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Boundary spike-layer solutions of the multi-dimensional singular Keller–Segel system: Existence, profiles and stability

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Abstract

This paper investigates boundary-layer solutions of the singular Keller–Segel system (proposed in Keller and Segel [J. Theor. Biol. **30** (1971), 377–380]) in multi-dimensional domains, which describes cells' chemotactic movement toward the concentration gradient of the nutrient they consume, subject to a zero-flux boundary condition for the cell density and a Dirichlet boundary condition for the nutrient. The steady-state problem of the system reduces to a scalar nonlocal Dirichlet elliptic problem with a singularity. By analyzing this nonlocal problem, we establish the existence of a unique steady-state solution which forms a boundary spike-layer profile as the nutrient diffusion coefficient $\varepsilon \to 0$. For radially symmetric domains, we derive explicit expansions for the boundary-layer steepness and thickness in terms of the domain radius (for small $\varepsilon > 0$), which quantifies the influence of radius on the profile and thickness. Additionally, we prove the nonlinear exponential stability of this boundary-layer steady state

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in radially symmetric domains. The key challenge in our analysis is the emergence of a singularity for small ε in both stationary and time-dependent problems. To address this, we reduce the nonlocal steady-state problem to a local one and conduct a refined analysis via the barrier method and Fermi coordinates, yielding sharp estimates for the local steady-state solution near the boundary. This approach enables us to determine the asymptotic profile of the nonlocal problem's solution as $\varepsilon \to 0$, accurately capturing and properly resolving the singularity to establish our main results. For the time-dependent problem in radially symmetric domains, we employ a variable transformation to eliminate the singularity, ultimately proving the nonlinear stability of the unique steady-state solution. Our analysis leverages the equation governing the radial mass distribution function relative to the steady state, along with delicate time-weighted energy estimates.

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1 | INTRODUCTION

Proposed in [19], the well-known Keller–Segel model with logarithmic chemotactic sensitivity in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ reads as

$$\begin{cases} u_t = \Delta u - \nabla \cdot (pu\nabla \log w) & \text{in } \Omega, \\ w_t = \varepsilon \Delta w - uw^{\beta} & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $u(x,t)$ denotes the bacterial density and $w(x,t)$ the chemical (oxygen or nutrient) concentration at position $x \in \Omega$ and time $t \geqslant 0$. $\varepsilon \geqslant 0$ is the chemical diffusion coefficient, $p > 0$ denotes the chemotactic coefficient and $\beta \geqslant 0$ the chemical consumption rate. The most prominent feature of the Keller–Segel model (1.1) lies in the logarithmic sensitivity $\log w$ which leads to a singularity at $w = 0$. The logarithmic sensitivity was used by Keller and Segel in [19] based on the Weber–Fechner law to explain the propagation of traveling bands driven by the *Escherichia coli* bacterial chemotaxis observed in the celebrated experiment of Adler [1], but later was employed to describe many other important biological processes such as the initiation of angiogenesis [23, 24], boundary movement of chemotactic bacteria [33], reinforced random walks [22, 34], boundary-layer formation of bacterial chemotaxis [4, 41], and so on. The mathematical derivation of logarithmic sensitivity was given in [22, 34] based on the random-walk framework. It was experimentally confirmed in [18] that *E. coli* bacteria use logarithmic sensing for chemotactic movement.

Mathematically the singular logarithmic sensitivity was necessary to generate traveling wave solutions from the system (1.1) (cf. [19, 31]). This singularity brings various challenges to analysis but attracts immense attention due to the interest in its own right. Up to now, most studies are limited to the case $\beta = 1$ except the existence of traveling wave solutions (cf. [42]) and a recent work [4] on the boundary-layer solutions in one dimension. When $\beta = 1$, a clever Cole–Hopf type transformation [22] can be used to remove the singularity. This stimulated massive interesting works, for example, the stability of traveling waves, see [5, 6, 9, 17, 25, 29] for instance, global well-posedness of solutions, see [8, 27, 28, 30, 32, 36, 40, 44, 45] in one-dimensional bounded domain with various boundary conditions or $\mathbb{R}$, and [10, 13, 26, 27, 37, 38, 43] in multi-dimensional spaces, just to mention a few. Among other things, this paper will be focused on the boundary-layer solutions of (1.1) and hence will only review the relevant results in this direction.

The observation of boundary-layer formation driven by chemotaxis was first reported in [41] where the chemotaxis model was coupled to fluid dynamics, with numerical studies followed in [7]. The analytical result of boundary-layer solutions of (1.1) was not available until in [14–16] where the Neumann boundary condition was imposed. It was shown therein that the (spatial) gradient of the solution instead of the solution itself possessed boundary-layer profile near the boundary. This is not consistent with the experimental observation of [41] where the model was imposed by the zero-flux boundary condition for u while Dirichlet boundary condition for w. Therefore, the authors [4] later considered the Keller–Segel system (1.1) with $\beta \geqslant 0$ in the half-space $\mathbb{R}^+ = [0, \infty)$ endowed with the zero-flux boundary condition for u while Dirichlet boundary condition for w at the boundary $x = 0$. The unique steady-state boundary-layer solution was explicitly obtained and shown to be locally asymptotically stable. The work [4] took advantage of the fact that the steady-state problem of (1.1) in $\Omega = [0, \infty)$ can be explicitly solved and hence the vanishing limit of the solution as $\varepsilon \to 0$ can be determined. In addition, the technique of taking anti-derivative or working at the level of the mass distribution function can be used in one dimension to establish the stability of the boundary-layer solution. All these advantages can only be used for one-dimensional space, and therefore the multi-dimensional problem still remains open. The main goal of this work is to fill this gap and consider the singular Keller–Segel system with physical mixed zero-flux and Dirichlet boundary conditions

$$\begin{cases} u_t = \Delta u - \nabla \cdot (pu\nabla \log w) & \text{in } \Omega, \\ w_t = \varepsilon\Delta w - uw & \text{in } \Omega, \\ (\nabla u - pu\nabla \log w)\cdot \nu = 0, \ w = b & \text{on } \partial\Omega, \\ (u, w)(x, t) = (u_0, w_0)(x) & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $b > 0$ is a positive constant denoting the boundary value of w, and ν is the unit outer normal vector of $\partial\Omega$. We note that in this paper we consider the case $\beta = 1$ only to avoid excessive technicalities. We start with the steady-state (stationary) problem of (1.2). First we remark that the integration of the first equation of (1.2) immediately gives

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx := m$$

which entails that the mass of u is preserved, denoted by $m > 0$, where $u_0 \geqslant (\not\equiv 0)$ denotes the initial value of u.

Then the steady-state solutions of (1.2), denoted by (U, W), satisfy

$$\begin{cases} \Delta U - \nabla \cdot (pU\nabla \log W) = 0 & \text{in } \Omega, \\ \varepsilon \Delta W - UW = 0 & \text{in } \Omega, \\ (\nabla U - pU\nabla \log W) \cdot \nu = 0, \ W = b & \text{on } \partial\Omega, \\ \int_\Omega U(x)dx = m. \end{cases} \tag{1.3}$$

Multiplying the first equation of (1.3) by $\log U - p \log W$, and integrating the equation on Ω, we have

$$\int_\Omega U|\nabla(\log U - p \log W)|^2 dx = 0. \tag{1.4}$$

Since we are interested in the nonnegative solutions, we have $U(x) \geqslant 0$ and $W(x) \geqslant 0$ for any $x \in \overline{\Omega}$. Applying the strong maximum principle to the second equation of (1.3), we have $W(x) > 0$ for any $x \in \overline{\Omega}$. We next write the first equation of (1.3) as

$$-\Delta U + p\nabla U \frac{\nabla W}{W} + \frac{p}{\varepsilon} U^2 = pU\frac{|\nabla W|^2}{W^2} \geqslant 0.$$

Then by the strong maximum principle and Hopf's boundary point lemma along with the fact $\int_\Omega U dx = m$, one has $U(x) > 0$ for all $x \in \overline{\Omega}$. Thus, it follows from (1.4) that

$$\log U - p \log W = c_0$$

for an arbitrary constant c_0. Therefore, we get a constant $\lambda = e^{c_0} > 0$ such that

$$U = \lambda W^p, \ \lambda = \frac{m}{\int_\Omega W^p dx}, \tag{1.5}$$

where the constant $\lambda = \frac{m}{\int_\Omega W^p dx}$ is obtained by the mass constraint in (1.3). Then the second equation of (1.3) can be rewritten as a nonlocal problem as follows:

$$\begin{cases} \varepsilon \Delta W = \dfrac{m}{\int_\Omega W^p dx} W^{p+1} & \text{in } \quad \Omega, \\ W = b > 0 & \text{on } \quad \partial\Omega. \end{cases} \tag{1.6}$$

As such, the steady-state problem (1.3) is reduced to a scalar nonlocal problem (1.6) with (1.5). That is, the existence of solutions to (1.3) is equivalent to that of (1.5)–(1.6). The Keller–Segel system (1.2) was considered in a one-dimensional half space $\Omega = [0, \infty)$ in our previous work [4], and an explicit expression of the unique solution W of (1.6) was found. Making use of this explicit formula, the steady-state solution (U, W) was shown to be of a boundary spike-layer profile (i.e., U is a Dirac mass concentrated at the boundary $x = 0$ and W is a boundary-layer profile) as $\varepsilon \to 0$, which is nonlinearly asymptotically stable as time tends to infinity. However, the explicit solution of (1.6) cannot be obtained in a multi-dimensional domain. As we know, the time-dependent problem (1.2) and nonlocal steady-state problem (1.6) in general multi-dimensional domains largely remain unexplored. Before proceeding, we recall a related nonlocal problem considered in [21] given by

$$\begin{cases} \varepsilon\Delta W = \dfrac{m}{\int_\Omega \exp(pW)dx} W\exp(pW) & \text{in} \quad \Omega, \\ W = b > 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.7}$$

which results from the steady-state problem of the Keller–Segel model with linear sensitivity (i.e., replacing $\log w$ in (1.2) by w). It was shown in [21] that the nonlocal problem (1.7) admits a unique solution which forms a boundary-layer profile as $\varepsilon \to 0$. When Ω is radially symmetric, the expansion of the boundary-layer profile and thickness in terms of ε was further explicitly identified.

Compared to (1.7), the nonlocal problem (1.6) is significantly more difficult to study. A major difference in the analysis lies in the nonlocal term in (1.6) which may become singular (i.e., $\int_\Omega W^p dx \to 0$) as $\varepsilon \to 0$ and hence brings considerable difficulties to study the ε-vanishing limit of solutions, while the nonlocal term $\int_\Omega \exp(pW)dx$ in (1.7) inherently has a positive lower bound. Nevertheless, some ideas developed in [21] like the barrier method and the use of Fermi coordinates strongly inspire a part of the present analysis. This paper has multiple goals as outlined below.

- (G1) Establish the existence and uniqueness of the solution to (1.6) in any dimension $n \geqslant 1$, denoted by $W_\varepsilon(x)$, which along with (1.5) yields a unique steady-state solution $(U_\varepsilon, W_\varepsilon)$ of (1.2) satisfying (1.3) (see Theorem 2.1).
- (G2) Prove that the solution $(U_\varepsilon, W_\varepsilon)$ is a boundary spike-layer profile as $\varepsilon \to 0$, namely U_ε converges to a Dirac mass concentrate at the boundary and W_ε to a boundary-layer profile with boundary-layer thickness at the order of ε as $\varepsilon \to 0$ (see Theorem 2.2).
- (G3) Explore how the boundary curvature affects the boundary-layer profile and thickness when the domain is radially symmetric (see Theorem 2.3).
- (G4) Prove the nonlinear stability of the unique boundary-layer profile $(U_\varepsilon, W_\varepsilon)$ of Keller–Segel system (1.2) (see Theorem 2.4) in the radially symmetric domain.

Strategy of achieving our goals and structure of the paper. As already mentioned, our current polynomial nonlinearity in (1.6) is much more difficult to deal with compared to the exponential nonlinearity, since estimating the nonlocal term $\int_\Omega W^p dx$ is significantly more challenging as it is not bounded from below by a positive constant. To overcome this difficulty, we first relegate the nonlocal problem to a local problem and compute the integral $\int_\Omega W^p dx$ based on the constructed sub- and super-solutions for the local problem. It is crucial to observe that as the diffusion coefficient ε tends to zero, the integrals of the sub- and super-solutions share the same leading-order term. This allows us to determine the leading-order behavior of the nonlocal integral for the small $\varepsilon > 0$. Once this relationship is established, we can effectively treat the nonlocal problem as a local one and extract further quantitative estimates on the solution needed for our purpose. We briefly further elaborate these ideas below.

For our goal (G1), inspired by the idea of [21], we first consider the nonlocal problem (1.6) by replacing the term $m/\int_\Omega W^p dx$ with a constant λ and reducing (1.6) to a local problem parameterized by λ. Then we establish the existence and uniqueness of the solution to the local problem by the standard monotone iteration scheme and comparison principle, a similar strategy used in [3]. By studying the continuity and the asymptotic limits of $\int_\Omega W^p dx$ with respect to the parameter λ, we finally obtain the unique solution for the nonlocal elliptic problem (1.6) via a fixed point argument in the parameter λ (see Theorem 2.1 and its proof in Section 3). This unique solution, along with (1.5), gives a unique solution to (1.3). We remark that it was recently shown in [2] that (1.6) admits a unique positive solution when $\Omega \subset \mathbb{R}^2$ or Ω is a ball in three or higher dimensions. Here we establish the existence and uniqueness of solutions for the nonlocal elliptic problem (1.6) in a general domain Ω across all dimensions,

To study the asymptotic behavior of the solution to (1.6) near the boundary as $\varepsilon \to 0$, that is, our goal (G2), we treat the product of the diffusion coefficient ε and the integral $\int_\Omega W^p dx$ as a unified diffusion coefficient and reformulate the nonlocal problem as a local one. Then we derive some quantitative results that describe the asymptotic behavior of the reformulated local problem near the boundary by constructing sharp sub-solutions and super-solutions based on the barrier method (cf. [39]) and a representation of Laplacian operator in terms of Fermi coordinates proved in our previous work [21]. Then based on these results, we can find how the integral $\int_\Omega W^p dx$ in the nonlocal problem (1.6) depends on ε and finally identify the asymptotic profile of W as $\varepsilon \to 0$, which is shown to be a boundary-layer profile with boundary-layer thickness of the order of ε. Further with (1.5), we show that U_ε is a Dirac mass concentrated at the boundary as $\varepsilon \to 0$ (see Theorem 2.2 with its proof in Section 4 and Remark 2.1).

In (G3), we are devoted to investigating how the boundary curvature affects the boundary-layer profile and thickness. This turns out to be a very difficult problem due to the nonlocal singular term. Hence we compromise to consider a radially symmetrical domain where the boundary curvature is the reciprocal of the radius. For the radially symmetric domain, we can gain more detailed estimates on the integral $\int_\Omega W^p dx$ and find the expansion for the slope of the solution on the boundary in terms of the radius as $\varepsilon > 0$ is small, and further identify how the boundary-layer thickness depends on the radius. It turns out such expansion is not a regular one as expected, but a nonregular one (see Theorem 2.3(i) with its proof in Section 5). This is due to the singularity of the nonlocal term in (1.6). In the radially symmetric case, we further can find the explicit asymptotic behavior of the solution (see Theorem 2.3(ii)) as $p \to \infty$ (strong chemotaxis) which was not found for the general domain. Based on our findings for the radially symmetrical domain, we conjecture that the boundary-layer thickness for the general domain is proportional to the boundary curvature times the volume of the domain, which remains unproved in our present work but has been verified by numerical simulations (see Figure 2 and Figure 3).

Finally, we consider our goal (G4) concerning the stability of the unique boundary spike-layer profile $(U_\varepsilon, W_\varepsilon)$ for the time-dependent problem (1.2). There are two challenging issues to study the stability. The first one is the singularity at $w = 0$, and the second one is that the Dirichet boundary condition on w is insufficient to gain the regularity of gradient of w required by the first equation of (1.2). To overcome the first barrier, we use a change of variable to transform the problem (1.2) into a new one (6.6) without singularity. To overcome the second one, we consider the radially symmetrical domain to employ the technique of taking anti-derivative or the radial mass distribution function reducing the order of the first equation of (6.6) so that the Dirichet boundary condition can be fully used. Then we perform sophisticated time-weighted energy estimates to obtain the nonlinear and exponential stability of $(U_\varepsilon, W_\varepsilon)$ for the radially symmetric domain (see Theorem 2.4 and its proof in Section 6).

2 | MAIN RESULTS AND CONJECTURES

The first result concerning the existence and uniqueness of solutions is the following.

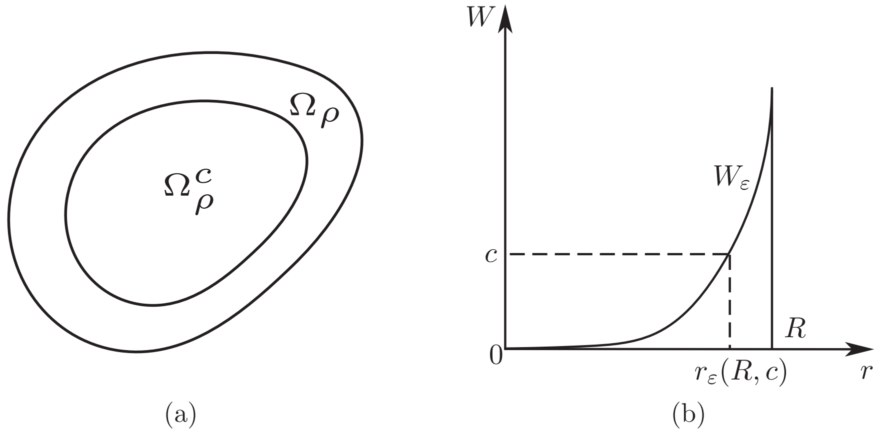

FIGURE 1 (a) A schematic of domain Ω_ρ. (b) Illustration of radial boundary-layer thickness.

Theorem 2.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^n$ $(n \geqslant 1)$ with smooth boundary and let m and b be given positive constants. Then for any $\varepsilon > 0$, the nonlocal problem (1.6) admits a unique positive classical solution $W_\varepsilon \in C^1(\overline{\Omega}) \cap C^\infty(\Omega)$ and hence the Keller–Segel system (1.2) admits a unique positive classical steady-state $(U_\varepsilon, W_\varepsilon) \in [C^1(\overline{\Omega}) \cap C^\infty(\Omega)]^2$ satisfying (1.5).*

Next we shall characterize the asymptotic profile of the steady-state solution $(U_\varepsilon, W_\varepsilon)$ as $\varepsilon \to 0$, which is a tricky problem since ε is a singular parameter. We shall show that U_ε is a Dirac measure while W_ε is a boundary-layer profile near the boundary as $\varepsilon \to 0$. To state our results, for any constant $\rho > 0$, we define Ω_ρ as (see an illustration in Figure 1a)

$$\Omega_\rho = \{x \in \Omega \mid 0 < \mathrm{dist}(x, \partial\Omega) < \rho\}, \tag{2.1}$$

and we denote by Ω_ρ^c its complement in Ω.

