|^2 + (\beta(x)+\eta(x))w^2\right\}\mathrm{d}x}. \tag{1.6}$$

When the infected individuals take random movement (i.e. $\gamma(S)=1$), we denote the basic reproduction number by $\hat{R}_0$ given in [31]. Below we present some qualitative properties of R_0 in terms of d_I, which can be readily proved by the proofs of [2, lemma 2.2] and [36, lemma 3.1]. We skip the details here for brevity.

Proposition 1.2. *Let $q_1(x) := \alpha(x)\gamma^{-1}(\widetilde{S})$, $q_2(x) := [\beta(x)+\eta(x)]\gamma^{-1}(\widetilde{S})$ and $q(x) := \frac{q_1(x)}{q_2(x)}$ with $\gamma^{-1}(\widetilde{S}) = 1/\gamma(\widetilde{S})$. Under hypotheses (H0)-(H1), the following results hold.*

(i) *R_0 is strictly decreasing in d_I provided that Ω^- and Ω^+ are nonempty. Moreover, $R_0\to\max\{q(x):\ x\in\overline{\Omega}\}$ as $d_I\to 0$ and $R_0\to\int_\Omega q_1(x)dx/\int_\Omega q_2(x)\mathrm{d}x$ as $d_I\to\infty$;*
(ii) *If $\int_\Omega q_1(x)dx > \int_\Omega q_2(x)dx$, then $R_0>1$ for all $d_I>0$;*
(iii) *If $\int_\Omega q_1(x)dx < \int_\Omega q_2(x)dx$, then there admits a unique positive constant d_I^* such that $R_0>1$ (resp. $R_0<1$) for $d_I<d_I^*$ (reps. $d_I>d_I^*$) when Ω^- and Ω^+ are nonempty.*

Remark 1.2. If Ω^- and Ω^+ are nonempty and $\Lambda(x)$ is a constant, then $\widetilde{S}>0$ is a constant. This along with the monotonicity of R_0 in proposition 1.2-(i) yields $R_0<\hat{R}_0$, and hence implies that the cross-diffusion can reduce the value of R_0. In other words, the intervention measures are effective for controlling the spread of diseases (see more discussion in section 5).

Remark 1.3. If Ω is a high-risk domain, namely $\int_\Omega\alpha(x)\mathrm{d}x\geqslant\int_\Omega[\beta(x)+\eta(x)]\mathrm{d}x$, we can choose a rate function $\gamma(S)$ such that $\int_\Omega\alpha(x)\gamma^{-1}(\widetilde{S})\mathrm{d}x < \int_\Omega[\beta(x)+\eta(x)]\gamma^{-1}(\widetilde{S})\mathrm{d}x$ (see a specific example in section 5). By proposition 1.2-(iii), there exists a unique d_I^* such that $R_0<1$

whenever $d_I > d_I^*$, which is substantially different from the well-known results with random diffusion (i.e. $\gamma(S) = 1$) for which the basic reproduction number $\hat{R}_0 > 1$ for all $d_I > 0$ (e.g. [31, proposition 3.2 (c)], [43, theorem 2.5 (a)]).

The basic reproduction number R_0 normally can determine threshold dynamics. Specifically, if $R_0 > 1$ (resp. $R_0 < 1$), the disease persists (resp. becomes extinct). The following theorem indicates that R_0 defined in (1.6) can determine the threshold dynamics locally.

Theorem 1.3. *Let hypotheses (H0)-(H2) hold. Then the following statements hold.*

(i) *If $R_0 < 1$, then DFE $(\widetilde{S}, 0)$ is linearly stable;*
(ii) *If $R_0 > 1$, then DFE $(\widetilde{S}, 0)$ is linearly unstable and (1.1) admits at least one EE.*

Remark 1.4. The uniqueness of non-trivial EE in general and the existence of non-trivial EE when $R_0 \leqslant 1$ remain open.

Finally we prove the global stability of DFE and EE depending on the sign of $1 - R_0$.

Theorem 1.4. *Let (S, I) be the solution obtained in theorem 1.1. The following statements hold.*

(i) *If $\alpha(x) \leqslant \beta(x) + \varepsilon\eta(x)$ with any fixed constant $0 \leqslant \varepsilon < 1$, then $R_0 < 1$ and DFE is globally asymptotically stable with*

$$\|S - \widetilde{S}(x)\|_{L^\infty} + \|I\|_{L^\infty} \leqslant M_1 \mathrm{e}^{-\kappa_1 t} \text{ for all } t > 1. \tag{1.7}$$

(ii) *Assume that Λ, α, β, η are all positive constants. If $\alpha > \beta + \eta$ (i.e. $R_0 > 1$), then the unique constant EE $(\hat{S}, \hat{I})$ defined in (4.8) is globally asymptotically stable provided that $\theta = \eta$ and*

$$2d_S d_I + 4d_S d_I M_2 > d_I^2 \mathcal{K}_0^2 + d_S^2 + d_I \mathcal{K}_1 H(\mathcal{K}_1), \tag{1.8}$$

where $H(\mathcal{K}_1) = M(\mathcal{K}_1)[1 + M_2(M(\mathcal{K}_1) + 1)^2]\{2d_S + \mathcal{K}_1 M(\mathcal{K}_1) d_I [1 + M_2(M(\mathcal{K}_1) + 1)^2]\}$ with $M_2 := \frac{4\eta[\Lambda(\alpha-\beta-\eta)+\eta\alpha]}{(\alpha-\beta-\eta)^2(\Lambda+\eta)}$.

Remark 1.5. If $\gamma(S) = 1$, the global stability of DEF can be proved with the mere condition $R_0 < 1$ (see [31]). Here we give a more sufficient condition. If Λ, α, β, η are positive constant, it follows from proposition 1.2-(i) that $\mathcal{R}_0 = \frac{\alpha}{\beta+\eta}$. Thus, $\alpha > \beta + \eta$ is equivalent to $\mathcal{R}_0 > 1$. In addition, $\theta = \eta$ is a technical assumption, which is not needed in the case $\gamma(S) = 1$ and $d_I = d_S$ (see [31]).

Remark 1.6. Since $M(\mathcal{K}_1) > 0$ is an increasing function of $\mathcal{K}_1$ and M_2 is independent of $\mathcal{K}_1$, the condition (1.8) can be achieved by choosing $\mathcal{K}_1$ small. For example, fixing $\Lambda = \eta = \theta = d_S = 1$, $\alpha = d_I = 2.5$, $\beta = 0.5$ and taking $\gamma(S) = \mathcal{K}_0 - \frac{\mathcal{K}_0 - 1}{S+1}$ with $1 < \mathcal{K}_0 \ll 2$, then $\gamma'(S) = \frac{\mathcal{K}_0 - 1}{(S+1)^2} \leqslant (\mathcal{K}_0 - 1) =: \mathcal{K}_1$. Let $\mathcal{K}_0$ be close to 1 (i.e. $\mathcal{K}_1$ be close to 0) such that $d_I \mathcal{K}_1 H(\mathcal{K}_1) \ll 1$, then $2d_S d_I + 4d_S d_I M_2 = 75 > d_I^2 2^2 + d_S^2 + 1 = 27 > d_I^2 \mathcal{K}_0^2 + d_S^2 + d_I \mathcal{K}_1 H(\mathcal{K}_1)$, and thus (1.8) holds.

The rest of this paper is organized as follows. In section 2, we shall establish the global existence and boundedness of solutions to (1.1). Then we are devoted to introducing the basic reproduction number R_0 and discussing the properties of R_0 as well as the threshold dynamics, especially the existence of EE in section 3. In section 4, we focus on exploring the globally asymptotical stability of solutions to (1.1). Finally, in section 5, we use numerical simulations to illustrate some applications of our analytical results and speculate some possible results for future study.

2. Global boundedness and existence: proof of theorem 1.1

In this section, we will study the global existence and boundedness of solutions to (1.1). Throughout this paper, we abbreviate $\int_\Omega f \mathrm{d}x$ and $\|f\|_{L^p(\Omega)}$ as $\int_\Omega f$ and $\|f\|_{L^p}$, respectively. c_i and $C_i\,(i=1,2,3,\cdots)$ are used to denote generic positive constants, which may vary in the context and are independent of t and $\mathcal{K}_1$.

2.1. Local existence and preliminaries

We first show the local solvability of (1.1).

Lemma 2.1 (local existence). *Let the conditions in theorem 1.1 hold. Then there admits a $T_{\max}\in(0,\infty]$ such that (1.1) has a unique classical solution $(S,I)\in[C(\overline{\Omega}\times[0,T_{\max}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{\max}))]^2$ with $S,I>0$ on $\overline{\Omega}\times(0,T_{\max})$. Moreover,*

$$\text{if } T_{\max}<\infty,\ \text{then}\ \lim_{t\to T_{\max}}\left(\|S\|_{W^{1,\infty}}+\|I\|_{L^\infty}\right)=\infty. \tag{2.1}$$

Proof. Denote $U:=(S,I)^{\mathcal{T}}$, where $\mathcal{T}$ represents the transpose. Then (1.1) can be rewritten as

$$\begin{cases} U_t=\nabla\cdot(\mathbb{M}(U)\nabla U)+\mathbb{F}(U), & x\in\Omega,t>0,\\ \partial_\nu U=0, & x\in\partial\Omega,t>0,\\ U(x,0)=(S_0,I_0)^{\mathcal{T}}, & x\in\Omega,\end{cases}$$

where

$$\mathbb{M}(U)=\begin{pmatrix} d_S & 0\\ d_I I\gamma'(S) & d_I\gamma(S)\end{pmatrix}\ \text{and}\ \mathbb{F}(U)=\begin{pmatrix}\Lambda(x)-\theta S-\alpha(x)\frac{SI}{S+I}+\beta(x)I\\ \alpha(x)\frac{SI}{S+I}-[\beta(x)+\eta(x)]I\end{pmatrix}.$$

By (1.3) and the assumption (H1), we see that the matrix $\mathbb{M}(U)$ is positive definite, which implies that (1.1) is uniformly parabolic. Moreover, $\mathbb{M}(U)$ is a lower triangular matrix. Then the existence and uniqueness of local classical solutions and (2.1) follow from [4–6], while the positivity of S and I follows from the strong maximum principle, see e.g. [25, lemma 2.1]. □

In the rest of this paper, we denote

$$g_*=\min_{x\in\overline{\Omega}} g \text{ and } g^*=\max_{x\in\overline{\Omega}} g \text{ for } g\in\{\Lambda(x),\alpha(x),\beta(x),\eta(x)\}. \tag{2.2}$$

Lemma 2.2. *Let (S, I) be the solution obtained in lemma 2.1. Then there exists a constant $C_1>0$ such that*

$$\|S(\cdot,t)\|_{L^1}+\|I(\cdot,t)\|_{L^1}\leqslant C_1 \text{ for all } t\in(0,T_{\max}).$$

Proof. Adding the first two equations of (1.1) and integrating the result by parts, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(S+I)+\min\{\theta,\eta_*\}\int_\Omega(S+I)\leqslant\Lambda^*|\Omega|.$$

This along with Grönwall's inequality indicates

$$\int_\Omega(S+I)\leq\frac{\Lambda^*|\Omega|}{\min\{\theta,\eta_*\}}+\int_\Omega(S_0+I_0)=:C_1,$$

where η_* and Λ^* are defined in (2.2). Hence, the proof of lemma 2.2 is completed. □

