Cauchy Problem of a System of Parabolic Conservation Laws Arising From the Singular Keller-Segel Model in Multi-Dimensions

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ABSTRACT. In this paper, we study the qualitative behavior of solutions to the Cauchy problem of a system of parabolic conservation laws, derived from a Keller-Segel type chemotaxis model with singular sensitivity, in multiple space dimensions. Assuming H^2 initial data, it is shown that under the assumption that only some fractions of the total energy associated with the initial perturbation around a prescribed constant ground state are small, the Cauchy problem admits a unique global-in-time solution, and the solution converges to the prescribed ground state as time goes to infinity. In addition, it is shown that solutions of the fully dissipative model converge to that of the corresponding partially dissipative model with certain convergence rates as a specific system parameter tends to zero.

1. INTRODUCTION

Chemotaxis, the movement of an organism in response to a chemical stimulus, has been an important mechanism of various biological phenomena/processes, such as aggregation of bacteria, slime mould formation, fish pigmentation, tumor angiogenesis, blood vessel formation, wound healing (cf. [29]). The prototypical chemotaxis model, known as the Keller-Segel model because of the pioneering works of [14–16], reads in its general form as

$$\begin{cases} p_t = \nabla \cdot (D\nabla p - \chi p \nabla \varphi(q)), \\ q_t = \varepsilon \Delta q + g(p, q), \end{cases} \tag{1.1}$$

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where $p(x,t)$ and $q(x,t)$ denote the cell density and chemical (signal) concentration at position $x \in \mathbb{R}^n$ and time t, respectively. The function $\varphi(q)$ is called the chemotactic sensitivity accounting for the signal response mechanism, and $g(p,q)$ is the chemical kinetics (growth and degradation). Also, $D > 0$ and $\varepsilon \geq 0$ are cell and chemical diffusion coefficients, respectively, and $\chi \neq 0$ is referred to as the chemotactic coefficient, where the chemotaxis is said to be attractive if $\chi > 0$ and repulsive if $\chi < 0$. The model (1.1) has generic applications depending on the specific forms of $\varphi(q)$ and $g(p,q)$. There are two major classes of chemotactic response function: linear response $\varphi(q) = q$ and logarithmic response $\varphi(q) = \ln q$. The former was originally used by Keller and Segel in [15, 16] to model the self-aggregation of *Dictyostelium discoideum* in response to cyclic adenosine monophosphate (cAMP) that it secreted, while the latter was used in [14] to model the wave propagation of bacterial chemotaxis. The prototypical Keller-Segel model with logarithmic sensitivity reads as

$$\begin{cases} p_t = \nabla \cdot (D\nabla p - \chi p \nabla \ln q), \\ q_t = \varepsilon \Delta q - \mu p q^k - \sigma q, \end{cases} \tag{1.2}$$

where $\mu \in \mathbb{R}$ and $\sigma \geq 0$ are constants. As $\chi, \mu > 0$, $0 \leq k < 1$, and $\sigma = 0$, the model (1.2) was proposed by Keller-Segel in [14] to explain the wave band propagation observed in the experiment by Adler [1]. Later, the same model with $k = 1$ was used in [18] to describe the dynamical interactions between vascular endothelial cells and signaling molecules vascular endothelial growth factor in the onset of tumor angiogenesis. It was particularly mentioned in [18] that the chemical diffusion coefficient ε was small or negligible since it is far less important than the interaction between vascular endothelial cells and vascular endothelial growth factors. As $\chi, \mu < 0$, $\sigma > 0$, the model (1.2) was derived in [17, 30] to model the chemotactic movement of reinforced random walkers (denoted by p) which deposit a non-diffusive or slowly moving (i.e., $0 \leq \varepsilon \ll 1$) signal q that modifies the local environment for succeeding passages. If $\chi > 0$ and $\mu < 0$, the model will exhibit blow-up behavior even in one dimension [17, 40]. In this paper we are concerned with the case $\chi\mu > 0$.

Though the logarithmic sensitivity plays an indispensable role in generating traveling wave solutions (cf. [14]) which can be obtained directly from the model (1.2), its singularity at $q = 0$ sets up a great obstacle to further understanding of the model dynamics such as stability of traveling wave solutions, well-posedness of the model, and so on. Therefore, the results of the Keller-Segel model (1.2) with logarithmic sensitivity are much less compared to the linear sensitivity (e.g., see [2, 9, 11, 32]). However, in the case $k = 1$, the logarithmic singularity can be resolved by the Cole-Hopf type transformation ([17, 26])

$$\mathbf{q} = -\frac{\sqrt{\chi\mu}}{\mu}\nabla \ln(\exp(\sigma t) q) = -\frac{\sqrt{\chi\mu}}{\mu}\frac{\nabla q}{q},$$

which converts the model (1.2) into a non-singular system of conservation laws

$$
\begin{cases} p_t - \nabla \cdot (p\mathbf{q}) = \Delta p, \\ \mathbf{q}_t - \nabla \left(p - \frac{\varepsilon}{\chi} |\mathbf{q}|^2 \right) = \frac{\varepsilon}{D} \Delta \mathbf{q}, \end{cases} \tag{1.3}
$$

where we have used the temporal-spatial re-scalings

$$
\tilde{t} = \frac{\chi\mu}{D} t, \quad \tilde{x} = \frac{\sqrt{\chi\mu}}{D} x
$$

and then dropped tildes for convenience. Though the transformed system (1.3) has no singularity and appears to be easier to analyze than (1.2), it creates a quadratic nonlinearity (i.e., $\varepsilon \nabla |\mathbf{q}|^2$) resembling the nonlinearity in the Navier-Stokes equations, and brings various difficulties for analysis. Many results have been obtained for the transformed system (1.3) in one dimension (to be recalled later), but the results in multi-dimensions are very limited; in particular, the existence of large-data solutions of (1.3) in multi-dimensions still remains open. Moreover, the parameter ε, which is the diffusion coefficient in the original Keller-Segel model, now acts as coefficient of both diffusion and advection. Since ε is small/negligible in applications mentioned above, the limit of solutions as $\varepsilon \to 0$ is a relevant but delicate question because of the dual role of ε. These features distinguish the transformed system (1.3) from other known hyperbolic systems (e.g., see [3, 10, 33]). The purpose of this paper is to establish the global existence of solutions to the transformed model (1.3) in multi-dimensions with very mild smallness assumptions on the initial data, and show the convergence of solutions as $\varepsilon \to 0$. For brevity, we assume that $\chi = -1$ and $D = 1$ since their specific values are not of importance in our analysis. That is we consider the following system of parabolic conservation laws:

$$
\begin{cases} \partial_t p - \nabla \cdot (p\mathbf{q}) = \Delta p, \\ \partial_t \mathbf{q} - \nabla (p + \varepsilon |\mathbf{q}|^2) = \varepsilon \Delta \mathbf{q}. \end{cases} \tag{1.4}
$$

The one-dimensional version of (1.4) has been well studied in the literature, and we recall the pertaining results below:

- explicit and numerical solutions on finite intervals [17]
- shock wave formation for the Riemann problem on $\mathbb{R}$ [35]
- global well-posedness and long-time behavior of small-amplitude classical solutions on finite intervals [41]
- local nonlinear stability of one-dimensional traveling wave solutions on $\mathbb{R}$ [13, 22, 24–27]
- global well-posedness of large-amplitude classical solutions on $\mathbb{R}$ [7]
- global well-posedness of large-amplitude classical solutions on finite intervals [6]

- long-time behavior and chemical diffusion limit of large-amplitude classical solutions on finite intervals [21, 23, 34, 37]
- long-time behavior, chemical diffusion limit and spatial analyticity of large-amplitude classical solutions on $\mathbb{R}$ [20, 28]
- boundary layer formation and characterization of large-amplitude classical solutions on finite intervals [12, 21].

Next, we point out the facts that motivate the current work, and state the specific goals to be achieved in this paper.

Motivation and goals. The current work is primarily motivated by the energy criticality of the model because of dimensionality. Let us first take a look at the scaling invariant property enjoyed by the model. Indeed, by a direct calculation, we can show that (1.4) holds its form under the scaling

$$(p, \mathbf{q}) \to (p^\lambda, \mathbf{q}^\lambda) := (\lambda^2 p(\lambda \mathbf{x}, \lambda^2 t), \lambda \mathbf{q}(\lambda \mathbf{x}, \lambda^2 t)).$$

Under this scaling, when the initial data are perturbed around the zero ground state, it holds that

$$\begin{aligned} \|p_0^\lambda\|_{L^2(\mathbb{R}^n)}^2 &= \lambda^{4-n}\|p_0\|_{L^2(\mathbb{R}^n)}^2 \\ \|\mathbf{q}_0^\lambda\|_{L^2(\mathbb{R}^n)}^2 &= \lambda^{2-n}\|\mathbf{q}_0\|_{L^2(\mathbb{R}^n)}^2, \end{aligned}$$

which reveals that norm-inflation (especially for the $\mathbf{q}$-component) is possible only when $n = 1$.

Next, we note that the weak Lyapunov functional associated with (1.4) reads

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^n} E(p, \bar{p})\,\mathrm{d}\mathbf{x} + \|\mathbf{q}\|_{L^2(\mathbb{R}^n)}^2\right) + \int_{\mathbb{R}^n} \frac{|\nabla p|^2}{p}\,\mathrm{d}\mathbf{x} + \varepsilon\|\nabla \mathbf{q}\|_{L^2(\mathbb{R}^n)}^2 \\ &\quad = \varepsilon \int_{\mathbb{R}^n} |\mathbf{q}|^2 \nabla \cdot \mathbf{q}\,\mathrm{d}\mathbf{x}, \end{aligned}$$

where $\bar{p} > 0$ is a constant ground state and the "entropy expansion" is defined by

$$E(p, \bar{p}) = [p\ln(p) - p] - [\bar{p}\ln(\bar{p}) - \bar{p}] - \ln(\bar{p})(p - \bar{p}),$$

which has been observed in many works dealing with the one-dimensional version of (1.4).

Because of the scaling property of the $\mathbf{q}$-component and the fact that the righthand side of the weak Lyapunov functional is zero only when $n = 1$, from the point of view of energy criticality we then see that the global well-posedness of large-data solutions to (1.4) is *sub-critical* when $n = 1$, *critical* when $n = 2$, and *super-critical* when $n \geq 3$. The observation partially explains why the model is globally well posed in one space dimension, as was observed in many previous works, while the problem is still widely open in the multi-dimensional case.

To the authors' knowledge, the following results have been established for the Cauchy problem of (1.4) in $\mathbb{R}^n$ ($n \geq 2$), where $\bar{p} > 0$ is a constant:

- local well-posedness and blowup criteria of large-amplitude classical solutions [5, 19]
- global well-posedness and long-time behavior of small-amplitude classical solutions [8, 19]
- global well-posedness of classical solutions when only

$$\|p_0 - \bar{p}\|_{L^2(\mathbb{R}^3)} + \|\mathbf{q}_0\|_{H^1(\mathbb{R}^3)}$$

is small, and long-time behavior when $\|p_0 - \bar{p}\|_{H^2(\mathbb{R}^3)} + \|\mathbf{q}_0\|_{H^1(\mathbb{R}^3)}$ is small [4]
- global well-posedness, long-time behavior and chemical diffusion limit of classical solutions when only $\|(p_0 - \bar{p}, \mathbf{q}_0)\|_{L^2(\mathbb{R}^3)}$ is small [31]
- global well-posedness, long-time behavior and chemical diffusion limit of strong solutions when only $\|(p_0 - \bar{p}, \mathbf{q}_0)\|_{H^1(\mathbb{R}^n)}$ $(n = 2, 3)$ is small [36]
- global generalized (weak) solutions on bounded domains in $\mathbb{R}^2$ with Neumann boundary conditions [38], followed with a work addressing the eventual smoothness of solutions [39].

A close inspection shows that although the above list of results provides useful information for the understanding of global well-posedness, long-time behavior, and diffusion limit of solutions to (1.4) in multi-dimensional spaces, none of them gives a positive answer to such questions when the initial data carry a potentially large L^2 norm of the zeroth frequency of the perturbation.

Throughout this paper, we consider the Cauchy problem of (1.4) subject to the initial condition

$$(p, \mathbf{q})(\mathbf{x}, 0) = (p_0, \mathbf{q}_0)(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \ n = 2, 3. \tag{1.5}$$

The primary goal of this paper is to settle the aforementioned issue by constructing global-in-time solutions to (1.4) and (1.5) under minimal smallness requirements on the initial data, and studying their long-time behavior and zero diffusion limits. To be precise, let us recall the entropic energy:

$$\int_{\mathbb{R}^n} \{[p\ln(p) - p] - [\bar{p}\ln(\bar{p}) - \bar{p}] - \ln(\bar{p})(p - \bar{p})\}\, \mathrm{d}\mathbf{x} + \frac{1}{2}\|\mathbf{q}\|^2_{L^2(\mathbb{R}^n)}. \tag{1.6}$$

We will establish the global well-posedness of strong solutions to (1.4) and (1.5) in the following situations:

- In $\mathbb{R}^2$ when (1.6) is small and $\varepsilon > 0$,
- In $\mathbb{R}^3$ when (1.6) is small and $\varepsilon \geq 0$.

We comment that assuming the smallness of the spatial integral of the first order Taylor expansion of the anti-logarithmic function of p allows the usual Sobolev norm of the perturbation to be potentially large (see Remark 2.3). As a consequence of global well-posedness, we also identify the long-time behavior of the solutions, and study the zero chemical diffusion limits and convergence rate of solutions as $\varepsilon \to 0$. In addition, we prove the similar results for the following case:

- In $\mathbb{R}^2$ when $\|(p_0-\bar{p},\mathbf{q}_0)\|_{L^2(\mathbb{R}^2)}+\|p_0-\bar{p}\|_{L^4(\mathbb{R}^2)}$ is small and $\varepsilon \geq 0$.

This situation has not been studied before.

We achieve our goals by developing L^p-based energy methods. Since we only assume the smallness of individual components of the total Sobolev norm of the initial data, the major technical difficulty consists in closing the energy estimates for each individual frequency of the solution, without combining low and high frequencies. Because of the lack of the Poincaré inequality in the whole space, the energy estimates for the zeroth frequency part of the solution is challenging, especially when the zeroth frequency part is allowed to be potentially large. Moreover, because the Gagliardo-Nirenberg interpolation inequalities generate less powers of high frequencies of a function in $\mathbb{R}^2$ than in $\mathbb{R}^3$, the proof in the two-dimensional case is considerably more complicated than the three-dimensional case. We break the walls by terminating low frequencies through creating higher-order nonlinearities, taking full advantage of the dissipation mechanisms and the smallness assumptions on individual frequencies, and using various Gagliardo-Nirenberg interpolation inequalities.

The rest of this paper is organized as follows. In Section 2, we state and comment on the main results. We then prove the main results in Sections 3–4. The paper ends with some concluding remarks.

2. STATEMENT OF MAIN RESULTS

We first state the common assumptions to be satisfied by the initial functions:

- For $n=2$ or 3, we assume universally that

$$(p_0-\bar{p},\mathbf{q}_0)\in H^2(\mathbb{R}^n), \tag{2.1}$$

where $\bar{p}>0$ is a constant.

- Because p represents the cell density, and $\mathbf{q}=\nabla \ln q$, we assume

$$p_0(\mathbf{x})\geq 0 \quad \text{and} \quad \nabla\times\mathbf{q}_0(\mathbf{x})=\mathbf{0}, \tag{2.2}$$

for any $\mathbf{x}\in\mathbb{R}^n$.

- We assume that one of the following quantities is sufficiently small:

$$2\int_{\mathbb{R}^n}[(p_0\ln p_0-p_0)-(\bar{p}\ln\bar{p}-\bar{p})-\ln\bar{p}(p_0-\bar{p})]\,\mathrm{d}\mathbf{x}+||\mathbf{q}_0||^2_{L^2(\mathbb{R}^n)}, \tag{2.3}$$

$$||p_0-\bar{p}||^2_{L^2(\mathbb{R}^n)}+||p_0-\bar{p}||^4_{L^4(\mathbb{R}^n)}+||\mathbf{q}_0||^2_{L^2(\mathbb{R}^n)}. \tag{2.4}$$

Remark 2.1. We underline that in the assumption (2.3), $\|p_0-\bar{p}\|_{L^2}$ can be potentially large due to the following inequality:

$$||p_0-\bar{p}||^2_{L^2}\geq\frac{\bar{p}}{2}\int_{\mathbb{R}^n}[(p_0\ln p_0-p_0)-(\bar{p}\ln\bar{p}-\bar{p})-\ln\bar{p}(p_0-\bar{p})]\,\mathrm{d}\mathbf{x}.$$

Indeed, let us consider the function

$$F(w) = (w-\bar{p})^2 - \frac{\bar{p}}{2}[(w\ln w - w) - (\bar{p}\ln\bar{p} - \bar{p}) - \ln\bar{p}(w-\bar{p})], \quad w \geq 0.$$

It is straightforward to check that

$$F(\bar{p}) = 0,\ F'(\bar{p}) = 0,\ F''(w) = 2 - \frac{\bar{p}}{2w},$$

which imply that $F(w) \geq 0$ for $w \in [\bar{p}/4, \infty)$. Moreover, since

$$\begin{aligned} F(0) &= \frac{(\bar{p})^2}{2}, \\ F\left(\frac{\bar{p}}{4}\right) &= \left(\frac{3}{16} + \frac{\ln 4}{8}\right)(\bar{p})^2, \\ F''(w) &< 0 \quad \text{for } w \in \left[0, \frac{\bar{p}}{4}\right), \end{aligned}$$

we have $F(w) > 0$ for $w \in [0, \bar{p}/4)$. Therefore, $F(w) \geq 0$ for all $w \in [0, \infty)$. In Appendix A, we provide explicit examples of initial functions whose p-component can have arbitrarily small entropic energy, but arbitrarily large H^2 energy.

2.1. Small initial entropy. The first result addresses the global well-posedness and long-time behavior of solutions to (1.4) and (1.5) when the initial entropy is small.

Theorem 2.2. *Let $n = 2, 3$ and consider the Cauchy problem* (1.4)—(1.5). *Suppose the initial data satisfy* (2.1) *and* (2.2), *and the initial entropy* (2.3) *is sufficiently small, where the smallness depends on the other components of the H^2 norm of the initial functions. Then, there exists a unique solution to* (1.4)—(1.5), *such that the following hold:*

(1) *When $n = 2$, for any fixed value of $\varepsilon > 0$, it holds that*

$$\begin{aligned} &\|(p-\bar{p})(t)\|_{L^2}^2 + \bar{p}\|\mathbf{q}(t)\|_{L^2}^2 \\ &\qquad + \int_0^t \left(\|\nabla p(\tau)\|_{L^2}^2 + \varepsilon\bar{p}\|\nabla\cdot\mathbf{q}(\tau)\|_{L^2}^2\right) \mathrm{d}\tau \leq C_1; \\ &\|\nabla p(t)\|_{H^1}^2 + \bar{p}\|\nabla\cdot\mathbf{q}(t)\|_{H^1}^2 \\ &\qquad + \int_0^t \left(\|\nabla p(\tau)\|_{H^2}^2 + \varepsilon\bar{p}\|\nabla\cdot\mathbf{q}(\tau)\|_{H^2}^2\right) \mathrm{d}\tau \leq C_2, \end{aligned}$$

where the time-independent constant C_1 depends only on $\|p_0\|$, $\|\mathbf{q}_0\|$, and $\bar{p}$, while C_2 depends on $\|p_0\|_{H^2}$, $\|\mathbf{q}_0\|_{H^2}$, $\bar{p}$, and $1/\varepsilon$, and $C_2 \to \infty$ as $\varepsilon \to 0$;

(2) *When $n = 3$, for any fixed value of $\varepsilon \geq 0$, it holds that*

$$\begin{aligned} &\|(p - \bar{p})(t)\|_{H^2}^2 + \bar{p}\|\mathbf{q}(t)\|_{H^2}^2 \\ &\qquad + \int_0^t (\|\nabla p(\tau)\|_{H^2}^2 + \varepsilon\bar{p}\|\nabla \cdot \mathbf{q}(\tau)\|_{H^2}^2)\, \mathrm{d}\tau \leq C_3, \\ &\int_0^t \|\nabla \cdot \mathbf{q}(\tau)\|_{H^1}^2\, \mathrm{d}\tau \leq C_4(1 + \varepsilon), \end{aligned}$$

where the constants C_3 and C_4 depend only on $\|p_0 - \bar{p}\|_{H^2}$, $\|\mathbf{q}_0\|_{H^2}$, and $\bar{p}$. In addition, the convergence

$$\lim_{t\to\infty} (\|(p - \bar{p})(t)\|_{L^\infty}^2 + \|\mathbf{q}(t)\|_{L^\infty}^2 + \|\nabla p(t)\|_{H^1}^2 + \|\nabla \cdot \mathbf{q}(t)\|_{H^1}^2) = 0 \tag{2.5}$$

holds in both cases.

Remark 2.3. We comment that the smallness of the quantities in (2.3)–(2.4) depends (relatively) on the other components of the H^2-norm of the initial functions. As the conditions are lengthy, we refer to the proofs for details. However, the reader will see from the proofs that we require the *products* of individual frequencies of the initial functions to be smaller than some absolute constants. Roughly speaking, this parallels to a scenario in which one assumes the product of two positive numbers to be sufficiently small, while allowing either one to be potentially large.

The second theorem establishes the consistency and convergence rate between the chemically diffusible and non-diffusible models in $\mathbb{R}^3$.

Theorem 2.4. *Let $n = 3$, and let $(p^\varepsilon, \mathbf{q}^\varepsilon)$ and $(p^0, \mathbf{q}^0)$ be the solutions to* (1.4)–(1.5) *obtained in Theorem* 2.2 *with $\varepsilon > 0$ and $\varepsilon = 0$, respectively, for the same initial data. Then, there are positive constants d_i ($i = 1, \ldots, 4$) such that, for any $t > 0$,*

$$\begin{aligned} &\|(p^\varepsilon - p^0)(t)\|_{L^2}^2 + \|(\mathbf{q}^\varepsilon - \mathbf{q}^0)(t)\|_{L^2}^2 \leq d_1 t e^{d_2 t}\varepsilon^2, \\ &\|(\nabla p^\varepsilon - \nabla p^0)(t)\|_{L^2}^2 + \|(\nabla \cdot \mathbf{q}^\varepsilon - \nabla \cdot \mathbf{q}^0)(t)\|_{L^2}^2 \leq d_3 e^{d_4 t}(1 + t\varepsilon)\varepsilon, \end{aligned} \tag{2.6}$$

where the constants d_i depend only on $\|p_0 - \bar{p}\|_{H^2}$, $\|\mathbf{q}_0\|_{H^2}$ and $\bar{p}$.