We will say that two functions of one real variable are similar $f \sim g$ if and only if $r_1 f < g < r_2 g$ for all values of the variable with some constants $r_1, r_2 > 0$. We now give the definition of boundary-layer thickness as a function of the positive parameter ε.

Definition 2.1. Let $\mu(\varepsilon)$ be a nonnegative function satisfying $\mu(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $W_\varepsilon(x)$ be the solution of (1.6). Denoting $\ell_\varepsilon = \mathrm{dist}(x_{in}, \partial\Omega)$ for any interior point of Ω, we say the boundary-layer thickness of $W_\varepsilon(x)$ is the same order of ε as $\mu(\varepsilon)$ if the following conditions are fulfilled:

(1) If $\lim\limits_{\varepsilon\to 0} \frac{\ell_\varepsilon}{\mu(\varepsilon)} = 0$, then $\lim_{\varepsilon\to 0} W_\varepsilon(x_{in}) = b$.

(2) If $\lim\limits_{\varepsilon\to 0} \frac{\ell_\varepsilon}{\mu(\varepsilon)} = L \in (0, \infty)$, then $\lim_{\varepsilon\to 0} W_\varepsilon(x_{in}) \in (0, b)$.

(3) If $\lim\limits_{\varepsilon\to 0} \frac{\ell_\varepsilon}{\mu(\varepsilon)} = \infty$, then $\lim_{\varepsilon\to 0} W_\varepsilon(x_{in}) = 0$.

Then our second main result is stated below.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^n$ $(n \geqslant 1)$ be a bounded domain with smooth boundary. Then for any small fixed constant $\delta > 0$, the solution obtained in Theorem 2.1 satisfies*

$$U_\varepsilon(x) \sim \frac{1}{\varepsilon\left(1+\frac{\operatorname{dist}(x,\partial\Omega)}{\varepsilon}\right)^2}, \quad W_\varepsilon(x) \sim \left(1+\frac{\operatorname{dist}(x,\partial\Omega)}{\varepsilon}\right)^{-\frac{2}{p}} \quad in \quad \Omega_\delta, \tag{2.2}$$

and there are some constants $c_1, c_2 > 0$ independent of ε such that

$$\|U_\varepsilon(x)\|_{L^\infty} \leqslant c_2\varepsilon \quad and \quad \|W_\varepsilon(x)\|_{L^\infty} \leqslant c_1\varepsilon^{\frac{2}{p}} \quad in \quad \Omega_\delta^c. \tag{2.3}$$

Moreover, the boundary-layer thickness is of order of ε.

Remark 2.1. From (2.3), we can see that $U_\varepsilon(x) \to 0$ as $\varepsilon \to 0$ provided x is away from $\partial\Omega$. With the fact $\int_\Omega U_\varepsilon dx = m$, we get that in the sense of distribution

$$U_\varepsilon \rightharpoonup m\delta_{\partial\Omega} \text{ as } \varepsilon \to 0,$$

where $\delta_{\partial\Omega}$ denotes the Dirac mass located in the $(n-1)$th Hausdorff dimension set $\partial\Omega$. From (2.3), we also have that

$$\lim_{\varepsilon\to 0} \|W_\varepsilon\|_{L^\infty(\overline{\Omega_\delta^c})} = 0,$$

but $W_\varepsilon = b > 0$ on $\partial\Omega$. Hence $W_\varepsilon(x)$ is a boundary-layer profile as $\varepsilon \to 0$.

In Theorem 2.1, the existence and uniqueness of nontrivial steady-state solutions of the Keller–Segel system (1.2) are established. In Theorem 2.2, we further show that the nontrivial steady-state solution $(U_\varepsilon, W_\varepsilon)$ obtained in Theorem 1.1 is a boundary spike-layer profile as $\varepsilon \to 0$ and find the explicit asymptotic profile of $(U_\varepsilon, W_\varepsilon)$ near the boundary as $\varepsilon > 0$ is small as given in (2.2). Now we proceed to investigate how the boundary curvature affects the boundary-layer profile and thickness, for which we can only give an answer when $\Omega = B_R(0)$ with radius $R > 0$ where the boundary curvature is $\frac{1}{R}$. In this case, we are able to find how the slope of boundary-layer solution at the boundary $r = R$ depends on the boundary curvature $1/R$ and hence quantify the boundary-layer thickness in terms of the radius. Below we sketch this idea. We first show the radial boundary-layer profiles $U_\varepsilon(r)$ and $W_\varepsilon(r)$ are strictly increasing with respect to r, and derive the expansion of their slopes at the boundary $r = R$ in terms of R for small $\varepsilon > 0$. Then for a given level set such that $W_\varepsilon(r_\varepsilon) = c \in (0, b)$, the distance from boundary $r = R$ to the point r_ε varies with respect to R. To be precise, for any $c \in (0, b)$, we define

$$r_\varepsilon(R,c) := W_\varepsilon^{-1}(c) \quad \text{and} \quad \Gamma_\varepsilon(R,c) := \{r \in [0,R] : W_\varepsilon \in [c,b]\} = [r_\varepsilon(R,c), R] \tag{2.4}$$

as functions of R and c, where $\Gamma_\varepsilon(R,c)$ is a closed interval with width $R - r_\varepsilon(R,c) = O(\varepsilon)$ which is nothing but the boundary-layer thickness (see an illustration in Figure 1b). Then we shall quantitatively expand $R - r_\varepsilon(R,c)$ in terms of R and ε up to the first-order expansion to see how the boundary-layer thickness depends on R and ε. Precisely we have the following results.

Theorem 2.3. *Let $\Omega = B_R(0)$, where $B_R(0)$ denotes a ball in $\mathbb{R}^n (n \geqslant 1)$ with radius $R > 0$. Then (1.3) admits a unique steady-state $(U_\varepsilon, W_\varepsilon)(r)$ with $r = |x|$, which is radially symmetric and satisfies $U_\varepsilon'(r) > 0, W_\varepsilon'(r) > 0$. Furthermore, the following conclusions hold.*

(i) *As* $\varepsilon \to 0$, $(U_\varepsilon, W_\varepsilon)(r)$ *has the following expansion at the boundary:*

$$\begin{aligned} U_\varepsilon'(R) &= \frac{p^4 m^3}{2(2+p)^2\omega_n^3 R^{3(n-1)}}\frac{1}{\varepsilon^2} + O\left(\frac{\log\varepsilon}{\varepsilon}\right), \\ W_\varepsilon'(R) &= \frac{pmb}{(2+p)\omega_n R^{n-1}}\frac{1}{\varepsilon} + O(\log\varepsilon), \end{aligned} \tag{2.5}$$

where ω_n *denotes the surface area of the unit ball in* $\mathbb{R}^n$. *Furthermore, the boundary-layer thickness has the following expansion:*

$$R - r_\varepsilon(R,c) = \left(\left(b/c\right)^{\frac{p}{2}} - 1\right)\frac{2n(p+2)}{mp^2}\frac{\alpha_n(R)\varepsilon}{R} + o_\varepsilon(1)\varepsilon, \tag{2.6}$$

where $r_\varepsilon(R,c)$ *is defined in (2.4) and* $\alpha_n(R) = \frac{\omega_n}{n}R^n$ *denotes the volume of* $B_R(0)$.

(ii) *As* $p \to \infty$, *it holds that for any* $\varepsilon > 0$

$$\omega_n r^{n-1} U_\varepsilon(r) \to m\delta(r-R) \text{ in the sense of distribution,} \tag{2.7}$$

$$W_\varepsilon(r) \to b \text{ in } C(\overline{B_R}), \tag{2.8}$$

where $\delta(r-R)$ *is the Dirac function centered at* $r = R$.

Remark 2.2. In the expansion of $W_\varepsilon'(R)$ given in (2.5), one generally expects that the second term should be of constant order, but surprisingly it is of order $\log\varepsilon$. Consequently, the second order term of $U_\varepsilon'(R)$ is $\frac{\log\varepsilon}{\varepsilon}$ instead of $\frac{1}{\varepsilon}$. These unexpected results stem from the singularity in (1.6) (i.e., $\frac{m}{\int_\Omega W^p dx} \to \infty$) as $\varepsilon \to 0$.

Remark 2.3. From the expansion (2.6), it seems that the boundary-layer thickness increases with the boundary curvature $1/R$. We suspect that this extends to general domains: The boundary-layer thickness increases with the boundary (mean) curvature. While we have not yet proven this conjecture, we present two sets of numerical simulations to illustrate its validity. In the first set, we consider two domains, where one is a disk and the other is an ellipse, with distinct geometries but identical areas. We then numerically solve (1.2) in these domains using the same parameter values and initial data, with the resulting boundary-layer patterns plotted in Figure 2. From these results, we observe that the boundary-layer thickness increases with boundary curvature. In the second set, we consider an arbitrary domain exhibiting significantly varying boundary curvatures across different regions of its boundary. The numerical profiles are shown in Figure 3, where clear boundary-layer structures are observed, and the boundary layer is thicker in regions of the boundary with greater curvature. All these numerical results are consistent with our speculation on the influence of the boundary curvature on the boundary-layer thickness.

Finally, we state the nonlinear stability of the unique radial steady-state solution, which forms a boundary-layer profile as $\varepsilon > 0$ is small. This result is valid for $1 \leqslant n \leqslant 3$ by using the Sobolev embedding theorem in the corresponding a priori estimates in Proposition 6.6.

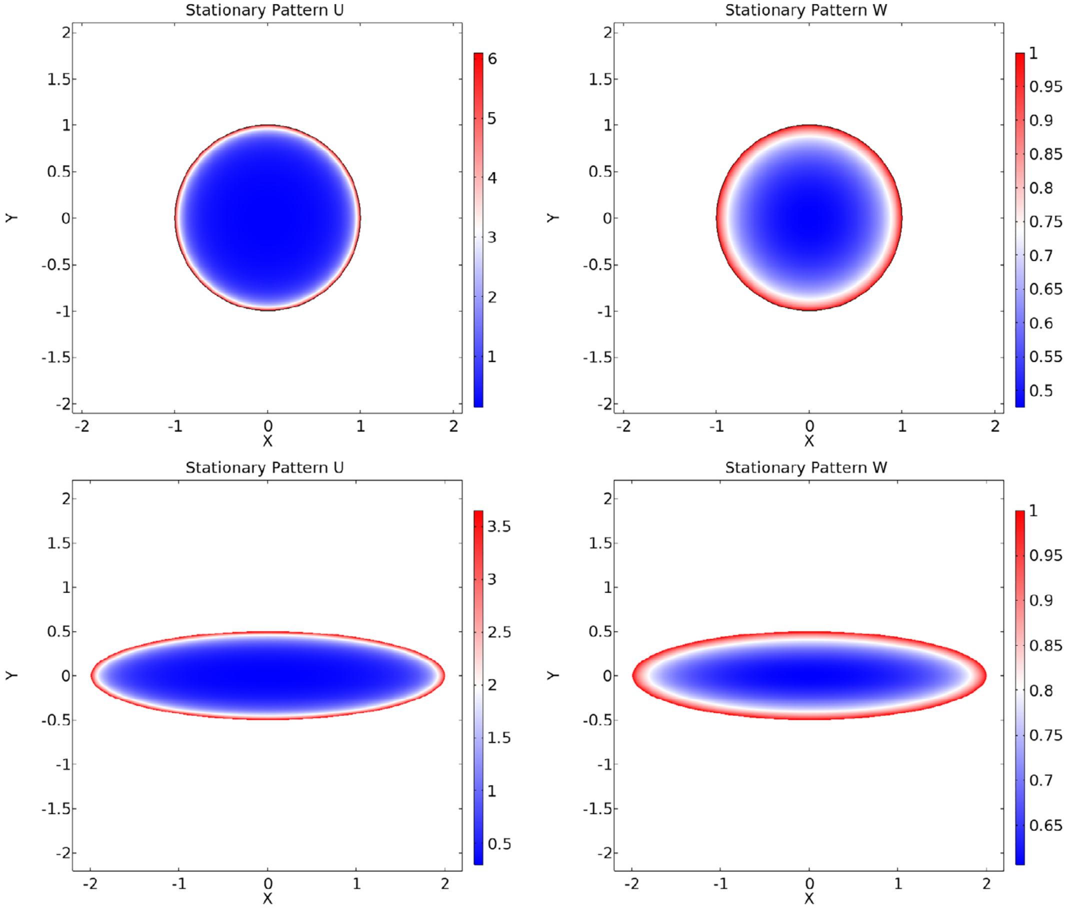

FIGURE 2 Numerical simulations of steady-state boundary-layer profiles of (1.2) in a disk (first row) and in an ellipse (second row) with the same area, where the parameter values are $p = 5, \varepsilon = 0.1, b = 1$ and initial value $(u_0, w_0) = (1, 1)$.

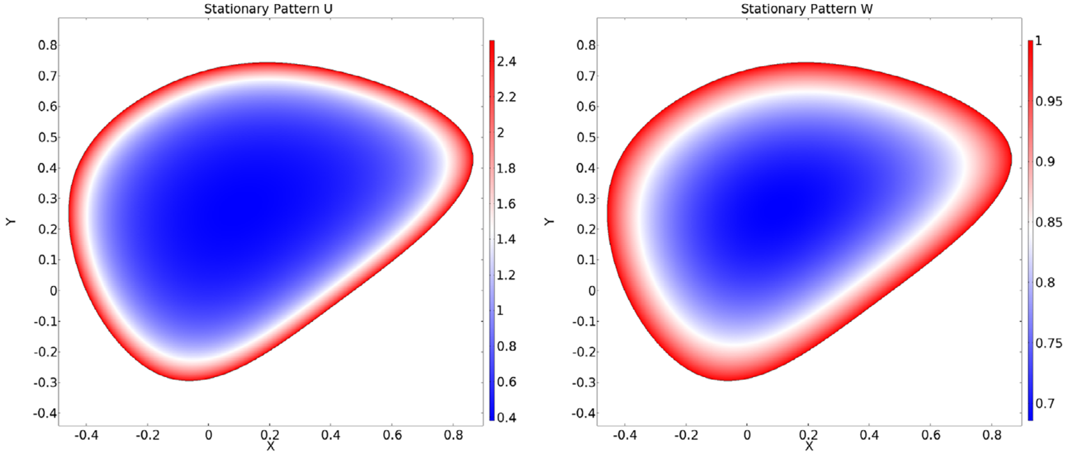

FIGURE 3 Numerical simulations of steady-state boundary-layer profiles of (1.2) in a two-dimensional general domain, where the parameter values are $p = 5, \varepsilon = 0.1, b = 1$ and initial value $(u_0, w_0) = (1, 1)$.

Theorem 2.4 (Nonlinear stability of the radial steady state). *Let $\Omega = B_R(0)$ in $\mathbb{R}^n (1 \leqslant n \leqslant 3)$ and the initial data (u_0, w_0) be radially symmetric with $u_0 > 0$ and $w_0 > 0$, and that $u_0 \in H^2(B_R)$, $w_0 - b \in H_0^1(B_R) \cap H^2(B_R)$. Let (U, W) be the unique steady state obtained in Theorem 2.3 with $m = \int_{B_R} u_0(x)dx$. Then there exists a constant $\delta_1 > 0$ such that if the initial datum satisfies*

$$\|(u_0 - U, w_0 - W)\|_{H^2(B_R)} \leqslant \delta_1,$$

the system (1.2) admits a unique global radial solution $(u, w) \in C([0, +\infty); H^2(B_R))$ satisfying

$$\|(u - U, w - W)(\cdot, t)\|_{L^\infty(B_R)} \leqslant Ce^{-\mu t}, \tag{2.9}$$

where C and μ are positive constants independent of t.

3 | EXISTENCE AND UNIQUENESS

This section is devoted to proving the existence and uniqueness of nonnegative solutions to (1.3) stated in Theorem 2.1. It suffices to prove the existence and uniqueness of solutions to the nonlocal problem (1.6) due to the relation given in (1.5).

3.1 | Existence

In the sequel, without confusion, we shall denote W_ε by W. Then the problem (1.6) is equivalent to the following local Dirichlet problem:

$$\begin{cases} \varepsilon\Delta W = \lambda W^{1+p} & \text{in} \quad \Omega, \\ W = b > 0, & \text{on} \quad \partial\Omega \end{cases} \tag{3.1}$$

subject to the constraint

$$\lambda \int_\Omega W^p dx = m \tag{3.2}$$

for a given constant $m > 0$. Note that we are concerned with positive solution for W only (see the discussion in the Introduction). Clearly $W_{\text{super}} = b$ and $W_{\text{sub}} = 0$ are a super-solution and sub-solution of (3.1), respectively. Since the function $f(W) = W^{p+1}$ is increasing, by the method of standard super-sub solutions, we immediately get that for any $\lambda > 0$, Equation (3.1) admits a unique classical positive solution depending on λ, denoted by W_λ, which must be nonconstant. Now it remains to show that there is a unique $\lambda > 0$ satisfying (3.2), namely $\lambda \int_\Omega W_\lambda^p dx = m$. It turns out this is difficult since how W_λ depends on λ is unknown. Here we overcome this barrier by introducing a change of variable

$$v_\lambda = \lambda^{\frac{1}{p}} W$$

and rewrite (3.1) as

$$\begin{cases} \varepsilon\Delta v_\lambda = v_\lambda^{1+p} & \text{in} \quad \Omega, \\ v_\lambda = \lambda^{\frac{1}{p}} b > 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.3}$$

and (3.2) as

$$\int_\Omega v_\lambda^p dx = m. \tag{3.4}$$

In the new transformed problem (3.3)–(3.4), we see that the parameter λ only appears in the boundary condition. The existence and uniqueness of classical solutions to (3.3) for each given $\lambda > 0$ is clearly obtained by the method of super-sub solutions (or directly from the existence of W). The crucial point is to find a unique λ such that the constraint (3.4) holds for given $m > 0$. Next we prove this in the following two steps.