Lemma 2.3. *Let (S,I) be the solution obtained in lemma 2.1. Then there exists a constant $C_2>0$ such that*

$$\int_t^{t+\tau}\int_\Omega I^2 \leqslant C_2 \text{ for all } t\in\left(0,\widetilde{T}_{\max}\right), \tag{2.3}$$

where τ is a constant such that

$$0<\tau<\min\{1,T_{\max}\} \text{ and } \widetilde{T}_{\max}:=T_{\max}-\tau. \tag{2.4}$$

Proof. We add the first equation of (1.1) with the second one to get

$$(S+I)_t=\Delta\left(d_S S+d_I\gamma(S)I\right)+\Lambda(x)-\theta S-\eta(x)I,$$

which, along with hypothesis (H2), can be rewritten as

$$\begin{aligned}(S+I)_t+\mathcal{A}\left(d_S S+d_I\gamma(S)I\right)&=\left(\delta d_I\gamma(S)-\eta(x)\right)I+\left(\delta d_S-\theta\right)S+\Lambda(x)\\&\leqslant\left(\delta d_I\mathcal{K}_0-\eta_*\right)I+\left(\delta d_S-\theta\right)S+\Lambda^*\leqslant\Lambda^*,\end{aligned} \tag{2.5}$$

where $\delta:=\min\left\{\frac{\eta_*}{d_I\mathcal{K}_0},\frac{\theta}{d_S}\right\}>0$, and $\mathcal{A}$ is the self-adjoint realisation of $-\Delta+\delta$ subject to homogeneous Neumann boundary conditions in $L^2(\Omega)$. Then $\mathcal{A}$ is invertible with bounded inverse by the Fredholm alternative theorem. Hence there is a constant $c_1>0$ such that

$$\|\mathcal{A}^{-1}\phi\|_{L^2}\leqslant c_1\|\phi\|_{L^2} \text{ for all } \phi\in L^2(\Omega), \tag{2.6}$$

and

$$\|\mathcal{A}^{-\frac{1}{2}}\phi\|_{L^2}^2=\int_\Omega\phi\cdot\mathcal{A}^{-1}\phi\,\mathrm{d}x\leqslant c_1\|\phi\|_{L^2}^2 \text{ for all } \phi\in L^2(\Omega). \tag{2.7}$$

We multiply (2.5) by $\mathcal{A}^{-1}(S+I)\geqslant 0$ to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\mathcal{A}^{-\frac{1}{2}}(S+I)|^2+\int_\Omega\left(d_S S+d_I\gamma(S)I\right)(S+I)\leqslant\Lambda^*\int_\Omega\mathcal{A}^{-1}(S+I),$$

which together with $\gamma(S)\geqslant 1$ (see hypothesis (H1)) gives a constant $c_2:=\min\{d_S,d_I\}$ such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\mathcal{A}^{-\frac{1}{2}}(S+I)|^2+c_2\int_\Omega(S+I)^2\leqslant\Lambda^*\int_\Omega\mathcal{A}^{-1}(S+I). \tag{2.8}$$

Using (2.6) and (2.7) along with Hölder inequality and Young's inequality yields

$$\Lambda^*\int_\Omega\mathcal{A}^{-1}(S+I)\leqslant\Lambda^*|\Omega|^{\frac{1}{2}}c_1\|S+I\|_{L^2}\leqslant\frac{c_2}{4}\|S+I\|_{L^2}^2+\frac{(\Lambda^*)^2|\Omega|c_1^2}{c_2}, \tag{2.9}$$

and

$$\frac{c_2}{4c_1}\int_\Omega|\mathcal{A}^{-\frac{1}{2}}(S+I)|^2\leqslant\frac{c_2}{4}\|S+I\|_{L^2}^2. \tag{2.10}$$

We substitute (2.9) and (2.10) into (2.8) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\mathcal{A}^{-\frac{1}{2}}(S+I)|^2+\frac{c_2}{2c_1}\int_\Omega|\mathcal{A}^{-\frac{1}{2}}(S+I)|^2+c_2\int_\Omega(S+I)^2\leqslant\frac{2(\Lambda^*)^2|\Omega|c_1^2}{c_2}=:c_3. \tag{2.11}$$

Applying Grönwall's inequality to (2.11) and using (2.7) again, one has

$$\int_\Omega|\mathcal{A}^{-\frac{1}{2}}(S+I)|^2\leq\frac{2c_1c_3}{c_2}+c_1\left(\|S_0\|_{L^2}^2+\|I_0\|_{L^2}^2\right)=:c_4. \tag{2.12}$$

We integrate (2.11) over $(t,t+\tau)$ and apply (2.12) to get

$$c_2 \int_t^{t+\tau} \int_\Omega I^2 \leqslant c_2 \int_t^{t+\tau} \int_\Omega (S+I)^2 \leqslant c_3 + c_4,$$

which yields (2.3) by letting $C_2 := (c_3+c_4)/c_2$. This finishes the proof of lemma 2.3. □

Lemma 2.4. *Let (S,I) be the solution obtained in lemma 2.1. Then there exist two positive constants C_3 and C_4 such that*

$$\|\nabla S(\cdot,t)\|_{L^2} \leqslant C_3 \text{ for all } t \in (0,T_{\max}), \tag{2.13}$$

and

$$\int_t^{t+\tau} \int_\Omega |\Delta S|^2 \leqslant C_4 \text{ for all } t \in \left(0,\widetilde{T}_{\max}\right), \tag{2.14}$$

where τ and $\widetilde{T}_{\max}$ are defined in (2.4).

Proof. We multiply the first equation of (1.1) by $-\Delta S$ and apply Young's inequality to get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla S|^2 + d_S \int_\Omega |\Delta S|^2 &= -\int_\Omega \Lambda(x)\Delta S + \theta\int_\Omega S\Delta S + \int_\Omega \left(\alpha(x)\frac{SI}{S+I} - \beta(x) I\right)\Delta S \\ &\leqslant \Lambda^* \int_\Omega |\Delta S| - \theta \int_\Omega |\nabla S|^2 + (\alpha^*+\beta^*)\int_\Omega I|\Delta S| \\ &\leqslant \frac{d_S}{2}\int_\Omega |\Delta S|^2 - \theta\int_\Omega |\nabla S|^2 + \frac{(\alpha^*+\beta^*)^2}{d_S}\int_\Omega I^2 + \frac{(\Lambda^*)^2|\Omega|}{d_S},\end{aligned}$$

which indicates

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla S|^2 + 2\theta\int_\Omega |\nabla S|^2 + d_S\int_\Omega |\Delta S|^2 \leqslant c_1 \int_\Omega I^2 + c_2 =: h(t), \tag{2.15}$$

where $c_1 = \frac{2(\alpha^*+\beta^*)^2}{d_S}$ and $c_2 = \frac{2(\Lambda^*)^2|\Omega|}{d_S}$. Moreover, it follows from (2.3) that $\int_t^{t+\tau} h(s)ds \leqslant c_1 C_2 + c_2 =: c_3$. This along with [23, lemma 2.4] gives

$$\int_\Omega |\nabla S|^2 \leqslant c_3 + 2\left(\|\nabla S_0\|_{L^2}^2 + 3c_3 + 6\theta\tau + c_3/2\theta\tau + 1\right) =: C_3^2,$$

which implies (2.13) directly. Integrating (2.15) over $(t,t+\tau)$ yields

$$d_S \int_t^{t+\tau}\int_\Omega |\Delta S|^2 \leqslant c_3 + C_3^2.$$

Thus (2.14) holds with $C_4 := (c_3 + C_3^2)/d_S$ and the proof of lemma 2.4 is finished. □

2.2. Boundedness of solutions

We first derive the *a priori* L^2-estimate of I.

Lemma 2.5. *Let (S,I) be the solution obtained in lemma 2.1. Then there exists a constant $C_5 > 0$ such that*

$$\|I(\cdot,t)\|_{L^2} \leqslant e^{C_5\left(\mathcal{K}_1^2+1\right)^2} \text{ for all } t \in (0,T_{\max}). \tag{2.16}$$

Proof. Multiplying the second equation of (1.1) by I and integrating the result by parts, one has

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega I^2 &= -d_I\int_\Omega \gamma(S)|\nabla I|^2 - d_I\int_\Omega I\gamma'(S)\nabla S\cdot\nabla I + \int_\Omega \alpha(x)\frac{SI^2}{S+I} - \int_\Omega[\beta(x)+\eta(x)]I^2\\ &\leqslant -d_I\int_\Omega\gamma(S)|\nabla I|^2 + d_I\int_\Omega\gamma'(S)I|\nabla S||\nabla I| + \alpha^*\int_\Omega I^2 - (\beta_*+\eta_*)\int_\Omega I^2,\end{aligned}$$

which, along with hypotheses (H1) and (H2), gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega I^2 + 2d_I\int_\Omega|\nabla I|^2 + 2(\beta_*+\eta_*)\int_\Omega I^2 \leqslant 2d_I\mathcal{K}_1\int_\Omega I|\nabla S||\nabla I| + 2\alpha^*\int_\Omega I^2. \tag{2.17}$$

With Young's inequality and Hölder inequality, we have

$$2d_I\mathcal{K}_1\int_\Omega I|\nabla S||\nabla I| \leqslant d_I\int_\Omega|\nabla I|^2 + d_I\mathcal{K}_1^2\|I\|_{L^4}^2\|\nabla S\|_{L^4}^2,$$

which, substituted into (2.17), gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega I^2 + d_I\int_\Omega|\nabla I|^2 + 2(\beta_*+\eta_*)\int_\Omega I^2 \leqslant d_I\mathcal{K}_1^2\|I\|_{L^4}^2\|\nabla S\|_{L^4}^2 + 2\alpha^*\|I\|_{L^2}^2. \tag{2.18}$$

On the other hand, we use Gagliardo–Nirenberg inequality in two dimensions to get

$$\|I\|_{L^4}^2 \leqslant c_1\left(\|\nabla I\|_{L^2}\|I\|_{L^2} + \|I\|_{L^2}^2\right), \tag{2.19}$$

and the estimate (see [24, lemma 2.5])

$$\|\nabla S\|_{L^4}^2 \leqslant c_2\left(\|\Delta S\|_{L^2}\|\nabla S\|_{L^2} + \|\nabla S\|_{L^2}^2\right) \leqslant c_2C_3\left(\|\Delta S\|_{L^2} + C_3\right), \tag{2.20}$$

where we have used (2.13). The combination of (2.19) with (2.20) yields

$$\begin{aligned}d_I\mathcal{K}_1^2\|I\|_{L^4}^2\|\nabla S\|_{L^4}^2 \leqslant& d_I\mathcal{K}_1^2c_1c_2C_3\left(\|\nabla I\|_{L^2}\|I\|_{L^2} + \|I\|_{L^2}^2\right)\left(\|\Delta S\|_{L^2} + C_3\right)\\ \leqslant& d_I\mathcal{K}_1^2c_1c_2C_3\|\nabla I\|_{L^2}\|I\|_{L^2}\|\Delta S\|_{L^2} + d_I\mathcal{K}_1^2c_1c_2C_3^2\|\nabla I\|_{L^2}\|I\|_{L^2}\\ &+ d_I\mathcal{K}_1^2c_1c_2C_3\|I\|_{L^2}^2\|\Delta S\|_{L^2} + d_I\mathcal{K}_1^2c_1c_2C_3^2\|I\|_{L^2}^2\\ \leqslant& d_I\|\nabla I\|_{L^2}^2 + c_3\mathcal{K}_1^4\|I\|_{L^2}^2\|\Delta S\|_{L^2}^2 + c_4\left(1+\mathcal{K}_1^2\right)^2\|I\|_{L^2}^2\end{aligned} \tag{2.21}$$

with $c_3 := d_Ic_1^2c_2^2C_3^2$ and $c_4 := \frac{d_I(1+c_1c_2C_3^2)^2}{2}$. Substituting (2.21) into (2.18) gives a constant $c_5 := c_4 + 2\alpha^*$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|I\|_{L^2}^2 \leqslant \left[c_3\mathcal{K}_1^4\|\Delta S\|_{L^2}^2 + c_5\left(1+\mathcal{K}_1^2\right)^2\right]\|I\|_{L^2}^2. \tag{2.22}$$