2.2. Small initial energy. In [31], the global well-posedness, long-time behavior, and diffusion limit of classical solutions to (1.4)–(1.5) were established in $\mathbb{R}^3$ when $(p_0 - \bar{p}, \mathbf{q}_0) \in H^3$, assuming that $\|(p_0 - \bar{p}, \mathbf{q}_0)\|_{L^2}$ is small. Next, we establish a similar result in $\mathbb{R}^2$ under lower regularity requirements on the initial data.

Theorem 2.5. *Let $n = 2$ and consider the Cauchy problem* (1.4)—(1.5). *Suppose the initial data satisfy* (2.1) *and* (2.2)*, and the initial energy* (2.4) *is sufficiently small, where the smallness depends on the other components of the H^2 norm of the* *initial functions. Then, there exists a unique solution to* (1.4) *and* (1.5)*, such that for any fixed value of* $\varepsilon \geq 0$*, it holds that*

$$\begin{aligned} &||(p-\bar{p})(t)||_{H^1}^2 + ||\mathbf{q}(t)||_{H^1}^2 + \int_0^t (||\nabla p(\tau)||_{H^1}^2 + \varepsilon\bar{p}||\nabla\cdot\mathbf{q}(\tau)||_{H^1}^2) \leq C_5, \\ &\|\Delta p(t)\|^2 + \|\Delta\mathbf{q}(t)\|^2 + \int_0^t (\|\nabla\Delta p(\tau)\|^2 + \varepsilon\bar{p}\|\Delta\nabla\cdot\mathbf{q}(\tau)\|^2) \leq C_6(1+\varepsilon), \\ &\int_0^t ||\nabla\cdot\mathbf{q}(\tau)||_{H^1}^2 \leq C_7(1+\varepsilon), \end{aligned}$$

where the constants C_5*,* C_6 *and* C_7 *depend only on* $\|p_0-\bar{p}\|_{H^2}$*,* $\|\mathbf{q}_0\|_{H^2}$*, and* $\bar{p}$*. In addition, results similar to those recorded in* (2.5) *and* (2.6) *hold.*

Remark 2.6. We finally comment that the global well-posedness, long-time behavior, and chemical diffusion limit of strong solutions to (1.4) and (1.5) with small initial entropy in $\mathbb{R}^2$ are still elusive when $\varepsilon = 0$; they cannot be proved by using the energy method developed in this paper. We leave the investigation for the future.

Notation 2.7. Throughout the rest of the paper, we use $\|\cdot\|$ to denote $\|\cdot\|_{L^2}$. Unless specified, we use c to denote a generic constant which is independent of the unknown functions, t, ε, and initial data. The value of the constant may vary line by line according to the context.

3. Small Entropic Solutions

In this section, we present the proofs for Theorems 2.2–2.4. To this end, we first set $\tilde{p} = p - \bar{p}$ and reformulate the Cauchy problem of (1.4) with initial data satisfying (2.1)–(2.2) as

$$\begin{cases} \partial_t p - \nabla\cdot(p\mathbf{q}) - \bar{p}\,\nabla\cdot\mathbf{q} = \Delta p, & \mathbf{x}\in\mathbb{R}^n,\ t>0, \\ \partial_t\mathbf{q} - \nabla p = \varepsilon\Delta\mathbf{q} - \varepsilon\nabla(|\mathbf{q}|^2), & \varepsilon>0, \\ p_0+\bar{p}\geq 0,\ \nabla\times\mathbf{q}_0 = \mathbf{0}, & (p_0,\mathbf{q}_0)\in H^2(\mathbb{R}^n), \end{cases} \tag{3.1}$$

where we have suppressed tilde for simplicity. In the sequel, $(p,\mathbf{q})$ always denotes the perturbation of the original solution around $(\bar{p},\mathbf{0})$ unless otherwise specified.

First, we note that by the initial conditions and maximum principle, one can show that the function $p+\bar{p}\geq 0$. In addition, because of the initial curl free condition and the equation $\partial_t(\nabla\times\mathbf{q}) = \varepsilon\Delta(\nabla\times\mathbf{q})$, the function $\mathbf{q}$ is curl free as time evolves. Hence, it suffices to deal with the divergence of $\mathbf{q}$, that is, $\nabla\cdot\mathbf{q}$, in order to estimate the spatial derivatives of $\mathbf{q}$. Moreover, under the curl free condition, we have $\Delta\mathbf{q} = \nabla(\nabla\cdot\mathbf{q})$. The existence of local solutions of (3.1) can be obtained by the standard argument (see, e.g., [36]).

Lemma 3.1 (Local existence). *There is a* $T_0 = T_0(\|p_0\|_{H^2(\mathbb{R}^n)}, \|\mathbf{q}_0\|_{H^2(\mathbb{R}^n)})$ *such that the Cauchy problem* (3.1) *has a unique solution*

$$(p, \mathbf{q}) \in C([0, T_0); H^2(\mathbb{R}^n)) \quad \text{with } p + \bar{p} \geq 0 \text{ and } \nabla \times \mathbf{q} = 0.$$

To extend the local solution to a global one, it suffices to derive the *a priori* estimates for the solution obtained in Lemma 3.1.

3.1. Global well-posedness in 2D. To this end, we first make an *a priori* assumption; that is, the following inequalities hold true for some finite $T > 0$:

$$\sup_{0 \leq t \leq T} \|\mathbf{q}(t)\|^2 \leq \delta_1, \qquad \sup_{0 \leq t \leq T} \|p(t)\|^2 \leq M_1, \tag{3.2}$$

where $\delta_1, M_1 > 0$ are constants to be determined later. Next, we shall derive the *a priori* estimates to obtain the global solution and show that the obtained solution satisfies the above *a priori* assumption.

Lemma 3.2. *Let the solution* $(p, \mathbf{q})$ *of* (3.1) *with* $n = 2$ *satisfy* (3.2). *Suppose that the initial data satisfy* (2.1) *and* (2.2), *and the initial entropy* (2.3) *is sufficiently small. Then, for any given constant* $M_1 > 0$ *and any fixed value of* $\varepsilon > 0$, *if* δ_1 *is suitably small, there are positive constants* γ_i $(i = 1, 2)$ *which are independent of* t, *such that*

$$\|p(t)\|^2 + \bar{p}\|\mathbf{q}(t)\|^2 + \int_0^t (\|\nabla p(\tau)\|^2 + \varepsilon\bar{p}\|\nabla \cdot \mathbf{q}(\tau)\|^2)\,\mathrm{d}\tau \leq \gamma_1,$$

$$\|\nabla p(t)\|_{H^1}^2 + \bar{p}\|\nabla \cdot \mathbf{q}(t)\|_{H^1}^2 + \int_0^t (\|\nabla p(\tau)\|_{H^2}^2 + \varepsilon\bar{p}\|\nabla \cdot \mathbf{q}(\tau)\|_{H^2}^2)\,\mathrm{d}\tau \leq \gamma_2,$$

and γ_1 *depends only on* $\|p_0\|$, $\|\mathbf{q}_0\|$, *and* $\bar{p}$, *while* γ_2 *depends on* $\|p_0\|_{H^2}$, $\|\mathbf{q}_0\|_{H^2}$, $\bar{p}$, *and* $1/\varepsilon$, *and* $\gamma_2 \to \infty$ *as* $\varepsilon \to 0$.

We proceed to prove Lemma 3.2 and close the *a priori* assumption (3.2) (i.e., the realization of (3.2)) where appropriate along the proof. The proof consists of four estimates given in the following Sections 3.1.1–3.1.4.

3.1.1. Entropy estimate. Testing the first equation of (3.1) by the expression $\ln(p + \bar{p}) - \ln(\bar{p})$ and the second equation by $\mathbf{q}$, then adding the results, we can show that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^2} [\eta(p + \bar{p}) - \eta(\bar{p}) - \eta'(\bar{p})p]\,\mathrm{d}\mathbf{x} + \frac{1}{2}\|\mathbf{q}\|^2\right) \\ &\qquad + \int_{\mathbb{R}^2} \frac{|\nabla p|^2}{p + \bar{p}}\,\mathrm{d}\mathbf{x} + \varepsilon\|\nabla \cdot \mathbf{q}\|^2 = \varepsilon \int_{\mathbb{R}^2} |\mathbf{q}|^2(\nabla \cdot \mathbf{q})\,\mathrm{d}\mathbf{x}, \end{aligned} \tag{3.3}$$

where $\eta(z) = z\ln z - z$, and the righthand side of (3.3) can be estimated by using the Gagliardo-Nirenberg inequality: $\|f\|_{L^4}^2 \lesssim \|f\|_{L^2}\|\nabla f\|_{L^2}$, as

$$\left|\varepsilon \int_{\mathbb{R}^2} |\mathbf{q}|^2(\nabla \cdot \mathbf{q})\,\mathrm{d}\mathbf{x}\right| \leq \varepsilon\|\mathbf{q}\|_{L^4}^2\|\nabla \cdot \mathbf{q}\| \leq c\varepsilon\|\mathbf{q}\|\,\|\nabla \cdot \mathbf{q}\|^2 \leq c\varepsilon\delta_1^{1/2}\|\nabla \cdot \mathbf{q}\|^2.$$

Hence, when δ_1 is smaller than some absolute constant, we update (3.3) as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{\mathbb{R}^2}[\eta(p+\bar{p})-\eta(\bar{p})-\eta'(\bar{p})p]\,\mathrm{d}\mathbf{x}+\frac{1}{2}\|\mathbf{q}\|^2\Big)\\ +\int_{\mathbb{R}^2}\frac{|\nabla p|^2}{p+\bar{p}}\,\mathrm{d}\mathbf{x}+\frac{\varepsilon}{2}\|\nabla\cdot\mathbf{q}\|^2\le 0,\end{aligned}$$

which implies

$$\begin{aligned}&\int_{\mathbb{R}^2}[\eta(p+\bar{p})-\eta(\bar{p})-\eta'(\bar{p})p]\,\mathrm{d}\mathbf{x}+\frac{1}{2}\|\mathbf{q}\|^2\\ &\quad+\int_0^t\Big(\int_{\mathbb{R}^2}\frac{|\nabla p|^2}{p+\bar{p}}\,\mathrm{d}\mathbf{x}+\frac{\varepsilon}{2}\|\nabla\cdot\mathbf{q}\|^2\Big)\,\mathrm{d}\tau\\ &\qquad\le\int_{\mathbb{R}^2}[\eta(p_0+\bar{p})-\eta(\bar{p})-\eta'(\bar{p})p_0]\,\mathrm{d}\mathbf{x}+\frac{1}{2}\|\mathbf{q}_0\|^2.\end{aligned}$$

In particular, we have

$$\begin{aligned}\|\mathbf{q}(t)\|^2+\varepsilon\int_0^t\|\nabla\cdot\mathbf{q}(\tau)\|^2\,\mathrm{d}\tau\\ \le 2\int_{\mathbb{R}^2}[\eta(p_0+\bar{p})-\eta(\bar{p})-\eta'(\bar{p})p_0]\,\mathrm{d}\mathbf{x}+\|\mathbf{q}_0\|^2.\end{aligned}\tag{3.4}$$

Therefore, we can realize the smallness of δ_1 by choosing the righthand side of (3.4) to be sufficiently small. Next, we go through the regular energy estimates.

3.1.2. L^2-estimate. Taking the L^2 inner products of the equations in (3.1) with the targeting functions and applying the same Gagliardo-Nirenberg inequality as above, we end up with

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|p\|^2+\bar{p}\|\mathbf{q}\|^2)+\|\nabla p\|^2+\varepsilon\bar{p}\|\nabla\cdot\mathbf{q}\|^2\\ &\quad=-\int_{\mathbb{R}^2}p(\mathbf{q}\cdot\nabla p)\,\mathrm{d}\mathbf{x}+\varepsilon\bar{p}\int_{\mathbb{R}^2}|\mathbf{q}|^2\nabla\cdot\mathbf{q}\,\mathrm{d}\mathbf{x}\\ &\le\|p\|_{L^4}\|\mathbf{q}\|_{L^4}\|\nabla p\|+\varepsilon\big|\|\mathbf{q}\|\big|_{L^4}^2\|\nabla\cdot\mathbf{q}\|\\ &\le c\big(\|p\|^{1/2}\|\mathbf{q}\|^{1/2}\|\nabla\cdot\mathbf{q}\|^{1/2}\|\nabla p\|^{3/2}+\varepsilon\bar{p}\|\mathbf{q}\|\,\|\nabla\cdot\mathbf{q}\|^2\big)\\ &\le c\big((\delta_1M_1)^{1/4}\|\nabla\cdot\mathbf{q}\|^{1/2}\|\nabla p\|^{3/2}+\varepsilon\bar{p}\delta_1^{1/2}\|\nabla\cdot\mathbf{q}\|^2\big)\\ &\le c(\delta_1M_1)^{1/4}\Big(\frac{\varepsilon}{4}\|\nabla\cdot\mathbf{q}\|^2+\frac{3}{4\varepsilon^{1/3}}\|\nabla p\|^2\Big)+c\varepsilon\bar{p}\delta_1^{1/2}\|\nabla\cdot\mathbf{q}\|^2\\ &\le c\varepsilon\bar{p}\left(\frac{(\delta_1M_1)^{1/4}}{4\bar{p}}+\delta_1^{1/2}\right)\|\nabla\cdot\mathbf{q}\|^2+c(\delta_1M_1)^{1/4}\frac{3}{4\varepsilon^{1/3}}\|\nabla p\|^2.\end{aligned}\tag{3.5}$$

Hence, when $\delta_1 M_1$ and δ_1 are smaller than some absolute constants (depending on ε), it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|p\|^2 + \bar{p}\|\mathbf{q}\|^2) + \|\nabla p\|^2 + \varepsilon\bar{p}\|\nabla\cdot\mathbf{q}\|^2 \le 0,$$

which yields, after integrating with respect to time,

$$\begin{aligned} \|p(t)\|^2 + \bar{p}\|\mathbf{q}(t)\|^2 &+ \int_0^t (\|\nabla p(\tau)\|^2 + \varepsilon\bar{p}\|\nabla\cdot\mathbf{q}(\tau)\|^2)\,\mathrm{d}\tau \\ &\le \|p_0\|^2 + \bar{p}\|\mathbf{q}_0\|^2, \quad \forall\, t\in[0,T]. \end{aligned} \tag{3.6}$$

Thus, we can realize the second assumption of (3.2) by choosing

$$M_1 = \|p_0\|^2 + \bar{p}\|\mathbf{q}_0\|^2 + 1.$$

Next, we estimate the first-order spatial derivatives of the solution.

3.1.3. ***H^1-estimate.*** Taking the L^2 inner products of the equations in (3.1) with $-\Delta$ of the targeting functions, we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla p\|^2 + \bar{p}\|\nabla\cdot\mathbf{q}\|^2) &+ \|\Delta p\|^2 + \varepsilon\bar{p}\|\Delta\mathbf{q}\|^2 \\ &= -\int_{\mathbb{R}^2}\nabla\cdot(p\mathbf{q})\Delta p\,\mathrm{d}\mathbf{x} + \varepsilon\bar{p}\int_{\mathbb{R}^2}\nabla(|\mathbf{q}|^2)\Delta\mathbf{q}\,\mathrm{d}\mathbf{x} \\ &\le \|p\|_{L^4}\|\nabla\cdot\mathbf{q}\|_{L^4}\|\Delta p\| + \|\nabla p\|_{L^4}\|\mathbf{q}\|_{L^4}\|\Delta p\| \\ &\quad + 2\varepsilon\bar{p}\|\mathbf{q}\|_{L^4}\|\nabla\mathbf{q}\|_{L^4}\|\Delta\mathbf{q}\|, \end{aligned} \tag{3.7}$$

where the first term on the righthand side can be estimated as

$$\begin{aligned} \|p\|_{L^4}\|\nabla\cdot\mathbf{q}\|_{L^4}\|\Delta p\| &\le c\|p\|^{1/2}\|\nabla p\|^{1/2}\|\nabla\cdot\mathbf{q}\|^{1/2}\|\Delta\mathbf{q}\|^{1/2}\|\Delta p\| \\ &\le \frac{1}{4}\|\Delta p\|^2 + c\|p\|\,\|\nabla p\|\,\|\nabla\cdot\mathbf{q}\|\,\|\Delta\mathbf{q}\| \\ &\le \frac{1}{4}\|\Delta p\|^2 + \frac{\varepsilon\bar{p}}{4}\|\Delta\mathbf{q}\|^2 + \frac{c}{\varepsilon\bar{p}}\|p\|^2\|\nabla p\|^2\|\nabla\cdot\mathbf{q}\|^2 \\ &\le \frac{1}{4}\|\Delta p\|^2 + \frac{\varepsilon\bar{p}}{4}\|\Delta\mathbf{q}\|^2 + \frac{cM_1}{\varepsilon\bar{p}}\|\nabla p\|^2\|\nabla\cdot\mathbf{q}\|^2. \end{aligned} \tag{3.8}$$

For the second term on the righthand side of (3.7), we have

$$\begin{aligned} \|\nabla p\|_{L^4}\|\mathbf{q}\|_{L^4}\|\Delta p\| &\le c\|\nabla p\|^{1/2}\|\Delta p\|^{3/2}\|\mathbf{q}\|^{1/2}\|\nabla\cdot\mathbf{q}\|^{1/2} \\ &\le \frac{1}{4}\|\Delta p\|^2 + c\|\nabla p\|^2\|\mathbf{q}\|^2\|\nabla\cdot\mathbf{q}\|^2 \\ &\le \frac{1}{4}\|\Delta p\|^2 + c\delta_1\|\nabla p\|^2\|\nabla\cdot\mathbf{q}\|^2. \end{aligned} \tag{3.9}$$

In a completely similar fashion, we can show that

$$
\begin{aligned}
2\varepsilon\bar{p}\|\mathbf{q}\|_{L^4}\|\nabla\mathbf{q}\|_{L^4}\|\Delta\mathbf{q}\| &\le 2\varepsilon\bar{p}c\|\mathbf{q}\|^{1/2}\|\nabla\cdot\mathbf{q}\|\|\Delta\mathbf{q}\|^{3/2} \\
&\le \frac{\varepsilon\bar{p}}{4}\|\Delta\mathbf{q}\|^2 + c\varepsilon\bar{p}\|\mathbf{q}\|^2\|\nabla\cdot\mathbf{q}\|^4 \\
&\le \frac{\varepsilon\bar{p}}{4}\|\Delta\mathbf{q}\|^2 + c\varepsilon\bar{p}\delta_1\|\nabla\cdot\mathbf{q}\|^2\|\nabla\cdot\mathbf{q}\|^2.
\end{aligned}
\tag{3.10}
$$

Feeding (3.8)–(3.10) into (3.7), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla p\|^2 + \bar{p}\|\nabla\cdot\mathbf{q}\|^2) + \|\Delta p\|^2 + \varepsilon\bar{p}\|\Delta\mathbf{q}\|^2 \\
&\le \left(\frac{cM_1}{\varepsilon\bar{p}} + c\delta_1\right)\|\nabla p\|^2\|\nabla\cdot\mathbf{q}\|^2 + c\varepsilon\bar{p}\delta_1\|\nabla\cdot\mathbf{q}\|^2\|\nabla\cdot\mathbf{q}\|^2.
\end{aligned}
$$

When δ_1 is smaller than some absolute constant, it holds that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla p\|^2 + \bar{p}\|\nabla\cdot\mathbf{q}\|^2) + \|\Delta p\|^2 + \varepsilon\bar{p}\|\Delta\mathbf{q}\|^2 \\
&\le \left(\frac{cM_1}{\varepsilon\bar{p}} + 1\right)\|\nabla p\|^2\|\nabla\cdot\mathbf{q}\|^2 + \varepsilon\bar{p}\|\nabla\cdot\mathbf{q}\|^2\|\nabla\cdot\mathbf{q}\|^2 \\
&\le \frac{1}{\bar{p}}\left(\frac{cM_1}{\varepsilon\bar{p}} + 1\right)(\|\nabla p\|^2 + \varepsilon\bar{p}\|\nabla\cdot\mathbf{q}\|^2)(\|\nabla p\|^2 + \bar{p}\|\nabla\cdot\mathbf{q}\|^2).
\end{aligned}
\tag{3.11}
$$

Applying the Gronwall inequality to (3.11) and using (3.6), we have

$$
\|\nabla p(t)\|^2 + \bar{p}\|\nabla\cdot\mathbf{q}(t)\|^2 \le M_2, \tag{3.12}
$$

where

$$
M_2 = (\|\nabla p_0\|^2 + \bar{p}\|\nabla\cdot\mathbf{q}_0\|^2)\exp\left\{\frac{1}{\bar{p}}\left(\frac{cM_1}{\varepsilon\bar{p}} + 1\right)(\|p_0\|^2 + \bar{p}\|\mathbf{q}_0\|^2)\right\}.
$$

Plugging (3.12) into (3.11), then integrating the result with respect to t, we have

$$
\begin{aligned}
&\int_0^t (\|\Delta p(\tau)\|^2 + \varepsilon\bar{p}\|\Delta\mathbf{q}(\tau)\|^2)\,\mathrm{d}\tau \\
&\le (\|\nabla p_0\|^2 + \bar{p}\|\nabla\cdot\mathbf{q}_0\|^2) \\
&\qquad + \frac{M_2}{\bar{p}}\left(\frac{cM_1}{\varepsilon\bar{p}} + 1\right)\int_0^t (\|\nabla p(\tau)\|^2 + \varepsilon\bar{p}\|\nabla\cdot\mathbf{q}(\tau)\|^2)\,\mathrm{d}\tau \\
&\le \left[\frac{M_2}{\bar{p}}\left(\frac{cM_1}{\varepsilon\bar{p}} + 1\right) + 1\right](\|\nabla p_0\|^2 + \bar{p}\|\nabla\cdot\mathbf{q}_0\|^2),
\end{aligned}
\tag{3.13}
$$

where we have used (3.6). It is clear that the energy bounds in (3.12) and (3.13) are not uniform in ε. Indeed, they will blow up as $\varepsilon \to 0$. This explains why the vanishing chemical diffusion coefficient limit cannot be realized in the 2D case.