Step 1 (continuity and monotonicity of $\int_\Omega v_\lambda^p dx$ with respect to λ). We denote the solution of (3.3) by v_λ and prove that $\int_\Omega v_\lambda^p dx$ is continuous with respect to λ. We start by showing that $v_\lambda(x)$ is nondecreasing with respect to λ. Indeed, for any two positive numbers $0 < \lambda_1 < \lambda_2$, we claim that $v_{\lambda_1} \leqslant v_{\lambda_2}$. If it is false, then $v_{\lambda_1} - v_{\lambda_2}$ admits an interior global maximum point $q \in \Omega$ due to the boundary conditions in (3.3) and the maximal value is positive. Then we have

$$\varepsilon\Delta(v_{\lambda_1} - v_{\lambda_2}) = v_{\lambda_1}^{1+p} - v_{\lambda_2}^{1+p} > 0 \tag{3.5}$$

in a small neighborhood of q, which is a contradiction for q to be an interior maximum. Thus the claim is proved. Next, we observe again that denoting by $v_{\lambda_1,\lambda_2} = v_{\lambda_2} - v_{\lambda_1}$, then v_{λ_1,λ_2} satisfies

$$\begin{cases} \varepsilon\Delta v_{\lambda_1,\lambda_2} = v_{\lambda_2}^{1+p} - v_{\lambda_1}^{1+p} \geqslant 0 & \text{in} \quad \Omega, \\ v_{\lambda_1,\lambda_2} = \left(\lambda_2^{\frac{1}{p}} - \lambda_1^{\frac{1}{p}}\right) b & \text{on} \quad \partial\Omega. \end{cases} \tag{3.6}$$

By the strong maximum principle either v_{λ_1,λ_2} is constant, which is false due to (3.6), or

$$0 \leqslant v_{\lambda_1,\lambda_2} < \left(\lambda_2^{\frac{1}{p}} - \lambda_1^{\frac{1}{p}}\right) b \quad \text{for} \quad x \in \Omega \tag{3.7}$$

which yields the continuity of v_λ with respect to λ. As a consequence, the continuity of $\int_\Omega v_\lambda^p dx$ with respect to λ is proved. Moreover, the function $\int_\Omega v_\lambda^p dx$ is increasing in λ. Assume otherwise $\int_\Omega v_{\lambda_1}^p dx = \int_\Omega v_{\lambda_2}^p dx$, since $v_{\lambda_1} \leqslant v_{\lambda_2}$ for some $0 < \lambda_1 < \lambda_2$, then it follows that $v_{\lambda_1} = v_{\lambda_2}$ in $\bar{\Omega}$, which is false in view of their continuity and the boundary condition in (3.6). In fact, one can additionally prove that $v_{\lambda_1} < v_{\lambda_2}$, this requires an argument using the Hopf-boundary point lemma that we do not detail here for brevity.

Step 2. We claim that $\int_\Omega v_\lambda^p dx$ can take any value in $[0, \infty)$ as λ ranges in $[0, +\infty)$. First notice that $\int_\Omega v_\lambda dx \to 0$ as $\lambda \to 0$ since $v_\lambda \to 0$ as $\lambda \to 0$. Next, we study the case that $\lambda > 0$ is large. We set

$$\Theta(y) = \lambda^{-\frac{1}{p}} v_\lambda(\lambda^{-\frac{1}{2}} y). \tag{3.8}$$

Then $\Theta(y)$ satisfies

$$\begin{cases} \varepsilon\Delta\Theta = \Theta^{1+p} & \text{in} \quad \Omega^\lambda, \\ \Theta = b & \text{on} \quad \partial\Omega^\lambda, \end{cases} \tag{3.9}$$

where Ω^λ is defined as

$$\Omega^\lambda = \{y \mid \lambda^{-\frac{1}{2}} y \in \Omega\}.$$

By standard elliptic regularity theory, we gain that $\|\Theta\|_{C^1(\Omega^\lambda)} \leqslant C_b$ for some uniform positive constant C_b. We choose ℓ such that $C_b \ell < \frac{b}{2}$. Then in the following set

$$\Omega^\lambda_\ell = \{y \mid \mathrm{dist}(y, \partial\Omega^\lambda) < \ell\}, \tag{3.10}$$

we have

$$\Theta(y) \geqslant b - C_b \ell \geqslant \frac{b}{2}. \tag{3.11}$$

It is not difficult to check that

$$|\Omega^\lambda_\ell| = \ell C_\Omega \lambda^{\frac{n-1}{2}} \text{ for some constant } C_\Omega > 0 \text{ depending only on } \Omega.$$

Consequently,

$$\begin{aligned} \int_\Omega v_\lambda^p dx &= \lambda^{1-\frac{n}{2}} \int_{\Omega^\lambda} \Theta^p(y) dy \geqslant \lambda^{1-\frac{n}{2}} \int_{\Omega^\lambda_\ell} \Theta^p(y) dy \\ &\geqslant \left(\frac{b}{2}\right)^p \ell C_\Omega \lambda^{1-\frac{n}{2}} \lambda^{\frac{n-1}{2}} = \ell C_\Omega \lambda^{\frac{1}{2}} \left(\frac{b}{2}\right)^p. \end{aligned} \tag{3.12}$$

Therefore, $\int_\Omega v_\lambda^p dx \to \infty$ as $\lambda \to +\infty$, and the claim is proved.

Combining the conclusions in Step 1 and Step 2, by the mean value theorem, we can find a λ such that $\int_\Omega v^p dx = m$ for the given m. Then we obtain a solution for (3.3)–(3.4), which gives a solution to (3.1)–(3.2) and hence to (1.6).

3.2 | Uniqueness

In Section 3.1, the existence of solutions to the nonlocal problem (1.6) has been obtained. Now we prove the uniqueness of solutions to (1.6). Supposing there are two distinct solutions W_1, W_2, we shall prove $W_1 \equiv W_2$ by the argument of contradiction and divide our analysis into two steps.

Step 1. We prove that either $W_1 \geqslant W_2$ or $W_1 \leqslant W_2$. Without loss of generality, we may assume $\int_\Omega W_1^p dx \geqslant \int_\Omega W_2^p dx$. Under this assumption, we claim that $W_1 \geqslant W_2$. If it is false, then there exists a point $q \in \Omega$, such that

$$(W_1 - W_2)|_q = \min_\Omega (W_1 - W_2) < 0.$$

As a consequence, we have

$$\left.\left(\frac{W_1^{1+p}}{\int_\Omega W_1^p dx} - \frac{W_2^{1+p}}{\int_\Omega W_2^p dx}\right)\right|_q < 0,$$

which yields

$$\left[\varepsilon\Delta(W_1 - W_2) - m\left(\frac{W_1^{1+p}}{\int_\Omega W_1^p dx} - \frac{W_2^{1+p}}{\int_\Omega W_2^p dx}\right)\right]\Bigg|_q > 0.$$

Contradiction arises. Thus, the claim holds. Therefore, for any two solutions W_1 and W_2, either $W_1 \geqslant W_2$ or $W_1 \leqslant W_2$.

Step 2. Next we prove that if $W_1 \geqslant W_2$, then $W_1 = W_2$. Set $Q = \frac{W_1}{W_2}$. It is obvious that

$$Q \geqslant 1 \text{ in } \Omega \quad \text{and} \quad Q = 1 \text{ on } \partial\Omega.$$

Suppose that $Q \neq 1$ and

$$Q(q_0) = \max_\Omega Q > 1.$$

Then

$$\frac{W_1^p(q_0)}{W_2^p(q_0)} \geqslant \frac{W_1^p}{W_2^p} \quad \text{in} \quad \Omega.$$

It implies that

$$\frac{W_1^p(q_0)}{W_2^p(q_0)} > \frac{\int_\Omega W_1^p dx}{\int_\Omega W_2^p dx}, \quad \text{and} \quad \left(\frac{W_1^p}{\int_\Omega W_1^p dx} - \frac{W_2^p}{\int_\Omega W_2^p dx}\right)\Bigg|_{q_0} > 0. \tag{3.13}$$

On the other hand, it is known that

$$\begin{aligned} 0 \geqslant \Delta Q|_{q_0} &= \nabla \cdot \left(\frac{\nabla W_1 \cdot W_2 - W_1 \cdot \nabla W_2}{W_2^2}\right)\Bigg|_{q_0} \\ &= \left(\frac{\Delta W_1 \cdot W_2 - W_1 \cdot \Delta W_2}{W_2^2} - 2\nabla Q \cdot \frac{\nabla W_2}{W_2}\right)\Bigg|_{q_0} \\ &= \frac{\Delta W_1(q_0) \cdot W_2(q_0) - W_1(q_0) \cdot \Delta W_2(q_0)}{W_2^2(q_0)} \\ &= \frac{mW_1(q_0)W_2(q_0)}{\varepsilon W_2^2(q_0)}\left(\frac{W_1^p}{\int_\Omega W_1^p dx} - \frac{W_2^p}{\int_\Omega W_2^p dx}\right)\Bigg|_{q_0}, \end{aligned} \tag{3.14}$$

where we have used $\nabla Q(q_0) = 0$ since q_0 is the maximal point of Q in Ω. Using (3.13) we see that the right-hand side of (3.14) is positive, then contradiction arises and $Q \equiv 1$ holds. Thus, we finish the proof.

Proof of Theorem 2.1. The existence and uniqueness of the solution to (1.6) has been proved in Section 3.1 and Section 3.2. Since the existence of (1.3) and (1.6) has one-to-one correspondence via (1.5), we obtain the existence of unique positive solution for (1.3) and complete the proof. □

4 | BOUNDARY-LAYER PROFILE AND THICKNESS

This section is devoted to the proof of Theorem 2.2. We start with the following auxiliary problem:

$$\begin{cases} \varepsilon\Delta v = v^{1+p} & \text{in} \quad \Omega, \\ v = b & \text{on} \quad \partial\Omega. \end{cases} \tag{4.1}$$

Lemma 4.1. *Let $v_\varepsilon \in C^\infty(\overline{\Omega})$ be the unique solution of (4.1). For any compact subset $K \subset \Omega$ and sufficiently small $\varepsilon > 0$, there exists a positive constant C_K independent of ε such that*

$$\max_K v_\varepsilon \leqslant C_K \varepsilon^{\frac{1}{p}}. \tag{4.2}$$

Proof. Let us first remark that we can reduce to analyze the case $b = 1$ in (4.1) by a simple scaling argument. In fact, take $\bar{v} = v/b$, it is simple to show that it satisfies

$$\begin{cases} \varepsilon b^{-p}\Delta\bar{v} = \bar{v}^{1+p} & \text{in} \quad \Omega, \\ \bar{v} = 1 & \text{on} \quad \partial\Omega. \end{cases} \tag{4.3}$$

Then proving the estimate (4.2) for the problem (4.1) with $b > 0$ is equivalent to prove

$$\max_K \bar{v}_\varepsilon \leqslant C(K)\varepsilon^{\frac{1}{p}} b^{-1}$$

for the solution of (4.3) or equivalently to prove (4.2) for the problem (4.1) with $b = 1$. Thus, in the rest of this proof, we assume $b = 1$.

When $n = 1$, without loss of generality we can assume that $\Omega = [-1, 1]$. It is straightforward to check that the following function:

$$v_{\varepsilon,s}(x) = \left(1 + \frac{x+1}{c_p\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}} + \left(1 + \frac{1-x}{c_p\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}}, \quad c_p = \sqrt{\frac{2}{p}\left(\frac{2}{p}+1\right)},$$

provides a super-solution to the above equation. Indeed, by direct computation

$$\varepsilon\Delta v_{\varepsilon,s} = \left(1 + \frac{x+1}{c_p\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2+2p}{p}} + \left(1 + \frac{1-x}{c_p\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2+2p}{p}} \leqslant v_{\varepsilon,s}^{1+p}.$$

Together with the trivial fact $v_{\varepsilon,s} > 1$ at $x = \pm 1$, one can easily conclude that $v_\varepsilon < v_{\varepsilon,s}$ by the strong maximum principle. Then (4.2) follows easily.

Now we give the proof for $n \geqslant 2$. For any $q \in \Omega$, we select R_q such that $B_{R_q}(q) \subset \Omega$ and $\partial B_{R_q}(q) \cap \partial\Omega \neq \emptyset$. Then we consider the solution $\hat{v}_\varepsilon$ of the following intermediate problem:

$$\begin{cases} \varepsilon\Delta\hat{v}_\varepsilon = \hat{v}_\varepsilon^{1+p} & \text{in} \quad B_{R_q}(q), \\ \hat{v}_\varepsilon = 1 & \text{on} \quad \partial B_{R_q}(q). \end{cases} \tag{4.4}$$

By the standard comparison argument, we obtain that $v_\varepsilon \leqslant \hat{v}_\varepsilon$ in $B_{R_q}(q)$. Next, using the method of moving planes, we see that $\hat{v}_\varepsilon(x)$ is a radially symmetric function with respect to q. We write $\hat{v}_\varepsilon(x) = \tilde{v}_\varepsilon(r)$ with $r = |x-q|$. Then it is not difficult to find that $\tilde{v}_\varepsilon(r)$ is a nondecreasing function in r and verifies that

$$\begin{cases} \varepsilon\left(\tilde{v}_\varepsilon'' + \frac{n-1}{r}\tilde{v}_\varepsilon'\right) - (\tilde{v}_\varepsilon)^{1+p} = 0, \\ \tilde{v}_\varepsilon(R_q) = 1, \quad \tilde{v}_\varepsilon'(0) = 0. \end{cases} \tag{4.5}$$

We claim that

$$\tilde{v}_\varepsilon \leqslant \begin{cases} 2^{\max\{n-2, \frac{n-1}{2}, \frac{2}{p}\}}\left(1 + \frac{R_q}{2C\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}}, & \text{for} \quad r \in [0, R_q/2], \\ 2^{\max\{n-2, \frac{n-1}{2}, \frac{2}{p}\}}\left(1 + \frac{R_q - r}{C\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}}, & \text{for} \quad r \in [R_q/2, R_q], \end{cases} \tag{4.6}$$

where C is some positive constant independent of ε to be determined later. It suffices to prove the claim for $r \in [R_q/2, R_q]$ while the rest one for $r \in [0, R_q/2]$ follows easily by the nondecreasing property of the function. We define a barrier function for $r \in (0, R_q]$ by

$$\tilde{v}_{\varepsilon,l} = \left(\frac{R_q}{r}\right)^a \left(1 + \frac{R_q - r}{C\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}},$$

where a is a constant determined later. By a straightforward computation, we have

$$\begin{aligned} \varepsilon\Delta\tilde{v}_{\varepsilon,l} - (\tilde{v}_{\varepsilon,l})^{1+p} &= \varepsilon\left(\tilde{v}_{\varepsilon,l}'' + \frac{n-1}{r}\tilde{v}_{\varepsilon,l}'\right) - (\tilde{v}_{\varepsilon,l})^{1+p} \\ &= \varepsilon a(a+2-n)\frac{R_q^a}{r^{a+2}}\left(1 + \frac{R_q - r}{C\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}} \\ &\quad + \varepsilon^{\frac{1}{2}}\frac{2}{Cp}(n-1-2a)\frac{R_q^a}{r^{a+1}}\left(1 + \frac{R_q - r}{C\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2+p}{p}} \\ &\quad + \left(\frac{c_p^2}{C^2} - \left(\frac{R_q}{r}\right)^{ap}\right)\frac{R_q^a}{r^a}\left(1 + \frac{R_q - r}{C\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2+2p}{p}}. \end{aligned} \tag{4.7}$$

We choose $a = \max\{n-2, \frac{n-1}{2}, \frac{2}{p}\}$. Then the second term on the right-hand side of (4.7) is nonpositive. For the first and third terms, we can rewrite them as

$$\begin{aligned} &\varepsilon a(a+2-n)\frac{R_q^a}{r^{a+2}}\left(1 + \frac{R_q - r}{C\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}} + \left(\frac{c_p^2}{C^2} - \left(\frac{R_q}{r}\right)^{ap}\right)\frac{R_q^a}{r^a}\left(1 + \frac{R_q - r}{C\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2+2p}{p}} \\ &= F(\varepsilon, r)\frac{R_q^a}{r^a}\left(1 + \frac{R_q - r}{C\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2+2p}{p}}, \end{aligned}$$

where
$$F(\varepsilon, r) = \frac{1}{r^2} a(a+2-n)\left(\varepsilon + \frac{2\varepsilon^{\frac{1}{2}}}{C}(R_q - r) + \frac{1}{C^2}(R_q - r)^2\right) + \frac{c_p^2}{C^2} - \left(\frac{R_q}{r}\right)^{ap}.$$
Considering the function $F(\varepsilon, r)$, in the limit case $\varepsilon = 0$ we have
$$\begin{aligned} r^{ap} F(0, r) &= \frac{1}{C^2}\left(r^{ap-2} a(a+2-n)(R_q - r)^2 + c_p^2 r^{ap}\right) - R_q^{ap} \\ &\leqslant \frac{1}{C^2}\left(a(a+2-n) + c_p^2 - C^2\right) R_q^{ap}, \end{aligned}$$
where we have used the fact that $a + 2 - n \geqslant 0$ and $r^{ap-2} \leqslant R_q^{ap-2}$ by the definition of a. Now taking $C > \sqrt{a(a+2-n) + c_p^2 + 1}$, we see that $F(0, r)$ is negative for $r \in (0, R_q]$. Thus, if ε is sufficiently small, we see that $F(\varepsilon, r) < 0$ for $r \in (0, R_q]$. On the other hand, it is straightforward to check that
$$\tilde{v}_{\varepsilon,l}(R_q) = 1 \quad \text{and} \quad \lim_{r \to 0} \tilde{v}_{\varepsilon,l}(r) = \infty. \tag{4.8}$$
By the standard comparison argument we obtain that $\tilde{v}_\varepsilon \leqslant \tilde{v}_{\varepsilon,l}$ for $r \in (0, R_q]$. Then the claim (4.6) follows directly. As a consequence, for any point $x \in B_{R_q/2}(q)$ we have
$$v_\varepsilon(x) \leqslant \hat{v}_\varepsilon(x) \leqslant 2^{\max\{n-2, \frac{n-1}{2}, \frac{2}{p}\}} \left(1 + \frac{R_q}{2C\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}}. \tag{4.9}$$
Since K is a compact subset of Ω, there exist finitely many open balls
$$B_{R_{q_j}/2}(q_j) \subsetneq B_{R_{q_j}}(q_j) \subset \Omega \text{ with } q_j \in K, \quad j = 1, \dots, m,$$
such that $K \subset \bigcup_{j=1}^m B_{R_{q_j}/2}(q_j)$. Let $R_0 = \min_{1 \leqslant j \leqslant m} R_{q_j}$. Then by (4.9), we have
$$v_\varepsilon(x) \leqslant 2^{\max\{n-2, \frac{n-1}{2}, \frac{2}{p}\}} \left(1 + \frac{R_0}{2C\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}}, \quad \forall x \in K,$$
which implies (4.2). Hence we finish the proof. □

From (4.2), we shall deduce that $W_\varepsilon \to 0$ as $\varepsilon \to 0$ for any fixed compact subset K of Ω (see the proof of Theorem 2.2 later). To capture the behavior of W_ε near $\partial\Omega$, we introduce the Fermi coordinates for any $x \in \Omega_\delta$, that is,
$$X : (y, z) \in \partial\Omega \times \mathbb{R}^+ \longmapsto x = X(y, z) = y + z\nu(y) \in \Omega_\delta,$$
where ν is the unit normal vector on $\partial\Omega$, and Ω_δ is defined in (2.1). There is a number $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$, the map X is from Ω_δ to a subset of $\mathcal{O}$ (cf. [20, Remark 8.1], where
$$\mathcal{O} = \{(y, z) \in \partial\Omega \times (0, 2\delta)\}.$$
It follows that X is actually a diffeomorphism onto its image $\mathcal{N} = X(\mathcal{O})$. For any fixed z, we set
$$\Gamma_z(y) = \{p \in \Omega \mid p = y + z\nu(y)\}.$$

It is straightforward to check that the distance between any point of $\Gamma_z(y)$ and $\partial\Omega$ is $|z|$. Hence we have the following results for the Laplacian operator in terms of Fermi coordinate shown in [21, Lemma 6.1] motivated by [35, Lemma 10.5].