Furthermore, (2.3) motivates us to find a positive constant $t_1 \in [(t-\tau)_+, t)$ for any $t \in (0, T_{\max})$ such that

$$\|I(\cdot,t_1)\|_{L^2}^2 \leqslant \max\left\{\|I_0\|_{L^2}^2, C_2/\tau\right\} =: c_6. \tag{2.23}$$

It then follows from (2.14) that

$$\int_{t_1}^{t_1+\tau}\int_\Omega|\Delta S|^2 \leqslant C_4. \tag{2.24}$$

Noting $t_1 < t \leqslant t_1 + \tau \leqslant t_1 + 1$, (2.23) and (2.24), we integrate (2.22) over (t_1, t) and get

$$\|I(\cdot,t)\|_{L^2}^2 \leqslant \|I(\cdot,t_1)\|_{L^2}^2 \, \mathrm{e}^{c_3\mathcal{K}_1^4 \int_{t_1}^t \|\Delta S(\cdot,s)\|_{L^2}^2 ds + c_5\left(1+\mathcal{K}_1^2\right)^2}$$
$$\leqslant c_6 \mathrm{e}^{c_3\mathcal{K}_1^4 C_4 + c_5\left(1+\mathcal{K}_1^2\right)^2} \leqslant \mathrm{e}^{(c_6+c_3C_4+c_5)\left(1+\mathcal{K}_1^2\right)^2}.$$

Hence (2.16) follows by letting $C_5 := (c_6 + c_3C_4 + c_5)/2$ and the proof is finished. □

Lemma 2.6. *Let (S, I) be the solution obtained in lemma 2.1. Then there exists a constant $C_6 > 0$ such that*

$$\|\nabla S(\cdot,t)\|_{L^4} \leqslant e^{C_6\left(1+\mathcal{K}_1^2\right)^2} \text{ for all } t \in (0, T_{\max}). \tag{2.25}$$

Proof. We rewrite the first equation of (1.1) as

$$S_t - d_S\Delta S + \theta S = \Lambda(x) - \alpha(x)\frac{SI}{S+I} + \beta(x) I =: H(x,t). \tag{2.26}$$

Applying (2.16) gives

$$\|H(\cdot,t)\|_{L^2} \leq \|\Lambda^* + \alpha^* I + \beta^* I\|_{L^2} \leqslant c_1 \mathrm{e}^{C_5\left(1+\mathcal{K}_1^2\right)^2}, \tag{2.27}$$

where $c_1 = \Lambda^*|\Omega|^{\frac{1}{2}} + (\alpha^* + \beta^*)$. By $(\mathrm{e}^{\Delta t})_{t>0}$ we denote the Neumann heat semigroup in Ω. Then applying Duhamel's principle to (2.26) yields that

$$S(\cdot,t) = \mathrm{e}^{t(d_S\Delta - \theta)} S_0 + \int_0^t \mathrm{e}^{(t-s)(d_S\Delta-\theta)} H(\cdot,s)\,\mathrm{d}s, \tag{2.28}$$

which, along with (2.27) and well-known semigroup estimates (see e.g. [10, lemma 2.1]), gives

$$\begin{aligned}
\|\nabla S(\cdot,t)\|_{L^4} &\leqslant \|\nabla \mathrm{e}^{t(d_S\Delta-\theta)} S_0\|_{L^4} + \int_0^t \|\nabla \mathrm{e}^{(t-s)(d_S\Delta-\theta)} H(\cdot,s)\|_{L^4}\mathrm{d}s \\
&\leqslant \kappa_1 \mathrm{e}^{-d_S\lambda_1 t}\|\nabla S_0\|_{L^4} + \kappa_2 \int_0^t \left(1 + (t-s)^{-\frac{3}{4}}\right) \mathrm{e}^{-d_S\lambda_1(t-s)} \|H(\cdot,s)\|_{L^2}\mathrm{d}s \\
&\leqslant \kappa_1 \|\nabla S_0\|_{L^4} + \kappa_2 c_1 \mathrm{e}^{C_5\left(1+\mathcal{K}_1^2\right)^2} \left[1 + \Gamma(1/4)(d_S\lambda_1)^{\frac{3}{4}}\right] \\
&\leqslant c_2 \mathrm{e}^{C_5\left(1+\mathcal{K}_1^2\right)^2},
\end{aligned}$$

where positive constants κ_i $(i = 1, 2)$ and λ_1 are independent of $\mathcal{K}_1$, and $c_2 := \kappa_1 \|\nabla S_0\|_{L^4} + \kappa_2 c_1[1 + \Gamma(1/4)(d_S\lambda_1)^{\frac{3}{4}}]$. Here $\Gamma(\cdot)$ denotes the Gamma function defined by $\Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-t}\mathrm{d}t$. Hence, (2.25) follows by letting $C_6 := c_2 + C_5$ and we complete the proof of lemma 2.6. □

Lemma 2.7. *Let (S, I) be the solution obtained in lemma 2.1. Then there exists a constant $C_7 > 0$ such that*

$$\|I(\cdot,t)\|_{L^3} \leqslant \left(1 + \mathcal{K}_1^2\right) e^{C_7\left(1+\mathcal{K}_1^2\right)^2} \text{ for all } t \in (0, T_{\max}). \tag{2.29}$$

Proof. We multiply the second equation of (1.1) by I^2 and integrate the result to get

$$\begin{aligned}
\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega I^3 = &-2d_I \int_\Omega \gamma(S) I|\nabla I|^2 - 2d_I \int_\Omega I^2 \gamma_1'(S)\nabla S \cdot \nabla I \\
&+ \int_\Omega \alpha(x)\frac{SI^3}{S+I} - \int_\Omega [\beta(x) + \eta(x)] I^3,
\end{aligned}$$

which, together with hypotheses (H1) and (H2), gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega I^3 + 6d_I\int_\Omega I|\nabla I|^2 + 3\left(\beta_* + \eta_*\right)\int_\Omega I^3 \leqslant 6d_I\mathcal{K}_1\int_\Omega I^2|\nabla S||\nabla I| + 3\alpha^*\int_\Omega I^3. \tag{2.30}$$

Applying Young's inequality, Hölder inequality and (2.25), one has

$$\begin{aligned}6d_I\mathcal{K}_1\int_\Omega I^2|\nabla S||\nabla I| + 3\alpha^*\int_\Omega I^3 &\leqslant 3d_I\int_\Omega I|\nabla I|^2 + 3d_I\mathcal{K}_1^2\int_\Omega I^3|\nabla S|^2 + 3\alpha^*\int_\Omega I^3\\ &\leqslant 3d_I\int_\Omega I|\nabla I|^2 + 3d_I\mathcal{K}_1^2\|I\|_{L^6}^3\|\nabla S\|_{L^4}^2 + 3\alpha^*|\Omega|^{\frac12}\|I\|_{L^6}^3\\ &\leqslant 3d_I\int_\Omega I|\nabla I|^2 + c_1\sigma_1\left(\mathcal{K}_1\right)\|I\|_{L^6}^3,\end{aligned}$$

which substituted into (2.30) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega I^3 + 3d_I\int_\Omega I|\nabla I|^2 + 3\left(\beta_* + \eta_*\right)\int_\Omega I^3 \leqslant c_1\sigma_1\left(\mathcal{K}_1\right)\|I\|_{L^6}^3, \tag{2.31}$$

where $c_1 := 3d_I + 3\alpha^*|\Omega|^{\frac12}$ and $\sigma_1(\mathcal{K}_1) := 1 + \mathcal{K}_1^2\mathrm{e}^{2C_6(1+\mathcal{K}_1^2)^2} > 1$.

From (2.16), we have $\|I^{\frac32}(\cdot,t)\|_{L^{\frac43}} = \|I(\cdot,t)\|_{L^2}^{\frac32} \leqslant \mathrm{e}^{\frac32 C_5(1+\mathcal{K}_1^2)^2}$. Then using Gagliardo–Nirenberg inequality in two dimensions and Young's inequality, one derives

$$\begin{aligned}c_1\sigma_1\left(\mathcal{K}_1\right)\|I\|_{L^6}^3 = c_1\sigma_1\left(\mathcal{K}_1\right)\|I^{\frac32}\|_{L^4}^2 &\leqslant c_1c_2\sigma_1\left(\mathcal{K}_1\right)\left(\|\nabla I^{\frac32}\|_{L^2}^{\frac43}\|I^{\frac32}\|_{L^{\frac43}}^{\frac23} + \|I^{\frac32}\|_{L^{\frac43}}^2\right)\\ &\leqslant c_3\sigma_2\left(\mathcal{K}_1\right)\left(\|\nabla I^{\frac32}\|_{L^2}^{\frac43} + 1\right)\\ &\leqslant \frac{4d_I}{3}\|\nabla I^{\frac32}\|_{L^2}^2 + \frac{c_3^3\sigma_2^3\left(\mathcal{K}_1\right)}{12d_I^2} + c_3\sigma_2\left(\mathcal{K}_1\right),\end{aligned} \tag{2.32}$$

where $c_3 := c_1c_2$ and $\sigma_2(\mathcal{K}_1) := (1+\mathcal{K}_1^2)\mathrm{e}^{(2C_6+3C_5)(1+\mathcal{K}_1^2)^2} > 1$. The combination of (2.32) with (2.31) implies

$$\frac{d}{dt}\|I\|_{L^3}^3 + 3\left(\beta_* + \eta_*\right)\|I\|_{L^3}^3 \leqslant c_4\sigma_2^3\left(\mathcal{K}_1\right), \tag{2.33}$$

where $c_4 := c_3 + \frac{c_3^3}{12d_I^2}$. Then (2.33) gives

$$\|I\|_{L^3}^3 \leqslant \mathrm{e}^{-3(\beta_*+\eta_*)t}\|I_0\|_{L^3}^3 + \frac{c_4\sigma_2^3\left(\mathcal{K}_1\right)}{3\left(\beta_* + \eta_*\right)}\left(1 - \mathrm{e}^{-3(\beta_*+\eta_*)t}\right) \leqslant c_8\sigma_2^3\left(\mathcal{K}_1\right)$$

with $c_8 := c_4/(3\beta_* + 3\eta_*) + \|I_0\|_{L^3}^3$. Therefore, (2.29) follows by letting $C_7 := c_8^{\frac13} + 2C_6 + 3C_5$ and the proof of lemma 2.7 is completed. □