3.1.4. ***H^2-estimate.*** Next, we estimate the second-order spatial derivatives of the solution. Taking the spatial gradient of the first equation and the spatial divergence of the second equation of (3.1), we get

$$\begin{cases} \partial_t \nabla p - \nabla(\nabla \cdot (p\mathbf{q})) - \bar{p}\,\nabla(\nabla \cdot \mathbf{q}) = \nabla \Delta p, \\ \partial_t \nabla \cdot \mathbf{q} - \Delta p = \varepsilon \Delta(\nabla \cdot \mathbf{q}) - \varepsilon \Delta(|\mathbf{q}|^2). \end{cases} \tag{3.14}$$

Computing the L^2 inner products of the first equation of (3.14) with $-\nabla\Delta p$ and the second one with $-\bar{p}\Delta(\nabla \cdot \mathbf{q})$, respectively, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Delta p\|^2 + \bar{p}\|\Delta \mathbf{q}\|^2) + \|\nabla\Delta p\|^2 + \varepsilon\bar{p}\|\Delta(\nabla \cdot \mathbf{q})\|^2 \\ &\qquad = -\int_{\mathbb{R}^2} \nabla(\nabla \cdot (p\mathbf{q})) \cdot \nabla(\Delta p)\,\mathrm{d}\mathbf{x} + \varepsilon\bar{p}\int_{\mathbb{R}^2} \Delta(|\mathbf{q}|^2)\Delta(\nabla \cdot \mathbf{q})\,\mathrm{d}\mathbf{x}. \end{aligned} \tag{3.15}$$

The first term on the righthand side of (3.15) can be estimated, by means of the Hölder, Gagliardo-Nirenberg, and Young inequalities, as

$$\begin{aligned} &\left| -\int_{\mathbb{R}^2} \nabla(\nabla \cdot (p\mathbf{q})) \cdot \nabla(\Delta p)\,\mathrm{d}\mathbf{x} \right| \\ &\quad \le (\|p\|_{L^4}\|\Delta\mathbf{q}\|_{L^4} + \|\nabla p\|_{L^4}\|\nabla \cdot \mathbf{q}\|_{L^4} + \|\Delta p\|_{L^4}\|\mathbf{q}\|_{L^4})\|\nabla\Delta p\| \\ &\quad \le c\Big(\|p\|^{1/2}\|\nabla p\|^{1/2}\|\Delta\mathbf{q}\|^{1/2}\|\Delta\nabla \cdot \mathbf{q}\|^{1/2} \\ &\qquad\quad + \|\nabla p\|^{1/2}\|\Delta p\|^{1/2}\|\nabla \cdot \mathbf{q}\|^{1/2}\|\Delta\mathbf{q}\|^{1/2} \\ &\qquad\quad + \|\Delta p\|^{1/2}\|\nabla\Delta p\|^{1/2}\|\mathbf{q}\|^{1/2}\|\nabla \cdot \mathbf{q}\|^{1/2}\Big)\|\nabla\Delta p\| \\ &\le \frac{1}{2}\|\nabla\Delta p\|^2 + c\Big(\|p\|\,\|\nabla p\|\,\|\Delta\mathbf{q}\|\,\|\Delta\nabla \cdot \mathbf{q}\| \\ &\qquad\quad + \|\nabla p\|\,\|\Delta p\|\,\|\nabla \cdot \mathbf{q}\|\,\|\Delta\mathbf{q}\| + \|\Delta p\|^2\|\mathbf{q}\|^2\|\nabla \cdot \mathbf{q}\|^2\Big) \\ &\quad \le \frac{1}{2}\|\nabla\Delta p\|^2 + \frac{\varepsilon\bar{p}}{4}\|\Delta(\nabla \cdot \mathbf{q})\|^2 + \frac{c}{\varepsilon\bar{p}}M_1\|\nabla p\|^2\|\Delta\mathbf{q}\|^2 \\ &\qquad\quad + c(\|\nabla p\|^2\|\Delta p\|^2 + \|\nabla \cdot \mathbf{q}\|^2\|\Delta\mathbf{q}\|^2) + c\delta_1\|\Delta p\|^2\|\nabla \cdot \mathbf{q}\|^2 \\ &\quad \le \frac{1}{2}\|\nabla\Delta p\|^2 + \frac{\varepsilon\bar{p}}{4}\|\Delta(\nabla \cdot \mathbf{q})\|^2 + cM_2\left(1 + \frac{\delta_1}{\bar{p}}\right)\|\Delta p\|^2 \\ &\qquad\quad + \frac{cM_2}{\varepsilon\bar{p}^2}\left(\frac{M_1}{\varepsilon} + 1\right)\varepsilon\bar{p}\|\Delta\mathbf{q}\|^2, \end{aligned}$$

where we used (3.2) and (3.12). For the second term on the righthand side of (3.15), we can show that

$$
\begin{aligned}
&\left|\varepsilon\bar{p}\int_{\mathbb{R}^2}\Delta(|\mathbf{q}|^2)\Delta(\nabla\cdot\mathbf{q})\,\mathrm{d}\mathbf{x}\right| \\
&\quad\le 2\varepsilon\bar{p}\left(\|\nabla\cdot\mathbf{q}\|_{L^4}^2+\|\mathbf{q}\|_{L^4}\|\Delta\mathbf{q}\|_{L^4}\right)\|\Delta(\nabla\cdot\mathbf{q})\| \\
&\quad\le c\varepsilon\bar{p}\left(\|\nabla\cdot\mathbf{q}\|\,\|\Delta\mathbf{q}\|+\|\mathbf{q}\|^{1/2}\|\nabla\cdot\mathbf{q}\|^{1/2}\|\Delta\mathbf{q}\|^{1/2}\|\Delta\nabla\cdot\mathbf{q}\|^{1/2}\right)\|\Delta\nabla\cdot\mathbf{q}\| \\
&\quad\le \frac{\varepsilon\bar{p}}{4}\|\Delta(\nabla\cdot\mathbf{q})\|^2+c\varepsilon\bar{p}\left(\|\nabla\cdot\mathbf{q}\|^2\|\Delta\mathbf{q}\|^2+\|\mathbf{q}\|^2\|\nabla\cdot\mathbf{q}\|^2\|\Delta\mathbf{q}\|^2\right) \\
&\quad\le \frac{\varepsilon\bar{p}}{4}\|\Delta(\nabla\cdot\mathbf{q})\|^2+\frac{cM_2}{\bar{p}}(1+\delta_1)\varepsilon\bar{p}\|\Delta\mathbf{q}\|^2.
\end{aligned}
$$

Plugging the above estimates into (3.15), we have

$$
\begin{aligned}
\text{(3.16)}\qquad &\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Delta p\|^2+\bar{p}\|\Delta\mathbf{q}\|^2\right)+\|\nabla\Delta p\|^2+\varepsilon\bar{p}\|\Delta(\nabla\cdot\mathbf{q})\|^2 \\
&\qquad\le M_3\left(\|\Delta p\|^2+\varepsilon\bar{p}\|\Delta\mathbf{q}\|^2\right),
\end{aligned}
$$

where

$$
M_3=2\max\left\{cM_2\left(1+\frac{\delta_1}{\bar{p}}\right),\frac{cM_2}{\varepsilon\bar{p}^2}\left(\frac{M_1}{\varepsilon}+1\right)+\frac{cM_2}{\bar{p}}(1+\delta_1)\right\}.
$$

Integrating (3.16) with respect to time and using (3.13), we get

$$
\begin{aligned}
\text{(3.17)}\qquad &\|\Delta p(t)\|^2+\bar{p}\,\|\Delta\mathbf{q}(t)\|^2+\int_0^t\left(\|\nabla\Delta p(\tau)\|^2+\varepsilon\bar{p}\|\Delta(\nabla\cdot\mathbf{q})\|^2\right)\mathrm{d}\tau \\
&\le \|\Delta p_0\|^2+\bar{p}\|\Delta\mathbf{q}_0\|^2 \\
&\qquad + M_3\left[\frac{M_2}{\bar{p}}\left(\frac{c\,M_1}{\varepsilon\,\bar{p}}+1\right)+1\right]\left(\|\nabla p_0\|^2+\bar{p}\,\|\nabla\cdot\mathbf{q}_0\|^2\right).
\end{aligned}
$$

This completes the proof of Lemma 3.2, and so the global well-posedness of (3.1) when $n=2$, $\varepsilon>0$. Next, we prove a similar result for the 3D case when $\varepsilon\ge 0$.

3.2. Global well-posedness in 3D. Similar to 2D, we first assume that the following hold true for some finite $T>0$:

$$
\text{(3.18a)}\qquad \sup_{0\le t\le T}\|\mathbf{q}(t)\|^2\le\delta_2,
$$

$$
\text{(3.18b)}\qquad \sup_{0\le t\le T}\|p(t)\|^2\le N_1,
$$

$$
\text{(3.18c)}\qquad \sup_{0\le t\le T}\left(\|\nabla p(t)\|^2+\|\nabla\cdot\mathbf{q}(t)\|^2\right)\le N_2,
$$

$$
\text{(3.18d)}\qquad \sup_{0\le t\le T}\left(\|\Delta p(t)\|^2+\|\Delta\mathbf{q}(t)\|^2\right)\le N_3,
$$

where $\delta_2, N_1, N_2, N_3>0$ are constants to be determined later.

We shall prove the following *a priori* estimates for the solution of (3.1) when $n=3$.

Lemma 3.3. *Let the solution* $(p, \mathbf{q})$ *of* (3.1) *with* $n = 3$ *satisfy* (3.18a)–(3.18d), *and assume that the initial entropy* (2.3) *is sufficiently small. Then, for any constants* $N_i > 0$ $(i = 1, 2, 3)$ *and any fixed value of* $\varepsilon \geq 0$, *if* δ_2 *is suitably small, there are positive constants* γ_i $(i = 3, 4)$ *which are independent of* t *and* ε, *such that*

$$||p(t)||_{H^2}^2 + \bar{p}||\mathbf{q}(t)||_{H^2}^2 + \int_0^t (||\nabla p(\tau)||_{H^2}^2 + \varepsilon\bar{p}||\nabla \cdot \mathbf{q}(\tau)||_{H^2}^2) \leq \gamma_3,$$

$$\int_0^t ||\nabla \cdot \mathbf{q}(\tau)||_{H^1}^2 \leq \gamma_4(1 + \varepsilon),$$

and γ_3 *and* γ_4 *depend only on* $\|p_0\|_{H^2}$, $\|\mathbf{q}_0\|_{H^2}$, *and* $\bar{p}$.

Next, we shall prove Lemma 3.3 in the following sections where the realization of the *a priori* assumptions (3.18a)–(3.18d) will be discussed when appropriate along the estimates.

3.2.1. Entropy estimate. Note that we still have the entropy estimate as in Section 3.1.1:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{\mathbb{R}^3} [\eta(p + \bar{p}) - \eta(\bar{p}) - \eta'(\bar{p})p]\,\mathrm{d}\mathbf{x} + \frac{1}{2}\|\mathbf{q}\|^2\Big) \\ + \int_{\mathbb{R}^3} \frac{|\nabla p|^2}{p + \bar{p}}\,\mathrm{d}\mathbf{x} + \varepsilon\,\|\nabla \cdot \mathbf{q}\|^2 \\ = \varepsilon\int_{\mathbb{R}^3} |\mathbf{q}|^2(\nabla \cdot \mathbf{q})\,\mathrm{d}\mathbf{x}, \end{aligned} \tag{3.19}$$

where the righthand side can be estimated by using the Gagliardo-Nirenberg interpolation inequality as

$$\begin{aligned} \left|\varepsilon\int_{\mathbb{R}^3} |\mathbf{q}|^2(\nabla \cdot \mathbf{q})\,\mathrm{d}\mathbf{x}\right| &\leq \varepsilon\|\mathbf{q}\|_{L^3}\,\|\mathbf{q}\|_{L^6}\,\|\nabla \cdot \mathbf{q}\| \\ &\leq c\varepsilon\|\mathbf{q}\|^{1/2}\,\|\nabla \cdot \mathbf{q}\|^{1/2}\,\|\nabla \cdot \mathbf{q}\|^2 \\ &\leq c\varepsilon(\delta_2 N_2)^{1/4}\,\|\nabla \cdot \mathbf{q}\|^2. \end{aligned}$$

Hence, when $\delta_2 N_2$ is smaller than some absolute constant, we update (3.19) as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{\mathbb{R}^3} [\eta(p + \bar{p}) - \eta(\bar{p}) - \eta'(\bar{p})p]\,\mathrm{d}\mathbf{x} + \frac{1}{2}\|\mathbf{q}\|^2\Big) \\ + \int_{\mathbb{R}^3} \frac{|\nabla p|^2}{p + \bar{p}}\,\mathrm{d}\mathbf{x} + \frac{\varepsilon}{2}\|\nabla \cdot \mathbf{q}\|^2 \leq 0, \end{aligned}$$

which implies that

$$\begin{aligned} \int_{\mathbb{R}^3} [\eta(p + \bar{p}) - \eta(\bar{p}) - \eta'(\bar{p})p]\,\mathrm{d}\mathbf{x} + \frac{1}{2}\|\mathbf{q}\|^2 \\ + \int_0^t \Big(\int_{\mathbb{R}^3} \frac{|\nabla p|^2}{p + \bar{p}}\,\mathrm{d}\mathbf{x} + \frac{\varepsilon}{2}\|\nabla \cdot \mathbf{q}\|^2\Big)\,\mathrm{d}\tau \\ \leq \int_{\mathbb{R}^3} [\eta(p_0 + \bar{p}) - \eta(\bar{p}) - \eta'(\bar{p})p_0]\,\mathrm{d}\mathbf{x} + \frac{1}{2}\|\mathbf{q}_0\|^2. \end{aligned}$$

In particular, we have

$$\|\mathbf{q}(t)\|^2 \le 2\int_{\mathbb{R}^3} [\eta(p_0+\bar{p}) - \eta(\bar{p}) - \eta'(\bar{p})p_0]\,\mathrm{d}\mathbf{x} + \|\mathbf{q}_0\|^2, \tag{3.20}$$

from which we can realize the smallness of δ_2 by choosing the righthand side of (3.20) to be sufficiently small. Next, we carry out regular energy estimates for the individual frequencies of the solution for up to the second order. We comment that the energy estimates in this section rely heavily on the Gagliardo-Nirenberg-Sobolev inequality $\|f\|_{L^6} \lesssim \|\nabla f\|$, which enables us to obtain the global well-posedness result for all values of $\varepsilon \ge 0$ and establish the consistency between the chemically diffusible and non-diffusible models in the process of a vanishing diffusion limit. This is one of the main features distinguishing the problems in the 2D and 3D cases.

3.2.2. ***L^2-estimate.*** By testing the equations in (3.1) with the targeting functions and using Gagliardo-Nirenberg interpolation inequalities in $\mathbb{R}^3$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|p\|^2 + \bar{p}\|\mathbf{q}\|^2) + \|\nabla p\|^2 + \varepsilon\bar{p}\|\nabla\cdot\mathbf{q}\|^2 \\
&\quad = -\int_{\mathbb{R}^3} p(\mathbf{q}\cdot\nabla p)\,\mathrm{d}\mathbf{x} + \varepsilon\bar{p}\int_{\mathbb{R}^3} |\mathbf{q}|^2\nabla\cdot\mathbf{q}\,\mathrm{d}\mathbf{x} \\
&\quad \le \|p\|_{L^6}\|\mathbf{q}\|_{L^3}\|\nabla p\| + \varepsilon\bar{p}\|\mathbf{q}\|_{L^3}\|\mathbf{q}\|_{L^6}\|\nabla\cdot\mathbf{q}\| \\
&\quad \le c(\|\nabla p\|\,\|\nabla\cdot\mathbf{q}\|^{1/2}\|\mathbf{q}\|^{1/2}\|\nabla p\| + \varepsilon\bar{p}\|\nabla\cdot\mathbf{q}\|^{1/2}\|\mathbf{q}\|^{1/2}\|\nabla\cdot\mathbf{q}\|^2) \\
&\quad \le c(\delta_2 N_2)^{1/4}(\|\nabla p\|^2 + \varepsilon\bar{p}\|\nabla\cdot\mathbf{q}\|^2).
\end{aligned}$$

Therefore, when $\delta_2 N_2$ is smaller than some absolute constant, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|p\|^2 + \bar{p}\|\mathbf{q}\|^2) + \|\nabla p\|^2 + \varepsilon\bar{p}\|\nabla\cdot\mathbf{q}\|^2 \le 0, \tag{3.21}$$

which yields

$$\begin{aligned}
\|p(t)\|^2 + \bar{p}\|\mathbf{q}(t)\|^2 + \int_0^t (\|\nabla p(\tau)\|^2 + \varepsilon\bar{p}\|\nabla\cdot\mathbf{q}(\tau)\|^2)\,\mathrm{d}\tau& \\
\le \|p_0\|^2 + \bar{p}\|\mathbf{q}_0\|^2.&
\end{aligned} \tag{3.22}$$

Hence, we can realize the second *a priori* assumption, (3.18b), by choosing

$$N_1 = \|p_0\|^2 + \bar{p}\|\mathbf{q}_0\|^2 + 1.$$

Next, we estimate the first-order spatial derivatives of the solution.

3.2.3. H^1***-estimate.*** Taking the L^2 inner products of the equations in (3.1) with the $-\Delta$ of the targeting functions and using Hölder, Gagliardo-Nirenberg and Young inequalities, we can show that

$$
\begin{aligned}
(3.23)\quad & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla p\|^2 + \bar{p}\|\nabla\cdot\mathbf{q}\|^2) + \|\Delta p\|^2 + \varepsilon\bar{p}\|\Delta\mathbf{q}\|^2 \\
&= -\int_{\mathbb{R}^3} \nabla\cdot(p\mathbf{q})\Delta p\,\mathrm{d}\mathbf{x} + \varepsilon\bar{p}\int_{\mathbb{R}^3}\nabla(|\mathbf{q}|^2)\cdot\Delta\mathbf{q}\,\mathrm{d}\mathbf{x} \\
&\le \|p\|_{L^6}\|\nabla\cdot\mathbf{q}\|_{L^3}\|\Delta p\| + \|\nabla p\|_{L^6}\|\mathbf{q}\|_{L^3}\|\Delta p\| + 2\varepsilon\bar{p}\|\mathbf{q}\|_{L^3}\|\nabla\mathbf{q}\|_{L^6}\|\Delta\mathbf{q}\| \\
&\le c\Big(\|\nabla p\|\,\|\mathbf{q}\|^{1/4}\|\Delta\mathbf{q}\|^{3/4}\|\Delta p\| \\
&\qquad + \|\mathbf{q}\|^{1/2}\|\nabla\cdot\mathbf{q}\|^{1/2}\|\Delta p\|^2 + \varepsilon\bar{p}\|\nabla\cdot\mathbf{q}\|^{1/2}\|\mathbf{q}\|^{1/2}\|\Delta\mathbf{q}\|^2\Big) \\
&\le \left(\frac{1}{4} + c(\delta_2 N_2)^{1/4}\right)\|\Delta p\|^2 + c(\delta_2(N_3)^3)^{1/4}\|\nabla p\|^2 + c\varepsilon\bar{p}(\delta_2 N_2)^{1/4}\|\Delta\mathbf{q}\|^2.
\end{aligned}
$$

Thus, when $\delta_2 N_2$ and $\delta_2(N_3)^3$ are smaller than some absolute constants, we have

$$
(3.24)\qquad \frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla p\|^2 + \|\nabla\cdot\mathbf{q}\|^2) + \|\Delta p\|^2 + \varepsilon\|\Delta\mathbf{q}\|^2 \le \|\nabla p\|^2.
$$

Integrating (3.24) with respect to time, we see that

$$
\begin{aligned}
(3.25)\qquad & \|\nabla p(t)\|^2 + \bar{p}\|\nabla\cdot\mathbf{q}(t)\|^2 + \int_0^t (\|\Delta p(\tau)\|^2 + \varepsilon\bar{p}\|\Delta\mathbf{q}(\tau)\|^2)\,\mathrm{d}\tau \\
&\le \|\nabla p_0\|^2 + \bar{p}\|\nabla\cdot\mathbf{q}_0\|^2 + \int_0^t \|\nabla p(\tau)\|^2\,\mathrm{d}\tau \\
&\le \|\nabla p_0\|^2 + \bar{p}\|\nabla\cdot\mathbf{q}_0\|^2 + (\|p_0\|^2 + \bar{p}\|\mathbf{q}_0\|^2),
\end{aligned}
$$

where we have used (3.22). Hence, we can realize the third *a priori* assumption, (3.18c), by choosing

$$
N_2 = \left(1 + \frac{1}{\bar{p}}\right)(||p_0||^2_{H^1} + \bar{p}||\mathbf{q}_0||^2_{H^1}) + 1.
$$

Next, we move on to the estimate of the second-order spatial derivatives of the solution.