Lemma 4.2. *The Euclidean Laplacian Δ can be computed by a formula in terms of the coordinate $(y, z) \in \mathcal{O}$ as*

$$\Delta_x = \partial_z^2 - H_{\Gamma_z(y)}\partial_z + \Delta_{\Gamma_z}, \quad x = X(y, z), \quad (y, z) \in \mathcal{O},$$

where $\Gamma_z(y)$ is the submanifold

$$\Gamma_z(y) = \{y + z\nu(y) \mid y \in \partial\Omega\},$$

and $H_{\Gamma_z(y)}$ is the mean curvature at the point in $\Gamma_z(y)$ and $\Delta_{\Gamma_z(y)}$ stands for the Beltrami–Laplacian operator on $\Gamma_z(y)$.

Lemma 4.3. *Let Ω be a smooth domain in $\mathbb{R}^n$ $(n \geqslant 1)$ and $v_\varepsilon \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ be the unique solution of (4.1). Then there exist positive constants ε_0 and δ_0 such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\delta \in \left(0, \min\{\frac{1}{2}, \delta_0\}\right)$, it holds that*

$$b_1\left(1 + b_2\frac{\operatorname{dist}(x, \partial\Omega)}{\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}} \leqslant v_\varepsilon \leqslant b_3\left(1 + b_4\frac{\operatorname{dist}(x, \partial\Omega)}{\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}} \quad in \quad \Omega_\delta, \tag{4.10}$$

where the definition of Ω_δ is given in (2.1) and $b_1, \ldots, b_4$ are positive constants independent of ε.

Proof. When $n = 1$, we can repeat almost the same arguments of Lemma 4.1 for $b = 1$ to derive the upper bound, just replacing the barrier function by the following one:

$$v_{\varepsilon,s} = b\left(\left(1 + \frac{b^{\frac{p}{2}}(x+1)}{c_p\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}} + \left(1 + \frac{b^{\frac{p}{2}}(1-x)}{c_p\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}}\right).$$

While for the lower bound, we set the barrier function by

$$v_{\varepsilon,l} = \frac{b}{2}\left(\left(1 + \frac{b^{\frac{p}{2}}(x+1)}{c_p\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}} + \left(1 + \frac{b^{\frac{p}{2}}(1-x)}{c_p\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}}\right).$$

Then

$$\begin{aligned}\varepsilon\Delta v_{\varepsilon,l} - v_{\varepsilon,l}^{1+p} = \frac{b^{1+p}}{2^{1+p}}\Bigg(2^p\left[\left(1 + \frac{b^{\frac{p}{2}}(x+1)}{c_p\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2+2p}{p}} + \left(1 + \frac{b^{\frac{p}{2}}(1-x)}{c_p\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2+2p}{p}}\right] \\ - \left[\left(1 + \frac{b^{\frac{p}{2}}(x+1)}{c_p\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}} + \left(1 + \frac{b^{\frac{p}{2}}(1-x)}{c_p\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}}\right]^{1+p}\Bigg).\end{aligned}$$

Using the classical inequality $(a_1 + a_2)^{1+p} \leqslant 2^p(a_1^{1+p} + a_2^{1+p})$ with $a_1, a_2 > 0$, we directly see that the right-hand side of the above equation is positive. Then by the fact that $v_{\varepsilon,l} \leqslant b$ at $x = \pm 1$ and comparison argument, it follows that $v_{\varepsilon,l}$ is a sub-solution, and the lower bound for v_ε is provided.

Now we give the proof for $n \geqslant 2$. Without loss of generality, we may assume that Ω is a simply connected domain for simplicity, while the case for multiply connected domain can be proved similarly. We first derive the lower bound for v_ε. Since Ω is simply connected, $\partial\Omega$ is a smooth connected manifold of dimension $n-1$. We set $v_{\varepsilon,l}$ by

$$v_{\varepsilon,l}(x) = (2\delta - \mathrm{dist}(x, \partial\Omega))\left(1 + \frac{\mathrm{dist}(x,\partial\Omega)}{c_p \varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}} \quad \text{for} \quad x \in \Omega_{2\delta}.$$

It is easy to see that

$$v_{\varepsilon,l}(x) = \begin{cases} 2\delta, & \text{on} \quad \partial\Omega, \\ 0, & \text{on} \quad \partial\Omega_{2\delta} \setminus \partial\Omega. \end{cases} \tag{4.11}$$

A straightforward computation based on Lemma 4.2 gives

$$\begin{aligned} \varepsilon\Delta v_{\varepsilon,l} - v_{\varepsilon,l}^{p+1} &= (\varepsilon\partial_z^2 - \varepsilon H_{\Gamma_z(y)}\partial_z + \varepsilon\Delta_{\Gamma_z(y)})(2\delta - z)\left(1 + \frac{z}{c_p\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}} \\ &\quad - (2\delta - z)^{p+1}\left(1 + \frac{z}{c_p\varepsilon^{\frac{1}{2}}}\right)^{-2-\frac{2}{p}} \\ &= \varepsilon^{\frac{1}{2}} H_{\Gamma_z(y)}\left(\frac{4}{pc_pH_{\Gamma_z(y)}} + \frac{2(2\delta - z)}{pc_p} + \varepsilon^{\frac{1}{2}} + \frac{z}{c_p}\right)\left(1 + \frac{z}{c_p\varepsilon^{\frac{1}{2}}}\right)^{-1-\frac{2}{p}} \\ &\quad + (2\delta - z)(1 - (2\delta - z)^p)\left(1 + \frac{z}{c_p\varepsilon^{\frac{1}{2}}}\right)^{-2-\frac{2}{p}}, \end{aligned} \tag{4.12}$$

where we have used the fact that $\frac{2(2+p)}{p^2c_p^2} = 1$ by recalling $c_p = \sqrt{\frac{2}{p}\left(\frac{2}{p}+1\right)}$. We can choose δ and ε small enough such that

$$\frac{4}{pc_p} + \frac{2(2\delta - z)}{pc_p} H_{\Gamma_z(y)} + \varepsilon^{\frac{1}{2}} H_{\Gamma_z(y)} + \frac{z}{c_p} H_{\Gamma_z(y)} \geqslant 0$$

and $1 - (2\delta - z)^p \geqslant 0$ for $z \in (0, 2\delta)$. Then

$$\varepsilon\Delta v_{\varepsilon,l} - v_{\varepsilon,l}^{p+1} \geqslant 0 \quad \text{in} \quad \Omega_{2\delta}.$$

Together with that $v_\varepsilon \geqslant v_{\varepsilon,l}$ on $\partial\Omega_{2\delta}^c$ and the classical comparison argument we have

$$v_\varepsilon \geqslant v_{\varepsilon,l} \quad \text{in} \quad \overline{\Omega_{2\delta}},$$

which implies that

$$v_\varepsilon \geqslant \delta\left(1+\frac{z}{c_p\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}} \quad \text{in} \quad \overline{\Omega_\delta}.$$

While for the upper bound, we set

$$v_{\varepsilon,b} = b\left(1+\frac{b^{\frac{p}{2}}\operatorname{dist}(x,\partial\Omega)}{\Lambda c_p\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}},$$

where Λ is a large constant to be determined later. A direct computation yields that

$$\begin{aligned}
&\varepsilon\Delta v_{\varepsilon,b} - v_{\varepsilon,b}^{p+1}\\
&= b(\varepsilon\partial_z^2 - \varepsilon H_{\Gamma_z(y)}\partial_z + \varepsilon\Delta_{\Gamma_z(y)})\left(1+\frac{b^{\frac{p}{2}}z}{\Lambda c_p\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}} - b^{1+p}\left(1+\frac{b^{\frac{p}{2}}z}{\Lambda c_p\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2+2p}{p}}\\
&= b^{1+p}\left(\frac{1}{\Lambda^2}-1\right)\left(1+\frac{b^{\frac{p}{2}}z}{\Lambda c_p\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2+2p}{p}} + \varepsilon^{\frac{1}{2}}b^{1+\frac{1}{2}p}\frac{2}{p\Lambda c_p}H_{\Gamma_z(y)}\left(1+\frac{b^{\frac{p}{2}}z}{\Lambda c_p\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2+p}{p}}\\
&= \left(\frac{1}{\Lambda^2} + \frac{2\varepsilon^{\frac{1}{2}}}{b^{\frac{1}{2}p}p\Lambda c_p}H_{\Gamma_z(y)} + \frac{pzH_{\Gamma_z(y)}}{(p+2)\Lambda^2} - 1\right)b^{1+p}\left(1+\frac{b^{\frac{p}{2}}z}{\Lambda c_p\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2+2p}{p}}.
\end{aligned} \tag{4.13}$$

Thus, for any $\varepsilon \in (0,1)$ and $z \in (0, 2\delta)$, we can always choose $C_{\delta,1}$ sufficiently large such that for $\Lambda \in (C_{\delta,1}, +\infty)$, the term in the bracket of the right-hand side is negative. For the behavior of $v_{\varepsilon,b}$ on $\partial\Omega_\delta$, we have $v_{\varepsilon,b} = v_\varepsilon$ on $\partial\Omega$, while on $\partial\Omega_\delta \setminus \partial\Omega$, we have $v_\varepsilon \leqslant C_{\overline{\Omega_\delta^c}}\varepsilon^{1/p}$ due to Lemma 4.1. Then we choose $C_{\delta,2}$ large enough such that for any $\Lambda > C_{\delta,2}$, it holds that

$$b\left(1+\frac{b^{\frac{p}{2}}\delta}{\Lambda c_p\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}} \geqslant C_{\overline{\Omega_\delta^c}}\varepsilon^{\frac{1}{p}}. \tag{4.14}$$

Then we choose $\Lambda = \max\{C_{\delta,1}, C_{\delta,2}\}$ in the definition of $v_{\varepsilon,b}$ and using (4.13)–(4.14) we see that $v_{\varepsilon,b}$ provides a super-solution to (4.1) in Ω_δ. We finally choose

$$b_1 = \delta, \quad b_2 = c_p, \quad b_3 = b, \quad b_4 = \frac{b^{\frac{p}{2}}}{\Lambda c_p}$$

to finish the proof. □

Returning the original nonlocal problem (1.6) which can be written as

$$\begin{cases} \varepsilon\lambda_\varepsilon \Delta W_\varepsilon = W_\varepsilon^{p+1}, & x \in \Omega, \\ W_\varepsilon = b > 0, & x \in \partial\Omega, \end{cases} \tag{4.15}$$

where $\lambda_\varepsilon = \frac{1}{m}\int_\Omega W_\varepsilon^p dx > 0$ is a constant. Then we have the following result.

Lemma 4.4. *Let W_ε be the solution to (4.15). Then there exist two positive constants $d_1,\ d_2$ such that*

$$d_1\varepsilon \leqslant \int_\Omega W_\varepsilon{}^p dx \leqslant d_2\varepsilon.$$

Moreover, for any $\delta > 0$ as in Lemma 4.3, we obtain a precise behavior at the boundary given by

$$b_5\left(1 + b_6\frac{\text{dist}(x,\partial\Omega)}{\varepsilon}\right)^{-\frac{2}{p}} \leqslant W_\varepsilon \leqslant b_7\left(1 + b_8\frac{\text{dist}(x,\partial\Omega)}{\varepsilon}\right)^{-\frac{2}{p}} \quad in \quad \Omega_\delta, \tag{4.16}$$

for some positive constants b_5, b_6, b_7, b_8 independent of ε.

Proof. By Lemma 4.3, we can find four positive constants b_5, b_6, b_7, b_8 which are independent of ε, such that

$$b_5\left(1 + b_6\frac{\text{dist}(x,\partial\Omega)}{\varepsilon^{\frac{1}{2}}\lambda_\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}} \leqslant W_\varepsilon \leqslant b_7\left(1 + b_8\frac{\text{dist}(x,\partial\Omega)}{\varepsilon^{\frac{1}{2}}\lambda_\varepsilon^{\frac{1}{2}}}\right)^{-\frac{2}{p}} \quad \text{in} \quad \Omega_\delta, \tag{4.17}$$

while in $\overline{\Omega_\delta^c}$, by the Equation (4.2) we can find a positive constant $C_{\overline{\Omega_\delta^c}}$ such that

$$\max_{\overline{\Omega_\delta^c}} W_\varepsilon(x) \leqslant C_{\overline{\Omega_\delta^c}}\varepsilon^{\frac{1}{p}}\lambda_\varepsilon^{\frac{1}{p}}. \tag{4.18}$$

By (4.17)–(4.18), we can get a lower and upper bound for the term $\int_\Omega W_\varepsilon{}^p dx$, that is,

$$\begin{aligned} m\lambda_\varepsilon = \int_\Omega W_\varepsilon{}^p dx &\geqslant \int_{\Omega_\delta} b_5^p\left(1 + b_6\frac{\text{dist}(x,\partial\Omega)}{\varepsilon^{\frac{1}{2}}\lambda_\varepsilon^{\frac{1}{2}}}\right)^{-2} dx \\ &\geqslant \int_0^\delta \int_{\Gamma_z(y)} b_5^p\left(1 + b_6\frac{z}{\varepsilon^{\frac{1}{2}}\lambda_\varepsilon^{\frac{1}{2}}}\right)^{-2} dydz \\ &\geqslant b_5^p\varepsilon^{\frac{1}{2}}\lambda_\varepsilon^{\frac{1}{2}} \min_{z\in(0,\ \delta)} |\Gamma_z(y)| + O(\varepsilon\lambda_\varepsilon), \end{aligned} \tag{4.19}$$

and

$$m\lambda_\varepsilon = \int_\Omega {W_\varepsilon}^p dx \leqslant \int_{\Omega_\delta} b_7^p \left(1 + b_8 \frac{\mathrm{dist}(x, \partial\Omega)}{\varepsilon^{\frac{1}{2}} \lambda_\varepsilon^{\frac{1}{2}}}\right)^{-2} dx + \int_{\Omega_\delta^c} C^p_{\Omega_\delta^c} \varepsilon\lambda_\varepsilon$$

$$\leqslant \int_0^\delta \int_{\Gamma_z(y)} b_7^p \left(1 + b_8 \frac{z}{\varepsilon^{\frac{1}{2}} \lambda_\varepsilon^{\frac{1}{2}}}\right)^{-2} dydz + O(\varepsilon\lambda_\varepsilon) \tag{4.20}$$

$$\leqslant b_7^p \varepsilon^{\frac{1}{2}} \lambda_\varepsilon^{\frac{1}{2}} \max_{z\in(0,\,\delta)} |\Gamma_z(y)| + O(\varepsilon\lambda_\varepsilon).$$

As a consequence of (4.19)–(4.20), we derive that $\varepsilon \sim \lambda_\varepsilon$ since $m\lambda_\varepsilon \leqslant b^p|\Omega|$ by the maximum principle comparing to the constant supersolution $b > 0$. Then the conclusion follows directly. □

With the above preparation, we give the proof of Theorem 2.2.

Proof of Theorem 2.2.. By (4.16) in Lemma 4.4, we get the profile for W_ε given in (2.2) for any small constant $\delta > 0$. From (1.5), we know that

$$U_\varepsilon = m \frac{W_\varepsilon^p}{\int_\Omega W_\varepsilon^p dx}. \tag{4.21}$$

Using Lemma 4.4 we obtain the profile for U_ε given in (2.2). Applying Lemma 4.1 to (4.15), we immediately observe that

$$\|W_\varepsilon\|_{L^\infty(\Omega_\delta^c)} \leqslant C\varepsilon^{\frac{2}{p}}.$$

Then using (4.21) and Lemma 4.4 again, we derive that

$$U_\varepsilon \leqslant C\varepsilon \quad \text{for} \quad x \in \Omega_\delta^c.$$

Now it remains to show that the boundary-layer thickness is the order $O(\varepsilon)$ to finish the proof. Let us denote $\ell_\varepsilon = \mathrm{dist}(x_{in}, \partial\Omega)$ for any interior point x_{in} of Ω. We just need to check the conditions of Definition 2.1 with $\mu(\varepsilon) \sim O(\varepsilon)$.

Case 1: If $\lim_{\varepsilon\to 0} \frac{\ell_\varepsilon}{\varepsilon} = 0$, we set $w^\varepsilon(y) = W_\varepsilon(\varepsilon y)$. Then w^ε satisfies

$$\Delta_y w^\varepsilon(y) = \frac{m\varepsilon}{\int_\Omega W_\varepsilon^p(x)dx} (w^\varepsilon(y))^{p+1}, \tag{4.22}$$

in $\Omega^\varepsilon = \frac{1}{\varepsilon}\Omega$. Recall that, by maximum principle, we have

$$\|w^\varepsilon(y)\|_{L^\infty(\Omega^\varepsilon)} = \|W_\varepsilon(x)\|_{L^\infty(\Omega)} \leqslant b.$$

Following the standard elliptic estimate and the fact that the right-hand side of (4.22) is uniformly bounded in Ω by Lemma 4.4, we get

$$|w^\varepsilon(y)|_{L^\infty(\Omega^\varepsilon)} + |D_y w^\varepsilon(y)|_{L^\infty(\Omega^\varepsilon)} \leqslant C, \tag{4.23}$$

where $C > 0$ is a universal constant independent of ε. This implies that

$$|D_x W_\varepsilon(x)| \leqslant C\varepsilon^{-1}.$$

Let $x_0 \in \partial\Omega$ be the boundary point such that $|x_0 - x_{\text{in}}| = \text{dist}(x_{\text{in}}, \partial\Omega)$. We get that $|x_0 - x_{\text{in}}| = o(\varepsilon)$ from $\lim_{\varepsilon\to 0} \frac{\ell_\varepsilon}{\varepsilon} = 0$, then

$$|W_\varepsilon(x_0) - W_\varepsilon(x_{\text{in}})| \leqslant C|D_x W_\varepsilon||x_0 - x_{\text{in}}| \leqslant C\varepsilon^{-1}|x_0 - x_{\text{in}}| = o_\varepsilon(1). \tag{4.24}$$

This implies that $\lim_{\varepsilon\to 0} W_\varepsilon(x_{\text{in}}) = b$, verifying the statement in Definition 2.1-(1).