Lemma 2.8. *Let (S, I) be the solution obtained in lemma 2.1. Then there exist two positive constants C_8 and C_9 such that*

$$\|S(\cdot,t)\|_{W^{1,\infty}} \leqslant \left(1 + \mathcal{K}_1^2\right)e^{C_8\left(1+\mathcal{K}_1^2\right)^2} \text{ for all } t \in (0, T_{\max}), \tag{2.34}$$

and

$$\|I(\cdot,t)\|_{L^\infty} \leqslant \left(1 + \mathcal{K}_1^2\right)^6 e^{C_9\left(1+\mathcal{K}_1^2\right)^2} \text{ for all } t \in (0, T_{\max}). \tag{2.35}$$

Proof. By (2.29), we conclude from (2.26) that

$$\|H(\cdot,t)\|_{L^3} \le \|\Lambda^* + \alpha^* I + \beta^* I\|_{L^3} \leqslant c_1\left(1+\mathcal{K}_1^2\right)\mathrm{e}^{C_7\left(1+\mathcal{K}_1^2\right)^2} =: c_1\sigma_3(\mathcal{K}_1), \tag{2.36}$$

where $c_1 := \Lambda^*|\Omega|^{\frac{1}{3}} + \alpha^* + \beta^*$. Applying the semigroup estimates to (2.28) and using (2.36), one has

$$\begin{aligned}\|S(\cdot,t)\|_{L^\infty} &\leqslant \kappa_3 \mathrm{e}^{-\theta t}\|S_0\|_{L^\infty} + \kappa_4\int_0^t\left(1+(t-s)^{-\frac{1}{3}}\right)\mathrm{e}^{-\theta(t-s)}\|H(\cdot,s)\|_{L^3}\mathrm{d}s \\ &\leqslant \kappa_3\|S_0\|_{L^\infty} + \kappa_4 c_1\sigma_3(\mathcal{K}_1)\int_0^t\left(1+(t-s)^{-\frac{1}{3}}\right)\mathrm{e}^{-\theta(t-s)}\mathrm{d}s \\ &\leqslant c_2\sigma_3(\mathcal{K}_1),\end{aligned} \tag{2.37}$$

where $c_2 := \kappa_3\|S_0\|_{L^\infty} + \kappa_4 c_1[1+\Gamma(2/3)\theta^{\frac{1}{3}}]$ with constants κ_3 and κ_4 independent of $\mathcal{K}_1$. Similarly, (2.36) along with the semigroup estimates yields

$$\begin{aligned}\|\nabla S(\cdot,t)\|_{L^\infty} &\leqslant c_3\|S_0\|_{W^{1,\infty}} + \kappa_2\int_0^t\left(1+(t-s)^{-\frac{5}{6}}\right)\mathrm{e}^{-d_S\lambda_1(t-s)}\|H(\cdot,s)\|_{L^3}\mathrm{d}s \\ &\leqslant c_4\sigma_3(\mathcal{K}_1).\end{aligned} \tag{2.38}$$

Here $c_4 := c_3\|S_0\|_{W^{1,\infty}} + \kappa_2 c_1[1+\Gamma(1/6)(d_S\lambda)^{\frac{5}{6}}]$. Then, (2.38) alongside (2.37) gives (2.34) by letting $C_8 := c_2 + c_4 + C_7$.

Multiplying the second equation of (1.1) by $I^{p-1}(p \geqslant 2)$ and integrating the result, one derives

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega I^p = &-d_I(p-1)\int_\Omega \gamma(S)I^{p-2}|\nabla I|^2 - d_I(p-1)\int_\Omega I^{p-1}\gamma'(S)\nabla S\cdot\nabla I \\ &+\int_\Omega \alpha(x)\frac{SI^p}{S+I} - \int_\Omega[\beta(x)+\eta(x)]I^p,\end{aligned}$$

which, along with hypotheses (H1) and (H2) and (2.38), gives

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega I^p + d_I(p-1)\int_\Omega I^{p-2}|\nabla I|^2 + (\beta_* + \eta_*)\int_\Omega I^p &\leqslant d_I(p-1)\sigma_4(\mathcal{K}_1)\int_\Omega I^{p-1}|\nabla I| + \alpha^*\int_\Omega I^p \\ &\leqslant \frac{d_I(p-1)}{2}\int_\Omega I^{p-2}|\nabla I|^2 + \sigma_5(\mathcal{K}_1)\int_\Omega I^p,\end{aligned}$$

where $\sigma_4(\mathcal{K}_1) := c_4\mathcal{K}_1(1+\mathcal{K}_1^2)\mathrm{e}^{C_7(1+\mathcal{K}_1^2)^2}$ and $\sigma_5(\mathcal{K}_1) := \frac{d_I(p-1)\sigma_4^2(\mathcal{K}_1)+2\alpha^*}{2}$. Hence, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega I^p + \frac{p(p-1)d_I}{2}\int_\Omega I^{p-2}|\nabla I|^2 \le \sigma_5(\mathcal{K}_1)p\int_\Omega I^p \leqslant c_5\sigma_6(\mathcal{K}_1)p(p-1)\int_\Omega I^p, \tag{2.39}$$

where $\sigma_6(\mathcal{K}_1) := (1+\mathcal{K}_1^2)^3\mathrm{e}^{2C_7(1+\mathcal{K}_1^2)^2} > 1$ and $c_5 := \frac{d_I c_4^2 + 2\alpha^*}{2}$ are independent of p. We add $p(p-1)\int_\Omega I^p$ to the both sides of (2.39) and denote $c_6 := c_5 + 1$. Then the inequality (2.39) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega I^p + p(p-1)\int_\Omega I^p \leqslant -\frac{p(p-1)d_I}{2}\int_\Omega I^{p-2}|\nabla I|^2 + c_6\sigma_6(\mathcal{K}_1)p(p-1)\int_\Omega I^p. \tag{2.40}$$

Based on (2.40), we can proceed with the same procedure as the proof in [12, lemma 3.6] to find a constant $c_7 > 0$ only depending on Ω such that

$$\|I(\cdot,t)\|_{L^\infty} \leqslant 2^6 c_8 \max\{C_1, \|I_0\|_{L^\infty}\} \leqslant c_9\left(1+\mathcal{K}_1^2\right)^6\mathrm{e}^{4C_7\left(1+\mathcal{K}_1^2\right)^2}$$

with $c_8 := c_6\sigma_6(\mathcal{K}_1)c_7 \max\left\{1, \frac{c_6\sigma_6(\mathcal{K}_1)}{2d_I}\right\} + |\Omega| + 1$ and $c_9 := 2^6(C_1 + \|I_0\|_{L^\infty})(c_6c_7 + \frac{c_6^2c_7}{2d_I} + |\Omega| + 1)$. Hence (2.35) holds with $C_9 := c_9 + 4C_7$, and we finish the proof of lemma 2.8. □

Proof of theorem 1.1. The combination of lemma 2.8 with lemma 2.1 yields theorem 1.1. □

3. Basic reproduction number R_0: proof of theorem 1.3

In this section, we study the properties of R_0 and the threshold dynamics of (1.1) in terms of R_0. Below we always suppose that hypotheses (H0)-(H1) hold.

3.1. Properties of R_0 and stability of DFE

Motivated by the ideas in [2], we consider the linearized eigenvalue problem of (1.1) at $(\widetilde{S}, 0)$:

$$\begin{cases} d_S\Delta\phi - \theta\phi + [\beta(x) - \alpha(x)]\psi + \lambda\phi = 0, & x \in \Omega, \\ d_I\Delta\left[\gamma(\widetilde{S})\psi\right] + [\alpha(x) - \beta(x) - \eta(x)]\psi + \lambda\psi = 0, & x \in \Omega, \\ \partial_\nu\phi = \partial_\nu\psi = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

Obviously, the differential operator defined in (3.1) is not self-adjoint and hence inconvenient to be studied by the conventional variational approach. To treat (3.1) variationally, we introduce a change of variable

$$u = \gamma(\widetilde{S})\psi,$$

which, along with the fact that the mapping $\psi \mapsto \gamma(\widetilde{S})\psi$ is bijective due to $1 \leq \gamma(\widetilde{S}) \leqslant \gamma\left(\frac{\Lambda^*}{\theta}\right)$, reformulates (3.1) as

$$d_S\Delta\phi - \theta\phi + [\beta(x) - \alpha(x)]\gamma^{-1}(\widetilde{S})u + \lambda\phi = 0, \quad x \in \Omega, \tag{3.2}$$

$$d_I\Delta u + [\alpha(x) - \beta(x) - \eta(x)]\gamma^{-1}(\widetilde{S})u + \lambda\gamma^{-1}(\widetilde{S})u = 0, \quad x \in \Omega, \tag{3.3}$$

$$\partial_\nu\phi = \partial_\nu u = 0, \quad x \in \partial\Omega, \tag{3.4}$$

where we denote $\gamma^{-1}(\widetilde{S}) = 1/\gamma(\widetilde{S})$ hereafter. The reformulated eigenvalue problem (3.2)–(3.4) is an elliptic system with self-adjoint operators and a weight function $\gamma^{-1}(\widetilde{S})$. For the weighted eigenvalue problem (3.3) with $\partial_\nu u = 0$, it follows from [30, remark 1.3.8] that there exists a principal eigenvalue $\lambda^* \in \mathbb{R}$, which is simple and corresponds to a unique positive eigenfunction u^* up to a constant multiple. Since the weight function $\gamma^{-1}(\widetilde{S})$ is strictly positive, we may use the variational formula (e.g. [9, pp. 102] and [13]) to characterize λ^* as

$$\lambda^* = \inf_{0 \neq w \in H^1(\Omega)} \frac{\int_\Omega d_I|\nabla w|^2 + [\beta(x) + \eta(x) - \alpha(x)]\gamma^{-1}(\widetilde{S})w^2 \mathrm{d}x}{\int_\Omega \gamma^{-1}(\widetilde{S})w^2 \mathrm{d}x}.$$

This inspires us to define the basic reproduction number

$$R_0 = \sup_{0 \neq w \in H^1(\Omega)} \frac{\int_\Omega \alpha(x)\gamma^{-1}(\widetilde{S})w^2 \mathrm{d}x}{\int_\Omega \left[d_I|\nabla w|^2 + (\beta(x) + \eta(x))\gamma^{-1}(\widetilde{S})w^2\right] \mathrm{d}x} > 0, \tag{3.5}$$

which is equivalent to (1.6). The above transformation makes the analysis on the properties of R_0 more tractable. To explore the threshold dynamics in terms of R_0, we establish the following property of R_0 in addition to those stated in proposition 1.2:

$$R_0 > 1 \ \text{ iff } \lambda^* < 0,\ R_0 = 1 \text{ iff } \lambda^* = 0 \text{ and } R_0 < 1 \text{ iff } \lambda^* > 0, \tag{3.6}$$

which can be proved by the same argument of the proof of [2, lemma 2.3].

Next, we shall show that the linear stability of DFE $(\widetilde{S},0)$ can be classified by the value of R_0.

Lemma 3.1. *The DFE $(\widetilde{S},0)$ is linearly stable if $R_0 < 1$, and unstable if $R_0 > 1$.*

Proof. We first show the linear stability of $(\widetilde{S},0)$ under the assumption $R_0 < 1$. This amounts to show that if (λ,ϕ,u) is a solution to (3.2)–(3.4) with $\phi \not\equiv 0$ or $u \not\equiv 0$, then $Re(\lambda) > 0$. We have two cases to proceed.