3.2.4. H^2***-estimate.*** Computing the second order L^2 inner products, we can show that

$$
\begin{aligned}
(3.26)\quad & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Delta p\|^2 + \bar{p}\|\Delta\mathbf{q}\|^2) + \|\nabla\Delta p\|^2 + \varepsilon\bar{p}\|\Delta(\nabla\cdot\mathbf{q})\|^2 \\
&= -\int_{\mathbb{R}^3}\nabla(\nabla\cdot(p\mathbf{q}))\cdot\nabla(\Delta p)\,\mathrm{d}\mathbf{x} + \varepsilon\bar{p}\int_{\mathbb{R}^3}\Delta(|\mathbf{q}|^2)\Delta(\nabla\cdot\mathbf{q})\,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

For the first term on the right-hand side of (3.26), by using the Hölder, Gagliardo-Nirenberg, and Young inequalities, we deduce that

$$\begin{aligned}
&\left| - \int_{\mathbb{R}^3} \nabla(\nabla \cdot (p\mathbf{q})) \cdot \nabla(\Delta p)\,\mathrm{d}\mathbf{x} \right| \\
&\quad \le (\|p\|_{L^\infty} \|\Delta \mathbf{q}\| + \|\nabla p\|_{L^3} \|\nabla \mathbf{q}\|_{L^6} + \|\Delta p\|_{L^6} \|\mathbf{q}\|_{L^3}) \|\nabla \Delta p\| \\
&\quad \le c\Big(\|\nabla p\|^{1/2} \|\Delta p\|^{1/2} \|\Delta \mathbf{q}\| + \|\nabla p\|^{1/2} \|\Delta p\|^{1/2} \|\Delta \mathbf{q}\| \\
&\qquad + \|\nabla \Delta p\| \|\mathbf{q}\|^{1/2} \|\nabla \cdot \mathbf{q}\|^{1/2} \Big) \|\nabla \Delta p\| \\
&\quad \le \left(\frac{1}{4} + c(\delta_2 N_2)^{1/4} \right) \|\nabla \Delta p\|^2 + c\|\nabla p\| \|\Delta p\| \|\Delta \mathbf{q}\|^2 \\
&\quad \le \left(\frac{1}{4} + c(\delta_2 N_2)^{1/4} \right) \|\nabla \Delta p\|^2 + c(\|\nabla p\|^2 + \|\Delta p\|^2) \|\Delta \mathbf{q}\|^2,
\end{aligned}$$

where we interpolated $\|p\|_{L^\infty}$ as

$$\|p\|_{L^\infty} \lesssim ||\Delta p||_{L^2}^{1/2} ||p||_{L^6}^{1/2} \lesssim ||\Delta p||_{L^2}^{1/2} \|\nabla p\|_{L^2}^{1/2}.$$

In a similar fashion, we can show that

$$\begin{aligned}
&\left| \varepsilon \bar{p} \int_{\mathbb{R}^3} \Delta(|\mathbf{q}|^2) \Delta(\nabla \cdot \mathbf{q})\,\mathrm{d}\mathbf{x} \right| \\
&\le 2\varepsilon \bar{p} (\|\nabla \mathbf{q}\|_{L^3} \|\nabla \mathbf{q}\|_{L^6} + \|\mathbf{q}\|_{L^3} \|\nabla^2 \mathbf{q}\|_{L^6}) \|\Delta(\nabla \cdot \mathbf{q})\| \\
&\le c\varepsilon \bar{p} (\|\nabla \cdot \mathbf{q}\|^{1/2} \|\Delta \mathbf{q}\|^{3/2} + \|\mathbf{q}\|^{1/2} \|\nabla \cdot \mathbf{q}\|^{1/2} \|\Delta(\nabla \cdot \mathbf{q})\|) \|\Delta(\nabla \cdot \mathbf{q})\| \\
&\le \varepsilon \bar{p} \left(\frac{1}{4} + c(\delta_2 N_2)^{1/4} \right) \|\Delta(\nabla \cdot \mathbf{q})\|^2 + c\varepsilon \bar{p} \|\nabla \cdot \mathbf{q}\| \|\Delta \mathbf{q}\| \|\Delta \mathbf{q}\|^2 \\
&\le \varepsilon \bar{p} \left(\frac{1}{4} + c(\delta_2 N_2)^{1/4} \right) \|\Delta(\nabla \cdot \mathbf{q})\|^2 + c(\varepsilon \bar{p} \|\nabla \cdot \mathbf{q}\|^2 + \varepsilon \bar{p} \|\Delta \mathbf{q}\|^2) \|\Delta \mathbf{q}\|^2.
\end{aligned}$$

Hence, when $\delta_2 N_2$ is smaller than some absolute constant, it holds that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \left(\|\Delta p\|^2 + \bar{p}\, \|\Delta \mathbf{q}\|^2 \right) &+ \|\nabla \Delta p\|^2 + \varepsilon\, \bar{p}\, \|\Delta(\nabla \cdot \mathbf{q})\|^2 \\
&\le \frac{c}{\bar{p}} \left(\|\nabla p\|^2 + \|\Delta p\|^2 + \varepsilon\, \bar{p}\, \|\nabla \cdot \mathbf{q}\|^2 + \varepsilon\, \bar{p}\, \|\Delta \mathbf{q}\|^2 \right) \\
&\qquad \times \left(\|\Delta p\|^2 + \bar{p}\, \|\Delta \mathbf{q}\|^2 \right).
\end{aligned} \tag{3.27}$$

Applying Gronwall's inequality to (3.27) and using (3.22) and (3.25), we have

$$\begin{aligned}
\|\Delta p(t)\|^2 + \bar{p} \|\Delta \mathbf{q}(t)\|^2 \le \exp &\left\{ \frac{c}{\bar{p}} (||p_0||_{H^1}^2 + \bar{p} ||\mathbf{q}_0||_{H^1}^2) \right\} \\
&\times (\|\Delta p_0\|^2 + \bar{p} \|\Delta \mathbf{q}_0\|^2).
\end{aligned} \tag{3.28}$$

Therefore, we can realize the fourth *a priori* assumption, (3.18d), by choosing

$$N_3 = \left(1 + \frac{1}{\bar{p}}\right) \exp\left\{\frac{c}{\bar{p}}(\|p_0\|_{H^1}^2 + \bar{p}\|\mathbf{q}_0\|_{H^1}^2)\right\}(\|\Delta p_0\|^2 + \bar{p}\|\Delta \mathbf{q}_0\|^2) + 1.$$

In addition, by plugging (3.28) into (3.27), we can show that

$$\begin{aligned}\int_0^t (\|\nabla\Delta p(\tau)\|^2 + \varepsilon\bar{p}\|\Delta(\nabla\cdot\mathbf{q})\|^2)\,\mathrm{d}\tau \\ \le (\|\Delta p_0\|^2 + \bar{p}\|\Delta\mathbf{q}_0\|^2) + \frac{cN_3}{\bar{p}}(\|p_0\|_{H^1}^2 + \bar{p}\|\mathbf{q}_0\|_{H^1}^2),\end{aligned} \tag{3.29}$$

where the constant on the righthand side is independent of t and ε.

***3.2.5. Uniform temporal integrability for* $\mathbf{q}$.** We see from above estimates (3.22), (3.25), and (3.29) that the temporal integral of the spatial derivatives of $\mathbf{q}$ is inversely proportional to ε. In this section, we derive the ε-independent temporal integrability for the spatial derivatives of $\mathbf{q}$, which will be used later for proving the zero chemical diffusion limit result. For this purpose, we take the divergence of the second equation of (3.1), and combine the result with the first equation to get

$$\partial_t(\nabla\cdot\mathbf{q}) + \bar{p}\nabla\cdot\mathbf{q} = \varepsilon\Delta(\nabla\cdot\mathbf{q}) + \partial_t p - \varepsilon\Delta(|\mathbf{q}|^2) - \nabla\cdot(p\mathbf{q}). \tag{3.30}$$

Taking the L^2 inner product of (3.30) with $\nabla\cdot\mathbf{q}$, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\cdot\mathbf{q}\|^2 + \bar{p}\|\nabla\cdot\mathbf{q}\|^2 + \varepsilon\|\Delta\mathbf{q}\|^2 \\ = \int_{\mathbb{R}^3}(\partial_t p)(\nabla\cdot\mathbf{q})\,\mathrm{d}\mathbf{x} - \varepsilon\int_{\mathbb{R}^3}\Delta(|\mathbf{q}|^2)(\nabla\cdot\mathbf{q})\,\mathrm{d}\mathbf{x} \\ - \int_{\mathbb{R}^3}(\nabla\cdot(p\mathbf{q}))(\nabla\cdot\mathbf{q})\,\mathrm{d}\mathbf{x}.\end{aligned} \tag{3.31}$$

We note that

$$\begin{aligned}\int_{\mathbb{R}^3}(\partial_t p)(\nabla\cdot\mathbf{q})\,\mathrm{d}\mathbf{x} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}p(\nabla\cdot\mathbf{q})\,\mathrm{d}\mathbf{x} - \int_{\mathbb{R}^3}p(\partial_t\nabla\cdot\mathbf{q})\,\mathrm{d}\mathbf{x} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}p(\nabla\cdot\mathbf{q})\,\mathrm{d}\mathbf{x} - \int_{\mathbb{R}^3}p(\Delta p)\,\mathrm{d}\mathbf{x} - \int_{\mathbb{R}^3}p(\varepsilon\Delta(\nabla\cdot\mathbf{q}) - \varepsilon\Delta(|\mathbf{q}|^2))\,\mathrm{d}\mathbf{x} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}p(\nabla\cdot\mathbf{q})\,\mathrm{d}\mathbf{x} + \|\nabla p\|^2 - \int_{\mathbb{R}^3}p(\varepsilon\Delta(\nabla\cdot\mathbf{q}) - \varepsilon\Delta(|\mathbf{q}|^2))\,\mathrm{d}\mathbf{x},\end{aligned}$$

where we have used the second equation of (3.1). Then, we update (3.31) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla\cdot\mathbf{q}\|^2 - \int_{\mathbb{R}^3}p(\nabla\cdot\mathbf{q})\,\mathrm{d}\mathbf{x}\right) + \bar{p}\|\nabla\cdot\mathbf{q}\|^2 + \varepsilon\|\Delta\mathbf{q}\|^2 \tag{3.32}$$

$$
\begin{aligned}
&= \|\nabla p\|^2 - \varepsilon \int_{\mathbb{R}^3} \Delta(|\mathbf{q}|^2)(\nabla\cdot\mathbf{q})\,\mathrm{d}\mathbf{x} - \int_{\mathbb{R}^3} (\nabla\cdot(p\mathbf{q}))(\nabla\cdot\mathbf{q})\,\mathrm{d}\mathbf{x} \\
&\quad - \int_{\mathbb{R}^3} p(\varepsilon\Delta(\nabla\cdot\mathbf{q}) - \varepsilon\Delta(|\mathbf{q}|^2))\,\mathrm{d}\mathbf{x} \\
&= \|\nabla p\|^2 + \varepsilon \int_{\mathbb{R}^3} \nabla(|\mathbf{q}|^2)\cdot(\Delta\mathbf{q})\,\mathrm{d}\mathbf{x} - \int_{\mathbb{R}^3} (\nabla\cdot(p\mathbf{q}))(\nabla\cdot\mathbf{q})\,\mathrm{d}\mathbf{x} \\
&\quad + \int_{\mathbb{R}^3} \nabla p\cdot(\varepsilon\nabla(\nabla\cdot\mathbf{q}) - \varepsilon\nabla(|\mathbf{q}|^2))\,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

For the second term on the righthand side of (3.32), according to (3.23), we have the following estimate:

$$
\left|\varepsilon \int_{\mathbb{R}^3} \nabla(|\mathbf{q}|^2)\cdot(\Delta\mathbf{q})\,\mathrm{d}\mathbf{x}\right| \le c\varepsilon\|\mathbf{q}\|^{1/2}\,\|\nabla\cdot\mathbf{q}\|^{1/2}\,\|\Delta\mathbf{q}\|^2 \le c\varepsilon(\delta_2 N_2)^{1/4}\|\Delta\mathbf{q}\|^2.
$$

Using similar arguments as in (3.23), we estimate the third term on the righthand side of (3.32) as

$$
\begin{aligned}
&\left| - \int_{\mathbb{R}^3} (\nabla\cdot(p\mathbf{q}))(\nabla\cdot\mathbf{q})\,\mathrm{d}\mathbf{x}\right| \\
&\quad \le c\big(\|\nabla p\|^{1/2}\,\|\Delta p\|^{1/2}\,\|\nabla\cdot\mathbf{q}\|^2 + \|\mathbf{q}\|^{1/2}\,\|\nabla\cdot\mathbf{q}\|^{1/2}\,\|\Delta p\|\,\|\nabla\cdot\mathbf{q}\|\big) \\
&\quad \le \frac{c}{\bar{p}}(\|\nabla p\|\,\|\Delta p\|\,\|\nabla\cdot\mathbf{q}\|^2 + \|\Delta p\|^2\,\|\mathbf{q}\|\,\|\nabla\cdot\mathbf{q}\|) + \frac{\bar{p}}{2}\|\nabla\cdot\mathbf{q}\|^2 \\
&\quad \le \frac{c}{\bar{p}}(N_1\|\nabla p\|\,\|\Delta p\| + \sqrt{\delta_2 N_2}\|\Delta p\|^2) + \frac{\bar{p}}{2}\|\nabla\cdot\mathbf{q}\|^2 \\
&\quad \le \frac{c}{\bar{p}}(N_1\|\nabla p\|^2 + N_1\|\Delta p\|^2 + \sqrt{\delta_2 N_2}\|\Delta p\|^2) + \frac{\bar{p}}{2}\|\nabla\cdot\mathbf{q}\|^2.
\end{aligned}
$$

For the fourth term on the righthand side of (3.32), we can show that

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3} \nabla p\cdot(\varepsilon\nabla(\nabla\cdot\mathbf{q}) - \varepsilon\nabla(|\mathbf{q}|^2))\,\mathrm{d}\mathbf{x}\right| \\
&\quad \le \varepsilon\|\nabla p\|\,\|\nabla(\nabla\cdot\mathbf{q})\| + 2\varepsilon\|\nabla p\|\,\|\mathbf{q}\|_{L^3}\,\|\nabla\mathbf{q}\|_{L^6} \\
&\quad \le 2\varepsilon\|\nabla p\|^2 + \frac{\varepsilon}{4}\|\Delta\mathbf{q}\|^2 + \varepsilon\|\mathbf{q}\|_{L^3}^2\,\|\nabla\mathbf{q}\|_{L^6}^2 \\
&\quad \le 2\varepsilon\|\nabla p\|^2 + \frac{\varepsilon}{4}\|\Delta\mathbf{q}\|^2 + c\varepsilon\|\mathbf{q}\|\,\|\nabla\cdot\mathbf{q}\|\,\|\Delta\mathbf{q}\|^2 \\
&\quad \le 2\varepsilon\|\nabla p\|^2 + \frac{\varepsilon}{4}\|\Delta\mathbf{q}\|^2 + c\varepsilon(\delta_2 N_2)\|\Delta\mathbf{q}\|^2.
\end{aligned}
$$

Hence, when $\delta_2 N_2$ is smaller than some absolute constant, we update (3.32) as

$$
\begin{aligned}
\text{(3.33)}\qquad &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla\cdot\mathbf{q}\|^2 - \int_{\mathbb{R}^3} p(\nabla\cdot\mathbf{q})\,\mathrm{d}\mathbf{x}\right) + \frac{\bar{p}}{2}\|\nabla\cdot\mathbf{q}\|^2 + \frac{\varepsilon}{2}\|\Delta\mathbf{q}\|^2 \\
&\quad \le \|\nabla p\|^2 + \frac{c}{\bar{p}}(N_1\|\nabla p\|^2 + N_1\|\Delta p\|^2 + \|\Delta p\|^2) + 2\varepsilon\|\nabla p\|^2.
\end{aligned}
$$

Multiplying (3.21) by 2, then adding the result to (3.33), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}[E(t)] + \frac{\bar{p}}{2}\|\nabla\cdot\mathbf{q}\|^2 + \frac{\varepsilon}{2}\|\Delta\mathbf{q}\|^2 + \|\nabla p\|^2 + 2\varepsilon\bar{p}\|\nabla\cdot\mathbf{q}\|^2 \leq \frac{c}{\bar{p}}(N_1\|\nabla p\|^2 + N_1\|\Delta p\|^2 + \|\Delta p\|^2) + 2\varepsilon\|\nabla p\|^2, \tag{3.34}$$

where

$$\begin{aligned} E(t) &= \frac{1}{2}\|\nabla\cdot\mathbf{q}\|^2 - \int_{\mathbb{R}^3} p(\nabla\cdot\mathbf{q})\,\mathrm{d}\mathbf{x} + 2\|p\|^2 + 2\bar{p}\|\mathbf{q}\|^2 \\ &= \frac{1}{4}\|\nabla\cdot\mathbf{q}\|^2 + \int_{\mathbb{R}^3}\left(\frac{1}{2}\nabla\cdot\mathbf{q} - p\right)^2\mathrm{d}\mathbf{x} + \|p\|^2 + 2\bar{p}\|\mathbf{q}\|^2. \end{aligned}$$

Integrating (3.34) with respect to time and using (3.22) and (3.25), we get, in particular, that

$$\begin{aligned} &\frac{\bar{p}}{2}\int_0^t \|\nabla\cdot\mathbf{q}(\tau)\|^2\,\mathrm{d}\tau \\ &\quad\leq E(0) + \int_0^t \left(\frac{c}{\bar{p}}(N_1\|\nabla p\|^2 + N_1\|\Delta p\|^2 + \|\Delta p\|^2) + 2\varepsilon\|\nabla p\|^2\right)\mathrm{d}\tau \\ &\quad\leq E(0) + \left(\frac{cN_1}{\bar{p}} + 2\varepsilon\right)(\|p_0\|^2 + \bar{p}\|\mathbf{q}_0\|^2) \\ &\qquad + \frac{c}{\bar{p}}(N_1+1)(||p_0||^2_{H^1} + \bar{p}||\mathbf{q}_0||^2_{H^1}), \end{aligned}$$

where the bound on the righthand side is independent of t, and is finite for any fixed $\varepsilon \geq 0$. In a similar fashion, we can show that the temporal integral of $\|\nabla(\nabla\cdot\mathbf{q})\|^2$ is bounded by a constant which is independent of t, and is finite for any fixed $\varepsilon \geq 0$. The results obtained in this subsection allow us to take the zero chemical diffusion limit of the solution.

3.3. Long-time behavior. In this section, we derive the long-time behavior of the solution obtained from previous sections. First, we would like to recall a fact: if $f(t) \in W^{1,1}(0,\infty)$, then $f(t) \to 0$ as $t \to \infty$. In what follows, we use such a fact, together with the energy estimates obtained in the previous subsections, to establish the decay estimate stated in Theorem 2.2. For brevity, we only present the proof for the decay of the first-order spatial derivatives of the solution, in order to illustrate the main idea. The proof for the second-order derivatives is in a completely similar fashion and we omit the details. In addition, we only present the proof for the 2D case, and the 3D case follows exactly in the same fashion.

First, we note that for any fixed $\varepsilon > 0$, it follows from (3.6) that

$$\|\nabla p(t)\|^2 + \|\nabla\cdot\mathbf{q}(t)\|^2 \in L^1(0,\infty). \tag{3.35}$$

Second, by following the arguments in the previous section (cf. (3.11)), we can show that

$$
\begin{aligned}
(3.36)\quad & \left|\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla p\|^2 + \bar{p}\|\nabla\cdot\mathbf{q}\|^2)\right| \\
& \lesssim \|\Delta p\|^2 + \varepsilon\bar{p}\|\Delta\mathbf{q}\|^2 + (\|\nabla p\|^2 + \varepsilon\bar{p}\|\nabla\cdot\mathbf{q}\|^2)(\|\nabla p\|^2 + \|\nabla\cdot\mathbf{q}\|^2) \\
& \lesssim \|\Delta p\|^2 + \varepsilon\bar{p}\|\Delta\mathbf{q}\|^2 + \|\nabla p\|^2 + \varepsilon\bar{p}\|\nabla\cdot\mathbf{q}\|^2,
\end{aligned}
$$

where we applied (3.12) for the uniform-in-time estimates of $\|\nabla p\|^2$ and $\|\nabla\cdot\mathbf{q}\|^2$. Integrating (3.36) with respect to t and applying (3.6) and (3.13), we see that

$$
\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla p(t)\|^2 + \bar{p}\|\nabla\cdot\mathbf{q}(t)\|^2) \in L^1(0,\infty).
$$

Combining (3.35) and (3.36), we conclude that

$$
\|\nabla p(t)\|^2 + \bar{p}\|\nabla\cdot\mathbf{q}(t)\|^2 \in W^{1,1}(0,\infty),
$$

which implies

$$
\lim_{t\to\infty}(\|\nabla p(t)\|^2 + \bar{p}\|\nabla\cdot\mathbf{q}(t)\|^2) = 0.
$$

By the same argument, we can use (3.13), (3.16), and (3.17) to show that

$$
\lim_{t\to\infty}(\|\Delta p(t)\|^2 + \bar{p}\|\Delta\mathbf{q}(t)\|^2) = 0.
$$

By the Gagliardo-Nirenberg inequality $\|f\|_{L^\infty(\mathbb{R}^2)} \lesssim ||f||^{1/2}_{L^2(\mathbb{R}^2)}||\Delta f||^{1/2}_{L^2(\mathbb{R}^2)}$, and noting that p is a perturbation of the original variable around $\bar{p}$, we get (2.5) for the two-dimensional case ($n = 2$). Using the results in Section 3.2 for the three-dimensional case ($n = 3$), and the same argument as above for the two-dimensional case, we can obtain the same result for the 3D case for $\varepsilon \geq 0$.

3.4. Diffusion limit in 3D. In the last part of Section 3, we prove the chemical diffusion limit and identify the convergence rate for the solution obtained in Theorem 2.2 when $n = 3$. For this purpose, we let $(p^\varepsilon, \mathbf{q}^\varepsilon)$ and $(p^0, \mathbf{q}^0)$ be the solutions to (3.1) with $\varepsilon > 0$ and $\varepsilon = 0$, respectively, for the same initial data, and set $\tilde{p} = p^\varepsilon - p^0$ and $\tilde{\mathbf{q}} = \mathbf{q}^\varepsilon - \mathbf{q}^0$. Then, $(\tilde{p}, \tilde{\mathbf{q}})$ satisfies

$$
(3.37)\qquad \begin{cases}
\partial_t\tilde{p} - \nabla\cdot\tilde{\mathbf{q}} = \Delta\tilde{p} + \nabla\cdot(\tilde{p}\mathbf{q}^\varepsilon + p^0\tilde{\mathbf{q}}), \\
\partial_t\tilde{\mathbf{q}} - \nabla\tilde{p} = \varepsilon\Delta\mathbf{q}^\varepsilon - \varepsilon\nabla(|\mathbf{q}^\varepsilon|^2), \\
(\tilde{p}_0, \tilde{\mathbf{q}}_0) = (0, \mathbf{0}),
\end{cases}
$$

where for simplicity, we took $\bar{p} = 1$. We begin with the zeroth frequency estimate.

Step 1. Taking the L^2 inner products, we find

$$
\begin{aligned}
(3.38)\qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\tilde{p}\|^2 + \|\tilde{\mathbf{q}}\|^2) + \|\nabla\tilde{p}\|^2 = & -\int_{\mathbb{R}^3}(\tilde{p}\mathbf{q}^\varepsilon + p^0\tilde{\mathbf{q}})\cdot\nabla\tilde{p}\,\mathrm{d}\mathbf{x} \\
& + \int[\varepsilon\Delta\mathbf{q}^\varepsilon - \varepsilon\nabla(|\mathbf{q}^\varepsilon|^2)]\cdot\tilde{\mathbf{q}}\,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

For the first term on the righthand side of (3.38), by applying Young's inequality, we have

$$\begin{aligned}(3.39)\qquad &\left| -\int_{\mathbb{R}^3} (\tilde{p}\mathbf{q}^\varepsilon + p^0\tilde{\mathbf{q}}) \cdot \nabla\tilde{p}\,\mathrm{d}\mathbf{x} \right| \\ &\quad \le \frac{1}{2}\|\nabla\tilde{p}\|^2 + \|\mathbf{q}^\varepsilon\|_{L^\infty}^2 \|\tilde{p}\|^2 + \|p^0\|_{L^\infty}^2 \|\tilde{\mathbf{q}}\|^2 \\ &\quad \le \frac{1}{2}\|\nabla\tilde{p}\|^2 + c(\|\mathbf{q}^\varepsilon\|_{H^2}^2 \|\tilde{p}\|^2 + \|p^0\|_{H^2}^2 \|\tilde{\mathbf{q}}\|^2) \\ &\quad \le \frac{1}{2}\|\nabla\tilde{p}\|^2 + L_1(\|\tilde{p}\|^2 + \|\tilde{\mathbf{q}}\|^2),\end{aligned}$$

where we applied Sobolev embedding and the constant L_1 is independent of t and ε according to Lemma 3.3. The second term on the righthand side of (3.38) is estimated as

$$\begin{aligned}(3.40)\qquad &\left| \int [\varepsilon\Delta\mathbf{q}^\varepsilon - \varepsilon\nabla(|\mathbf{q}^\varepsilon|^2)] \cdot \tilde{\mathbf{q}}\,\mathrm{d}\mathbf{x} \right| \\ &\quad \le \frac{1}{2}\|\tilde{\mathbf{q}}\|^2 + \varepsilon^2\|\Delta\mathbf{q}^\varepsilon\|^2 + 4\varepsilon^2\|\tilde{\mathbf{q}}^\varepsilon\|_{L^\infty}^2 \|\nabla\mathbf{q}^\varepsilon\|^2 \\ &\quad \le \frac{1}{2}\|\tilde{\mathbf{q}}\|^2 + L_2\varepsilon^2,\end{aligned}$$

where again we applied Sobolev embedding and the constant L_2 is independent of t and ε according to Lemma 3.3. Plugging (3.39) and (3.40) into (3.38), we have

$$(3.41)\qquad \frac{\mathrm{d}}{\mathrm{d}t}(\|\tilde{p}\|^2 + \|\tilde{\mathbf{q}}\|^2) + \|\nabla\tilde{p}\|^2 \le 2L_1(\|\tilde{p}\|^2 + \|\tilde{\mathbf{q}}\|^2) + 2L_2\varepsilon^2.$$

Applying Gronwall's inequality to (3.41), we have

$$\|\tilde{p}(t)\|^2 + \|\tilde{\mathbf{q}}(t)\|^2 \le (2L_2te^{2L_1t})\varepsilon^2.$$

Next, we consider the convergence of the first-order spatial derivatives of the perturbation.