Case 2: $\lim_{\varepsilon\to 0} \frac{\ell_\varepsilon}{\varepsilon} = L$. In this case, we first show that $\lim\limits_{\varepsilon\to 0} W_\varepsilon(x_{\text{in}}) > 0$. Indeed, by Lemma 4.4 and $\lim_{\varepsilon\to 0} \ell_\varepsilon/\varepsilon = L$, we have

$$\lim_{\varepsilon\to 0} W_\varepsilon(x_{\text{in}}) \geqslant b_5\left(1 + b_6\frac{m}{d_1}L\right)^{-\frac{2}{p}} > 0.$$

To show $\lim\limits_{\varepsilon\to 0} W_\varepsilon(x_{\text{in}}) < b$, we claim that

$$W_\varepsilon(x) \leqslant b\left(1 + C_0\frac{\text{dist}(x, \partial\Omega)}{\varepsilon}\right)^{-\frac{2}{p}} \quad \text{in} \quad \Omega_\delta \tag{4.25}$$

for some suitable positive constant C_0. Let d_2 be defined in Lemma 4.4 and $W_{\varepsilon,b}$ be the solution of the following equation:

$$\begin{cases} \varepsilon^2 \Delta W_{\varepsilon,b} = \frac{m}{d_2} W_{\varepsilon,b}^{1+p} & \text{in} \quad \Omega, \\ W_{\varepsilon,b} = b & \text{on} \quad \partial\Omega. \end{cases} \tag{4.26}$$

By the maximum principle, we get that $W_\varepsilon \leqslant W_{\varepsilon,b}$. Now we shall prove that

$$W_{\varepsilon,b} \leqslant W_\delta := b\left(1 + b_9\frac{\text{dist}(x, \partial\Omega)}{\varepsilon}\right)^{-\frac{2}{p}} \quad \text{in} \quad \Omega_\delta$$

for some suitable positive constant b_9. Indeed first we can always choose b_9 small enough such that

$$W_{\varepsilon,b} \leqslant b\left(1 + b_9\frac{\text{dist}(x, \partial\Omega)}{\varepsilon}\right)^{-\frac{2}{p}} \quad \text{on} \quad \partial\Omega_\delta \setminus \partial\Omega. \tag{4.27}$$

To see this, we denote $\Omega_* = \Omega \setminus \Omega_\delta$. Then by Lemma 4.1 applied to (4.26), we get

$$\max_{\overline{\Omega_*}} W_{\varepsilon,b} \leqslant C_{\overline{\Omega_*}}\varepsilon^{\frac{2}{p}}. \tag{4.28}$$

Then for each $x \in \partial\Omega_\delta \setminus \partial\Omega$, it has $\delta = \text{dist}(x, \partial\Omega)$. By choosing b_9 sufficiently small, we will have

$$C_{\overline{\Omega_*}}\varepsilon^{\frac{2}{p}} \leqslant b\left(1 + b_9\frac{\text{dist}(x, \partial\Omega)}{\varepsilon}\right)^{-\frac{2}{p}}. \tag{4.29}$$

Then (4.27) follows from (4.28) and (4.29). Since $W_{\varepsilon,b} = W_\delta = b$ at $\partial\Omega$, we conclude that

$$W_{\varepsilon,b} \leqslant W_\delta := b\left(1 + b_9 \frac{\text{dist}(x, \partial\Omega)}{\varepsilon}\right)^{-\frac{2}{p}} \text{on} \quad \partial\Omega_\delta.$$

By a direct computation, we have

$$\begin{aligned} \varepsilon^2 \Delta W_\delta - \frac{m}{d_2} W_\delta^{1+p} = &\left(c_p^2 b_9^2 b + \frac{2}{p}\varepsilon b_9 H_{\Gamma_z(y)}\left(1 + b_9 \frac{\text{dist}(x,\partial\Omega)}{\varepsilon}\right) - \frac{m}{d_2} b^{1+p}\right) \\ &\times \left(1 + b_9 \frac{\text{dist}(x,\partial\Omega)}{\varepsilon}\right)^{-2-\frac{2}{p}} \quad \text{in} \quad \Omega_\delta. \end{aligned} \tag{4.30}$$

Since b_9 is chosen to be sufficiently small, the first bracket in the right-hand side of (4.30) can be made negative. By the comparison principle, we infer that

$$W_\varepsilon \leqslant W_{\varepsilon,b} \leqslant W_\delta \quad \text{in } \Omega_\delta.$$

Hence the claim (4.25) is proved by identifying C_0 with b_9 and we get that

$$\lim_{\varepsilon\to 0} W_\varepsilon(x_{\text{in}}) \leqslant b(1 + C_0 L)^{-\frac{2}{p}} < b.$$

Thus, Definition 2.1(2) is verified.

Case 3: $\lim_{\varepsilon\to 0} \frac{\ell_\varepsilon}{\varepsilon} = +\infty$. The conclusion in Definition 2.1(3) is a direct consequence of Lemma 4.1 in Ω_δ and Lemma 4.3 in Ω_δ^c.

Collecting the above three cases, we complete the proof. □

5 | THE RADIAL CASE

In this section, we consider the special case $\Omega = B_R(0) := B_R$ in $\mathbb{R}^n (n \geqslant 1)$ to find the refined solution structure near the boundary as $\varepsilon \to 0$. With this, we can explore how the radius of the domain (and hence the boundary curvature) affects the boundary-layer profile and thickness, and further show the asymptotic profile of the radial steady state as $p \to \infty$ (namely the chemotactic sensitivity is very strong).

5.1 | Asymptotic profile near the boundary

We first establish the following result.

Lemma 5.1. *Given $b > 0$, the system (1.3) has a unique smooth positive solution (U, W) that is radially symmetric in $B_R(0)$. Moreover, (U, W) satisfies $U_r > 0$ and $W_r > 0$ with $r := |x|$.*

Proof. The existence and uniqueness of smooth solutions come from Theorem 2.1 directly. The fact that the unique solution is radially symmetric is a consequence of Gidas–Ni–Nirenberg theorem

[11] applied to the following problem:

$$\begin{cases} \varepsilon\lambda_\varepsilon \Delta W = W^{p+1}, & x \in B_R(0), \\ W = b > 0, & x \in \partial B_R(0), \end{cases} \tag{5.1}$$

where $\lambda_\varepsilon = \frac{1}{m}\int_{B_R} W^p dx > 0$ is a constant. Note that (5.1) is the analogue of (4.15) for $\Omega = B_R(0)$. By Theorem 4.4, we see that $\lambda_\varepsilon \sim \varepsilon$. Next we prove the monotonicity of $(U, W)(r)$. Indeed, the steady-state problem (1.3) in the ball $B_R(0)$ can be written as

$$\begin{cases} U_r = pU\frac{W_r}{W}, & r \in (0, R), \\ \varepsilon W_{rr} + \frac{\varepsilon(n-1)}{r} W_r = UW, & r \in (0, R), \\ U_r(0) = W_r(0) = 0,\ W(R) = b, \\ \omega_n \int_0^R r^{n-1} U(r) dr = m. \end{cases} \tag{5.2}$$

Write the second equation of (5.2) as

$$\varepsilon(r^{n-1}W_r)_r = r^{n-1}UW. \tag{5.3}$$

Noting $W_r(0) = 0$, we integrate (5.3) over $(0, r)$ and get

$$W_r(r) > 0,\ \forall r \in (0, R]. \tag{5.4}$$

Using the first equation of (5.2), we further get $U_r(r) > 0$ for any $r \in (0, R]$. Hence $U(r)$ is monotonically increasing on $[0, R]$. □

In this section, we shall study the boundary expansion for the problem (1.3) in the ball. To start with our discussion, we shall first analyze (5.1). In the sequel, we set $\sigma = \varepsilon\lambda_\varepsilon$ for simplicity, we can rewrite (5.1) as

$$\begin{cases} \sigma\left(W'' + \frac{n-1}{r}W'\right) - W^{1+p} = 0, \quad \text{for} \quad r \in (0, R], \\ W(R) = b, \quad W'(0) = 0. \end{cases} \tag{5.5}$$

We remind the reader that $\sigma \sim \varepsilon^2$ by Theorem 4.4. Our aim is to derive sharper upper and lower bounds of the solution to (5.5) than in the previous section.

Lemma 5.2. *Let W_ε be a solution of (5.5). Then we have $W_{\varepsilon,l}(r) \leqslant W_\varepsilon(r) \leqslant W_{\varepsilon,u}(r)$ for all $r \in [0, R]$ with*

$$W_{\varepsilon,l}(r) := b\left(1 + \frac{b^{\frac{p}{2}}(R-r)}{c_p \sigma^{\frac{1}{2}}}\right)^{-\frac{2}{p}},\ W_{\varepsilon,u}(r) := \begin{cases} b\left(\frac{R}{r}\right)^{\frac{n-1}{2}}\left(1 + \frac{b^{\frac{p}{2}}(R-r)}{c_p\sigma^{\frac{1}{2}}}\right)^{-\frac{2}{p}}, & \text{if } n \geqslant 3, \\ b\left(\frac{R}{r}\right)^{a_p}\left(1 + \frac{b^{\frac{p}{2}}(R-r)}{c_{p,1}\sigma^{\frac{1}{2}}}\right)^{-\frac{2}{p}}, & \text{if } n = 2, \\ b\left(1 + \frac{b^{\frac{p}{2}}(R-r)}{c_p\sigma^{\frac{1}{2}}}\right)^{-\frac{2}{p}} + \frac{c_p^{\frac{2}{p}}\sigma^{\frac{1}{p}}}{R^{\frac{2}{p}}}, & \text{if } n = 1, \end{cases} \tag{5.6}$$

where $c_{p,1} = c_p(1 - \frac{a_p^2\sigma}{b^p R^2})^{-1/2}$ *and* $a_p = \max\{\frac{1}{2}, \frac{2}{p}\}$.

Proof. We denote the left-hand side and right-hand side functions of (5.6) by $W_{\varepsilon,1}$ and $W_{\varepsilon,2}$. First we show that $W_{\varepsilon,1} \leqslant W_\varepsilon$. By direct computation we have

$$\sigma\left(W''_{\varepsilon,1} + \frac{n-1}{r}W'_{\varepsilon,1}\right) - W^{1+p}_{\varepsilon,1} = \frac{2(n-1)\sigma^{\frac{1}{2}}}{pc_p r} b^{1+\frac{p}{2}}\left(1 + \frac{b^{\frac{p}{2}}(R-r)}{c_p\sigma^{\frac{1}{2}}}\right)^{-\frac{2+p}{p}} \geqslant 0. \tag{5.7}$$

Now we claim $W_{\varepsilon,1}$ is a sub-solution of (5.5). Indeed, $W_{\varepsilon,1}(R) = b = W_\varepsilon(R)$ and $W_{\varepsilon,1} - W_\varepsilon$ cannot possess an interior local positive maximal value due to maximum principle and (5.7). At the zero point, we have

$$W'_{\varepsilon,1}(0) - W'_\varepsilon(0) = W'_{\varepsilon,1}(0) > 0$$

which entails that 0 cannot be a local maximal point. Thus, we obtain the left-hand side inequality of (5.6).

While for the upper bound, we shall divide our discussion into three cases.

Case 1: $n \geqslant 3$, by a direct computation we have

$$\begin{aligned}
&\sigma\left(W''_{\varepsilon,2} + \frac{n-1}{r}W'_{\varepsilon,2}\right) - W^{1+p}_{\varepsilon,2} \\
&= \frac{\sigma(n-1)(3-n)}{4} bR^{\frac{n-1}{2}} r^{-\frac{n+3}{2}}\left(1 + \frac{b^{\frac{p}{2}}(R-r)}{c_p\sigma^{\frac{1}{2}}}\right)^{-\frac{2}{p}} \\
&\quad + b^{1+p}\left(1 - \left(\frac{R}{r}\right)^{\frac{(n-1)p}{2}}\right)\left(\frac{R}{r}\right)^{\frac{(n-1)}{2}}\left(1 + \frac{b^{\frac{p}{2}}(R-r)}{c_p\sigma^{\frac{1}{2}}}\right)^{-\frac{2+2p}{p}} \\
&\leqslant 0.
\end{aligned}$$

Together with that $W_{\varepsilon,2}(R) = W_\varepsilon(R) = b$ and $W_{\varepsilon,2}(r) \to +\infty$ as $r \to 0$, we see that $W_{\varepsilon,2}$ provides a super-solution.

Case 2: $n = 2$, following the computations as we did in (4.7) and using $a_p = \max\{\frac{1}{2}, \frac{2}{p}\}$ we have

$$\begin{aligned}
&\sigma\left(W''_{\varepsilon,2} + \frac{n-1}{r}W'_{\varepsilon,2}\right) - W^{1+p}_{\varepsilon,2} \\
&\leqslant b\left(\frac{\sigma a_p^2}{r^2}\left(1 + \frac{b^{\frac{p}{2}}(R-r)}{c_{p,1}\sigma^{\frac{1}{2}}}\right)^2 + \frac{c_p^2 b^p}{c_{p,1}^2} - b^p\left(\frac{R}{r}\right)^{pa_p}\right)\left(\frac{R}{r}\right)^{a_p}\left(1 + \frac{b^{\frac{p}{2}}(R-r)}{c_{p,1}\sigma^{\frac{1}{2}}}\right)^{-\frac{2+2p}{p}}.
\end{aligned}$$

Denoting

$$F_\varepsilon(r,R) := \frac{\sigma a_p^2}{r^2}\left(1 + \frac{b^{\frac{p}{2}}(R-r)}{c_{p,1}\sigma^{\frac{1}{2}}}\right)^2 + \frac{c_p^2 b^p}{c_{p,1}^2} - b^p\left(\frac{R}{r}\right)^{pa_p}.$$

Using the definition of $c_{p,1}$ we can rewrite $F_\varepsilon(r,R)$ as

$$F_\varepsilon(r,R) = b^p\left(\frac{a_p^2(R-r)^2}{c_{p,1}^2 r^2} + 1 - \left(\frac{R}{r}\right)^{pa_p}\right) + \frac{\sigma a_p^2}{r^2} - \frac{\sigma a_p^2}{R^2} + \frac{2\sigma^{\frac{1}{2}} a_p^2 b^{\frac{p}{2}}(R-r)}{c_{p,1} r^2}.$$

We claim that $F_\varepsilon(r,R)$ is strictly negative for sufficiently small ε (ε small implies that σ small). Using the fact $pa_p \geqslant 2$ and the well-known inequality

$$(1+x)^p \geqslant 1 + px + \frac{1}{2}p(p-1)x^2 \ \text{ for } \ x \geqslant 0, \quad \text{if} \quad p > 2.$$

Then

$$\begin{aligned} F_\varepsilon(r,R) \leqslant\ & b^p\left(\frac{a_p^2(R-r)^2}{c_{p,1}^2 r^2} + 1 - \frac{r^2 + pa_p(R-r)r + \frac{1}{2}pa_p(pa_p-1)(R-r)^2}{r^2}\right) \\ & + \frac{\sigma a_p^2}{r^2} - \frac{\sigma a_p^2}{R^2} + \frac{2\sigma^{\frac{1}{2}} a_p^2 b^{\frac{p}{2}}(R-r)}{c_{p,1} r^2} \\ =\ & \frac{b^p(R-r)^2}{r^2}\left(\frac{a_p^2}{c_{p,1}^2} - \frac{1}{2}pa_p(pa_p-1)\right) - pa_p b^p \frac{R-r}{r} + \frac{2\sigma^{\frac{1}{2}} a_p^2 b^{\frac{p}{2}}(R-r)}{c_{p,1} r^2} \\ & + \frac{\sigma a_p^2}{r^2} - \frac{\sigma a_p^2}{R^2}. \end{aligned}$$

For the coefficient $\frac{a_p^2}{c_{p,1}^2} - pa_p(pa_p-1)$, where $c_{p,1} = \sqrt{\frac{c_p^2}{1 - \frac{a_p^2 \varepsilon}{b^p R^2}}}$ and $c_p = \sqrt{\frac{2}{p}\left(\frac{2}{p}+1\right)}$, one can easily check that

$$\begin{aligned} \frac{a_p^2}{c_{p,1}^2} - \frac{1}{2}pa_p(pa_p-1) &= \frac{a_p^2}{c_p^2} - \frac{1}{c_p^2}\frac{a_p^4\sigma}{b^p R^2 \lambda_\varepsilon^2} - \frac{1}{2}pa_p(pa_p-1) \\ &< \frac{a_p^2}{c_p^2} - \frac{1}{2}pa_p(pa_p-1) = a_p\left(\frac{1}{2}p - \frac{p^2+p^3}{4+2p}a_p\right) \\ &< a_p\left(\frac{1}{2}p - \frac{p^2+p^3}{4+2p}\frac{2}{p}\right) = -\frac{p^2}{4+2p}a_p < -\frac{p}{2+p}. \end{aligned}$$

As a consequence, we have

$$F_\varepsilon(r,R) \leqslant -\frac{p}{2+p}\frac{b^p(R-r)^2}{r^2} - pa_p b^p \frac{R-r}{r} + \frac{2\sigma^{\frac{1}{2}} a_p^2 b^{\frac{p}{2}}(R-r)}{c_{p,1} r^2} + \frac{\sigma a_p^2}{r^2} - \frac{\sigma a_p^2}{R^2}.$$

Then we choose σ_0 as

$$\sigma_0 = \left(\min\left\{ \frac{pa_p b^p}{\frac{8a_p}{R^2} + \frac{8a_p^2 b^{\frac{p}{2}}}{c_{p,1}R}}, \frac{p}{2+p}\frac{b^p R^2 c_{p,1}}{4a_p^2 c_{p,1} + 8a_p^2 b^{\frac{p}{2}} R}, 1 \right\} \right)^2.$$

One can easily check that if $\sigma \in (0, \sigma_0)$, then $F_\varepsilon(r, R) \leqslant 0$ for $r \in (0, R]$. Thus, we see that $W_{\varepsilon,2}$ provides a super-solution of (5.5).

Case 3: $n = 1$. In this case, we consider the original problem for $x \in [-R, R]$ and set

$$W_{\varepsilon,b} := b\left(1 + \frac{b^{\frac{p}{2}}(R-x)}{c_p \sigma^{\frac{1}{2}}}\right)^{-\frac{2}{p}} + b\left(1 + \frac{b^{\frac{p}{2}}(R+x)}{c_p \sigma^{\frac{1}{2}}}\right)^{-\frac{2}{p}}.$$

As we have shown that in last section this provides a super-solution. Together with the trivial fact that $W_{\varepsilon,b} \leqslant W_{\varepsilon,2}$ for $x \in (0, R)$, we finish the proof for the right-hand side of (5.6) in this case. □

In the following lemma, we shall derive a useful expansion of the normal derivative for the solution W_ε of (5.5).