Case 1: $u \equiv 0$ and $\phi \not\equiv 0$. Hence (λ,ϕ) is an eigenpair of the following eigenvalue problem

$$d_S\Delta\phi - \theta\phi + \lambda\phi = 0,\ x \in \Omega;\ \ \partial_\nu\phi = 0,\ x \in \partial\Omega. \tag{3.7}$$

Since the Laplacian operator Δ in (3.7) is self-adjoint, λ is real. Multiplying the first equation of (3.7) by ϕ and integrating the result, we immediately get $\lambda \geqslant \theta > 0$.

Case 2: $u \not\equiv 0$. In this case, (λ,u) is an eigenpair of the eigenvalue problem (3.3) with $\partial_\nu u = 0$. It follows from (3.6) and $R_0 < 1$ that $\text{Re}(\lambda) \geqslant \lambda^* > 0$. Therefore, DFE $(\widetilde{S},0)$ is stable if $R_0 < 1$.

We now show that $(\widetilde{S},0)$ is linearly unstable if $R_0 > 1$. First (3.6) indicates that $\lambda^* < 0$. On the other hand, one can easily check that

$$d_S\Delta\phi - \theta\phi + [\beta(x) - \alpha(x)]\gamma^{-1}(\widetilde{S})u^* + \lambda^*\phi = 0,\ x \in \Omega;\quad \partial_\nu\phi = 0,\ x \in \partial\Omega$$

has a solution ϕ^*. Then (λ^*,ϕ^*,u^*) is a solution to (3.2)–(3.4) with $u^* > 0$ and $\lambda^* < 0$, which shows that $(\widetilde{S},0)$ is linearly unstable. □

3.2. *Existence of EE with $R_0 > 1$*

In this subsection, we shall establish the existence of EE for $R_0 > 1$. Usually the existence of EE can be established based on the uniform persistence theory. But this is inapplicable here due to the cross-diffusion structure in the I-equation. Below we shall directly explore the existence of positive solutions to the stationary problem (1.5). To this end, we introduce a change of variable

$$Z = \gamma(S)I,$$

and reformulate (1.5) into the following problem without cross-diffusion

$$\begin{cases} d_S\Delta S + \Lambda(x) - \theta S - \alpha(x)\frac{SZ\gamma^{-1}(S)}{S+Z\gamma^{-1}(S)} + \beta(x)Z\gamma^{-1}(S) = 0, & x \in \Omega, \\ d_I\Delta Z + \alpha(x)\frac{SZ\gamma^{-1}(S)}{S+Z\gamma^{-1}(S)} - [\beta(x) + \eta(x)]Z\gamma^{-1}(S) = 0, & x \in \Omega, \\ \partial_\nu S = \partial_\nu Z = 0, & x \in \partial\Omega. \end{cases} \tag{3.8}$$

Thus, (1.5) admits a positive solution if and only if (3.8) admits a positive solution. In the spatially homogeneous environment, it is easy to verify that (1.5) admits a unique constant EE if $R_0 > 1$. For the spatially inhomogeneous environment, to establish the existence of EE for $R_0 > 1$, we first prove (3.8) admits a positive solution by applying the index theory and principal eigenvalue theory.

We start by giving a result on the eigenvalue problem, which will be used later.

Lemma 3.2. **([15, 35, 46])** *Let $\lambda_1(d,r)$ be the principal eigenvalue of*

$$d\Delta u + r(x)u + \lambda u = 0,\ \ x \in \Omega;\ \ \partial_\nu u = 0,\ \ x \in \partial\Omega. \tag{3.9}$$

Consider the weighted eigenvalue problem

$$-d\Delta u + Mu = \mu(M+r)u,\ \ x \in \Omega;\ \ \partial_\nu u = 0,\ \ x \in \partial\Omega, \tag{3.10}$$

where function $r(x) \in C(\Omega)$, $d > 0$, $M > 0$ and $M + r > 0$ on Ω. Then the following statements hold:

(i) *If $\lambda_1(d,r) < 0$, (3.10) has an eigenvalue μ smaller than 1;*
(ii) *If $\lambda_1(d,r) > 0$, (3.10) has no eigenvalue μ smaller than or equal to 1.*

Next we derive *a priori* estimates for the positive solutions of (3.8).

Lemma 3.3. *Let (S,Z) be a positive solution of (3.8) and assumptions (H0)-(H2) hold. Then*

$$S \leqslant \frac{\Lambda^*}{c_0 d_S} =: C_S \text{ and } Z \leqslant \frac{\Lambda^*}{c_0 d_I} =: C_Z \text{ in } \Omega, \tag{3.11}$$

where the constant $c_0 := \min\left\{\frac{\theta}{d_S}, \frac{\eta_*}{\mathcal{K}_0 d_I}\right\}$.

Proof. Adding the first two equations of (3.8), one gets

$$\Delta(d_S S + d_I Z) + \Lambda(x) - \theta S - \eta(x) Z\gamma^{-1}(S) = 0,$$

which, along with hypotheses (H1)-(H2) and (3.8), gives

$$\begin{cases} \Delta(d_S S + d_I Z) + \Lambda^* - c_0(d_S S + d_I Z) \geqslant 0, & x \in \Omega, \\ \partial_\nu(d_S S + d_I Z) = 0, & x \in \partial\Omega. \end{cases} \tag{3.12}$$

Denoting $v := d_S S + d_I Z$ and applying the maximum principle [37, proposition 2.2] to equation (3.12), we get $\max_{\overline{\Omega}}(d_S S + d_I Z) = \max_{\overline{\Omega}} v \leqslant \frac{\Lambda^*}{c_0}$. This gives (3.11) and the proof of lemma 3.3 is finished. □

With lemma 3.3 in hand, we introduce some notations as in [46]:

$$\begin{aligned} &X = \left\{\phi \in C^1\left(\overline{\Omega}\right) \cap C^2\left(\Omega\right) \mid \partial_\nu \phi = 0 \text{ on } \partial\Omega\right\}, \\ &E = C\left(\overline{\Omega}\right) \times C\left(\overline{\Omega}\right), \\ &W = C^+\left(\overline{\Omega}\right) \times C^+\left(\overline{\Omega}\right) \text{ with } C^+\left(\overline{\Omega}\right) = \left\{\phi \in C\left(\overline{\Omega}\right) \mid \phi \geqslant 0\right\}, \\ &D = \{(S,Z) \in W \mid S < 1 + C_S,\ Z < 1 + C_Z\} \subset W. \end{aligned}$$

Then for any constant $\delta \in [0,1]$, we define a operator $T_\delta :\ D \to W$ by

$$T_\delta(S,\ Z) \triangleq \begin{pmatrix} \mathcal{T}_1^{-1}\left[\Lambda(x) - \alpha(x)\frac{SZ\gamma^{-1}(S)}{S+Z\gamma^{-1}(S)} + \beta(x)Z\gamma^{-1}(S) + (m-\theta)S\right] \\ \mathcal{T}_2^{-1}\left[mZ + \delta\alpha(x)\frac{SZ\gamma^{-1}(S)}{S+Z\gamma^{-1}(S)} - (\beta(x)+\eta(x))Z\gamma^{-1}(S)\right] \end{pmatrix},\ \ \forall (S,Z) \in D,$$

where $m > 0$ is a large constant such that $m - [\beta(x) + \eta(x)]\gamma^{-1}(S) - \theta > 0$ for all $(S,Z) \in D$, and $\mathcal{T}_i^{-1}$ $(i = 1,2)$ denote the inverse operators of $\mathcal{T}_i$ under homogeneous Neumann boundary conditions, respectively, with $\mathcal{T}_1(S) := -d_S\Delta S + mS$ for $S \in X$ and $\mathcal{T}_2(Z) := -d_I\Delta Z + mZ$ for $Z \in X$. Lemma 3.3 shows that (3.8) admits a positive solution if and only if T_1 has a positive fixed point on D. Moreover, one can check that the operator T_1 is compact and $T_1(D) \subseteq W$

by applying the elliptic regularity theory and compact embedding theorem, and $(\widetilde{S},0)$ is the unique non-positive fixed point of T_1 on D.

Then, we shall show that $\text{index}_W(T_1,(\widetilde{S},0))$, as defined in [27, definition I.2.1], exists and compute it.

Lemma 3.4. *Let the conditions in lemma 3.3 hold and assume $\lambda_1(d_I,m_2(x))\not\equiv 0$. Then*

$$\text{index}_W\left(T_1,\left(\widetilde{S},0\right)\right)=\begin{cases}0, & \text{if } \lambda_1(d_I,m_2(x))<0,\\ 1, & \text{if } \lambda_1(d_I,m_2(x))>0,\end{cases}$$

where $m_2(x):=[\alpha(x)-\beta(x)-\eta(x)]\gamma^{-1}(\widetilde{S})$.

Proof. By a straightforward calculation, the Fréchet derivative $DT_1(\widetilde{S},0)$ of T_1 at $(\widetilde{S},0)$ is given by

$$DT_1\left(\widetilde{S},0\right)(\phi,\psi)=\begin{pmatrix}\mathcal{T}_1^{-1}\left[(m-\theta)\phi+m_1(x)\psi\right]\\ \mathcal{T}_2^{-1}\left[(m+m_2(x))\psi\right]\end{pmatrix},$$

where $m_1(x):=[\beta(x)-\alpha(x)F(\widetilde{S},0)]\gamma^{-1}(\widetilde{S})$. We shall prove that $DT_1(\widetilde{S},0)$ has no non-zero fixed point in $C(\overline{\Omega})\times C^+(\overline{\Omega})$. If not, then we obtain

$$\begin{cases}d_S\Delta\phi-\theta\phi+m_1(x)\psi=0, & x\in\Omega,\\ d_I\Delta\psi+m_2(x)\psi=0, & x\in\Omega,\\ \partial_\nu\phi=\partial_\nu\psi=0, & x\in\partial\Omega.\end{cases}\tag{3.13}$$

It follows from the first equation of (3.13) that $\phi=0$ if $\psi=0$. Hence, $\psi\in C^+(\overline{\Omega})\backslash\{0\}$, this along with [30, theorem 1.3.6] gives $\lambda_1(d_I,m_2(x))\equiv 0$, which contradicts the assumption $\lambda_1(d_I,m_2(x))\not\equiv 0$. Therefore, $DT_1(\widetilde{S},0)$ has no non-zero fixed point in $C(\overline{\Omega})\times C^+(\overline{\Omega})$, this means that $\text{index}_W(T_1,(\widetilde{S},0))$ exists.