Step 2. Taking the L^2 inner products of the first two equations in (3.37) with the $-\Delta$ of the targeting functions, we deduce

$$\begin{aligned}(3.42)\qquad &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla\tilde{p}\|^2 + \|\nabla\cdot\tilde{\mathbf{q}}\|^2) + \|\Delta\tilde{p}\|^2 \\ &\quad = -\int_{\mathbb{R}^3} [\nabla\cdot(\tilde{p}\mathbf{q}^\varepsilon + p^0\tilde{\mathbf{q}})]\Delta\tilde{p}\,\mathrm{d}\mathbf{x} + \varepsilon\int_{\mathbb{R}^3} (\Delta\nabla\cdot\mathbf{q}^\varepsilon)(\nabla\cdot\tilde{\mathbf{q}})\,\mathrm{d}\mathbf{x} \\ &\qquad - \varepsilon\int_{\mathbb{R}^3} \Delta(|\mathbf{q}^\varepsilon|^2)(\nabla\cdot\tilde{\mathbf{q}})\,\mathrm{d}\mathbf{x}.\end{aligned}$$

For the first term on the righthand side of (3.42), applying Young's inequality gives

$$\begin{aligned}\left| -\int_{\mathbb{R}^3} [\nabla\cdot(\tilde{p}\mathbf{q}^\varepsilon + p^0\tilde{\mathbf{q}})]\Delta\tilde{p}\,\mathrm{d}\mathbf{x}\right| \\ \le \frac{1}{2}\|\Delta\tilde{p}\|^2 + \frac{1}{2}\|\nabla\cdot(\tilde{p}\mathbf{q}^\varepsilon + p^0\tilde{\mathbf{q}})\|^2,\end{aligned} \tag{3.43}$$

where the second term on the righthand side can be estimated as

$$\begin{aligned}&\frac{1}{2}\|\nabla\cdot(\tilde{p}\mathbf{q}^\varepsilon + p^0\tilde{\mathbf{q}})\|^2\\ &\le 2(\|\nabla\tilde{p}\cdot\mathbf{q}^\varepsilon\|^2 + \|\tilde{p}(\nabla\cdot\mathbf{q}^\varepsilon)\|^2 + \|\nabla p^0\cdot\tilde{\mathbf{q}}\|^2 + \|p^0(\nabla\cdot\tilde{\mathbf{q}})\|^2)\\ &\le 2\Big(\|\nabla\tilde{p}\|^2\,||\mathbf{q}^\varepsilon||_{L^\infty}^2 + ||\tilde{p}||_{L^6}^2\,||\nabla\cdot\mathbf{q}^\varepsilon||_{L^3}^2\\ &+ ||\nabla p^0||_{L^3}^2\,||\tilde{\mathbf{q}}||_{L^6}^2 + ||p^0||_{L^\infty}^2\,\|\nabla\cdot\tilde{\mathbf{q}}\|^2\Big)\\ &\le c(\|\nabla\tilde{p}\|^2\,||\mathbf{q}^\varepsilon||_{H^2}^2 + \|\nabla\cdot\tilde{\mathbf{q}}\|^2\,||p^0||_{H^2}^2)\\ &\le L_3(\|\nabla\tilde{p}\|^2 + \|\nabla\cdot\tilde{\mathbf{q}}\|^2),\end{aligned}$$

where we applied various Gagliardo-Nirenberg and Sobolev inequalities and the constant L_3 is independent of t and ε according to Lemma 3.3. Thus, we update (3.43) as

$$\begin{aligned}\left| -\int_{\mathbb{R}^3} [\nabla\cdot(\tilde{p}\mathbf{q}^\varepsilon + p^0\tilde{\mathbf{q}})]\Delta\tilde{p}\,\mathrm{d}\mathbf{x}\right| \\ \le \frac{1}{2}\|\Delta\tilde{p}\|^2 + L_3(\|\nabla\tilde{p}\|^2 + \|\nabla\cdot\tilde{\mathbf{q}}\|^2).\end{aligned} \tag{3.44}$$

For the second and third terms on the righthand side of (3.42), in a similar fashion, we can show that

$$\begin{aligned}&\left|\varepsilon\int_{\mathbb{R}^3}(\Delta\nabla\cdot\mathbf{q}^\varepsilon)(\nabla\cdot\tilde{\mathbf{q}})\,\mathrm{d}\mathbf{x} - \varepsilon\int_{\mathbb{R}^3}\Delta(|\mathbf{q}^\varepsilon|^2)(\nabla\cdot\tilde{\mathbf{q}})\,\mathrm{d}\mathbf{x}\right|\\ &\le \frac{1}{2}\|\nabla\cdot\tilde{\mathbf{q}}\|^2 + \varepsilon^2\|\Delta\nabla\cdot\mathbf{q}^\varepsilon\|^2 + \varepsilon^2\|\Delta(|\mathbf{q}^\varepsilon|^2)\|^2\\ &\le \frac{1}{2}\|\nabla\cdot\tilde{\mathbf{q}}\|^2 + \varepsilon^2\|\Delta\nabla\cdot\mathbf{q}^\varepsilon\|^2 + c\varepsilon^2(\|\Delta\mathbf{q}^\varepsilon\|^2\,||\mathbf{q}^\varepsilon||_{L^\infty}^2 + ||\nabla\mathbf{q}^\varepsilon||_{L^4}^4)\\ &\le \frac{1}{2}\|\nabla\cdot\tilde{\mathbf{q}}\|^2 + \varepsilon^2\|\Delta\nabla\cdot\mathbf{q}^\varepsilon\|^2 + c\varepsilon^2||\mathbf{q}^\varepsilon||_{H^2}^4\\ &\le \frac{1}{2}\|\nabla\cdot\tilde{\mathbf{q}}\|^2 + \varepsilon^2\|\Delta\nabla\cdot\mathbf{q}^\varepsilon\|^2 + L_4\varepsilon^2,\end{aligned} \tag{3.45}$$

where the constant L_4 is independent of t and ε according to Lemma 3.3. Plugging (3.44) and (3.45) into (3.42), we find

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla\tilde{p}\|^2 + \|\nabla\cdot\tilde{\mathbf{q}}\|^2) + \|\Delta\tilde{p}\|^2\\ &\le 2L_3(\|\nabla\tilde{p}\|^2 + \|\nabla\cdot\tilde{\mathbf{q}}\|^2) + 2\varepsilon^2\|\Delta\nabla\cdot\mathbf{q}^\varepsilon\|^2 + 2L_4\varepsilon^2.\end{aligned}$$

Applying Gronwall's inequality to (3.41), we deduce

$$\begin{aligned}\|\nabla\tilde{p}(t)\|^2+\|\nabla\cdot\tilde{\mathbf{q}}(t)\|^2 &\le e^{2L_3t}\left(2\varepsilon\int_0^t\varepsilon\|\Delta\nabla\cdot\mathbf{q}^\varepsilon(\tau)\|^2\,\mathrm{d}\tau+2L_4t\varepsilon^2\right)\\ &\le L_5e^{2L_3t}(1+t\varepsilon)\varepsilon,\end{aligned}$$

where the constant L_5 is independent of t and ε according to Lemma 3.3.

3.5. Proof of Theorem 2.2 and Theorem 2.4. Collecting the results obtained in Sections 3.1–3.3, we prove Theorem 2.2. Theorem 2.4 is a consequence of results in Section 3.4.

4. Small Energetic Solutions

In this section, we are devoted to proving Theorem 2.5. Similarly, we first assume for some finite time $T>0$ that

$$\sup_{0\le t\le T}(\|p(t)\|^2+\|\mathbf{q}(t)\|^2)\le\delta_3, \tag{4.1a}$$

$$\sup_{0\le t\le T}(\|\nabla p(t)\|^2+\|\nabla\cdot\mathbf{q}(t)\|^2)\le K_1, \tag{4.1b}$$

$$\sup_{0\le t\le T}(\|\Delta p(t)\|^2+\|\Delta\mathbf{q}(t)\|^2)\le K_2, \tag{4.1c}$$

where $\delta_3,K_1,K_2>0$ are constants to be determined later.

Then, we have the following *a priori* estimates for the solutions of (3.1).

Lemma 4.1. *Let the solution $(p,\mathbf{q})$ of (3.1) with $n=2$ satisfy (4.1a)–(4.1c), and the initial energy (2.4) be sufficiently small. Then, for any given constants $K_i>0$ $(i=1,2)$, if δ_3 is suitably small, there are positive constants γ_i $(i=5,6,7)$ which are independent of t and ε, such that*

$$||p(t)||_{H^1}^2+||\mathbf{q}(t)||_{H^1}^2+\int_0^t(||\nabla p(\tau)||_{H^1}^2+\varepsilon\bar{p}||\nabla\cdot\mathbf{q}(\tau)||_{H^1}^2)\le\gamma_5,$$

$$\|\Delta p(t)\|^2+\|\Delta\mathbf{q}(t)\|^2+\int_0^t(\|\nabla\Delta p(\tau)\|^2+\varepsilon\bar{p}\|\Delta\nabla\cdot\mathbf{q}(\tau)\|^2)\le\gamma_6(1+\varepsilon),$$

$$\int_0^t||\nabla\cdot\mathbf{q}(\tau)||_{H^1}^2\le\gamma_7(1+\varepsilon),$$

and γ_5, γ_6, and γ_7 depend only on $\|p_0\|_{H^2}$, $\|\mathbf{q}_0\|_{H^2}$ and $\bar{p}$.

In the following subsections, we prove Lemma 4.1 and realize *a priori* assumption (4.1) where appropriate along the proof.

4.1. ***L^2-estimate.*** Testing the equations in (3.1) by the targeting functions, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|p\|^2+\bar{p}\|\mathbf{q}\|^2)+\|\nabla p\|^2+\varepsilon\bar{p}\|\nabla\cdot\mathbf{q}\|^2 = -\int_{\mathbb{R}^2} p(\mathbf{q}\cdot\nabla p)\,\mathrm{d}\mathbf{x}+\varepsilon\bar{p}\int_{\mathbb{R}^2}|\mathbf{q}|^2\nabla\cdot\mathbf{q}\,\mathrm{d}\mathbf{x}. \tag{4.2}$$

We comment that, at the current stage of energy estimates, if one directly works on the righthand side of (4.2) as in deriving (3.5), then the inverse of ε will inevitably enter the energy bound, which is not desirable for the study of zero chemical diffusion limit. This is due to the Gagliardo-Nirenberg interpolation inequality in 2D: $\|f\|_{L^4}^2 \lesssim \|f\|\|\nabla f\|$ which does not generate enough powers of $\|\nabla p\|$ for the first term on the righthand side of (4.2) to be absorbed by the dissipation term on the lefthand side under the smallness assumption on $\|\mathbf{q}\|^2$. On the other hand, since the smallness of $\|p\|^2$ is assumed in (4.1), we can improve the energy estimate by taking advantage of such an assumption. The idea is to cancel the "bad" term and create higher-order nonlinearities through carrying out L^p $(p>2)$ level energy estimates. We begin the process by taking the L^2 inner product of the first equation in (3.1) with $-p^2$ to get

$$-\frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^2} p^3\,\mathrm{d}\mathbf{x}\right)-\int_{\mathbb{R}^2} p|\nabla p|^2\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^2} p(\mathbf{q}\cdot\nabla p)\,\mathrm{d}\mathbf{x}+\int_{\mathbb{R}^2} p^2(\mathbf{q}\cdot\nabla p)\,\mathrm{d}\mathbf{x}. \tag{4.3}$$

Taking the L^2 inner product of the first equation in (3.1) with p^3, we have

$$\frac{1}{12}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^2} p^4\,\mathrm{d}\mathbf{x}\right)+\int_{\mathbb{R}^2} p^2|\nabla p|^2\,\mathrm{d}\mathbf{x} = -\int_{\mathbb{R}^2} p^2(\mathbf{q}\cdot\nabla p)\,\mathrm{d}\mathbf{x}-\int_{\mathbb{R}^2} p^3(\mathbf{q}\cdot\nabla p)\,\mathrm{d}\mathbf{x}. \tag{4.4}$$

Summing up (4.2), (4.3), and (4.4), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\left(\int_{\mathbb{R}^2}\left(\frac{p^2}{2}-\frac{p^3}{6}+\frac{p^4}{12}\right)\mathrm{d}\mathbf{x}+\frac{\bar{p}}{2}\|\mathbf{q}\|^2\right)\\ &+\int_{\mathbb{R}^2}(|\nabla p|^2-p|\nabla p|^2+p^2|\nabla p|^2)\,\mathrm{d}\mathbf{x}+\varepsilon\bar{p}\|\nabla\cdot\mathbf{q}\|^2\\ &\quad=-\int_{\mathbb{R}^2} p^3(\mathbf{q}\cdot\nabla p)\,\mathrm{d}\mathbf{x}+\varepsilon\bar{p}\int_{\mathbb{R}^2}|\mathbf{q}|^2\nabla\cdot\mathbf{q}\,\mathrm{d}\mathbf{x},\end{aligned} \tag{4.5}$$

where

$$\int_{\mathbb{R}^2}\left(\frac{p^2}{2}-\frac{p^3}{6}+\frac{p^4}{12}\right)\mathrm{d}\mathbf{x}=\frac{1}{36}\|3p-p^2\|^2+\frac{1}{4}\|p\|^2+\frac{1}{18}||p||_{L^4}^4$$

and

$$\int_{\mathbb{R}^2} (|\nabla p|^2 - p|\nabla p|^2 + p^2|\nabla p|^2)\,\mathrm{d}\mathbf{x} = \frac{1}{2}\|\nabla p\|^2 + \frac{1}{2}\|\nabla p - p\nabla p\|^2 + \frac{1}{2}\|p\nabla p\|^2.$$

In addition, by applying the Gagliardo-Nirenberg interpolation inequalities in 2D, namely,

$$\|F\|_{L^8} \lesssim \|\nabla F\|^{3/4}\,\|F\|^{1/4}, \quad \|F\|_{L^4} \lesssim \|\nabla F\|^{1/2}\,\|F\|^{1/2}, \tag{4.6}$$

we can show that

$$\begin{aligned} \left| -\int_{\mathbb{R}^2} p^3(\mathbf{q}\cdot\nabla p)\,\mathrm{d}\mathbf{x} \right| &\le \|p\|_{L^8}^2\,\|\mathbf{q}\|_{L^4}\,\|p\nabla p\| \\ &\le c\|\nabla p\|^{3/2}\,\|p\|^{1/2}\,\|\nabla\cdot\mathbf{q}\|^{1/2}\,\|\mathbf{q}\|^{1/2}\,\|p\nabla p\| \\ &\le c\|\nabla p\|\,\|p\|\,\|\nabla p\|^2 + c\|\nabla\cdot\mathbf{q}\|\,\|\mathbf{q}\|\,\|p\nabla p\|^2 \\ &\le c(\delta_3 K_1)^{1/2}(\|\nabla p\|^2 + \|p\nabla p\|^2) \end{aligned}$$

and

$$\begin{aligned} \left| \varepsilon\bar{p}\int_{\mathbb{R}^2} |\mathbf{q}|^2\nabla\cdot\mathbf{q}\,\mathrm{d}\mathbf{x} \right| &\le \varepsilon\bar{p}\|\mathbf{q}\|_{L^4}^2\,\|\nabla\cdot\mathbf{q}\| \le c\varepsilon\bar{p}\|\mathbf{q}\|\,\|\nabla\cdot\mathbf{q}\|^2 \\ &\le c\varepsilon\bar{p}(\delta_3)^{1/2}\,\|\nabla\cdot\mathbf{q}\|^2. \end{aligned}$$

Hence, when $\delta_3 K_1$ is smaller than some absolute constant, we update (4.5) as

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{36}\|3p - p^2\|^2 + \frac{1}{4}\|p\|^2 + \frac{1}{18}\|p\|_{L^4}^4 + \frac{\bar{p}}{2}\|\mathbf{q}\|^2\right) \\ &\quad + \frac{1}{4}\|\nabla p\|^2 + \frac{1}{2}\|\nabla p - p\nabla p\|^2 + \frac{1}{4}\|p\nabla p\|^2 + \frac{\varepsilon\bar{p}}{2}\|\nabla\cdot\mathbf{q}\|^2 \le 0. \end{aligned} \tag{4.7}$$

By integrating (4.7) with respect to time, we obtain

$$\begin{aligned} &\left(\frac{1}{36}\|3p - p^2\|^2 + \frac{1}{4}\|p\|^2 + \frac{1}{18}\|p\|_{L^4}^4 + \frac{\bar{p}}{2}\|\mathbf{q}\|^2\right)(t) \\ &\quad + \int_0^t \left(\frac{1}{4}\|\nabla p\|^2 + \frac{1}{2}\|\nabla p - p\nabla p\|^2 \right. \\ &\qquad \left. + \frac{1}{4}\|p\nabla p\|^2 + \frac{\varepsilon\bar{p}}{2}\|\nabla\cdot\mathbf{q}\|^2\right)(\tau)\,\mathrm{d}\tau \le E_0, \end{aligned} \tag{4.8}$$

where

$$E_0 = \frac{1}{36}\|3p_0 - p_0^2\|^2 + \frac{1}{4}\|p_0\|^2 + \frac{1}{18}\|p_0\|_{L^4}^4 + \frac{\bar{p}}{2}\|\mathbf{q}_0\|^2.$$

Since $E_0 \cong \|p_0\|^2 + \|p_0\|_{L^4}^4 + \|\mathbf{q}_0\|^2$, the smallness of δ_3 can be realized by choosing $\|p_0\|^2 + \|p_0\|_{L^4}^4 + \|\mathbf{q}_0\|^2$ to be sufficiently small. Next, we deal with the estimate of the first order spatial derivatives of the solution.

4.2. ***H^1-estimate.*** Testing the equations in (3.1) by the $-\Delta$ of the targeting functions, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla p\|^2 + \bar{p}\|\nabla\cdot\mathbf{q}\|^2) + \|\Delta p\|^2 + \varepsilon\bar{p}\|\Delta\mathbf{q}\|^2 \\ = -\int_{\mathbb{R}^2}\nabla\cdot(p\mathbf{q})\Delta p\,\mathrm{d}\mathbf{x} + \varepsilon\bar{p}\int_{\mathbb{R}^2}\nabla(|\mathbf{q}|^2)\cdot(\Delta\mathbf{q})\,\mathrm{d}\mathbf{x} \\ = -\int_{\mathbb{R}^2}[p(\nabla\cdot\mathbf{q}) + \nabla p\cdot\mathbf{q}]\Delta p\,\mathrm{d}\mathbf{x} + \varepsilon\bar{p}\int_{\mathbb{R}^2}\nabla(|\mathbf{q}|^2)\cdot(\Delta\mathbf{q})\,\mathrm{d}\mathbf{x},\end{aligned}$$

which is equivalent to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla p\|^2 + \bar{p}\|\nabla\cdot\mathbf{q}\|^2) + 2\|\Delta p\|^2 + 2\varepsilon\bar{p}\|\Delta\mathbf{q}\|^2 \\ = -\int_{\mathbb{R}^2}[2p(\nabla\cdot\mathbf{q}) + 2\nabla p\cdot\mathbf{q}]\Delta p\,\mathrm{d}\mathbf{x} + 2\varepsilon\bar{p}\int_{\mathbb{R}^2}\nabla(|\mathbf{q}|^2)\cdot(\Delta\mathbf{q})\,\mathrm{d}\mathbf{x}.\end{aligned} \tag{4.9}$$

We comment that the first term on the righthand side of (4.9) is again a "trouble maker", because of the deficiency of the Gagliardo-Nirenberg interpolation inequalities in 2D. To terminate such a term, we multiply the first equation in (3.1) by $p\Delta p$ to get

$$\frac{\Delta p}{2}\partial_t(p^2) = p(\Delta p)[\Delta p + \nabla\cdot(p\mathbf{q}) + \bar{p}\,\nabla\cdot\mathbf{q}]. \tag{4.10}$$

Taking Δ to the first equation in (3.1), then multiplying the resulting equation by $p^2/2$, we get

$$\frac{p^2}{2}\partial_t(\Delta p) = \frac{p^2}{2}\Delta[\Delta p + \nabla\cdot(p\mathbf{q}) + \bar{p}\,\nabla\cdot\mathbf{q}]. \tag{4.11}$$

Summing up (4.10) and (4.11), then integrating the resulting equation over $\mathbb{R}^2$, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}p^2\Delta p\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^2}p\Delta p[\Delta p + \nabla\cdot(p\mathbf{q}) + \bar{p}\,\nabla\cdot\mathbf{q}]\,\mathrm{d}\mathbf{x} \\ + \int_{\mathbb{R}^2}\frac{p^2}{2}\Delta[\Delta p + \nabla\cdot(p\mathbf{q}) + \bar{p}\,\nabla\cdot\mathbf{q}]\,\mathrm{d}\mathbf{x}.\end{aligned} \tag{4.12}$$

After integrating by parts twice the second integral on the righthand side of (4.12), we get

$$\begin{aligned}\int_{\mathbb{R}^2}\frac{p^2}{2}\Delta[\Delta p + \nabla\cdot(p\mathbf{q}) + \bar{p}\,\nabla\cdot\mathbf{q}]\,\mathrm{d}\mathbf{x} \\ = \int_{\mathbb{R}^2}(p\Delta p + |\nabla p|^2)[\Delta p + \nabla\cdot(p\mathbf{q}) + \bar{p}\,\nabla\cdot\mathbf{q}]\,\mathrm{d}\mathbf{x}.\end{aligned}$$

Then, we update (4.12) as

$$
\begin{aligned}
(4.13)\qquad & -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} p|\nabla p|^2\,\mathrm{d}\mathbf{x} \\
&= \int_{\mathbb{R}^2}(2p\Delta p + |\nabla p|^2)[\Delta p + \nabla\cdot(p\mathbf{q}) + \bar{p}\,\nabla\cdot\mathbf{q}]\,\mathrm{d}\mathbf{x},
\end{aligned}
$$

where we have integrated the lefthand side by parts.