Lemma 5.3. *Let W_ε be a solution of (5.5). Then on the boundary $\partial B_R(0)$ we have for $0 < \sigma \ll 1$*

$$W_\varepsilon'(R) = \sqrt{\frac{2}{p+2}} b^{1+\frac{p}{2}} \sigma^{-\frac{1}{2}} - \frac{2(n-1)b}{(p+4)R} + o_\sigma(1).$$

Proof. First multiplying both sides of (5.5) by W_ε', and integrating the result from 0 to r, we get

$$\frac{1}{2}W_\varepsilon'(r)^2 = \frac{1}{(p+2)\sigma}\left(W_\varepsilon^{p+2}(r) - W_\varepsilon^{p+2}(0)\right) - \int_0^r \frac{n-1}{s} W_\varepsilon'(s)^2 ds. \tag{5.8}$$

Using (5.5), we have

$$\sigma r^{n-1} W_\varepsilon'(r) = \int_0^r s^{n-1} W_\varepsilon^{p+1}(s) ds. \tag{5.9}$$

So by the increasing property of W_ε, one has

$$\sigma \frac{W_\varepsilon'(r)}{r} = \frac{1}{r^n}\int_0^r s^{n-1} W_\varepsilon^{p+1}(s) ds \leqslant \frac{1}{n} W_\varepsilon^{p+1}(r). \tag{5.10}$$

Then for $r \in \left(\frac{R}{2}, R\right)$, it follows that

$$\begin{aligned}\sigma \int_0^r \frac{n-1}{s} W_\varepsilon'(s)^2 ds &= \sigma \int_0^{\frac{R}{4}} \frac{n-1}{s} W_\varepsilon'(s)^2 ds + \sigma \int_{\frac{R}{4}}^r \frac{n-1}{s} W_\varepsilon'(s)^2 ds \\ &\leqslant \frac{1}{p+2} \frac{n-1}{n} W_\varepsilon^{p+2}\left(\frac{R}{4}\right) + \frac{4(n-1)}{R} \int_{\frac{R}{4}}^r \left(\int_0^s W_\varepsilon^{p+1}(\tau) d\tau\right) W_\varepsilon'(s) ds,\end{aligned} \tag{5.11}$$

where (5.10) and (5.9) have been used in the first and second terms, respectively. For the second term on the right-hand side of (5.11), using Lemma 4.1 and Lemma 5.2, we have

$$\int_0^s W_\varepsilon^{p+1}(\tau) d\tau \leqslant \begin{cases} C\sigma^{1+\frac{1}{p}}, & \text{for } s \in \left(0, \frac{R}{4}\right), \\ C\sigma^{\frac{1}{2}}\left(1 + \frac{R-s}{C\sigma^{\frac{1}{2}}}\right)^{-1-\frac{2}{p}}, & \text{for } s \in \left(\frac{R}{4}, R\right), \end{cases} \tag{5.12}$$

where C is a generic positive constant. Particularly, there holds that

$$\int_0^R W_\varepsilon^{p+1} dr \leqslant C\sigma^{\frac{1}{2}}. \tag{5.13}$$

As a consequence, we have from (5.11) and (4.2) that

$$\sigma \int_0^R \frac{n-1}{r} W_\varepsilon'(r)^2 dr \leqslant C\sigma^{1+\frac{2}{p}} + C\sigma^{\frac{1}{2}} \int_{\frac{R}{4}}^R W_\varepsilon'(r) dr \leqslant C\sigma^{\frac{1}{2}}. \tag{5.14}$$

Returning to Equation (5.8), we can see that $\frac{1}{\sigma} W_\varepsilon^{p+2}(r)$ is the predominant term on the right-hand side for $R - r \leqslant C\sigma^{\frac{p+4}{4(p+2)}+\gamma}$ for some sufficiently small positive number γ. Indeed, with Lemma 5.2, we have

$$\frac{1}{\sigma} W_\varepsilon^{p+2} \geqslant \frac{1}{\sigma} b^{p+2}\left(1 + \frac{b^{\frac{p}{2}}}{c_p} \sigma^{-\frac{p}{4(p+2)}+\gamma}\right)^{-\frac{2(p+2)}{p}} \gg C\sigma^{-\frac{1}{2}}.$$

Hence for $r \in \left(R - C\sigma^{\frac{p+4}{4(p+2)}+\gamma}, R\right)$, we can apply the Taylor's expansion to rewrite (5.8) as

$$W_\varepsilon'(r) = \sigma^{-\frac{1}{2}} \sqrt{\frac{2}{p+2}} W_\varepsilon^{\frac{p+2}{2}}(r)\left(1 + O\left(\frac{W_\varepsilon^{p+2}(0)}{W_\varepsilon^{p+2}(r)}\right) + O\left(\frac{\sigma \int_0^r \frac{n-1}{s} W_\varepsilon'(s)^2 ds}{W_\varepsilon^{p+2}(r)}\right)\right),$$

where the leading coefficient in the second and third terms are negative. With this, we resort (5.14) to derive that

$$
\begin{aligned}
\sigma\int_0^R \frac{n-1}{r}W_\varepsilon'(r)^2dr &\geqslant \sigma\int_{R-C\sigma^{\frac{p+4}{4(p+2)}+\gamma}}^R \frac{n-1}{r}W_\varepsilon'(r)^2dr\\
&\geqslant \frac{(n-1)\sigma^{\frac12}}{R}\sqrt{\frac{2}{p+2}}\int_{R-C\sigma^{\frac{p+4}{4(p+2)}+\gamma}}^R W_\varepsilon^{\frac{p+2}{2}}(r)W_\varepsilon'(r)dr\\
&\quad + O(\sigma^{\frac12})\int_{R-C\sigma^{\frac{p+4}{4(p+2)}+\gamma}}^R \frac{W_\varepsilon^{p+2}(0)}{W_\varepsilon^{\frac{p+2}{2}}(r)}W_\varepsilon'(r)dr\\
&\quad + O(\sigma^{\frac12})\int_{R-C\sigma^{\frac{p+4}{4(p+2)}+\gamma}}^R \frac{\sigma\int_0^r\frac{n-1}{s}W_\varepsilon'(s)^2ds}{W_\varepsilon^{\frac{p+2}{2}}(r)}W_\varepsilon'(r)dr\\
&= \frac{(n-1)\sigma^{\frac12}}{R}\sqrt{\frac{2}{p+2}}\frac{2}{p+4}b^{2+\frac{p}{2}} + O\left(\sigma^{\frac12+\frac{p+4}{4p+8}-\frac{p+4}{p}\gamma}\right),
\end{aligned} \tag{5.15}
$$

where we have used that $W_\varepsilon(0)\leqslant C\sigma^{\frac{1}{p}}$ by applying Lemma 4.1 to (5.1) with ε replaced by $\sigma=\varepsilon\lambda_\varepsilon$ there, and $\sigma\int_0^r\frac{n-1}{s}W_\varepsilon'(s)^2ds\leqslant C\sigma^{\frac12}$ from (5.14). While on the other hand, using (5.8), we have

$$
\begin{aligned}
\sigma\int_0^R \frac{n-1}{r}W_\varepsilon'(r)^2dr &= \sigma\int_{R-C\sigma^{\frac{p+4}{4(p+2)}+\gamma}}^R \frac{n-1}{r}W_\varepsilon'(r)^2dr\\
&\quad + \sigma\int_0^{R-C\sigma^{\frac{p+4}{4(p+2)}+\gamma}} \frac{n-1}{r}W_\varepsilon'(r)^2dr\\
&\leqslant \frac{(n-1)\sigma^{\frac12}}{R-C\sigma^{\frac{p+4}{4(p+2)}+\gamma}}\sqrt{\frac{2}{p+2}}\frac{2}{p+4}b^{2+\frac{p}{2}}\\
&\quad + \sigma\int_0^{R-C\sigma^{\frac{p+4}{4(p+2)}+\gamma}} \frac{n-1}{r}W_\varepsilon'(r)^2dr.
\end{aligned} \tag{5.16}
$$

For the second term on the right-hand side of (5.16), using (5.11)–(5.12), we see that

$$
\begin{aligned}
\sigma\int_0^{R-C\sigma^{\frac{p+4}{4(p+2)}+\gamma}} \frac{n-1}{r}W_\varepsilon'(r)^2dr &\leqslant \frac{4C(n-1)\sigma^{\frac12}}{R}\int_{\frac{R}{4}}^{R-C\sigma^{\frac{p+4}{4(p+2)}+\gamma}} \frac{W_\varepsilon'(r)}{\left(1+\frac{R-r}{C\sigma^{\frac12}}\right)^{1+\frac{2}{p}}}dr\\
&\quad + \int_0^{\frac{R}{4}}\frac{n-1}{n}W^{p+1}(r)W'(r)dr\\
&\leqslant C\sigma^{\frac34-\frac{p+2}{p}\gamma}W\left(R-C\sigma^{\frac{p+4}{4(p+2)}+\gamma}\right)\\
&\quad + \frac{1}{p+2}\frac{n-1}{n}W_\varepsilon^{p+2}(R/4)\\
&\leqslant C\sigma^{\frac12+\frac{p+4}{4p+8}-\frac{p+4}{p}\gamma},
\end{aligned} \tag{5.17}
$$

where the lower bound of $W_\varepsilon(r)$ given in Lemma 5.2 has been used. From (5.16) and (5.17) we get

$$\sigma\int_0^R \frac{n-1}{r}W_\varepsilon'(r)^2dr \leqslant \frac{(n-1)\sigma^{\frac12}}{R}\sqrt{\frac{2}{p+2}}\frac{2}{p+4}b^{2+\frac{p}{2}} + O\left(\sigma^{\frac12+\frac{p+4}{4p+8}-\frac{p+4}{p}\gamma}\right).$$

Combined with (5.15), we finally arrive at

$$\sigma\int_0^R \frac{n-1}{r}W_\varepsilon'(r)^2dr = \frac{(n-1)\sigma^{\frac12}}{R}\sqrt{\frac{2}{p+2}}\frac{2}{p+4}b^{2+\frac{p}{2}} + O\left(\sigma^{\frac12+\frac{p+4}{4p+8}-\frac{p+4}{p}\gamma}\right).$$

Evaluating (5.8) at R and using Lemma 4.1 yields

$$(W_\varepsilon'(R))^2 = \frac{1}{\sigma}\frac{2}{p+2}b^{p+2} - \sigma^{-\frac12}\frac{n-1}{R}\sqrt{\frac{2}{p+2}}\frac{4}{p+4}b^{2+\frac{p}{2}} + o_\sigma(1)\sigma^{-\frac12}. \tag{5.18}$$

This implies the desired conclusion. □

Next we shall use Lemma 5.2 and the conclusions obtained in the last section to derive a more accurate estimate on the integration $\int_{B_R(0)} W_\varepsilon{}^p dx$ of the original nonlocal problem.

Lemma 5.4. *Let W_ε be a solution of the following nonlocal problem:*

$$\begin{cases} \varepsilon\Delta W = \frac{m}{\int_\Omega W^p dx}W^{1+p} & in \quad B_R(0), \\ W = b & on \quad \partial B_R(0), \end{cases} \tag{5.19}$$

and $\lambda_\varepsilon = \frac{\int_\Omega W_\varepsilon^p dx}{m}$. By ω_n we denote the surface area of the unit sphere in $\mathbb{R}^n$, then we have

$$\lambda_\varepsilon = \frac{\omega_n^2 b^p c_p^2 R^{2n-2}}{m^2}\varepsilon + O(\varepsilon^2\log\varepsilon).$$

Proof. In the following, we shall consider the case $n \geqslant 2$, while the case $n = 1$ can be treated similarly with simpler calculations. Using Lemma 5.2 for $n \geqslant 2$ (with σ replaced by $\varepsilon\lambda_\varepsilon$) and inequality (4.2) we have

$$b\left(1+\frac{b^{\frac{p}{2}}(R-r)}{c_p\varepsilon^{\frac12}\lambda_\varepsilon^{\frac12}}\right)^{-\frac{2}{p}} \leqslant W_\varepsilon(r) \leqslant \begin{cases} b\left(\frac{R}{r}\right)^\theta\left(1+\frac{b^{\frac{p}{2}}(R-r)}{c_p\varepsilon^{\frac12}\lambda_\varepsilon^{\frac12}}\right)^{-\frac{2}{p}}, & \text{if } r \geqslant \frac{R}{2}, \\ C_K\varepsilon^{\frac1p}\lambda_\varepsilon^{\frac1p}, & \text{if } r \leqslant \frac{R}{2}, \end{cases}$$

where $\theta = \frac{n-1}{2}, \tilde{c}_p = c_p$ for $n \geqslant 3$ and $\theta = a_p, \tilde{c}_p = c_{p,1}$ for $n = 2$ with $c_{p,1} = c_p(1-\frac{a_p^2\varepsilon\lambda_\varepsilon}{b^pR^2})^{-1/2}$ and $a_p = \max\{\frac12, \frac2p\}$. As a consequence, we deduce

$$\int_{B_R(0)} W_\varepsilon^p dx \geqslant \omega_n \int_0^R r^{n-1} b^p \left(1 + \frac{b^{\frac{p}{2}}(R-r)}{\tilde{c}_p \varepsilon^{\frac{1}{2}} \lambda_\varepsilon^{\frac{1}{2}}}\right)^{-2} dr$$

$$= \omega_n b^{\frac{p}{2}} \tilde{c}_p \varepsilon^{\frac{1}{2}} \lambda_\varepsilon^{\frac{1}{2}} \int_0^{\frac{b^{\frac{p}{2}} R}{\tilde{c}_p \varepsilon^{\frac{1}{2}} \lambda_\varepsilon^{\frac{1}{2}}}} (1+r)^{-2} \left(R - \frac{\tilde{c}_p \varepsilon^{\frac{1}{2}} \lambda_\varepsilon^{\frac{1}{2}}}{b^{\frac{p}{2}}} r\right)^{(n-1)} dr \tag{5.20}$$

$$= \omega_n b^{\frac{p}{2}} \tilde{c}_p \varepsilon^{\frac{1}{2}} \lambda_\varepsilon^{\frac{1}{2}} R^{n-1} - (n-1)\omega_n \tilde{c}_p^2 \varepsilon \lambda_\varepsilon \int_0^{\frac{b^{\frac{p}{2}} R}{c_p \varepsilon^{\frac{1}{2}} \lambda_\varepsilon^{\frac{1}{2}}}} \frac{R^{n-2} r}{(1+r)^2} dr + O(\varepsilon\lambda_\varepsilon)$$

$$= \omega_n b^{\frac{p}{2}} c_p R^{n-1} \varepsilon^{\frac{1}{2}} \lambda_\varepsilon^{\frac{1}{2}} + \frac{1}{2}(n-1)\omega_n c_p^2 R^{n-2} \varepsilon \lambda_\varepsilon \log(\varepsilon\lambda_\varepsilon) + O(\varepsilon\lambda_\varepsilon)$$

and

$$\int_{B_R(0)} W_\varepsilon^p dx = \omega_n \int_{R/2}^R r^{n-1} W_\varepsilon^p dr + \omega_n \int_0^{R/2} r^{n-1} W_\varepsilon^p dr$$

$$\leqslant \omega_n \int_{R/2}^R r^{n-1} b^p \left(\frac{R}{r}\right)^{\theta p} \left(1 + \frac{b^{\frac{p}{2}}(R-r)}{\tilde{c}_p \varepsilon^{\frac{1}{2}} \lambda_\varepsilon^{\frac{1}{2}}}\right)^{-2} dr + C\varepsilon\lambda_\varepsilon \tag{5.21}$$

$$= \omega_n b^{\frac{p}{2}} \tilde{c}_p \varepsilon^{\frac{1}{2}} \lambda_\varepsilon^{\frac{1}{2}} \int_0^{\frac{b^{\frac{p}{2}} R}{2\tilde{c}_p \varepsilon^{\frac{1}{2}} \lambda_\varepsilon^{\frac{1}{2}}}} \frac{\left(R - \frac{\tilde{c}_p \varepsilon^{\frac{1}{2}} \lambda_\varepsilon^{\frac{1}{2}}}{b^{\frac{p}{2}}} r\right)^{n-1-\frac{p(n-1)}{2}} R^{\theta p}}{(1+r)^2} dr + O(\varepsilon\lambda_\varepsilon)$$

$$= \omega_n b^{\frac{p}{2}} c_p \varepsilon^{\frac{1}{2}} \lambda_\varepsilon^{\frac{1}{2}} R^{n-1} + \frac{1}{2}(n-1-\theta p)\omega_n c_p^{2\, n-2} \varepsilon \lambda_\varepsilon \log(\varepsilon\lambda_\varepsilon) + O(\varepsilon\lambda_\varepsilon).$$

Therefore, we conclude that

$$\int_{B_R(0)} W_\varepsilon^p dx = \omega_n b^{\frac{p}{2}} c_p R^{n-1} \varepsilon^{\frac{1}{2}} \lambda_\varepsilon^{\frac{1}{2}} + O(\varepsilon\lambda_\varepsilon \log(\varepsilon\lambda_\varepsilon)).$$

This together with $\lambda_\varepsilon \sim \varepsilon$ implies that

$$m\lambda_\varepsilon^{\frac{1}{2}} = \omega_n b^{\frac{p}{2}} c_p R^{n-1} \varepsilon^{\frac{1}{2}} + O(\varepsilon^{\frac{3}{2}} \log \varepsilon).$$

As a consequence, we get

$$\int_{B_R(0)} W_\varepsilon^p dx = \frac{\omega_n^2 b^p c_p^2 R^{2n-2}}{m} \varepsilon + O(\varepsilon^2 \log \varepsilon) \tag{5.22}$$

and the desired conclusion of Lemma 5.4. □

5.2 | Asymptotic profile as $p \to \infty$

We can characterize the profile of the steady-state (U, W) as $p \to \infty$.

Lemma 5.5. *Let $(U_p, W_p)(r)$ be the unique radial solution of (1.3) in $B_R(0)$. Then as $p \to \infty$, U_p concentrates on the boundary $\partial B_R(0)$ and W_p converges to the boundary value b. That is as $p \to \infty$, it holds that*

$$\omega_n r^{n-1} U_p(r) \to m\delta(r-R) \text{ in the sense of distribution,} \tag{5.23}$$

$$W_p(r) \to b \text{ in } C(\overline{B_R}), \tag{5.24}$$

where $\delta(r-R)$ is the Dirac mass centered at $r = R$.

Proof. The proof consists of three steps.