To compute $\text{index}_W(T_1,(\widetilde{S},0))$, we shall employ principal eigenvalue result given in lemma 3.2 and the index theory (see [14, 45]), which is presented in [46, lemma 3.1]. Choose $W_{(\widetilde{S},0)}=C(\overline{\Omega})\times C^+(\overline{\Omega})$, $H_{(\widetilde{S},0)}=C(\overline{\Omega})\times\{0\}$, $E_{(\widetilde{S},0)}=\{0\}\times C(\overline{\Omega})$ such that $E=H_y\oplus E_y$ and $W_{(\widetilde{S},0)}$ is a generating cone. Then it follows from [46, lemma 3.1] that $P\circ DT_1(\widetilde{S},0)=\mathcal{T}_2^{-1}[m+m_2(x)]$, where $P:\ E\to\ E_y$ is a projection operator. If $\lambda_1(d_I,m_2(x))<0$, by lemma 3.2, we know that $\mathcal{T}_2^{-1}[m+m_2(x)]$ has an eigenvalue bigger than 1. This along with [46, lemma 3.1] gives $\text{index}_W(T_1,(\widetilde{S},0))=0$. If $\lambda_1(d_I,m_2(x))>0$, lemma 3.2 shows that all eigenvalues of the operator $\mathcal{T}_2^{-1}[m+m_2(x)]$ are smaller than 1. Thus, [46, lemma 3.1] yields

$$\text{index}_W\left(T_1,\left(\widetilde{S},0\right)\right)=(-1)^{\ell},$$

where ℓ denotes the sum of algebraic multiplicities of the eigenvalues of $DT_1(\widetilde{S},0)$ restricted in $H_{(\widetilde{S},0)}$ which are greater than 1.

We next prove that $DT_1(\widetilde{S},0)$ restricted in $H_{(\widetilde{S},0)}$ does not have eigenvalues greater than or equal to 1. Assume that $DT_1(\widetilde{S},0)$ has an eigenvalue $\mu_0\geqslant 1$ associated with eigenfunction $(\phi,\psi)=(\phi,0)\in H_{(\widetilde{S},0)}$ fulfilling $\|\phi\|_{L^2}=1$. Then we have

$$-d_S\Delta\phi+m\phi=\frac{1}{\mu_0}(m-\theta)\phi,\ \ x\in\Omega;\quad \partial_\nu\phi=0,\ \ x\in\partial\Omega.$$

Since $\lambda_1(d_S,-\theta)>0$, lemma 3.2 gives $\frac{1}{\mu_0}>1$. This contradicts $\mu_0\geqslant 1$. Hence $\text{index}_W(T,(\widetilde{S},0))=(-1)^{\ell}=(-1)^0=1$ and the proof of lemma 3.4 is completed. □

Lemma 3.5. *Let the conditions in lemma 3.3 hold. Then (3.8) admits at least one positive solution when* $\lambda_1(d_I, m_2(x)) < 0$.

Proof. Assume that (3.8) has no positive solution, then $(\widetilde{S}, 0)$ is the unique fixed point of T_1 on D. Lemma 3.3 indicates that T_1 has no fixed point on ∂D (i.e. $(I - T_1)(\partial D) \neq 0$), and thus $\deg_W(I - T_1, D, 0)$ is well-defined (see the definition in [27, definition II.2.2]). Then the excision property [3, corollary 11.2] shows that

$$\deg_W(I - T_1, D, 0) = \text{index}_W\big(T_1, (\widetilde{S}, 0)\big),$$

which, along with $\lambda_1(d_I, m_2(x)) < 0$ and lemma 3.4, gives

$$\deg_W(I - T_1, D, 0) = 0. \tag{3.14}$$

On the other hand, for each $\delta \in [0, 1]$, T_δ has a fixed point (S, Z) iff (S, Z) is a solution of the following problem

$$\begin{cases} d_S\Delta S + \Lambda(x) - \theta S - \alpha(x)\frac{SZ\gamma^{-1}(S)}{S+Z\gamma^{-1}(S)} + \beta(x) Z\gamma^{-1}(S) = 0, & x \in \Omega, \\ d_I\Delta Z + \delta\alpha(x)\frac{SZ\gamma^{-1}(S)}{S+Z\gamma^{-1}(S)} - [\beta(x) + \eta(x)] Z\gamma^{-1}(S) = 0, & x \in \Omega, \\ \partial_\nu S = \partial_\nu Z = 0, & x \in \partial\Omega. \end{cases} \tag{3.15}$$

Proceeding with the similar procedure as the proof in lemma 3.3, we get that all fixed points of T_δ satisfy (3.11) for each $\delta \in [0, 1]$, which means that $(I - T_\delta)(\partial D) \neq 0$. Hence, the homotopy invariance of the topological degree [3, theorem 11.1] implies

$$\deg_W(I - T_\delta, D, 0) = \deg_W(I - T_1, D, 0) = \deg_W(I - T_0, D, 0). \tag{3.16}$$

When $\delta = 0$, (3.15) only has a unique solution, which is denoted by $(\widetilde{S}^0, 0)$. Hence, the excision property implies that

$$\deg_W(I - T_0, D, 0) = \text{index}_W\big(T_0, (\widetilde{S}^0, 0)\big). \tag{3.17}$$

Following the same proof as in lemma 3.4, one can check that

$$\text{index}_W\big(T_0, (\widetilde{S}^0, 0)\big) = 1,$$

which, together with (3.16) and (3.17), gives $\deg_W(I - T_1, D, 0)) = \deg_W(I - T_0, D, 0)) = 1$. This contradicts (3.14). Hence (3.8) admits at least one positive solution and the proof of lemma 3.5 is completed. □

Using lemma 3.5, we further establish the existence of EE when $R_0 > 1$. To achieve this goal, we show that the principal eigenvalues of the weighted and unweighted eigenvalue problems have the same sign.

Lemma 3.6. *Assume that* $d > 0$, $r(x) \in C(\overline{\Omega})$, *and the positive function* $a(x) \in C(\overline{\Omega})$. *Let* $\lambda_1(d, r)$ *and* ς^* *be the principal eigenvalue of (3.9) and*

$$d\Delta u + r(x) u + \varsigma a(x) u = 0, \;\; x \in \Omega; \quad \partial_\nu u = 0, \;\; x \in \partial\Omega,$$

respectively. Then it follows that

$$\text{sign}(\varsigma^*) = \text{sign}[\lambda_1(d, r)]. \tag{3.18}$$

Proof. Denote the positive eigenfunctions associated with $\lambda_1(d,r)$ and ς^* by v^* and w^*, respectively, satisfying $\|v^*\|_{L^\infty} = \|w^*\|_{L^\infty} = 1$. Then we have

$$\begin{cases} d\Delta v^* + r(x)v^* + \lambda_1(d,r)v^* = 0, & x \in \Omega, \\ d\Delta w^* + r(x)w^* + \varsigma^* a(x)w^* = 0, & x \in \Omega, \\ \partial_\nu v^* = \partial_\nu w^* = 0, & x \in \partial\Omega. \end{cases} \tag{3.19}$$

We multiply the first equation of (3.19) by w^* and the second by v^*, and integrate the results by parts. Then subtract the resulting equation, we get

$$\varsigma^* \int_\Omega a(x)w^*v^* = \lambda_1(d,r)\int_\Omega w^*v^*.$$

This along with the fact that $a(x)$, w^*, v^* are positive gives (3.18) directly and hence completes the proof of lemma 3.6. ☐

Lemma 3.7. *Let the conditions in lemma 3.3 hold. Then (1.5) admits at least an EE when* $R_0 > 1$.

Proof. Taking $d = d_I$, $r(x) = m_2(x)$ and $a(x) = \gamma^{-1}(\widetilde{S})$ in lemma 3.6, then (3.18) along with (3.6) indicates that $\operatorname{sign}(\lambda_1(d_I, m_2(x))) = \operatorname{sign}(\lambda^*) = \operatorname{sign}(1 - R_0) < 0$. Thus, lemma 3.5 implies lemma 3.7 directly. ☐

Proof of theorem 1.3. Combining lemma 3.1 with lemma 3.7, we get theorem 1.3. ☐

4. Global stability: proof of theorem 1.4

In this section, we shall explore the globally asymptotical stability of non-negative steady states of (1.1). We first improve the regularity of the solution (S,I).

Lemma 4.1. *Let (S, I) be the solution obtained in theorem 1.1. Then there exist constants* $\kappa \in (0,1)$ *and* $C_{10} > 0$ *such that*

$$\|(S,I)(\cdot,t)\|_{C^{2+\kappa,1+\frac{\kappa}{2}}(\overline{\Omega}\times[t,t+1])} \leqslant C_{10}, \quad \forall t > 1. \tag{4.1}$$

Proof. Using (1.4) and Hölder estimates for quasilinear parabolic equations (see [44, theorem 1.3 and remark 1.4]) along with the parabolic Schauder theory [29], one obtains

$$\|S(\cdot,t)\|_{C^{2+\kappa,1+\frac{\kappa}{2}}(\overline{\Omega}\times[t,t+1])} + \|I(\cdot,t)\|_{C^{\kappa,\frac{\kappa}{2}}(\overline{\Omega}\times[t,t+1])} \leqslant c_1, \quad \forall t > 1. \tag{4.2}$$

Rewrite I-equations of (1.1) as

$$\begin{cases} I_t = d_I\gamma(S)\Delta I + 2d_I\gamma'(S)\nabla S\cdot\nabla I + G(x,t)I, & x \in \Omega, t > 0, \\ \partial_\nu I = 0, & x \in \partial\Omega, t > 0, \\ I(x,0) = I_0, & x \in \Omega, \end{cases} \tag{4.3}$$

where $G(x,t) := d_I\gamma'(S)\Delta S + d_I\gamma''(S)|\nabla S|^2 + \alpha(x)\frac{S}{S+I} - \beta(x) - \eta(x)$. Applying (4.2) and the parabolic Schauder theory [29] to (4.3) give (4.1) directly. ☐

For convenience, we cite the following result [40, lemma1] for later use.

Lemma 4.2. *Let a and b be positive constants. Assume that* $\varphi, \psi \in C^1([a,\infty)), \psi(t) \geqslant 0$ *and* φ *is bounded from below. If* $\varphi'(t) \leqslant -b\psi(t)$ *and* $\psi'(t) \leqslant K$ *in* $[a,\infty)$ *for some constant K, then* $\lim_{t\to\infty}\psi(t) = 0$.

Proof of theorem 1.4. We first prove the results claimed in theorem 1.4-(i). With the given condition, it is obvious $R_0 < 1$ by the definition (1.6). Integrating the second equation of (1.1) by parts yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega I + (1-\varepsilon)\int_\Omega \eta(x) I &= \int_\Omega \left[\alpha(x)\frac{S}{S+I} - \beta(x) - \varepsilon\eta(x)\right] I \\ &\leqslant \int_\Omega [\alpha(x) - \beta(x) - \varepsilon\eta(x)] I,\end{aligned}$$

which, along with $\alpha(x) \leqslant \beta(x) + \varepsilon\eta(x)$ and $\varepsilon \in [0,1)$, implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega I + (1-\varepsilon)\eta_* \int_\Omega I \leqslant 0.$$

This indicates that for all $t > 0$

$$\|I\|_{L^1} \leqslant \mathrm{e}^{-(1-\varepsilon)\eta_* t}\|I_0\|_{L^1}. \tag{4.4}$$

We utilize Gagliardo–Nirenberg inequality in two dimensions to find a constant $c_1 > 0$ such that

$$\|I\|_{L^\infty} \leqslant c_1\left(\|\nabla I\|_{L^\infty}^{\frac{2}{3}}\|I\|_{L^1}^{\frac{1}{3}} + \|I\|_{L^1}\right) \leqslant c_2 \mathrm{e}^{-\frac{(1-\varepsilon)\eta_*}{3}t}, \quad \forall t > 1, \tag{4.5}$$

where we have used (4.1) and (4.4).