In exactly the same fashion, we can show that

$$
\begin{aligned}
(4.14)\qquad & \frac{1}{\bar{p}}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} p^2|\nabla p|^2\,\mathrm{d}\mathbf{x} \\
&= \frac{2}{\bar{p}}\int_{\mathbb{R}^2}(p^2\Delta p + p|\nabla p|^2)[\Delta p + \nabla\cdot(p\mathbf{q}) + \bar{p}\,\nabla\cdot\mathbf{q}]\,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

Multiplying (4.9) by $\bar{p}$, then adding the result with (4.13) and (4.14), we can show that

$$
\begin{aligned}
(4.15)\qquad & \frac{\mathrm{d}}{\mathrm{d}t}\Big(\bar{p}\,\|\nabla p\|^2 - \int_{\mathbb{R}^2} p|\nabla p|^2\,\mathrm{d}\mathbf{x} \\
&\quad + \frac{1}{\bar{p}}\int_{\mathbb{R}^2} p^2|\nabla p|^2\,\mathrm{d}\mathbf{x} + (\bar{p})^2\,\|\nabla\cdot\mathbf{q}\|^2\Big) + \\
&\quad + 2\bar{p}\|\Delta p\|^2 - 2\int_{\mathbb{R}^2} p(\Delta p)^2\,\mathrm{d}\mathbf{x} \\
&\quad + \frac{2}{\bar{p}}\,\|p\Delta p\|^2 + 2\,\varepsilon\,(\bar{p})^2\|\Delta\mathbf{q}\|^2 = H(t),
\end{aligned}
$$

where

$$
\begin{aligned}
H(t) &= -2\bar{p}\int_{\mathbb{R}^2}(\nabla p\cdot\mathbf{q})\Delta p\,\mathrm{d}\mathbf{x} + 2\int_{\mathbb{R}^2} p\Delta p\nabla\cdot(p\mathbf{q})\,\mathrm{d}\mathbf{x} \\
&\quad + \int_{\mathbb{R}^2}|\nabla p|^2\Delta p\,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^2}|\nabla p|^2\nabla\cdot(p\mathbf{q})\,\mathrm{d}\mathbf{x} \\
&\quad + \bar{p}\int_{\mathbb{R}^2}|\nabla p|^2\nabla\cdot\mathbf{q}\,\mathrm{d}\mathbf{x} - \frac{2}{\bar{p}}\int_{\mathbb{R}^2} p^2\Delta p\nabla\cdot(p\mathbf{q})\,\mathrm{d}\mathbf{x} \\
&\quad - 2\int_{\mathbb{R}^2} p^2\Delta p\nabla\cdot\mathbf{q}\,\mathrm{d}\mathbf{x} - \frac{2}{\bar{p}}\int_{\mathbb{R}^2} p|\nabla p|^2\Delta p\,\mathrm{d}\mathbf{x} \\
&\quad - \frac{2}{\bar{p}}\int_{\mathbb{R}^2} p|\nabla p|^2\nabla\cdot(p\mathbf{q})\,\mathrm{d}\mathbf{x} - 2\int_{\mathbb{R}^2} p|\nabla p|^2\nabla\cdot\mathbf{q}\,\mathrm{d}\mathbf{x} \\
&\quad + 2\varepsilon(\bar{p})^2\int_{\mathbb{R}^2}\nabla(|\mathbf{q}|^2)\cdot(\Delta\mathbf{q})\,\mathrm{d}\mathbf{x} \\
&\equiv \sum_{k=1}^{11} I_k(t).
\end{aligned}
$$

Next, we carry out energy estimates for $H(t)$. For $I_1(t)$, by using the second interpolation inequality in (4.6), we can show that

$$
\begin{aligned}
(4.16)\qquad |I_1(t)| &= 2\bar{p}\left|\int_{\mathbb{R}^2}(\nabla p\cdot\mathbf{q})\Delta p\,\mathrm{d}\mathbf{x}\right| \\
&\le 2\bar{p}\|\nabla p\|_{L^4}\|\mathbf{q}\|_{L^4}\|\Delta p\| \\
&\le c\bar{p}\|\nabla p\|^{1/2}\|\Delta p\|^{3/2}\|\mathbf{q}\|^{1/2}\|\nabla\cdot\mathbf{q}\|^{1/2} \\
&\le \frac{\bar{p}}{12}\|\Delta p\|^2 + c\bar{p}\|\nabla p\|^2\|\mathbf{q}\|^2\|\nabla\cdot\mathbf{q}\|^2 \\
&\le \frac{\bar{p}}{12}\|\Delta p\|^2 + c\bar{p}\delta_3K_1\|\nabla p\|^2.
\end{aligned}
$$

For $I_2(t)$, by using the second interpolation inequality in (4.6), and also

$$
(4.17)\qquad \|F\|_{L^\infty} \lesssim \|F\|^{1/2}\|\Delta F\|^{1/2},
$$

we can show that

(4.18)

$$
\begin{aligned}
|I_2(t)| &= 2\left|\int_{\mathbb{R}^2} p\Delta p\nabla\cdot(p\mathbf{q})\,\mathrm{d}\mathbf{x}\right| = 2\left|\int_{\mathbb{R}^2} p\Delta p(p\nabla\cdot\mathbf{q}+\mathbf{q}\cdot\nabla p)\,\mathrm{d}\mathbf{x}\right| \\
&\le 2(\|\Delta p\|\,||p||^2_{L^\infty}\|\nabla\cdot\mathbf{q}\| + \|\Delta p\|\,\|p\|_{L^\infty}\|\mathbf{q}\|_{L^4}\|\nabla p\|_{L^4}) \\
&\le c(\|\Delta p\|^2\|p\|\,\|\nabla\cdot\mathbf{q}\| + \|\Delta p\|^2\|p\|^{1/2}\|\mathbf{q}\|^{1/2}\|\nabla\cdot\mathbf{q}\|^{1/2}\|\nabla p\|^{1/2}) \\
&\le c(\delta_3K_1)^{1/2}\|\Delta p\|^2.
\end{aligned}
$$

For $I_3(t)$, by using (4.17) and the interpolation inequality

$$
\|\nabla F\|_{L^4} \lesssim ||F||^{1/2}_{L^\infty}\|\Delta F\|^{1/2},
$$

we can show that

$$
\begin{aligned}
|I_3(t)| &= \left|\int_{\mathbb{R}^2}|\nabla p|^2\Delta p\,\mathrm{d}\mathbf{x}\right| \le ||\nabla p||^2_{L^4}\|\Delta p\| \le c\|p\|_{L^\infty}\|\Delta p\|^2 \\
&\le c\|p\|^{1/2}\|\Delta p\|^{1/2}\|\Delta p\|^2 \le c(\delta_3K_2)^{1/4}\|\Delta p\|^2.
\end{aligned}
$$

For $I_4(t)$, using arguments similar to those in (4.18) and using (4.17), we can show that

$$
\begin{aligned}
|I_4(t)| &= \left|\int_{\mathbb{R}^2}|\nabla p|^2\nabla\cdot(p\mathbf{q})\,\mathrm{d}\mathbf{x}\right| = 2\left|\int_{\mathbb{R}^2}(\nabla p\cdot\mathbb{H}(p))\cdot(p\mathbf{q})\,\mathrm{d}\mathbf{x}\right| \\
&\le 2\|\Delta p\|\,\|p\|_{L^\infty}\|\nabla p\|_{L^4}\|\mathbf{q}\|_{L^4} \\
&\le c\|\Delta p\|^2\|p\|^{1/2}\|\nabla p\|^{1/2}\|\mathbf{q}\|^{1/2}\|\nabla\cdot\mathbf{q}\|^{1/2} \\
&\le c(\delta_3K_1)^{1/2}\|\Delta p\|^2,
\end{aligned}
$$

where $\mathbb{H}(p)$ denotes the Hessian matrix of p. For $I_5(t)$, by using the second interpolation inequality in (4.6), we can show that

$$\begin{aligned}|I_5(t)| &= \bar{p}\left|\int_{\mathbb{R}^2}|\nabla p|^2\nabla\cdot\mathbf{q}\,\mathrm{d}\mathbf{x}\right| \le \bar{p}\|\nabla p\|_{L^4}^2\|\nabla\cdot\mathbf{q}\| \\ &\le c\bar{p}\|\nabla p\|\,\|\Delta p\|\,\|\nabla\cdot\mathbf{q}\| \le \frac{\bar{p}}{12}\|\Delta p\|^2 + c\bar{p}K_1\|\nabla p\|^2.\end{aligned}$$

For $I_6(t)$, much as with the estimate of $I_2(t)$, we can show that

$$\begin{aligned}|I_6(t)| &= \frac{2}{\bar{p}}\left|\int_{\mathbb{R}^2}p^2\Delta p\nabla\cdot(p\mathbf{q})\,\mathrm{d}\mathbf{x}\right| \\ &= \frac{2}{\bar{p}}\left|\int_{\mathbb{R}^2}p^2\Delta p(p\nabla\cdot\mathbf{q}+\mathbf{q}\cdot\nabla p)\,\mathrm{d}\mathbf{x}\right| \\ &\le \frac{2}{\bar{p}}(\|p\Delta p\|\,\|p\|_{L^\infty}^2\|\nabla\cdot\mathbf{q}\| + \|p\Delta p\|\,\|p\|_{L^\infty}\,\|\mathbf{q}\|_{L^4}\,\|\nabla p\|_{L^4}) \\ &\le \frac{c}{\bar{p}}\Big(\|p\Delta p\|\,\|\Delta p\|\,\|p\|\,\|\nabla\cdot\mathbf{q}\| \\ &\qquad + \|p\Delta p\|\,\|\Delta p\|\,\|p\|^{1/2}\,\|\mathbf{q}\|^{1/2}\,\|\nabla\cdot\mathbf{q}\|^{1/2}\,\|\nabla p\|^{1/2}\Big) \\ &\le \frac{c}{\bar{p}}(\delta_3K_1)^{1/2}(\|p\Delta p\|^2+\|\Delta p\|^2).\end{aligned}$$

For $I_7(t)$, we can show that

$$\begin{aligned}|I_7(t)| &= 2\left|\int_{\mathbb{R}^2}p^2\Delta p\nabla\cdot\mathbf{q}\,\mathrm{d}\mathbf{x}\right| \le 2\|p\|_{L^\infty}^2\,\|\Delta p\|\,\|\nabla\cdot\mathbf{q}\| \\ &\le c\|p\|\,\|\nabla\cdot\mathbf{q}\|\,\|\Delta p\|^2 \le c(\delta_3K_1)^{1/2}\|\Delta p\|^2.\end{aligned}$$

For $I_8(t)$, much as with the estimate of $I_3(t)$, we can show that

$$\begin{aligned}|I_8(t)| &= \frac{2}{\bar{p}}\left|\int_{\mathbb{R}^2}p|\nabla p|^2\Delta p\,\mathrm{d}\mathbf{x}\right| \le \frac{2}{\bar{p}}\|\nabla p\|_{L^4}^2\,\|p\Delta p\| \\ &\le \frac{c}{\bar{p}}\|p\|_{L^\infty}\,\|\Delta p\|\,\|p\Delta p\| \le \frac{c}{\bar{p}}\,\|p\|^{1/2}\,\|\Delta p\|^{1/2}\,\|\Delta p\|\,\|p\Delta p\| \\ &\le \frac{c}{\bar{p}}(\delta_3K_2)^{1/4}(\|\Delta p\|^2+\|p\Delta p\|^2).\end{aligned}$$

For $I_9(t)$, much as in the estimate of $I_4(t)$, we can show that

$$\begin{aligned}|I_9(t)| &= \frac{2}{\bar{p}}\left|\int_{\mathbb{R}^2}p|\nabla p|^2\nabla\cdot(p\mathbf{q})\,\mathrm{d}\mathbf{x}\right| \\ &= \frac{2}{\bar{p}}\left|\int_{\mathbb{R}^2}|\nabla p|^2\nabla p\cdot(p\mathbf{q})\,\mathrm{d}\mathbf{x} + 2\int_{\mathbb{R}^2}p(\nabla p\cdot\mathbb{H}(p))\cdot(p\mathbf{q})\,\mathrm{d}\mathbf{x}\right|\end{aligned}$$

$$\begin{aligned}
&\le \frac{c}{\bar{p}}(\|\nabla p\|_{L^4}^3 \|p\|_{L^\infty}\|\mathbf{q}\|_{L^4} + \|p\|_{L^\infty}^2 \|\Delta p\| \,\|\nabla p\|_{L^4} \|\mathbf{q}\|_{L^4}) \\
&\le \frac{c}{\bar{p}}\Big(\|\Delta p\|^2 \|p\|^{1/2} \|\nabla p\|^{3/2} \|\mathbf{q}\|^{1/2} \|\nabla\cdot\mathbf{q}\|^{1/2} \\
&\qquad + \|\Delta p\|^2 \|p\| \,\|\nabla p\|^{1/2} \|\Delta p\|^{1/2} \|\mathbf{q}\|^{1/2} \|\nabla\cdot\mathbf{q}\|^{1/2}\Big) \\
&\le \frac{c}{\bar{p}}[(\delta_3)^{1/2}K_1 + (\delta_3)^{3/4}(K_1)^{1/2}(K_2)^{1/4}] \,\|\Delta p\|^2.
\end{aligned}$$

For $I_{10}(t)$, much as in the estimate of $I_5(t)$, we can show that

$$\begin{aligned}
|I_{10}(t)| = 2\left| \int_{\mathbb{R}^2} p|\nabla p|^2 \nabla\cdot\mathbf{q}\,\mathrm{d}\mathbf{x}\right| &\le 2\|p\|_{L^\infty} \|\nabla p\|_{L^4}^2 \|\nabla\cdot\mathbf{q}\| \\
&\le c\|p\|^{1/2} \|\nabla p\| \,\|\Delta p\|^{3/2} \|\nabla\cdot\mathbf{q}\| \\
&\le \frac{\bar{p}}{12}\|\Delta p\|^2 + \frac{c}{(\bar{p})^3}\|p\|^2 \|\nabla p\|^4 \|\nabla\cdot\mathbf{q}\|^4 \\
&\le \frac{\bar{p}}{12}\|\Delta p\|^2 + \frac{c}{(\bar{p})^3}\delta_3 (K_1)^3 \|\nabla p\|^2.
\end{aligned}$$

For $I_{11}(t)$, by using the second interpolation inequality in (4.6), we can show that

$$\begin{aligned}
|I_{11}(t)| &= 2\varepsilon(\bar{p})^2 \left| \int_{\mathbb{R}^2} \nabla(|\mathbf{q}|^2)\cdot(\Delta\mathbf{q})\,\mathrm{d}\mathbf{x}\right| \\
&\le 2\varepsilon(\bar{p})^2\|\mathbf{q}\|_{L^4} \|\nabla\mathbf{q}\|_{L^4} \|\Delta\mathbf{q}\| \\
&\le c\varepsilon(\bar{p})^2\|\mathbf{q}\|^{1/2} \|\nabla\cdot\mathbf{q}\| \,\|\Delta\mathbf{q}\|^{3/2} \\
&\le \varepsilon(\bar{p})^2\|\Delta\mathbf{q}\|^2 + c\varepsilon(\bar{p})^2\|\mathbf{q}\|^2 \|\nabla\cdot\mathbf{q}\|^4 \\
&\le \varepsilon(\bar{p})^2\|\Delta\mathbf{q}\|^2 + c\varepsilon(\bar{p})^2\delta_3 K_1\|\nabla\cdot\mathbf{q}\|^2.
\end{aligned} \tag{4.19}$$

Combining (4.16) and (4.18)–(4.19), we can show that when $\delta_3 K_1$, $\delta_3 K_2$, and $\delta_3(K_1)^2$ are smaller than some absolute constants, it holds that

$$\begin{aligned}
|H(t)| \le \frac{\bar{p}}{2}\|\Delta p\|^2 + \frac{1}{2\bar{p}}\|p\Delta p\|^2 + c\left[\bar{p}\,(1+K_1) + \frac{(K_1)^2}{(\bar{p})^3}\right]\|\nabla p\|^2 \\
+ \varepsilon\,(\bar{p})^2\,\|\Delta\mathbf{q}\|^2 + \varepsilon\,(\bar{p})^2\,\|\nabla\cdot\mathbf{q}\|^2.
\end{aligned} \tag{4.20}$$

Plugging (4.20) into (4.15), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\bar{p}\|\nabla p\|^2 - \int_{\mathbb{R}^2} p|\nabla p|^2\,\mathrm{d}\mathbf{x} + \frac{1}{\bar{p}}\int_{\mathbb{R}^2} p^2|\nabla p|^2\,\mathrm{d}\mathbf{x} \\
&\qquad + (\bar{p})^2\|\nabla\cdot\mathbf{q}\|^2\Big) + \frac{3\bar{p}}{2}\|\Delta p\|^2 \\
&\qquad - 2\int_{\mathbb{R}^2} p(\Delta p)^2\,\mathrm{d}\mathbf{x} + \frac{3}{2\bar{p}}\|p\Delta p\|^2 + \varepsilon(\bar{p})^2\|\Delta\mathbf{q}\|^2 \\
&\le c\left[\bar{p}(1+K_1) + \frac{(K_1)^2}{(\bar{p})^3}\right]\|\nabla p\|^2 + \varepsilon(\bar{p})^2\|\nabla\cdot\mathbf{q}\|^2.
\end{aligned} \tag{4.21}$$

We observe that in (4.21),

$$
\begin{aligned}
X_1(t) &\equiv \bar{p}\,\|\nabla p\|^2 - \int_{\mathbb{R}^2} p|\nabla p|^2\,\mathrm{d}\mathbf{x} + \frac{1}{\bar{p}}\int_{\mathbb{R}^2} p^2|\nabla p|^2\,\mathrm{d}\mathbf{x} \\
&\qquad + (\bar{p})^2\,\|\nabla\cdot\mathbf{q}\|^2 \\
&\ge \frac{\bar{p}}{2}\|\nabla p\|^2 + (\bar{p})^2\|\nabla\cdot\mathbf{q}\|^2 + \frac{1}{2\bar{p}}\int_{\mathbb{R}^2} p^2|\nabla p|^2\,\mathrm{d}\mathbf{x},
\end{aligned}
\tag{4.22a}
$$

$$
\begin{aligned}
Y_1(t) &\equiv \frac{3\bar{p}}{2}\|\Delta p\|^2 - 2\int_{\mathbb{R}^2} p(\Delta p)^2\,\mathrm{d}\mathbf{x} + \frac{3}{2\bar{p}}\|p\Delta p\|^2 \\
&\ge \frac{\bar{p}}{2}\|\Delta p\|^2 + \frac{1}{2\bar{p}}\|p\Delta p\|^2.
\end{aligned}
\tag{4.22b}
$$

After integrating (4.21) with respect to time, we find that

$$
\begin{aligned}
&X_1(t) + \int_0^t (Y_1(\tau) + \varepsilon(\bar{p})^2\|\Delta\mathbf{q}(\tau)\|^2)\,\mathrm{d}\tau \\
&\le X_1(0) + c\left[\bar{p}(1+K_1) + \frac{(K_1)^2}{(\bar{p})^3}\right]\int_0^t \|\nabla p(\tau)\|^2\,\mathrm{d}\tau \\
&\qquad + \bar{p}\int_0^t \varepsilon\bar{p}\|\nabla\cdot\mathbf{q}(\tau)\|^2\,\mathrm{d}\tau \\
&\le X_1(0) + 4E_0\left(c\left[\bar{p}(1+K_1) + \frac{(K_1)^2}{(\bar{p})^3}\right] + \bar{p}\right),
\end{aligned}
\tag{4.23}
$$

where we have used (4.8). In view of (4.22a)–(4.22b), we see that

$$
\frac{\bar{p}}{2}\|\nabla p\|^2 + (\bar{p})^2\|\nabla\cdot\mathbf{q}\|^2 \le X_1(0) + 4E_0\left(c\left[\bar{p}(1+K_1) + \frac{(K_1)^2}{(\bar{p})^3}\right] + \bar{p}\right),
$$

which implies

$$
\begin{aligned}
&\|\nabla p\|^2 + \|\nabla\cdot\mathbf{q}\|^2 \\
&\qquad \le \left(\frac{2}{\bar{p}} + \frac{1}{(\bar{p})^2}\right)\left\{X_1(0) + 4E_0\left(c\left[\bar{p}\,(1+K_1) + \frac{(K_1)^2}{(\bar{p})^3}\right] + \bar{p}\right)\right\}.
\end{aligned}
$$

Hence, we can fulfill the second line of (4.1) by choosing

$$
K_1 = \left(\frac{2}{\bar{p}} + \frac{1}{(\bar{p})^2}\right)(X_1(0)+1) + 1
$$

and E_0 to be sufficiently small, such that

$$
4E_0\left(c\left[\bar{p}(1+K_1) + \frac{(K_1)^2}{(\bar{p})^3}\right] + \bar{p}\right) \le 1.
\tag{4.24}
$$

In addition, we see from (4.22a)–(4.22b), (4.23), and (4.24) that

$$\int_0^t \left(\frac{\bar{p}}{2} \|\Delta p(\tau)\|^2 + \varepsilon(\bar{p})^2 \|\Delta \mathbf{q}(\tau)\|^2 \right) \mathrm{d}\tau \le X_1(0) + 1,$$

which implies

$$\int_0^t (\|\Delta p(\tau)\|^2 + \varepsilon\bar{p}\|\Delta \mathbf{q}(\tau)\|^2)\, \mathrm{d}\tau \le \frac{2}{\bar{p}}(X_1(0) + 1). \tag{4.25}$$

Thus, the H^1-estimate is completed.