Step 1. Since $\omega_n \int_0^R r^{n-1} U_p(r) dr = m$ and $U_p(r)$ is monotonically increasing on $[0, R]$, by a contradiction argument, one can see that for any $0 < \eta < R$, there exists a constant $M_\eta > 0$ such that

$$U_p(r) < M_\eta \text{ for any } r \in [0, R-\eta]. \tag{5.25}$$

Hence by the Helly's compactness theorem and the diagonal argument, there exists a sequence $p_k \to \infty$ such that for any $0 < \eta < R$

$$U_{p_k}(r) \to \text{some } U_\infty(r) \text{ pointwise on } [0, R-\eta] \text{ as } p_k \to \infty,$$

and $U_\infty(r) \in L^1(0, R-\eta)$ is monotonically increasing.

Step 2. Set $F_k := \frac{(W_{p_k})_r}{W_{p_k}}$. Then (U_{p_k}, F_k) satisfies

$$\begin{cases} (U_{p_k})_r = pU_{p_k} F_k, & r \in (0, R), \\ \varepsilon F_{kr} + \varepsilon \cdot \frac{n-1}{r} F_k + \varepsilon F_k^2 = U_{p_k}, & r \in (0, R). \end{cases} \tag{5.26}$$

By (5.4), we get $F_k(r) > 0$ for any $r \in (0, R]$.

We claim $U_\infty \equiv 0$ on $[0, R)$. Otherwise, there exists $r_0 \in [0, R)$ such that $U_\infty(r_0) > 0$. By the monotonicity of U_∞, we assume $r_0 > 0$. Taking $r_1 \in (r_0, R)$, then

$$U_\infty(r) > U_\infty(r_0) := d_1 > 0 \text{ for } r \in [r_0, r_1]. \tag{5.27}$$

Write the second equation of (5.26) as

$$\varepsilon (r^{n-1} F_k)_r + \varepsilon r^{n-1} F_k^2 = r^{n-1} U_{p_k}.$$

Integrating this equation over $(0, r)$ for $r \in (r_0, r_1)$ yields

$$\varepsilon r^{n-1} F_k(r) \leqslant \int_0^R r^{n-1} U_{p_k}(r) dr = \frac{m}{\omega_n} \text{ for } r \in [r_0, r_1].$$

Thus, $F_k(r) \leqslant \frac{m}{\varepsilon r_0^{n-1}}$, $\forall r \in [r_0, r_1]$. When p_k is large enough, the second equation of (5.26) further gives

$$\varepsilon F_{kr}(r) \leqslant U_{p_k}(r) \leqslant 2U_\infty(r_1) \text{ for } r \in [r_0, r_1].$$

Thus, $F_k(r)$ is bounded in $C^1([r_0, r_1])$ with respect to p_k. Thanks to the Arzelá–Ascoli theorem, there exists $F_\infty \in C[r_0, r_1]$ such that after passing to a subsequence of $p_k \to \infty$,

$$F_k \to F_\infty \text{ in } C([r_0, r_1]).$$

We next claim

$$\text{there exists } \bar{r} \in (r_0, r_1) \text{ such that } F_\infty(\bar{r}) > 0. \tag{5.28}$$

Otherwise, $F_\infty \equiv 0$ on $[r_0, r_1]$. Multiplying the second equation of (5.26) by a test function $\phi \in C_0^\infty((r_0, r_1))$, then

$$\varepsilon \int_{r_0}^{r_1} F_k \phi_r dr + \varepsilon \int_{r_0}^{r_1} \frac{n-1}{r} F_k \phi dr + \varepsilon \int_{r_0}^{r_1} F_k^2 \phi dr = \int_{r_0}^{r_1} U_{p_k} \phi dr.$$

Sending $p_k \to \infty$ and using the Lebesgue Dominated Convergence Theorem, we get

$$\int_{r_0}^{r_1} U_\infty \phi dr = 0 \text{ for any } \phi \in C_0^\infty((r_0, r_1)).$$

Thus, $U_\infty \equiv 0$ on (r_0, r_1), which contradicts the assumption (5.27).

By (5.28) and the continuity of the function $F_\infty(r)$ on $[r_0, r_1]$, there exists an interval $[r_2, r_3] \subset [r_0, r_1]$ such that $F_\infty(r) > d_2$ on $[r_2, r_3]$ for some constant $d_2 > 0$. Now integrating the first equation of (5.26), in view of (5.27), we have

$$U_{p_k}(r) = U_{p_k}(r_0) e^{p_k \int_{r_0}^r F_k(\tau) d\tau} \geqslant \frac{d_1}{2} e^{p_k d_2 (r - r_2)} \text{ for } r > r_2.$$

It then follows that

$$m \geqslant \int_{r_2}^{r_3} r^{n-1} U_{p_k}(r) dr \geqslant \frac{d_1}{2} \int_{r_2}^{r_3} r^{n-1} e^{p_k d_2 (r - r_2)} dr \to \infty \text{ as } p_k \to \infty,$$

which is a contradiction. Therefore, $U_\infty \equiv 0$ on $[0, R)$, which implies

$$\omega_n r^{n-1} U_{p_k}(r) \to m\delta(r - R) \text{ in the sense of distribution,}$$

due to $\int_{B_R} U_{p_k}(x) dx = m > 0$.

Step 3. It remains to show the limit of W_p. On one hand, by the maximum principle, $0 < W_{p_k}(r) \leqslant b$ for $r \in [0, R]$. On the other hand, since $U_{p_k}\left(\frac{R}{2}\right) \leqslant M_{\frac{R}{2}}$, integrating (5.3) gives

$$0 \leqslant \varepsilon (W_{p_k})_r(r) = \frac{1}{r^{n-1}} \int_0^r s^{n-1} U_{p_k} W_{p_k} \leqslant b U_{p_k}\left(\frac{R}{2}\right) \frac{1}{r^{n-1}} \int_0^r s^{n-1} \leqslant C \text{ for } r \in \left[0, \frac{R}{2}\right]. \tag{5.29}$$

Moreover, for $r \in \left[\frac{R}{2}, R\right]$, there holds that

$$\begin{aligned} 0 \leqslant \varepsilon(W_{p_k})_r(r) &= \frac{1}{r^{n-1}} \int_{\frac{R}{2}}^{r} s^{n-1} U_{p_k} W_{p_k} ds + \varepsilon\left(\frac{R}{2r}\right)^{n-1} (W_{p_k})_r\left(\frac{R}{2}\right) \\ &\leqslant b\left(\frac{2}{R}\right)^{n-1} \int_0^R r^{n-1} U_{p_k} dr + \varepsilon\left(\frac{R}{2r}\right)^{n-1} (W_{p_k})_r\left(\frac{R}{2}\right) \\ &\leqslant b\left(\frac{2}{R}\right)^{n-1} \frac{m}{\omega_n} + \varepsilon(W_{p_k})_r\left(\frac{R}{2}\right) \leqslant C, \end{aligned}$$

where we have used $\frac{R}{2r} \leqslant 1$ in the second inequality. Hence $\|W_{p_k}\|_{C^1([0,R])} \leqslant C$. Thanks to the Arzelá–Ascoli theorem, there exists $W_\infty \in C([0, R])$ such that after passing to a subsequence of $p_k \to \infty$, $W_{p_k} \to W_\infty$ in $C([0, R])$, which also indicates that $W_\infty(R) = b$. Integrating (5.3) twice gives

$$\varepsilon W_{p_k}(r) - \varepsilon W_{p_k}\left(\frac{R}{2}\right) = \int_{\frac{R}{2}}^{r} \frac{1}{s^{n-1}} \int_0^s \tau^{n-1} U_{p_k} W_{p_k} d\tau ds.$$

Now sending $p_k \to \infty$ and recalling $U_{p_k} \to 0$ pointwise on $[0, R)$, by the Lebesgue Dominated Convergence Theorem, we have

$$W_\infty(r) = W_\infty\left(\frac{R}{2}\right) \text{ for any } r \in [0, R].$$

Therefore, $W_\infty(r) \equiv b$. Since the limit (U_∞, W_∞) is unique, all the convergence statements made above hold without passing to a subsequence. □

5.3 | Proof of Theorem 2.3

First, the existence and uniqueness of radially symmetric solutions with monotonicity follows from Lemma 5.1. By Lemma 5.3 and Lemma 5.4, we get

$$W_\varepsilon'(R) = \frac{pmb}{(2+p)\omega_n R^{n-1}} \frac{1}{\varepsilon} + O(\log \varepsilon) \tag{5.30}$$

which gives the expansion for $W_\varepsilon'(R)$ in (2.5) directly. With (1.5) and (5.22), the expansion for $U_\varepsilon'(R)$ in (2.5) is obtained. In the following, we shall use Lemma 5.2 (with σ replaced by $\varepsilon\lambda_\varepsilon$ there) to derive the first-order expansion of $R - r_\varepsilon(R, c)$ as $\varepsilon \to 0$ for a given $c \in (0, b)$. By Lemma 5.2, using the sub-solution of (5.19) given by $W_{\varepsilon,l}(r)$, we can solve $W_{\varepsilon,l}(r) = c$, to obtain we have

$$r_{1,\varepsilon}(c) := W_{\varepsilon,l}^{-1}(c) = R - \frac{b^{\frac{p}{2}} - c^{\frac{p}{2}}}{b^{\frac{p}{2}} c^{\frac{p}{2}}} c_p \varepsilon^{\frac{1}{2}} \lambda_\varepsilon^{\frac{1}{2}}.$$

Next, we use the supersolution of (5.19) given by $W_{\varepsilon,u}(r)$ in (5.6) (with ε replaced by $\varepsilon\lambda_\varepsilon$ there) and solve $W_{\varepsilon,u}(r) = c$, to get

$$r_{2,\varepsilon}(c) := W_{\varepsilon,u}^{-1}(c) = \begin{cases} R - \dfrac{b^{\frac{p}{2}} - c^{\frac{p}{2}}}{b^{\frac{p}{2}} c^{\frac{p}{2}}} c_p \varepsilon^{\frac{1}{2}} \lambda_\varepsilon^{\frac{1}{2}} + O(\varepsilon\lambda_\varepsilon), & \text{if } n \geqslant 2, \\ R - \dfrac{b^{\frac{p}{2}} - c^{\frac{p}{2}}}{b^{\frac{p}{2}} c^{\frac{p}{2}}} c_p \varepsilon^{\frac{1}{2}} \lambda_\varepsilon^{\frac{1}{2}} + O(\varepsilon^{2/p+1}), & \text{if } n = 1. \end{cases}$$

It is easy to see that $r_{2,\varepsilon}(c) \leqslant r_\varepsilon(R,c) \leqslant r_{1,\varepsilon}(c)$, then we have

$$r_\varepsilon(R,c) = R - \frac{b^{\frac{p}{2}} - c^{\frac{p}{2}}}{b^{\frac{p}{2}} c^{\frac{p}{2}}} c_p \varepsilon^{\frac{1}{2}} \lambda_\varepsilon^{\frac{1}{2}} + O(\varepsilon^{2/p+1}).$$

Using Lemma 5.4, we further get that

$$r_\varepsilon(R,c) = R - \frac{b^{\frac{p}{2}} - c^{\frac{p}{2}}}{c^{\frac{p}{2}}} \frac{\omega_n c_p^2 R^{n-1}}{m} \varepsilon + o_\varepsilon(1)\varepsilon$$

which yields (2.6) and hence completes the proof of Theorem 2.3(i).

The conclusion of Theorem 2.3(ii) comes from Lemma 5.5. The proof of Theorem 2.3 is completed. □

Remark 5.1. The expansion (5.30) differs from the linear sensitivity problem (1.7), where the second-order term in the expansion is of order one, as shown in [21].

6 | NONLINEAR STABILITY OF THE RADIAL BOUNDARY-LAYER STEADY STATE

6.1 | A preliminary result

The monotonicity of the radially symmetric steady-state solution (U, W) will be essentially used later to prove the stability result. We further show the following result concerning a sign property.

Lemma 6.1. *Set $V := \log W$. Then it follows that*

$$(r^{n-1}V_r)_r > 0 \ \textit{for any } r \in [0,R].$$

Proof. Since $W > 0$, dividing the second equation of (5.2) by W, we get

$$\varepsilon\left(\frac{W_r}{W}\right)_r + \varepsilon\left(\frac{W_r}{W}\right)^2 + \varepsilon\cdot\frac{n-1}{r}\cdot\frac{W_r}{W} = U. \tag{6.1}$$

Noting $V_r = \frac{W_r}{W}$, one can see from (6.1) that V satisfies

$$\varepsilon(r^{n-1}V_r)_r = r^{n-1}(U - \varepsilon V_r^2). \tag{6.2}$$

Since $U(0) > 0$ and $V_r(0) = 0$, there exists $r_0 > 0$ such that

$$(r^{n-1}V_r)_r > 0, \ \forall\, r \in (0, r_0].$$

We claim $(r^{n-1}V_r)_r > 0$ for all $r \in [0, R]$. Otherwise, denote by $r_1 \in (r_0, R]$ the smallest number such that $(r^{n-1}V_r)_r|_{r=r_1} = 0$. In other words $(r^{n-1}V_r)_r > 0$, $\forall r \in (0, r_1)$, and hence the function $r^{n-1}V_r(r)$ is monotonically increasing on $(0, r_1)$. In view of (6.2), we have

$$U(r_1) - \varepsilon V_r^2(r_1) = 0. \tag{6.3}$$

Moreover a simple calculation for (6.2) along with (6.3) gives

$$V_{rr}(r_1)r_1^{n-1} = -(n-1)r_1^{n-2}V_r(r_1). \tag{6.4}$$

Differentiating (6.2) in r gives

$$\varepsilon(r^{n-1}V_r)_{rr} = r^{n-1}(U_r - 2\varepsilon V_r V_{rr}) + (n-1)r^{n-2}(U - \varepsilon V_r^2).$$

Now by (6.3) and (6.4), we have

$$\begin{aligned}\varepsilon(r^{n-1}V_r)_{rr}|_{r=r_1} &= r_1^{n-1}(U_r - 2\varepsilon V_r V_{rr})|_{r=r_1}\\ &= r_1^{n-1}U_r(r_1) + 2\varepsilon(n-1)r_1^{n-2}V_r^2(r_1) > 0,\end{aligned}$$

where we have used the monotonicity of $U(r)$ in the last inequality (see Theorem 5.1). Thus, the function $r^{n-1}V_r(r)$ takes a local minimum at $r = r_1$, which contradicts the fact that it is monotonically increasing on $(0, r_1)$. Therefore, $(r^{n-1}V_r)_r > 0$ for all $r \in [0, R]$. □

6.2 | Nonlinear stability of $(U, V)(r)$ on $B_R(0)$

In this section, we study the asymptotic stability of the steady-state (U, W) to the system (1.2) under radial perturbations. Since the $n = 1$ case has been achieved in our previous work [4], we only consider the case $n = 2, 3$.

Theorem 6.2. *Let $n = 2, 3$. Assume that the initial datum (u_0, w_0) is radially symmetric with $u_0 > 0$ and $w_0 > 0$, and that $u_0 \in H^2(B_R)$, $w_0 - b \in H_0^1(B_R) \cap H^2(B_R)$. Let (U, W) be the unique steady state obtained in Lemma 5.1 with $\int_{B_R} U(x)dx = \int_{B_R} u_0(x)dx$. Then there exists a constant $\delta_1 > 0$ such that if the initial datum satisfies*

$$\|(u_0 - U, w_0 - W)\|_{H^2(B_R)} \leqslant \delta_1,$$

then the system (1.2) admits a unique global radial solution $(u, w) \in C([0, +\infty); H^2(B_R))$, which satisfies $u(x, t) > 0$, $w(x, t) > 0$ on $B_R \times (0, +\infty)$, and

$$\|(u - U, w - W)(\cdot, t)\|_{H^2(B_R)}^2 \leqslant C\|(u_0 - U, w_0 - W)\|_{H^2(B_R)}^2 e^{-\mu t}, \tag{6.5}$$

where C and μ are some positive constants independent of t.

Corollary 6.3. *By the Sobolev embedding theorem, we further have the following asymptotic convergence:*

$$\|(u-U, w-W)(\cdot,t)\|^2_{C(\overline{B_R})} \leqslant Ce^{-\mu t}.$$

Theorem 2.4 is a direct consequence of Theorem 6.2 and Corollary 6.3. To show Theorem 6.2, we first note that since $w|_{\partial B_R(0)} = b > 0$, if $w_0(x) > 0$ for all $x \in B_R(0)$, then by the maximum principle, we have $w(x,t) > 0$ for all $x \in B_R(0)$ and $t > 0$. Thus, one can remove the logarithmic nonlinearity of the sensitive function of (1.2) by setting $v := \log w$. Then the system (1.2) is transformed into

$$\begin{cases} u_t = \Delta u - p\nabla\cdot(u\nabla v), & x \in B_R(0),\ t > 0,\\ v_t = \varepsilon\Delta v + \varepsilon|\nabla v|^2 - u, & x \in B_R(0),\ t > 0,\\ u(x,0) = u_0(x),\ v(x,0) = v_0(x) := \log w_0(x), & x \in B_R(0),\\ \frac{\partial u}{\partial \nu} - pu\frac{\partial v}{\partial \nu} = 0,\ v = \log b, & x \in \partial B_R(0),\ t > 0. \end{cases} \tag{6.6}$$

Let (U,W) be the unique positive solution of (1.3) with $\Omega = B_R(0)$ and $m = \int_{B_R} u_0(x)dx$ (see Lemma 5.1). Set $V = \log W$. Then it is straightforward to check that (U,V) satisfies

$$\begin{cases} \Delta U - p\nabla\cdot(U\nabla V) = 0, & x \in B_R(0),\\ \varepsilon\Delta V + \varepsilon|\nabla V|^2 - U = 0, & x \in B_R(0),\\ \frac{\partial U}{\partial \nu} - pU\frac{\partial V}{\partial \nu} = 0,\ V = \log b, & x \in \partial B_R(0),\\ \int_{B_R} U dx = \int_{B_R} u_0(x)dx, \end{cases} \tag{6.7}$$

which implies that (U,V) is a steady state of (6.6). Using the uniqueness of positive solutions of (1.3), one can immediately show that (U,V) is the unique solution of (6.7).

We next investigate the stability of the steady state to (6.6). Applying the contraction mapping theorem, one can readily derive the local well-posedness of (6.6). The routine and tedious proof details are omitted for brevity.