It follows from the first equation of (1.1) that

$$\left(S - \widetilde{S}\right)_t = d_S\Delta\left(S - \widetilde{S}\right) - \theta\left(S - \widetilde{S}\right) - \alpha(x)\frac{SI}{S+I} + \beta(x) I. \tag{4.6}$$

Applying Duhamel's principle to (4.6), one has

$$S - \widetilde{S} = \mathrm{e}^{t(d_S\Delta - \theta)}\left[S(\cdot,1) - \widetilde{S}\right] + \int_1^t \mathrm{e}^{(t-z)(d_S\Delta-\theta)}\left[\beta(x) - \alpha(x)\frac{S}{S+I}\right] I(\cdot,z)\,\mathrm{d}z.$$

By the standard heat Neumann semigroup estimates (see e.g. [10, lemma 2.1]), we get from (4.5) that

$$\begin{aligned}\|S - \widetilde{S}\|_{L^\infty} &\leqslant c_3\mathrm{e}^{-\theta t}\|S(\cdot,1) - \widetilde{S}\|_{L^\infty} + c_3\int_1^t \mathrm{e}^{-\theta(t-z)}\left(1 + (t-z)^{-\frac{1}{2}}\right)\left\|\left(\beta(x) - \frac{\alpha(x)S}{S+I}\right) I(\cdot,z)\right\|_{L^2}\mathrm{d}z \\ &\leqslant c_3\mathrm{e}^{-\theta t}\|S(\cdot,1) - \widetilde{S}\|_{L^\infty} + c_3(\beta^* + \alpha^*)\int_1^t \mathrm{e}^{-\theta(t-z)}\left(1 + (t-z)^{-\frac{1}{2}}\right)\|I(\cdot,z)\|_{L^2}\mathrm{d}z \\ &\leqslant c_4\mathrm{e}^{-\theta t} + c_3c_2|\Omega|^{\frac{1}{2}}(\beta^* + \alpha^*)\int_1^t \mathrm{e}^{-\theta(t-z)}\left(1 + (t-z)^{-\frac{1}{2}}\right)\mathrm{e}^{-\frac{(1-\varepsilon)\eta_*}{3}z}\mathrm{d}z \\ &\leqslant c_4\mathrm{e}^{-\theta t} + c_3c_2|\Omega|^{\frac{1}{2}}(\beta^* + \alpha^*)\int_1^t \mathrm{e}^{-\theta(t-z)}\left(1 + (t-z)^{-\frac{1}{2}}\right)\mathrm{e}^{-c_5 z}\mathrm{d}z \\ &\leqslant c_6\mathrm{e}^{-c_5 t},\end{aligned} \tag{4.7}$$

where $c_5 := \frac{1}{2}\min\{\theta, (1-\varepsilon)\eta_*/3\}$. Therefore, combining (4.5) with (4.7) indicates (1.7) directly. This completes the proof of theorem 1.4-(i).

Next we proceed to prove theorem 1.4-(ii). When $\Lambda(x)$, $\alpha(x)$, $\beta(x)$ and $\eta(x)$ are positive constants, it follows from proposition 1.2-(i) that $R_0 = \frac{\alpha}{\beta+\eta}$. Clearly there exists a unique constant EE $(\hat{S}, \hat{I})$ iff $R_0 > 1$, where

$$\hat{S} = \frac{\Lambda(\beta+\eta)}{\eta(\alpha-\beta-\eta) + \theta(\beta+\eta)} \quad \text{and} \quad \hat{I} = \frac{\Lambda(\alpha-\beta-\eta)}{\eta(\alpha-\beta-\eta) + \theta(\beta+\eta)}. \tag{4.8}$$

We define

$$\mathcal{E}(t) := \int_\Omega \left\{ (S+I+1) - \left(\hat{S}+\hat{I}+1\right) - \left(\hat{S}+\hat{I}+1\right)\ln\frac{S+I+1}{\hat{S}+\hat{I}+1} \right\} + \frac{4\eta\alpha}{(\alpha-\beta-\eta)^2}\int_\Omega \left[(I+1) - \left(\hat{I}+1\right) - \left(\hat{I}+1\right)\ln\frac{I+1}{\hat{I}+1} \right].$$

By Taylor's expansion, one gets that for all $z, z_0 > 0$

$$z - z_0 - z_0 \ln\frac{z}{z_0} = \frac{z_0}{2\tilde{z}^2}(z-z_0)^2 \geqslant 0, \tag{4.9}$$

where $\tilde{z}$ is between z and z_0. With (4.9), one can directly check that $\mathcal{E}(t) \geqslant 0$ where '=' holds iff $(S,I) = (\hat{S},\hat{I})$. Next, we show that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \leqslant -c_1\mathcal{F}(t)$ for some $c_1 > 0$ and function $\mathcal{F}(t) \geqslant 0$. For simplicity, we denote

$$E := E(S,I) := (S+I+1) - \left(\hat{S}+\hat{I}+1\right) - \left(\hat{S}+\hat{I}+1\right)\ln\frac{S+I+1}{\hat{S}+\hat{I}+1} + \frac{4\eta\alpha}{(\alpha-\beta-\eta)^2}\left[(I+1) - \left(\hat{I}+1\right) - \left(\hat{I}+1\right)\ln\frac{I+1}{\hat{I}+1} \right],$$

and

$$h_1(S,I) = \Lambda - \theta S - \alpha\frac{SI}{S+I} + \beta I, \quad h_2(S,I) = \alpha\frac{SI}{S+I} - (\beta+\eta)I.$$

Hence, one gets

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) &= \int_\Omega E_S S_t + E_I I_t \\ &= \int_\Omega \left[E_S h_1(S,I) + E_I h_2(S,I)\right] + \int_\Omega \left[d_S E_S \Delta S + d_I E_I \Delta(\gamma(S)I)\right] =: J_1 + J_2, \end{aligned} \tag{4.10}$$

where $E_S := \frac{\partial E}{\partial S}$ and $E_I := \frac{\partial E}{\partial I}$. Noting $\Lambda = \theta\hat{S} + \eta\hat{I}$, $\beta + \eta = \frac{\alpha\hat{S}}{\hat{S}+\hat{I}}$ and $\theta = \eta$, we have

$$\begin{aligned} J_1 &= \int_\Omega \left(1 - \frac{\hat{I}+\hat{S}+1}{S+I+1}\right)h_1(S,I) + \int_\Omega \left[1 - \frac{\hat{I}+\hat{S}+1}{S+I+1} + \frac{4\eta\alpha}{(\alpha-\beta-\eta)^2}\left(1 - \frac{\hat{I}+1}{I+1}\right)\right]h_2(S,I) \\ &= \int_\Omega \left(1 - \frac{\hat{I}+\hat{S}+1}{S+I+1}\right)(\Lambda - \theta S - \eta I) + \frac{4\eta\alpha}{(\alpha-\beta-\eta)^2}\int_\Omega \left(1 - \frac{\hat{I}+1}{I+1}\right)\left(\frac{\alpha S}{S+I} - \beta - \eta\right)I \\ &= \int_\Omega \frac{S-\hat{S}+I-\hat{I}}{S+I+1}\left(\eta\hat{S} + \eta\hat{I} - \eta S - \eta I\right) + \frac{4\eta\alpha^2}{(\alpha-\beta-\eta)^2}\int_\Omega \frac{\left(I-\hat{I}\right)I}{I+1}\frac{\left(S\hat{I} - \hat{S}\hat{I} + \hat{S}\hat{I} - I\hat{S}\right)}{(S+I)\left(\hat{S}+\hat{I}\right)} \\ &= -\int_\Omega \Phi B_1 \Phi^{\mathcal{T}}, \end{aligned}$$

where $\Phi = \left(\frac{S-\hat{S}}{\sqrt{S+I+1}}, \frac{I-\hat{I}}{\sqrt{S+I+1}}\right)$, $\mathcal{T}$ represents the transpose, and

$$B_1 = \begin{pmatrix} \eta & \frac{1}{2}\left[2\eta - \frac{4\eta\alpha}{(\alpha-\beta-\eta)}\cdot\frac{I(S+I+1)}{(I+1)(S+I)}\right] \\ \frac{1}{2}\left[2\eta - \frac{4\eta\alpha}{(\alpha-\beta-\eta)}\cdot\frac{I(S+I+1)}{(I+1)(S+I)}\right] & \frac{4\eta\alpha(\beta+\eta)}{(\alpha-\beta-\eta)^2}\cdot\frac{I(S+I+1)}{(I+1)(S+I)} + \eta \end{pmatrix}.$$

A direct calculation gives that $|B_1| = \frac{4\eta^2\alpha^2}{(\alpha-\beta-\eta)^2} \cdot \frac{I(S+I+1)}{(I+1)(S+1)} \left(1 - \frac{I(S+I+1)}{(I+1)(S+I)}\right) > 0$, which yields a constant $c_1 > 0$ such that

$$J_1 = -\int_\Omega \Phi B_1 \Phi^{\mathcal{T}} \leqslant -c_1 \int_\Omega \left(\frac{(S-\hat{S})^2}{S+I+1} + \frac{(I-\hat{I})^2}{S+I+1} \right) \leqslant 0. \tag{4.11}$$

With simple calculations, we find

$$E_{SS} = E_{SI} = E_{IS} = \frac{\hat{S}+\hat{I}+1}{(S+I+1)^2},$$

and

$$\begin{aligned} E_{II} &= \frac{\hat{S}+\hat{I}+1}{(S+I+1)^2} \left[1 + \frac{4\eta\alpha}{(\alpha-\beta-\eta)^2} \frac{\hat{I}+1}{\hat{S}+\hat{I}+1} \left(\frac{S+I+1}{I+1} \right)^2 \right] \\ &=: \frac{\left(\hat{S}+\hat{I}+1\right)[1+M_0 f(S,I)]}{(S+I+1)^2}, \end{aligned}$$

where $M_0 := \frac{4\eta[\Lambda(\alpha-\beta-\eta)+\eta\alpha]}{(\alpha-\beta-\eta)^2(\Lambda+\eta)}$ and $f(S,I) := \left(\frac{S+I+1}{I+1}\right)^2$. Thus, J_2 can be rewritten as

$$\begin{aligned} J_2 = &-d_S \int_\Omega E_{SS} |\nabla S|^2 - d_S \int_\Omega E_{SI} \nabla I \cdot \nabla S - d_I \int_\Omega \gamma(S) E_{IS} \nabla I \cdot \nabla S \\ &- d_I \int_\Omega \gamma(S) E_{II} |\nabla I|^2 - d_I \int_\Omega I\gamma'(S) E_{IS} |\nabla S|^2 - d_I \int_\Omega I\gamma'(S) E_{II} \nabla I \cdot \nabla S \\ = &-\int_\Omega \Psi B_2 \Psi^{\mathcal{T}} \end{aligned}$$

with $\Psi = (\nabla S, \nabla I)$ and

$$B_2 = \begin{pmatrix} d_S E_{SS} + d_I I\gamma'(S) E_{IS} & \frac{d_S E_{SI} + d_I\gamma(S) E_{IS} + d_I I\gamma'(S) E_{II}}{2} \\ \frac{d_S E_{SI} + d_I\gamma(S) E_{IS} + d_I I\gamma'(S) E_{II}}{2} & d_I\gamma(S) E_{II} \end{pmatrix}.$$