4.3. *H^2**-estimate.*** We now estimate the second-order spatial derivatives of the solution. Applying Δ to the equations in (3.1), then taking the L^2 inner products of the resulting equations with Δ of the targeting functions, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Delta p\|^2 + \bar{p}\|\Delta \mathbf{q}\|^2) + \|\nabla\Delta p\|^2 + \varepsilon\bar{p}\|\Delta(\nabla\cdot\mathbf{q})\|^2 \\ &\quad = -\int_{\mathbb{R}^2} \nabla(\nabla\cdot(p\mathbf{q}))\cdot(\nabla\Delta p)\,\mathrm{d}\mathbf{x} + \varepsilon\bar{p}\int_{\mathbb{R}^3} \Delta(|\mathbf{q}|^2)\Delta(\nabla\cdot\mathbf{q})\,\mathrm{d}\mathbf{x} \\ &\quad \le (\|p\|_{L^\infty}\|\Delta\mathbf{q}\| + \|\nabla p\|_{L^4}\|\nabla\mathbf{q}\|_{L^4} + \|\Delta p\|_{L^4}\|\mathbf{q}\|_{L^4})\|\nabla\Delta p\| \\ &\qquad + 2\varepsilon\bar{p}(\|\nabla\mathbf{q}\|_{L^4}^2 + \|\mathbf{q}\|_{L^4}\|\nabla^2\mathbf{q}\|_{L^4})\,\|\Delta(\nabla\cdot\mathbf{q})\|. \end{aligned} \tag{4.26}$$

Note that, by the Gagliardo-Nirenberg and Young inequalities, it holds that

$$\begin{aligned} &\|\nabla p\|_{L^4}\|\nabla\mathbf{q}\|_{L^4}\|\nabla\Delta p\| \\ &\quad \le c\|\nabla p\|^{1/2}\|\Delta p\|^{1/2}\|\nabla\cdot\mathbf{q}\|^{1/2}\|\Delta\mathbf{q}\|^{1/2}\|\nabla\Delta p\| \\ &\quad \le \frac{c}{\bar{p}}\|\nabla p\|^2\|\Delta p\|^2\|\nabla\cdot\mathbf{q}\|^2 + \frac{(\bar{p})^{1/3}}{8}\|\Delta\mathbf{q}\|^{2/3}\|\nabla\Delta p\|^{4/3} \\ &\quad \le \frac{c(K_1)^2}{\bar{p}}\|\Delta p\|^2 + \frac{\bar{p}}{24}\|\Delta\mathbf{q}\|^2 + \frac{1}{12}\|\nabla\Delta p\|^2. \end{aligned}$$

Similarly, we can show that

$$\begin{aligned} &\|\Delta p\|_{L^4}\|\mathbf{q}\|_{L^4}\|\nabla\Delta p\| \le c\|\Delta p\|^{1/2}\|\mathbf{q}\|^{1/2}\|\nabla\cdot\mathbf{q}\|^{1/2}\|\nabla\Delta p\|^{3/2} \\ &\quad \le c\|\Delta p\|^2\|\mathbf{q}\|^2\|\nabla\cdot\mathbf{q}\|^2 + \frac{1}{12}\|\nabla\Delta p\|^2 \\ &\quad \le c\delta_3 K_1\|\Delta p\|^2 + \frac{1}{12}\|\nabla\Delta p\|^2; \\ &2\varepsilon\bar{p}\|\nabla\mathbf{q}\|_{L^4}^2\|\Delta(\nabla\cdot\mathbf{q})\| \le c\varepsilon\bar{p}\|\nabla\cdot\mathbf{q}\|\,\|\Delta\mathbf{q}\|\,\|\Delta(\nabla\cdot\mathbf{q})\| \\ &\quad \le c\varepsilon\bar{p}\|\nabla\cdot\mathbf{q}\|^2\|\Delta\mathbf{q}\|^2 + \frac{\varepsilon\bar{p}}{4}\|\Delta(\nabla\cdot\mathbf{q})\|^2 \\ &\quad \le c\varepsilon\bar{p}K_1\|\Delta\mathbf{q}\|^2 + \frac{\varepsilon\bar{p}}{4}\|\Delta(\nabla\cdot\mathbf{q})\|^2, \end{aligned}$$

and

$$
\begin{aligned}
&2\varepsilon\bar{p}\|\mathbf{q}\|_{L^4}\|\nabla^2\mathbf{q}\|_{L^4}\|\Delta(\nabla\cdot\mathbf{q})\| \\
&\quad\le \varepsilon\bar{p}\|\mathbf{q}\|^{1/2}\|\nabla\cdot\mathbf{q}\|^{1/2}\|\Delta\mathbf{q}\|^{1/2}\|\Delta(\nabla\cdot\mathbf{q})\|^{3/2} \\
&\quad\le c\varepsilon\bar{p}\|\mathbf{q}\|^2\|\nabla\cdot\mathbf{q}\|^2\|\Delta\mathbf{q}\|^2+\frac{\varepsilon\bar{p}}{4}\|\Delta(\nabla\cdot\mathbf{q})\|^2 \\
&\quad\le c\varepsilon\bar{p}\delta_3K_1\|\Delta\mathbf{q}\|^2+\frac{\varepsilon\bar{p}}{4}\|\Delta(\nabla\cdot\mathbf{q})\|^2.
\end{aligned}
$$

When δ_3K_1 is smaller than some absolute constant, we update (4.26) as

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Delta p\|^2+\bar{p}\|\Delta\mathbf{q}\|^2)+\frac{5}{6}\|\nabla\Delta p\|^2+\frac{\varepsilon\bar{p}}{2}\|\Delta(\nabla\cdot\mathbf{q})\|^2 \\
&\quad\le \|p\|_{L^\infty}\|\Delta\mathbf{q}\|\|\nabla\Delta p\|+\frac{\bar{p}}{24}\|\Delta\mathbf{q}\|^2 \\
&\qquad+c\left[\frac{(K_1)^2}{\bar{p}}+1\right]\|\Delta p\|^2+c(K_1+1)\varepsilon\bar{p}\|\Delta\mathbf{q}\|^2.
\end{aligned}\tag{4.27}
$$

In order to control the terms involving $\|\Delta\mathbf{q}\|^2$ on the righthand side of (4.27), we refer to (3.30):

$$
\partial_t(\nabla\cdot\mathbf{q})+\bar{p}\nabla\cdot\mathbf{q}=\varepsilon\Delta(\nabla\cdot\mathbf{q})+\partial_t p-\varepsilon\Delta(|\mathbf{q}|^2)-\nabla\cdot(p\mathbf{q}).\tag{4.28}
$$

By working with (4.28), we can show that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\Delta\mathbf{q}\|^2-\int_{\mathbb{R}^2}\nabla p\cdot\Delta\mathbf{q}\,\mathrm{d}\mathbf{x}\right)+\bar{p}\|\Delta\mathbf{q}\|^2+\varepsilon\|\Delta(\nabla\cdot\mathbf{q})\|^2 \\
&\quad=\|\Delta p\|^2-\int_{\mathbb{R}^2}\nabla(\nabla\cdot(p\mathbf{q}))\cdot\Delta\mathbf{q}\,\mathrm{d}\mathbf{x}+\varepsilon\int_{\mathbb{R}^2}\Delta(|\mathbf{q}|^2)\Delta(\nabla\cdot\mathbf{q})\,\mathrm{d}\mathbf{x} \\
&\qquad-\varepsilon\int_{\mathbb{R}^2}\Delta p\Delta(\nabla\cdot\mathbf{q})\,\mathrm{d}\mathbf{x}-\varepsilon\int_{\mathbb{R}^2}\Delta p\Delta(|\mathbf{q}|^2)\,\mathrm{d}\mathbf{x},
\end{aligned}\tag{4.29}
$$

where the righthand side can be estimated as follows. First of all, we have

$$
\begin{aligned}
&\left|-\int_{\mathbb{R}^2}\nabla(\nabla\cdot(p\mathbf{q}))\cdot\Delta\mathbf{q}\,\mathrm{d}\mathbf{x}\right| \\
&\quad\le(\|p\|_{L^\infty}\|\Delta\mathbf{q}\|+\|\nabla p\|_{L^4}\|\nabla\mathbf{q}\|_{L^4}+\|\Delta p\|_{L^4}\|\mathbf{q}\|_{L^4})\|\Delta\mathbf{q}\| \\
&\quad\le\|p\|_{L^\infty}\|\Delta\mathbf{q}\|^2+c\Big(\|\nabla p\|^{1/2}\|\Delta p\|^{1/2}\|\nabla\cdot\mathbf{q}\|^{1/2}\|\Delta\mathbf{q}\|^{3/2} \\
&\qquad+\|\Delta p\|^{1/2}\|\nabla\Delta p\|^{1/2}\|\mathbf{q}\|^{1/2}\|\nabla\cdot\mathbf{q}\|^{1/2}\|\Delta\mathbf{q}\|\Big) \\
&\quad\le\left(\|p\|_{L^\infty}+\frac{5\bar{p}}{24}\right)\|\Delta\mathbf{q}\|^2+\frac{c}{(\bar{p})^3}\|\nabla p\|^2\|\Delta p\|^2\|\nabla\cdot\mathbf{q}\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \frac{c}{\bar{p}}\|\mathbf{q}\|\,\|\nabla\cdot\mathbf{q}\|\,\|\nabla\Delta p\|\,\|\Delta p\| \\
&\leq \left(\|p\|_{L^\infty} + \frac{5\bar{p}}{24}\right)\|\Delta\mathbf{q}\|^2 + \frac{c}{(\bar{p})^3}\|\nabla p\|^2\,\|\Delta p\|^2\,\|\nabla\cdot\mathbf{q}\|^2 \\
&\quad + \frac{c}{(\bar{p})^2}\|\Delta p\|^2 + \|\mathbf{q}\|^2\,\|\nabla\cdot\mathbf{q}\|^2\,\|\nabla\Delta p\|^2 \\
&\leq \left(\|p\|_{L^\infty} + \frac{5\bar{p}}{24}\right)\|\Delta\mathbf{q}\|^2 + c\left(\frac{(K_1)^2}{(\bar{p})^3} + \frac{1}{(\bar{p})^2}\right)\|\Delta p\|^2 + \delta_3 K_1\|\nabla\Delta p\|^2.
\end{aligned}
$$

Secondly, much as in the last line of (4.27), we can show that

$$
\left|\varepsilon\int_{\mathbb{R}^2}\Delta(|\mathbf{q}|^2)\Delta(\nabla\cdot\mathbf{q})\,\mathrm{d}\mathbf{x}\right| \leq c(K_1 + \delta_3 K_1)\varepsilon\|\Delta\mathbf{q}\|^2 + \frac{\varepsilon}{4}\|\Delta(\nabla\cdot\mathbf{q})\|^2.
$$

Thirdly, by using the Young inequality, we can show that

$$
\left|-\varepsilon\int_{\mathbb{R}^2}\Delta p\Delta(\nabla\cdot\mathbf{q})\,\mathrm{d}\mathbf{x}\right| \leq \varepsilon\|\Delta p\|^2 + \frac{\varepsilon}{4}\|\Delta(\nabla\cdot\mathbf{q})\|^2.
$$

Lastly, we can show that

$$
\begin{aligned}
&\left|-\varepsilon\int_{\mathbb{R}^2}\Delta p\Delta(|\mathbf{q}|^2)\,\mathrm{d}\mathbf{x}\right| \\
&\leq 2\varepsilon(\|\Delta p\|_{L^4}\|\mathbf{q}\|_{L^4}\|\Delta\mathbf{q}\| + \|\Delta p\|\,\|\nabla\mathbf{q}\|_{L^4}^2) \\
&\leq c\varepsilon(\|\Delta p\|^{1/2}\|\nabla\Delta p\|^{1/2}\|\mathbf{q}\|^{1/2}\|\nabla\cdot\mathbf{q}\|^{1/2}\|\Delta\mathbf{q}\| + \|\Delta p\|\,\|\nabla\cdot\mathbf{q}\|\,\|\Delta\mathbf{q}\|) \\
&\leq \|\Delta p\|\,\|\nabla\Delta p\|\,\|\mathbf{q}\|\,\|\nabla\cdot\mathbf{q}\| + \|\Delta p\|^2\|\nabla\cdot\mathbf{q}\|^2 + c\varepsilon^2\|\Delta\mathbf{q}\|^2 \\
&\leq \frac{1}{12}\|\nabla\Delta p\|^2 + 3\|\mathbf{q}\|^2\|\nabla\cdot\mathbf{q}\|^2\|\Delta p\|^2 + \|\Delta p\|^2\|\nabla\cdot\mathbf{q}\|^2 + c\varepsilon^2\|\Delta\mathbf{q}\|^2 \\
&\leq \frac{1}{12}\|\nabla\Delta p\|^2 + (3\delta_3 + 1)K_1\|\Delta p\|^2 + c\varepsilon^2\|\Delta\mathbf{q}\|^2.
\end{aligned}
$$

Plugging the above estimates into (4.29), we have

$$
\begin{aligned}
\text{(4.30)}\qquad &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\Delta\mathbf{q}\|^2 - \int_{\mathbb{R}^2}\nabla p\cdot\Delta\mathbf{q}\,\mathrm{d}\mathbf{x}\right) + \frac{19\bar{p}}{24}\|\Delta\mathbf{q}\|^2 + \frac{\varepsilon}{2}\|\Delta(\nabla\cdot\mathbf{q})\|^2 \\
&\leq \|p\|_{L^\infty}\|\Delta\mathbf{q}\|^2 + \left(\frac{1}{12} + \delta_3 K_1\right)\|\nabla\Delta p\|^2 \\
&\quad + \left[c\left(\frac{(K_1)^2}{(\bar{p})^3} + \frac{1}{(\bar{p})^2}\right) + \varepsilon + (3\delta_3 + 1)K_1\right]\|\Delta p\|^2 \\
&\quad + c(K_1 + \delta_3 K_1 + \varepsilon)\varepsilon\|\Delta\mathbf{q}\|^2.
\end{aligned}
$$

Combining (4.27) and (4.30), we find that

$$
\begin{aligned}
(4.31)\qquad & \frac{\mathrm{d}}{\mathrm{d}t}X_2(t) + Y_2(t) \\
&\le \|p\|_{L^\infty}(\|\Delta\mathbf{q}\|\,\|\nabla\Delta p\| + \|\Delta\mathbf{q}\|^2) + \left(\frac{1}{12} + \delta_3 K_1\right)\|\nabla\Delta p\|^2 \\
&\quad + \left[c\left(\frac{(K_1)^2}{(\bar p)^3} + \frac{1}{(\bar p)^2}\right) + \varepsilon + (3\delta_3 + 1)K_1 + c\left(\frac{(K_1)^2}{\bar p} + 1\right)\right]\|\Delta p\|^2 \\
&\quad + \left(\frac{c}{\bar p}(K_1 + \delta_3 K_1 + \varepsilon) + c(K_1 + 1)\right)\varepsilon\bar p\|\Delta\mathbf{q}\|^2,
\end{aligned}
$$

where

$$
(4.32\text{a})\qquad X_2(t) = \frac{1}{2}\|\Delta p\|^2 + \frac{\bar p}{2}\|\Delta\mathbf{q}\|^2 + \frac{1}{2}\|\Delta\mathbf{q}\|^2 - \int_{\mathbb{R}^2}\nabla p\cdot\Delta\mathbf{q}\,\mathrm{d}\mathbf{x},
$$

$$
(4.32\text{b})\qquad Y_2(t) = \frac{5}{6}\|\nabla\Delta p\|^2 + \frac{3\bar p}{4}\|\Delta\mathbf{q}\|^2 + \frac{\varepsilon}{2}(\bar p + 1)\|\Delta(\nabla\cdot\mathbf{q})\|^2.
$$

Since $\|p\|_{L^\infty} \le c(\delta_3 K_2)^{1/4}$ because of (4.17), when $\delta_3 K_2$ and $\delta_3 K_1$ are smaller than some absolute constants, we update (4.31) and (4.32a)–(4.32b) as

$$
\begin{aligned}
(4.33)\qquad & \frac{\mathrm{d}}{\mathrm{d}t}X_2(t) + Y_3(t) \\
&\le \left[c\left(\frac{(K_1)^2}{(\bar p)^3} + \frac{1}{(\bar p)^2}\right) + \varepsilon + 1 + K_1 + c\left(\frac{(K_1)^2}{\bar p} + 1\right)\right]\|\Delta p\|^2 \\
&\quad + \left(\frac{c}{\bar p}(K_1 + 1 + \varepsilon) + c(K_1 + 1)\right)\varepsilon\bar p\|\Delta\mathbf{q}\|^2,
\end{aligned}
$$

where

$$
Y_3(t) = \frac{1}{2}\|\nabla\Delta p\|^2 + \frac{\bar p}{2}\|\Delta\mathbf{q}\|^2 + \frac{\varepsilon}{2}(\bar p + 1)\|\Delta(\nabla\cdot\mathbf{q})\|^2.
$$

We note that by definition, $X_2(t)$ may not be positive (cf. (4.32a)–(4.32b)). However, by combining (4.33) with (4.21) $\times\, 2/\bar p$, we obtain

$$
\begin{aligned}
(4.34)\qquad & \frac{\mathrm{d}}{\mathrm{d}t}X_3(t) + Y_4(t) \\
&\le \left[c\left(\frac{(K_1)^2}{(\bar p)^3} + \frac{1}{(\bar p)^2}\right) + \varepsilon + 1 + K_1 + c\left(\frac{(K_1)^2}{\bar p} + 1\right)\right]\|\Delta p\|^2 \\
&\quad + \left(\frac{c}{\bar p}(K_1 + 1 + \varepsilon) + c(K_1 + 1)\right)\varepsilon\bar p\|\Delta\mathbf{q}\|^2 \\
&\quad + c\left[1 + K_1 + \frac{(K_1)^2}{(\bar p)^4}\right]\|\nabla p\|^2 + \varepsilon\bar p\|\nabla\cdot\mathbf{q}\|^2,
\end{aligned}
$$

where

$$X_3(t) = X_2(t) + 2\|\nabla p\|^2 - \frac{2}{\bar{p}}\int_{\mathbb{R}^2} p|\nabla p|^2\,\mathrm{d}\mathbf{x} + \frac{2}{(\bar{p})^2}\int_{\mathbb{R}^2} p^2|\nabla p|^2\,\mathrm{d}\mathbf{x} + 2\bar{p}\|\nabla\cdot\mathbf{q}\|^2$$
$$\geq \frac{1}{2}\|\Delta p\|^2 + \frac{\bar{p}}{2}\|\Delta\mathbf{q}\|^2 + \frac{1}{2}\|\nabla p\|^2 + 2\bar{p}\|\nabla\cdot\mathbf{q}\|^2,$$
$$Y_4(t) = Y_3(t) + 3\|\Delta p\|^2 - \frac{4}{\bar{p}}\int_{\mathbb{R}^2} p(\Delta p)^2\,\mathrm{d}\mathbf{x} + \frac{3}{(\bar{p})^2}\|p\Delta p\|^2$$
$$\geq \frac{1}{2}\|\nabla\Delta p\|^2 + \frac{\bar{p}}{2}\|\Delta\mathbf{q}\|^2 + \frac{\varepsilon}{2}(\bar{p}+1)\|\Delta(\nabla\cdot\mathbf{q})\|^2 + \|\Delta p\|^2.$$

Integrating (4.34) with respect to time and using (4.8) and (4.25), we obtain

$$\frac{1}{2}\|\Delta p(t)\|^2 + \frac{\bar{p}}{2}\|\Delta\mathbf{q}(t)\|^2 + \int_0^t \left(\frac{1}{2}\|\nabla\Delta p(\tau)\|^2 + \frac{\varepsilon}{2}(\bar{p}+1)\|\Delta(\nabla\cdot\mathbf{q})(\tau)\|^2\right)\mathrm{d}\tau$$
$$\leq X_3(0) + \left[c\left(\frac{(K_1)^2}{(\bar{p})^3} + \frac{1}{(\bar{p})^2}\right) + \varepsilon + 1 + K_1 + c\left(\frac{(K_1)^2}{\bar{p}} + 1\right)\right]\frac{2}{\bar{p}}(X_1(0)+1)$$
$$+ \left(\frac{c}{\bar{p}}(K_1 + 1 + \varepsilon) + c(K_1+1)\right)\frac{2}{\bar{p}}(X_1(0)+1) + 2cE_0\left[1 + K_1 + \frac{(K_1)^2}{(\bar{p})^4}\right]$$
$$\equiv \bar{K}_2,$$

which yields

$$\|\Delta p(t)\|^2 + \|\Delta\mathbf{q}(t)\|^2 \leq 2\left(1 + \frac{1}{\bar{p}}\right)\bar{K}_2 \equiv K_2.$$

Thus, the H^2-estimate is completed.

4.4. Proof of Theorem 2.5. First, we obtain Lemma 4.1 by combining the results in Sections 4.1–4.3. Then, the global existence of solutions to (3.1) with $n = 2$ asserted in Theorem 2.5 results from Lemma 3.1 and Lemma 4.1. In addition, by working with (4.28) and arguing in a similar way as in Section 3.2.5, we can show that $\|\nabla\cdot\mathbf{q}\|_{H^1}^2$ is uniformly integrable with respect to time. Then, by repeating the arguments in Section 3.3 and Section 3.4, we can establish the long-time behavior and diffusion limit results for solutions with small initial energy in 2D. We omit the details for brevity. This completes the proof for Theorem 2.5.

5. CONCLUSION

We have studied the qualitative behavior of solutions to the Cauchy problem of a system of parabolic conservation laws (1.4) in multiple space dimensions. Using energy methods, we first showed that for any fixed value of $\varepsilon > 0$, the Cauchy problem is globally (with respect to time) well posed provided that either the initial entropy or initial energy around a constant state is sufficiently small, and the smallness of the specific frequency depends on the other components in the energy spectrum. Moreover, the solution converges to the constant state as time goes to infinity. Second, we showed that similar results hold in 3D for small initial entropy and in 2D for small initial energy when $\varepsilon = 0$. Based on this, we established the convergence of solutions with $\varepsilon > 0$ toward those with $\varepsilon = 0$ and identified the convergence rate for each case. Finally we note that the questions of global well-posedness and long-time behavior of solutions in 2D for small initial entropy when $\varepsilon = 0$ are still open at the present time. We leave the investigation for the future.

APPENDIX A. EXPLICIT EXAMPLES

In this appendix, we provide some explicit examples of initial data that fulfill the requirements of the main results in this paper.

1.1. Small entropy in 2D. First, we recall the *a priori* assumptions made in Section 3.1:

$$\sup_{0\le t\le T} \|\mathbf{q}(t)\|^2 \le \delta_1,$$
$$\sup_{0\le t\le T} \|p(t)-\bar{p}\|^2 \le M_1.$$

As the proof proceeded, we obtained the following estimates and choices of constants (see Sections 3.1.1–3.1.2):

$$\sup_{0\le t\le T} \|\mathbf{q}(t)\|^2 \le 2\int_{\mathbb{R}^2}[(p_0\ln(p_0)-p_0)-(\bar{p}\ln(\bar{p})-\bar{p})-\ln(\bar{p})(p_0-\bar{p})]\,\mathrm{d}\mathbf{x}+\|\mathbf{q}_0\|^2,$$
$$M_1 = \|p_0\|^2+\bar{p}\|\mathbf{q}_0\|^2+1.$$

We also required that δ_1 and $\delta_1 M_1$ be smaller than some absolute constants.