With direct computation, we can show that B_2 is positive definite iff

$$g_1(S,I) > g_2(S,I), \tag{4.12}$$

where

$$\begin{aligned} g_1(S,I) &= 2d_S d_I \gamma(S) + 2d_I^2 I\gamma'(S)\gamma(S)[1+M_0 f(S,I)] + 4d_S d_I \gamma(S) M_0 f(S,I), \\ g_2(S,I) &= d_I^2 [\gamma(S)]^2 + d_S^2 + d_I \gamma'(S) I [1+M_0 f(S,I)] \{2d_S + d_I I\gamma'(S)[1+M_0 f(S,I)]\}. \end{aligned}$$

By hypothesis (H2), (1.4) and $1 < f(S,I) < (S+I+1)^2$, (1.8) ensures that (4.12) holds. Therefore, there is a constant $c_2 > 0$ such that

$$J_2 = -\int_\Omega \Psi B_2 \Psi^{\mathcal{T}} \leqslant -c_2 \int_\Omega \left(|\nabla S|^2 + |\nabla I|^2 \right) \leqslant 0,$$

which along with (4.11) substituted into (4.10) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \leq -c_1 \int_\Omega \left(\frac{(S-\hat{S})^2}{S+I+1} + \frac{(I-\hat{I})^2}{S+I+1} \right) =: -c_1 \mathcal{F}(t).$$

Calculating directly, one derives

$$\mathcal{F}'(t) = -\int_\Omega \left[\frac{(S-\hat{S})^2(S_t+I_t)}{(S+I+1)^2} - \frac{2(S-\hat{S})S_t}{S+I+1}\right] - \int_\Omega \left[\frac{(I-\hat{I})^2(S_t+I_t)}{(S+I+1)^2} - \frac{2(I-\hat{I})I_t}{S+I+1}\right],$$

which along with (4.1) gives

$$\mathcal{F}'(t) \leqslant |\mathcal{F}'(t)| \leq \int_\Omega \left[(S-\hat{S})^2 + (I-\hat{I})^2\right]|S_t+I_t| + 2|(S-\hat{S})S_t| + 2|(I-\hat{I})I_t|$$
$$\leqslant c_2, \quad \forall t > 1.$$

for some constant $c_2 > 0$. Noticing that $\mathcal{E}(t) \geqslant 0$ for all $t \geqslant 0$, by lemma 4.2, we get

$$\lim_{t\to\infty}\left(\|S-\hat{S}\|_{L^2} + \|I-\hat{I}\|_{L^2}\right) = 0. \tag{4.13}$$

Applying the Gagliardo–Nirenberg inequality in two dimensions for any $u \in W^{1,\infty}$:

$$\|v\|_{L^\infty} \leqslant c_3\left(\|\nabla u\|_{L^\infty}^{\frac{1}{2}}\|u\|_{L^2}^{\frac{1}{2}} + \|u\|_{L^2}\right),$$

we obtain from (4.1) and (4.13) that $\lim_{t\to\infty}(\|S-\hat{S}\|_{L^\infty} + \|I-\hat{I}\|_{L^\infty}) = 0$. This finishes the proof of theorem 1.4-(ii).

□

5. Numerical simulations and discussion

This paper investigates an SIS model with cross-diffusion dispersal strategy for the infected individuals describing the public health intervention measures (like quarantine) during the outbreak of infectious diseases. The considered SIS model adopts the frequency-dependent transmission mechanism and includes demographic changes (i.e. population recruitment and death). Apart from the global boundedness of solutions established in theorem 1.1, we define the basic reproduction number R_0 by a variational formula and study the threshold dynamics of the model based on R_0 (see theorems 1.3 and 1.4). Below we shall use numerical simulations to illustrate the applications of our analytical results and speculate some results not proved in the paper. We set $\Omega = (0,2)$ in all simulations.

In a special case where the recruitment rate $\Lambda(x)$ of susceptible individuals is constant, we see that cross-diffusion dispersal strategy (see remark 1.2) reduces the value of R_0, namely the basic reproduction number R_0 for $\gamma'(S) \neq 0$ is less than $\hat{R}_0$, where $\hat{R}_0$ is the basic reproduction number when $\gamma(S) = 1$. We can see a numerical example shown in figure 1(a). This implies that public health intervention measures limiting the mobility of infected individuals is effective in controlling the spread of infectious diseases. However, if $\Lambda(x)$ is not constant, we are unable to prove $R_0 < \hat{R}_0$ analytically. Below we use an example to illustrate this conclusion numerically for non-constant $\Lambda(x)$. To this end, we take

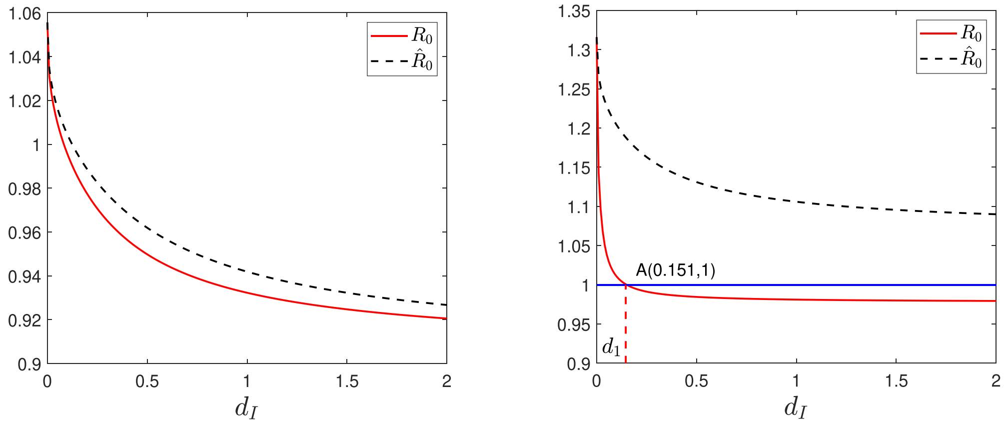
$$\gamma(S) = \mathrm{e}^S \tag{5.1}$$

satisfying hypothesis (H1) and

$$d_S = \theta = 1, \tag{5.2}$$

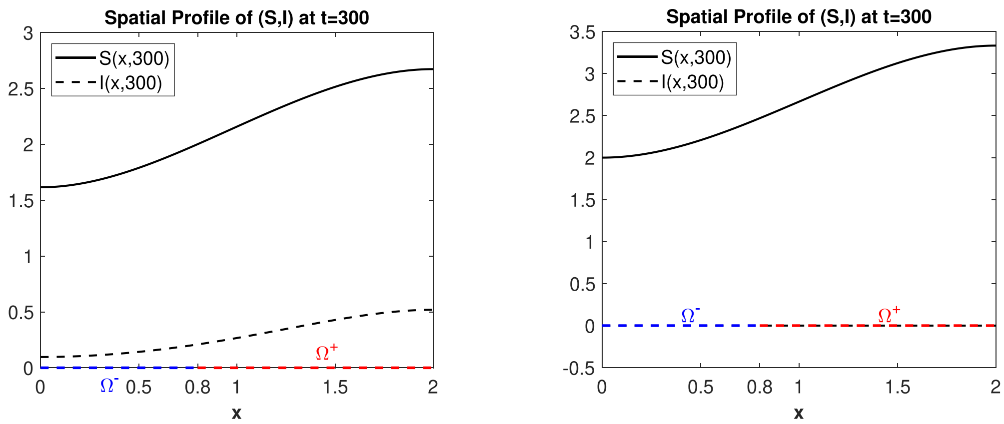
as well as

$$\Lambda(x) = -\frac{1}{3}x^3 + x^2 + 2x, \quad \alpha(x) = 2x+1, \quad \beta(x) = x, \quad \eta(x) = 1.8. \tag{5.3}$$



(a) (b)

Figure 1. Graphs of functions R_0 and $\hat{R}_0$ versus $d_I > 0$, where functions and parameters are taken as follows: (a) $\gamma(S) = 2 - (S+1)^{-1}, d_S = \theta = \Lambda(x) = 1$, and $\alpha(x) = x^2 + 2x + 1.5, \beta(x) = x^2 + 0.5, \eta(x) = x + 2.5$; (b) The functions and parameter values are given in (5.1)–(5.3).



(a) (b)

Figure 2. The profile of susceptible and infected populations with $d_I = 0.2$. (a): $\gamma(S) = 1$; (b): $\gamma(S) = \mathrm{e}^S$. Other functions and parameter values are given in (5.2) and (5.3). The initial value (S_0, I_0) is set as a small random perturbation of (2,1).

Then $\Lambda(x)$ is positive on Ω and one can check that $\widetilde{S} = -\frac{1}{3}x^3 + x^2 + 2 > 0$. The graphs of functions R_0 and $\hat{R}_0$ are numerically plotted in figure 1(b), where we observe that $R_0 < \hat{R}_0$. However, whether or not the cross-diffusion dispersal strategy reduces the basic reproduction number so that $R_0 < \hat{R}_0$ for all $\gamma(S)$ satisfying the hypothesis (H1) remains an outstanding theoretical question for future efforts.

When $\gamma(S)$ is constant, namely the infected individuals undergo random dispersal, the classical results showed that the disease would persist in the high-risk domain Ω (see [31, proposition 3.2, theorem 3.1], [43, theorems 2.5 and 3.3]), as numerically shown in figure 2(a) where we assume $\gamma(S) = 1$ and $d_I = 0.2$ while other functions and parameter values are given by (5.2) and (5.3). The results in proposition 1.2 along with theorems 1.3 and 1.4 indicate that the cross-diffusion dispersal strategy will help eradicate the infectious disease even in the high-risk domain. To illustrate this result, we use the functions and parameter values given in (5.1)–(5.3). With them, we can verify that

$$\int_0^2 [\alpha(x) - \beta(x) - \eta(x)]\,\mathrm{d}x = 0.4 > 0,$$

which means that Ω is a high-risk domain. In this case, the asymptotically stable spatial profile of (S, I) is numerically plotted in figure 2(b) which demonstrates that the disease will be eradicated in the whole domain. By (5.3), we find $\int_0^2 [\alpha(x) - \beta(x) - \eta(x)]\gamma^{-1}(\widetilde{S})\mathrm{d}x \approx -0.0083 < 0$, $\Omega^- = \{x :\ 0 < x < 0.8\}$ and $\Omega^+ = \{x :\ 0.8 < x < 2\}$ are nonempty. This alongside proposition 1.2-(iii) and the figure 1(b) show that $R_0 < 1$ if $d_I = 0.2 > d_1 \approx 0.151$. Therefore, it follows from theorem 1.3 that DFE is linearly stable, which implies that the disease may be eradicated. This is well supported by numerical results shown in figure 2(b). However, we can not conclude the global stability of DFE based on theorem 1.4-(i) since one can check that the condition $\alpha(x) \leqslant \beta(x) + \varepsilon\eta(x)$ for all $x \in \Omega$ with some $\varepsilon \in [0, 1)$ is not satisfied by the functions chosen in (5.3). The numerical simulation of the asymptotically stable spatial profile shown in figure 2(b) indicates that DFE may be globally asymptotically stable even if the condition in theorem 1.4-(i) is not fulfilled. Therefore how to relax the condition of theorem 1.4-(i) is another interesting question remaining open in this paper. The best situation we anticipate is to replace the condition of theorem 1.4-(i) by $R_0 < 1$, but this can not be proven based on the method in this paper.

Data availability statement

No new data were created or analysed in this study.

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