Now, let us consider the following initial data:

$$p_0(\mathbf{x}) = \begin{cases} m\left[\sin\left(r-\frac{\pi}{2}\right)+1\right]+f(m), & 2\pi\le r\le 4\pi, \\ \text{f(m)}, & r\in(-\infty,2\pi)\cup(4\pi,\infty); \end{cases} \tag{A.1a}$$

$$
\mathbf{q}_0(\mathbf{x}) = \begin{cases} \dfrac{[\sin(f(m)r - \pi/2) + 1]}{\sqrt{f(m)}} \cdot \dfrac{\mathbf{x}}{r}, \\ \qquad\qquad\qquad\qquad 2\pi \le r \le 4\pi, \\ \mathbf{0}, \qquad\qquad\qquad r \in (-\infty, 2\pi) \cup (4\pi, \infty), \end{cases} \tag{A.1b}
$$

where $m \in \mathbb{N}$, $r = |\mathbf{x}|$, and $m < f(m) \equiv \bar{p} \in \mathbb{N}$ is to be determined later. It is straightforward to check that $(p_0, \mathbf{q}_0) \in H^2(\mathbb{R}^2)$ and $\nabla \times \mathbf{q}_0 = \mathbf{0}$. By direct calculations, we can show that

$$
\|p_0 - \bar{p}\|^2 \cong m^2, \quad \|\mathbf{q}_0\|^2 \cong \frac{1}{f(m)}.
$$

In addition, we can show that

$$
\begin{aligned}
&\int_{\mathbb{R}^2} [(p_0 \ln(p_0) - p_0) - (\bar{p}\ln(\bar{p}) - \bar{p}) - \ln(\bar{p})(p_0 - \bar{p})]\,\mathrm{d}\mathbf{x} \\
&\quad = \int_{\mathbb{R}^2} \frac{1}{2p^*}(p_0 - \bar{p})^2\,\mathrm{d}\mathbf{x} \le \frac{1}{2f(m)}\|p_0 - \bar{p}\|^2,
\end{aligned}
$$

where p^* is between p_0 and $\bar{p}$. Hence, by taking

$$
\delta_1 = 2\int_{\Omega} [(p_0 \ln(p_0) - p_0) - (\bar{p}\ln(\bar{p}) - \bar{p}) - \ln(\bar{p})(p_0 - \bar{p})]\,\mathrm{d}\mathbf{x} + \|\mathbf{q}_0\|^2,
$$

we have that $\delta_1 \lesssim m^2/f(m)$. Moreover, it holds that $\delta_1 M_1 \lesssim m^4/f(m)$. Then, it is easy to see that $\delta_1 \to 0$ and $\delta_1 M_1 \to 0$ as $m \to \infty$, provided that $f(m) = O(m^{4+\varepsilon})$ for some $\varepsilon > 0$. Therefore, the smallness of δ_1 and $\delta_1 M_1$ can be realized as long as $m \ge m_0$ for some $m_0 \in \mathbb{N}$. Furthermore, from (A.1a)–(A.1b) we can show that $\|p_0 - \bar{p}\|_{H^2}^2 = O(m^2)$ and $\|\nabla \cdot \mathbf{q}_0\|^2 = O(f(m))$ and $\|\Delta \mathbf{q}_0\|^2 = O(f^3(m))$ for large m.

1.2. Small entropy in 3D. First, we recall the *a priori* assumptions made in Section 3.2:

$$
\begin{aligned}
&\sup_{0\le t\le T} \|\mathbf{q}(t)\|^2 \le \delta_2, \\
&\sup_{0\le t\le T} \|p(t) - \bar{p}\|^2 \le N_1, \\
&\sup_{0\le t\le T} (\|\nabla p(t)\|^2 + \|\nabla \cdot \mathbf{q}(t)\|^2) \le N_2, \\
&\sup_{0\le t\le T} (\|\Delta p(t)\|^2 + \|\Delta \mathbf{q}(t)\|^2) \le N_3.
\end{aligned}
$$

As the proof proceeded, we got the following estimates and choices of constants (see Sections 3.2.1–3.2.4):

$$\begin{aligned}
\sup_{0\le t\le T} &\|\mathbf{q}(t)\|^2 \\
&\le 2\int_{\mathbb{R}^3}[(p_0\ln(p_0)-p_0)-(\bar p\ln(\bar p)-\bar p)-\ln(\bar p)(p_0-\bar p)]\,\mathrm{d}\mathbf{x}+\|\mathbf{q}_0\|^2,
\end{aligned}$$

$$N_2=\left(1+\frac{1}{\bar p}\right)(\|p_0\|_{H^1}^2+\bar p\|\mathbf{q}_0\|_{H^1}^2)+1,$$

$$N_3=\left(1+\frac{1}{\bar p}\right)\exp\left\{\frac{c}{\bar p}(\|p_0\|_{H^1}^2+\bar p\|\mathbf{q}_0\|_{H^1}^2)\right\}(\|\Delta p_0\|^2+\bar p\|\Delta\mathbf{q}_0\|^2)+1.$$

We also required that $\delta_2 N_2$ and $\delta_2(N_3)^3$ be smaller than some absolute constants.

Now, let us consider the following initial functions:

$$p_0(\mathbf{x})=\begin{cases} m\left[\sin\left(r-\dfrac{\pi}{2}\right)+1\right]+g(m), & 2\pi\le r\le 4\pi,\\ g(m), & r\in(-\infty,2\pi)\cup(4\pi,\infty);\end{cases}$$

$$\mathbf{q}_0(\mathbf{x})=\begin{cases} \dfrac{m[\sin(r-\pi/2)+1]}{\sqrt{g(m)}}\cdot\dfrac{\mathbf{x}}{r}, & 2\pi\le r\le 4\pi,\\ \mathbf{0}, & r\in(-\infty,2\pi)\cup(4\pi,\infty),\end{cases}$$

where $m\in\mathbb{N}$, $r=|\mathbf{x}|$, and $\bar p\equiv g(m)>m$ is to be determined later. It is straightforward to check that $(p_0,\mathbf{q}_0)\in H^2(\mathbb{R}^3)$ and $\nabla\times\mathbf{q}_0=\mathbf{0}$. By direct calculations, we can show that

$$\text{(A.2a)}\qquad \|p_0-\bar p\|^2\cong m^2,\qquad \|\mathbf{q}_0\|^2\cong\frac{1}{g(m)}m^2,$$

$$\text{(A.2b)}\qquad \|\nabla p_0\|^2\cong m^2,\qquad \|\nabla\cdot\mathbf{q}_0\|^2\cong\frac{1}{g(m)}m^2,$$

$$\text{(A.2c)}\qquad \|\Delta p_0\|^2\cong m^2,\qquad \|\Delta\mathbf{q}_0\|^2\cong\frac{1}{g(m)}m^2,$$

which imply

$$N_2\cong\left(1+\frac{1}{g(m)}\right)n^2+1\equiv g_1(m)+1,$$

$$N_3\cong\exp\left\{\frac{c\,m^2}{g(m)}\right\}\left(1+\frac{1}{g(m)}\right)m^2+1\equiv g_2(m)e^{g_3(m)}+1.$$

In addition, we can show that

$$\begin{aligned}
\int_{\mathbb{R}^3}&[(p_0\ln(p_0)-p_0)-(\bar p\ln(\bar p)-\bar p)-\ln(\bar p)(p_0-\bar p)]\,\mathrm{d}\mathbf{x}\\
&=\int_{\mathbb{R}^3}\frac{1}{2p^*}(p_0-\bar p)^2\,\mathrm{d}\mathbf{x}\le\frac{1}{2g(m)}\|p_0-\bar p\|^2,
\end{aligned}$$

where p^* is between p_0 and $\bar{p}$. Hence, by taking

$$\delta_2 = 2\int_{\mathbb{R}^3} [(p_0 \ln(p_0) - p_0) - (\bar{p}\ln(\bar{p}) - \bar{p}) - \ln(\bar{p})(p_0 - \bar{p})]\,\mathrm{d}\mathbf{x} + \|\mathbf{q}_0\|^2,$$

we have that $\delta \cong m^2/f(m)$. Moreover, it holds that

$$\begin{aligned}
\delta_2 N_2 &\cong \frac{m^2}{g(m)} \cdot (g_1(m) + 1) \equiv g_4(m),\\
\delta_2 (N_3)^3 &\cong \frac{m^2}{g(m)} \cdot ([g_2(m)]^3 e^{3g_3(m)} + 3[g_2(m)]^2 e^{2g_3(m)} + 3g_2(m)e^{g_3(m)} + 1)\\
&\equiv g_5(m),
\end{aligned}$$

from which we see that $g_4(m) \to 0$ and $g_5(m) \to 0$ as $m \to \infty$, provided that $g(m) = O(m^{8+\varepsilon})$ for some $\varepsilon > 0$. Therefore, the smallness of $\delta_2 N_2$ and $\delta_2(N_3)^3$ can be realized as long as $m \geq m_0$ for some $m_0 \in \mathbb{N}$. Furthermore, from (A.2a)–(A.2c) we see that $\|p_0 - \bar{p}\|_{H^2} = O(m)$ for large m, while $\|\mathbf{q}_0\|_{H^2} \to 0$ as $m \to \infty$.

1.3. Small energy in 2D. First, let us recall the *a priori* assumptions made at the beginning of Section 4.1:

$$\begin{aligned}
&\sup_{0\leq t\leq T} (\|p(t) - \bar{p}\|^2 + \|\mathbf{q}(t)\|^2) \leq \delta_3,\\
&\sup_{0\leq t\leq T} (\|\nabla p(t)\|^2 + \|\nabla\cdot\mathbf{q}(t)\|^2) \leq K_1,\\
&\sup_{0\leq t\leq T} (\|\Delta p(t)\|^2 + \|\Delta\mathbf{q}(t)\|^2) \leq K_2.
\end{aligned}$$

During the proof of Theorem 2.5, we required $\delta_3 K_1$, δ_3, $\delta_3 K_2$, $\delta_3(K_1)^2$, and $\delta_3(K_1)^3$ to be smaller than some absolute constants.

Next, let us consider the initial functions

$$p_0(\mathbf{x}) = \begin{cases} m^{-3/2}\left[\sin\left(mr - \frac{\pi}{2}\right) + 1\right] + A, & 2\pi \leq r \leq 4\pi,\\ A, & r \in (-\infty, 2\pi) \cup (4\pi, \infty),\end{cases}$$

$$\mathbf{q}_0(\mathbf{x}) = \begin{cases} m^{-3/2}\left[\sin\left(mr - \frac{\pi}{2}\right) + 1\right] \cdot \frac{\mathbf{x}}{r}, & 2\pi \leq r \leq 4\pi,\\ \mathbf{0}, & r \in (-\infty, 2\pi) \cup (4\pi, \infty),\end{cases}$$

where $A > 0$ is any fixed constant, $m \in \mathbb{N}$, and $r = |\mathbf{x}|$. Then, direct calculations show that $(p_0, \mathbf{q}_0) \in H^2(\mathbb{R}^2)$, $\nabla \times \mathbf{q}_0 = \mathbf{0}$, and

(A.3a) $\qquad \|p_0 - A\|^2 \cong m^{-3}, \qquad \|\mathbf{q}_0\|^2 \cong m^{-3},$

(A.3b) $\qquad \|\nabla p_0\|^2 \cong m^{-1}, \qquad \|\nabla\cdot\mathbf{q}_0\|^2 \cong m^{-1},$

(A.3c) $\qquad \|\Delta p_0\|^2 \cong m, \qquad \|\Delta\mathbf{q}_0\|^2 \cong m.$

As the proof of Theorem 2.5 proceeded, we obtained the following qualitative relations:

$$\text{(A.4)}\qquad \begin{cases} \delta_3 \cong E_0 \cong \|p_0 - A\|^2 + ||p_0 - A||_{L^4}^4 + \|\mathbf{q}_0\|^2, \\ X_1(0) \cong \|\nabla p_0\|^2 + \|(p_0 - \bar{p})\nabla p_0\|^2 + \|\nabla \cdot \mathbf{q}_0\|^2, \\ K_1 \cong X_1(0) + 1, \\ E_0(K_1 + 1)^2 \ll 1, \\ K_2 \cong \|\Delta p_0\|^2 + \|\Delta \mathbf{q}_0\|^2 + X_1(0) \\ \qquad\qquad + (K_1 + 1)^2(X_1(0) + E_0 + 1). \end{cases}$$

From (A.3a)–(A.3c) we see that

$$\delta_3 \cong E_0 \cong m^{-3}, \quad K_1 \cong 1, \quad K_2 \cong m + 1,$$

from which we see that when $m \in \mathbb{N}$ is sufficiently large, the quantities $\delta_3 K_1$, δ_3, $\delta_3 K_2$, $\delta_3(K_1)^2$, and $\delta_3(K_1)^3$ are all small, and the fourth inequality in (A.4) can be realized.

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REFERENCES

[1] J. ADLER, *Chemotaxis in bacteria*, Science **153** (1966), no. 3737, 708–716.

[2] N. BELLOMO, A. BELLOUQUID, Y. TAO, and M. WINKLER, *Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues*, Math. Models Methods Appl. Sci. **25** (2015), no. 9, 1663–1763. http://dx.doi.org/10.1142/S021820251550044X. MR3351175

[3] A. BRESSAN, *Hyperbolic Systems of Conservation Laws: The One-dimensional Cauchy Problem*, Oxford Lecture Series in Mathematics and its Applications, vol. 20, Oxford University Press, Oxford, 2000. MR1816648

[4] C. DENG and T. LI, *Well-posedness of a 3D parabolic-hyperbolic Keller-Segel system in the Sobolev space framework*, J. Differential Equations **257** (2014), no. 5, 1311–1332. http://dx.doi.org/10.1016/j.jde.2014.05.014. MR3217041

[5] J. FAN and K. ZHAO, *Blow up criterion for a hyperbolic-parabolic system arising from chemotaxis*, J. Math. Anal. Appl. **394** (2012), no. 2, 687–695. http://dx.doi.org/10.1016/j.jmaa.2012.05.036. MR2927489

[6] M. A. FONTELOS, A. FRIEDMAN, and B. HU, *Mathematical analysis of a model for the initiation of angiogenesis*, SIAM J. Math. Anal. **33** (2002), no. 6, 1330–1355. http://dx.doi.org/10.1137/S0036141001385046. MR1920634

[7] J. GUO, J. XIAO, H. ZHAO, and C. ZHU, *Global solutions to a hyperbolic-parabolic coupled system with large initial data*, Acta Math. Sci. Ser. B (Engl. Ed.) **29** (2009), no. 3, 629–641. http://dx.doi.org/10.1016/S0252-9602(09)60059-X. MR2514366

[8] C. HAO, *Global well-posedness for a multidimensional chemotaxis model in critical Besov spaces*, Z. Angew. Math. Phys. **63** (2012), no. 5, 825–834. http://dx.doi.org/10.1007/s00033-012-0193-0. MR2991216

[9] T. HILLEN and K. J. PAINTER, *A user's guide to PDE models for chemotaxis*, J. Math. Biol. **58** (2009), no. 1–2, 183–217. http://dx.doi.org/10.1007/s00285-008-0201-3. MR2448428

[10] D. HOFF and K. ZUMBRUN, *Asymptotic behavior of multidimensional scalar viscous shock fronts*, Indiana Univ. Math. J. **49** (2000), no. 2, 427–474. http://dx.doi.org/10.1512/iumj.2000.49.1942. MR1793680

[11] D. HORSTMANN, *From 1970 until present: The Keller-Segel model in chemotaxis and its consequences. I*, Jahresber. Deutsch. Math.-Verein. **105** (2003), no. 3, 103–165. MR2013508

[12] Q. HOU, Z. WANG, and K. ZHAO, *Boundary layer problem on a hyperbolic system arising from chemotaxis*, J. Differential Equations **261** (2016), no. 9, 5035–5070. http://dx.doi.org/10.1016/j.jde.2016.07.018. MR3542967

[13] H. JIN, J. LI, and Z. WANG, *Asymptotic stability of traveling waves of a chemotaxis model with singular sensitivity*, J. Differential Equations **255** (2013), no. 2, 193–219. http://dx.doi.org/10.1016/j.jde.2013.04.002. MR3047400

[14] E. F. KELLER and L. A. SEGEL, *Initiation of slime mold aggregation viewed as an instability*, J. Theor. Biol. **26** (1970), no. 3, 399–415. http://dx.doi.org/10.1016/0022-5193(70)90092-5.

[15] ______, *Model for chemotaxis*, J. Theor. Biol. **30** (1971), no. 2, 225–234. http://dx.doi.org/10.1016/0022-5193(71)90050-6.

[16] ______, *Traveling bands of chemotactic bacteria: A theoretical analysis*, J. Theor. Biol. **30** (1971), no. 2, 235–248. http://dx.doi.org/10.1016/0022-5193(71)90051-8.

[17] H. A. LEVINE and B. D. SLEEMAN, *A system of reaction diffusion equations arising in the theory of reinforced random walks*, SIAM J. Appl. Math. **57** (1997), no. 3, 683–730. http://dx.doi.org/10.1137/S0036139995291106. MR1450846

[18] H. A. LEVINE, B. D. SLEEMAN, and M. NILSEN-HAMILTON, *A mathematical model for the roles of pericytes and macrophages in the initiation of angiogenesis. I: The role of protease inhibitors in preventing angiogenesis*, Math. Biosci. **168** (2000), no. 1, 77–115. http://dx.doi.org/10.1016/S0025-5564(00)00034-1. MR1788960

[19] D. LI, T. LI, and K. ZHAO, *On a hyperbolic-parabolic system modeling chemotaxis*, Math. Models Methods Appl. Sci. **21** (2011), no. 8, 1631–1650. http://dx.doi.org/10.1142/S0218202511005519. MR2826467

[20] D. LI, R. PAN, and K. ZHAO, *Quantitative decay of a one-dimensional hybrid chemotaxis model with large data*, Nonlinearity **28** (2015), no. 7, 2181–2210. http://dx.doi.org/10.1088/0951-7715/28/7/2181. MR3366640

[21] H. LI and K. ZHAO, *Initial-boundary value problems for a system of hyperbolic balance laws arising from chemotaxis*, J. Differential Equations **258** (2015), no. 2, 302–338. http://dx.doi.org/10.1016/j.jde.2014.09.014. MR3274760

[22] J. LI, T. LI, and Z. WANG, *Stability of traveling waves of the Keller-Segel system with logarithmic sensitivity*, Math. Models Methods Appl. Sci. **24** (2014), no. 14, 2819–2849. http://dx.doi.org/10.1142/S0218202514500389. MR3269780

[23] T. LI, R. PAN, and K. ZHAO, *Global dynamics of a hyperbolic-parabolic model arising from chemotaxis*, SIAM J. Appl. Math. **72** (2012), no. 1, 417–443. http://dx.doi.org/10.1137/110829453. MR2888351

[24] T. LI and Z. WANG, *Nonlinear stability of traveling waves to a hyperbolic-parabolic system modeling chemotaxis*, SIAM J. Appl. Math. **70** (2009/10), no. 5, 1522–1541. http://dx.doi.org/10.1137/09075161X. MR2578681

[25] ______, *Nonlinear stability of large amplitude viscous shock waves of a generalized hyperbolic-parabolic system arising in chemotaxis*, Math. Models Methods Appl. Sci. **20** (2010), no. 11, 1967–1998. http://dx.doi.org/10.1142/S0218202510004830. MR2740711

[26] ______, *Asymptotic nonlinear stability of traveling waves to conservation laws arising from chemotaxis*, J. Differential Equations **250** (2011), no. 3, 1310–1333. http://dx.doi.org/10.1016/j.jde.2010.09.020. MR2737206

[27] ______, *Steadily propagating waves of a chemotaxis model*, Math. Biosci. **240** (2012), no. 2, 161–168. http://dx.doi.org/10.1016/j.mbs.2012.07.003. MR3000369

[28] V. R. MARTINEZ, Z. WANG, and K. ZHAO, *Asymptotic and viscous stability of large-amplitude solutions of a hyperbolic system arising from biology*, Indiana Univ. Math. J. **67** (2018), no. 4, 1383–1424. http://dx.doi.org/10.1512/iumj.2018.67.7394. MR3853914

[29] J. D. MURRAY, *Mathematical Biology. I: An Introduction*, 3rd ed., Interdisciplinary Applied Mathematics, vol. 17, Springer-Verlag, New York, 2002. MR1908418

[30] H. G. OTHMER and A. STEVENS, *Aggregation, blowup, and collapse: The ABCs of taxis in reinforced random walks*, SIAM J. Appl. Math. **57** (1997), no. 4, 1044–1081. http://dx.doi.org/10.1137/S0036139995288976. MR1462051

[31] H. PENG, H. WEN, and C. ZHU, *Global well-posedness and zero diffusion limit of classical solutions to 3D conservation laws arising in chemotaxis*, Z. Angew. Math. Phys. **65** (2014), no. 6, 1167–1188. http://dx.doi.org/10.1007/s00033-013-0378-1. MR3279524

[32] B. PERTHAME, *Transport Equations in Biology*, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2007. MR2270822

[33] J. SMOLLER, *Shock Waves and Reaction-diffusion Equations*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 258, Springer-Verlag, New York, 1994. http://dx.doi.org/10.1007/978-1-4612-0873-0. MR1301779

[34] Y. TAO, L. WANG, and Z. WANG, *Large-time behavior of a parabolic-parabolic chemotaxis model with logarithmic sensitivity in one dimension*, Discrete Contin. Dyn. Syst. Ser. B **18** (2013), no. 3, 821–845. http://dx.doi.org/10.3934/dcdsb.2013.18.821. MR3007757

[35] Z. WANG and T. HILLEN, *Shock formation in a chemotaxis model*, Math. Methods Appl. Sci. **31** (2008), no. 1, 45–70. http://dx.doi.org/10.1002/mma.898. MR2373922

[36] Z. WANG, Z. XIANG, and P. YU, *Asymptotic dynamics on a singular chemotaxis system modeling onset of tumor angiogenesis*, J. Differential Equations **260** (2016), no. 3, 2225–2258. http://dx.doi.org/10.1016/j.jde.2015.09.063. MR3427665

[37] Z. WANG and K. ZHAO, *Global dynamics and diffusion limit of a one-dimensional repulsive chemotaxis model*, Commun. Pure Appl. Anal. **12** (2013), no. 6, 3027–3046. http://dx.doi.org/10.3934/cpaa.2013.12.3027. MR3060922

[38] M. WINKLER, *The two-dimensional Keller-Segel system with singular sensitivity and signal absorption: Global large-data solutions and their relaxation properties*, Math. Models Methods Appl. Sci. **26** (2016), no. 5, 987–1024. http://dx.doi.org/10.1142/S0218202516500238. MR3464427

[39] ______, *The two-dimensional Keller-Segel system with singular sensitivity and signal absorption: Eventual smoothness and equilibration of small-mass solutions*, preprint.

[40] Y. YANG, H. CHEN, and W. LIU, *On existence of global solutions and blow-up to a system of the reaction-diffusion equations modelling chemotaxis*, SIAM J. Math. Anal. **33** (2001), no. 4, 763–785. http://dx.doi.org/10.1137/S0036141000337796. MR1884721

[41] M. ZHANG and C. ZHU, *Global existence of solutions to a hyperbolic-parabolic system*, Proc. Amer. Math. Soc. **135** (2007), no. 4, 1017–1027. http://dx.doi.org/10.1090/S0002-9939-06-08773-9. MR2262902

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