Proposition 6.4 (Local well-posedness). *For any $\Xi > 0$, assume that the initial datum (u_0, v_0) satisfies*

$$0 < u_0 \in H^2(B_R),\ v_0 - \log b \in H^1_0(B_R) \cap H^2(B_R) \text{ and } \|u_0 - U\|_{H^2} + \|v_0 - V\|_{H^2} \leqslant \Xi,$$

where (U,V) is the unique solution of (6.7). Then, there exists a constant $T > 0$ only depending on Ξ such that the system (6.6) has a unique local solution $u \in C([0,T]; H^2(B_R))$, $v - \log b \in C([0,T]; H^1_0(B_R) \cap H^2(B_R))$, $(u,v) \in L^2((0,T); H^3(B_R))$, and for $t \in [0,T]$ it holds

$$\sup_{\tau\in[0,t]} \|(u-U, v-V)(\cdot,\tau)\|^2_{H^2} + \int_0^t \|(u-U, v-V)(\cdot,\tau)\|^2_{H^3} d\tau \leqslant 4\|(u_0-U, v_0-V)\|^2_{H^2}.$$

Noting that the system (6.6) is rotationally invariant, by the uniqueness of solutions, one can see that if the initial data $(u_0, v_0)(x)$ is radially symmetric, then the solution $(u,v)(x,t)$ is also radially symmetric. In the radial setting, we have the following stability result for the reformulated system (6.6).

Theorem 6.5. *Let $n = 2, 3$. Assume that the initial datum (u_0, v_0) is radially symmetric, and that $0 < u_0 \in H^2(B_R)$, $v_0 - \log b \in H_0^1(B_R) \cap H^2(B_R)$. Let (U, V) be the unique positive solution of (6.7). Then there exists a constant $\delta_2 > 0$ such that if the initial datum satisfies $\|(u_0 - U, v_0 - V)\|_{H^2} \leqslant \delta_2$, then the system (6.6) admits a unique global radial solution $(u, v) \in C([0, +\infty); H^2(B_R))$, which satisfies $u(x, t) > 0$ on $B_R \times (0, +\infty)$, and*

$$\|(u - U, v - V)(\cdot, t)\|_{H^2}^2 \leqslant C\|(u_0 - U, v_0 - V)\|_{H^2}^2 e^{-\mu t}, \tag{6.8}$$

where C and μ are positive constants independent of t.

To show the global existence result claimed in Theorem 6.5, by the local well-posedness result and the standard continuation argument, it suffices to establish the corresponding a priori estimates of $(u - U, v - V)$. In order to achieve the proofs of Theorems 6.5 and 6.2 at the end of this section, we introduce the notation

$$N(t) := \sup_{\tau\in[0,t]} \left\{\|(u - U)(\cdot, \tau)\|_{H^2} + \|(v - V)(\cdot, \tau)\|_{H^2}\right\}. \tag{6.9}$$

By the Sobolev embedding theorem, for $n = 2, 3$, we have

$$\sup_{\tau\in[0,t]} \left\{\|(u - U)(\cdot, \tau)\|_{L^\infty} + \|(v - V)(\cdot, \tau)\|_{L^\infty}\right\} \leqslant CN(t). \tag{6.10}$$

Then the following a priori estimates, proven below in a sequence of iterated lemmas, can be established.

Proposition 6.6 (A priori estimate). *Let $n = 2, 3$. Assume that $(u, v)(x, t)$ is a radially symmetric solution of the system (6.6) obtained in Proposition 6.4 on $[0, T]$ for some $T > 0$. Then there exists a constant $h > 0$ independent of T such that if $N(T) \leqslant h$, it holds that*

$$e^{\mu t}\|(u - U, v - V)(\cdot, t)\|_{H^2}^2 + \int_0^t e^{\mu\tau}\|(u - U, v - V)(\cdot, \tau)\|_{H^3}^2 d\tau \leqslant C\|(u_0 - U, v_0 - V)\|_{H^2}^2 \tag{6.11}$$

for any $t \in [0, T]$, where C and μ are positive constants independent of t and h.

We now write the system (6.6) in the radial coordinates as

$$\begin{cases} u_t = \frac{1}{r^{n-1}}[(r^{n-1}u_r)_r - p(r^{n-1}uv_r)_r], & r \in (0, R),\ t > 0,\\ v_t = \frac{\varepsilon}{r^{n-1}}(r^{n-1}v_r)_r + \varepsilon|v_r|^2 - u, & r \in (0, R),\ t > 0,\\ u(r, 0) = u_0(r),\ v(r, 0) = \log w_0(r), & r \in (0, R),\\ u_r - puv_r = 0,\ v = \log b, & r = R,\ t > 0,\\ u_r = 0,\ v_r = 0, & r = 0,\ t > 0. \end{cases} \tag{6.12}$$

Using the boundary condition of U, one can see that the steady-state (U, V) of (6.12) satisfies

$$\begin{cases} U_r = pUV_r, & r \in (0, R),\\ \frac{\varepsilon}{r^{n-1}}(r^{n-1}V_r)_r + \varepsilon|V_r|^2 - U = 0, & r \in (0, R),\\ U_r(0) = 0 = V_r(0),\ V(R) = \log b. \end{cases} \tag{6.13}$$

Thanks to the conservation of mass, it holds that $\int_{B_R} u(x,t)dx = \int_{B_R} u_0(x)dx$. Since

$$\int_{B_R} u_0(x)dx = \int_{B_R} U(x)dx,$$

we have

$$\int_{B_R} u(x,t)dx = \int_{B_R} U(x)dx \text{ for all } t \in [0,T],$$

that is,

$$\omega_n \int_0^R u(r,t)r^{n-1}dr = \omega_n \int_0^R U(r)r^{n-1}dr,$$

where we have used the fact $\int_{B_R} f(|x|,t)dx = \omega_n \int_0^R f(r,t)r^{n-1}dr$ with ω_n denoting the surface area of the unit sphere in $\mathbb{R}^n$. This inspires us to take the relative difference between the radial mass distribution function of any solution $u(r,t)$ to the unique steady-state $U(r)$ or "radius-dependent anti-derivative" to reformulate the problem. Precisely, if we set

$$\phi(r,t) := \frac{1}{r^{n-1}} \int_0^r (u(s,t) - U(s))s^{n-1}ds, \quad \psi(r,t) := v(r,t) - V(r). \tag{6.14}$$

Then by L'Hôpital's rule, we have

$$\phi(0,t) = \lim_{r\to 0} \frac{1}{r^{n-1}} \int_0^r (u(s,t) - U(s))s^{n-1}ds = \lim_{r\to 0} \frac{(u(r,t) - U(r)) \cdot r}{n-1} = 0,$$

and

$$\phi(R,t) = \int_0^R (u(s,t) - U(s))s^{n-1}ds = 0.$$

Substituting (6.14) into (6.12), one can find that (ϕ, ψ) satisfies

$$\begin{cases} \phi_t = \left(\frac{(r^{n-1}\phi)_r}{r^{n-1}}\right)_r - pV_r \frac{(r^{n-1}\phi)_r}{r^{n-1}} - pU\psi_r - p\frac{(r^{n-1}\phi)_r}{r^{n-1}}\psi_r, & r \in (0,R), t > 0, \\ \psi_t = \frac{\varepsilon}{r^{n-1}}(r^{n-1}\psi_r)_r + 2\varepsilon V_r \psi_r + \varepsilon\psi_r^2 - \frac{(r^{n-1}\phi)_r}{r^{n-1}}, & r \in (0,R), t > 0, \\ (\phi, \psi_r)(0,t) = (0,0), \ (\phi,\psi)(R,t) = (0,0), & t > 0, \end{cases} \tag{6.15}$$

with initial data

$$(\phi_0, \psi_0)(r) := \left(\frac{1}{r^{n-1}} \int_0^r (u_0(s) - U(s))s^{n-1}ds, \log w_0(r) - V(r) \right). \tag{6.16}$$

We next establish the a priori estimate (6.11). We begin with the basic L^2 estimate. In what follows, we shall abbreviate $\int_0^R f(r)dr$ as $\int_0^R f(r)$, $\int_0^t \int_0^R f(r,s)drds$ as $\int_0^t \int_0^R f(r,s)$, and $B_R(0)$ as B_R for the sake of notational simplicity.

Lemma 6.7. *Let the assumptions of Proposition 6.6 hold. If $N(T) \ll 1$, then there exist two constants $C > 0$ and $\mu > 0$ independent of t such that*

$$e^{\mu t}(\|\phi(\cdot,t)\|_{L^2}^2 + \|\psi(\cdot,t)\|_{L^2}^2) + \int_0^t e^{\mu\tau}(\|u(\cdot,\tau) - U(\cdot)\|_{L^2}^2 + \|\psi(\cdot,\tau)\|_{H^1}^2)d\tau \leqslant C(\|\phi_0\|_{L^2}^2 + \|\psi_0\|_{L^2}^2) \tag{6.17}$$

holds for any $t \in [0, T]$.

Proof. Integrating the sum of the first equation of (6.15) multiplied by $\frac{r^{n-1}\phi}{U}$ and the second one multiplied by $pr^{n-1}\psi$, we have alongside the integration by parts

$$\frac{1}{2}\frac{d}{dt}\int_0^R \left(\frac{r^{n-1}\phi^2}{U} + pr^{n-1}\psi^2\right) + \int_0^R \frac{|(r^{n-1}\phi)_r|^2}{r^{n-1}U} + p\varepsilon\int_0^R r^{n-1}\psi_r^2 + p\varepsilon\int_0^R (r^{n-1}V_r)_r\psi^2 + \int_0^R (r^{n-1}\phi)_r\phi\left[\left(\frac{1}{U}\right)_r + \frac{pV_r}{U}\right] = -p\int_0^R \frac{\phi(r^{n-1}\phi)_r\psi_r}{U} + p\varepsilon\int_0^R r^{n-1}\psi\psi_r^2. \tag{6.18}$$

A simple calculation from the first equation of (6.13) yields

$$\left(\frac{1}{U}\right)_r + \frac{pV_r}{U} = 0.$$

From Lemma 6.1, we get $(r^{n-1}V_r)_r > 0$. The last two terms of (6.18) can be estimated as follows. By Hölder's inequality, we can derive that

$$\left|\int_0^R \frac{\phi(r^{n-1}\phi)_r\psi_r}{U}\right| \leqslant \|\phi\|_{L^\infty}\int_0^R \frac{|(r^{n-1}\phi)_r|^2}{r^{n-1}U} + \|\phi\|_{L^\infty}\int_0^R \frac{r^{n-1}\psi_r^2}{U} \leqslant CN(t)\int_0^R \left(\frac{|(r^{n-1}\phi)_r|^2}{r^{n-1}U} + \frac{r^{n-1}\psi_r^2}{U}\right)$$

and

$$\left|\int_0^R r^{n-1}\psi\psi_r^2\right| \leqslant \|\psi\|_{L^\infty}\int_0^R r^{n-1}\psi_r^2 \leqslant CN(t)\int_0^R r^{n-1}\psi_r^2,$$

where we have used the fact based on (6.10)

$$\|\phi(\cdot,t)\|_{L^\infty} \leqslant C\|u(\cdot,t) - U(\cdot)\|_{L^\infty} \leqslant CN(t)$$

and

$$\|\psi(\cdot,t)\|_{L^\infty} \leqslant CN(t).$$

Since U is continuous and positive in $\overline{B_R}$, then $\max_{x\in\overline{B_R}} U(x) = U_{\max} \geqslant U \geqslant U_{\min} = \min_{x\in\overline{B_R}} U(x) > 0$ and hence $\frac{r^{n-1}\psi_r^2}{U} \leqslant \frac{r^{n-1}\psi_r^2}{U_{\min}}$, $\frac{|(r^{n-1}\phi)_r|^2}{r^{n-1}U} \geqslant \frac{|(r^{n-1}\phi)_r|^2}{r^{n-1}U_{\max}}$. Then substituting the above estimates into (6.18), we obtain

$$\frac{d}{dt}\int_0^R \left(\frac{r^{n-1}\phi^2}{U} + pr^{n-1}\psi^2\right) + \frac{2}{U_{\max}}(1 - pCN(t))\int_0^R \frac{|(r^{n-1}\phi)_r|^2}{r^{n-1}} \\ + 2[p\varepsilon - (p\varepsilon + p/U_{\min})CN(t)]\int_0^R r^{n-1}\psi_r^2 \leqslant 0. \tag{6.19}$$

Recalling the transformation (6.14), we get

$$(r^{n-1}\phi)_r = (u - U)r^{n-1}, \tag{6.20}$$

from which we further have

$$\int_0^R \frac{|(r^{n-1}\phi)_r|^2}{r^{n-1}} = \int_0^R r^{n-1}|u - U|^2 dr = \frac{1}{\omega_n}\int_{B_R} |u - U|^2 dx.$$

Noticing that $\int_0^R r^{n-1}\psi_r^2 dr = \frac{1}{\omega_n}\int_{B_R} |\nabla\psi|^2 dx$, we have from (6.19) that

$$\frac{d}{dt}\int_0^R \omega_n\left(\frac{r^{n-1}\phi^2}{U} + pr^{n-1}\psi^2\right) + p\varepsilon\int_{B_R} |\nabla\psi|^2 dx + \frac{1}{U_{\max}}\int_{B_R} |u - U|^2 dx \leqslant 0, \tag{6.21}$$

where we have used the assumption that $N(t)$ is sufficiently small such that, for instance, $1 - pCN(t) > 1/2$ and $[p\varepsilon - (p\varepsilon + p/U_{\min})CN(t)] > p\varepsilon/2$. By (6.14), we also have

$$\begin{aligned}\int_{B_R} \phi^2 dx = \omega_n\int_0^R r^{n-1}\phi^2 &= \omega_n\int_0^R \frac{1}{r^{n-1}}\left(\int_0^r (u - U)s^{n-1}ds\right)^2 \\ &\leqslant \omega_n\int_0^R (u - U)^2 r^{n-1}dr\int_0^R \frac{1}{r^{n-1}}\int_0^r s^{n-1}dsdr \\ &= \frac{\omega_n R^2}{2n}\int_{B_R} (u - U)^2 dx.\end{aligned}$$

Noting that $\psi|_{\partial B_R} = 0$ from the equations in (6.15), it follows from Poincaré's inequality that

$$\omega_n\int_0^R r^{n-1}\psi^2 = \int_{B_R} \psi^2 dx \leqslant C\int_{B_R} |\nabla\psi|^2 dx.$$

Now, multiplying (6.21) by $e^{\mu t}$, where $\mu > 0$ is a constant to be determined, we get

$$\begin{aligned}& e^{\mu t}\left(\int_{B_R}\frac{\phi^2}{U} + p\int_{B_R}\psi^2\right) + \int_0^t e^{\mu\tau}\left(\frac{1}{U_{\max}}\int_{B_R}(u - U)^2 + p\varepsilon\int_{B_R}|\nabla\psi|^2\right) \\ &\leqslant \mu\int_0^t e^{\mu\tau}\left(\int_{B_R}\frac{\phi^2}{U} + p\int_{B_R}\psi^2\right) + \int_{B_R}\frac{\phi_0^2}{U} + p\int_{B_R}\psi_0^2 \\ &\leqslant \mu\int_0^t e^{\mu\tau}\left(\frac{1}{U_{\min}}\int_{B_R}\phi^2 + p\int_{B_R}\psi^2\right) + \frac{1}{U_{\min}}\int_{B_R}\phi_0^2 + p\int_{B_R}\psi_0^2 \\ &\leqslant C\mu\int_0^t e^{\mu\tau}\left(\int_{B_R}(u - U)^2 + p\int_{B_R}|\nabla\psi|^2\right) + C\int_{B_R}(\phi_0^2 + \psi_0^2).\end{aligned}$$

Thus, the desired estimate (6.17) holds by choosing $\mu > 0$ to be suitably small. □

We next establish the L^2 estimate for $u - U$.

Lemma 6.8. *Let the assumptions of Proposition 6.6 hold. If $N(T) \ll 1$, then it holds that*

$$\begin{aligned} &e^{\mu t}\|u(\cdot,t)-U(\cdot)\|_{L^2}^2 + \int_0^t e^{\mu\tau}\|\nabla(u(\cdot,\tau)-U(\cdot))\|_{L^2}^2 \\ &\leqslant C(\|u_0-U\|_{L^2}^2 + \|\phi_0\|_{L^2}^2 + \|\psi_0\|_{L^2}^2), \end{aligned} \tag{6.22}$$

for any $t \in [0,T]$, where $C > 0$ is a constant independent of t.

Proof. Using (6.20), we write the first equation of (6.15) as

$$\begin{cases} \phi_t = (u-U)_r - pV_r(u-U) - pU\psi_r - p(u-U)\psi_r, & r \in (0,R), t > 0, \\ \phi(0,t) = \phi(R,t) = 0, & t > 0, \end{cases} \tag{6.23}$$

with the following identity:

$$\begin{aligned} \phi_t r^{n-1}(u-U)_r &= (r^{n-1}\phi_t(u-U))_r - (u-U)(r^{n-1}\phi)_{tr} \\ &= (r^{n-1}(u-U)\phi_t)_r - \frac{1}{2}(r^{n-1}(u-U)^2)_t. \end{aligned}$$

Then multiplying (6.23) by $r^{n-1}(u-U)_r$ and integrating the result by parts along with Young's inequality, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^R r^{n-1}(u-U)^2 + \int_0^R r^{n-1}|(u-U)_r|^2 \\ &= p\int_0^R r^{n-1}V_r(u-U)(u-U)_r + p\int_0^R r^{n-1}U\psi_r(u-U)_r \\ &\quad + p\int_0^R r^{n-1}\psi_r(u-U)(u-U)_r \\ &\leqslant \frac{1}{2}\int_0^R r^{n-1}|(u-U)_r|^2 + C\int_0^R r^{n-1}[V_r^2(u-U)^2 + U^2\psi_r^2] \\ &\quad + C\int_0^R r^{n-1}(u-U)^2|\psi_r|^2. \end{aligned} \tag{6.24}$$

Using (6.10), we have

$$\|u(\cdot,t)-U(\cdot)\|_{L^\infty} \leqslant CN(t) \tag{6.25}$$

and hence get

$$\int_0^R r^{n-1}(u-U)^2|\psi_r|^2 \leqslant \|u-U\|_{L^\infty}^2 \int_0^R r^{n-1}|\psi_r|^2 \leqslant CN(t)^2\int_0^R r^{n-1}|\psi_r|^2. \tag{6.26}$$

Substituting (6.26) into (6.24), since $\|\nabla V\|_{L^\infty} = \|V_r\|_{L^\infty} \leqslant C$ and $\|U\|_{L^\infty} \leqslant C$, we have

$$\frac{d}{dt}\|u(\cdot,t)-U(\cdot)\|_{L^2}^2 + \|\nabla(u(\cdot,t)-U(\cdot))\|_{L^2}^2 \leqslant C\|u(\cdot,t)-U(\cdot)\|_{L^2}^2 + C\|\psi(\cdot,t)\|_{H^1}^2. \tag{6.27}$$

Now, multiplying (6.27) by $e^{\mu t}$ and integrating the resulting inequality in t, from Lemma 6.7, we obtain the desired estimate (6.22). □

We proceed to derive the H^1 estimate for ψ.

Lemma 6.9. *Let the assumptions of Proposition 6.6 hold. If $N(T) \ll 1$, then it holds*

$$e^{\mu t}\|\nabla\psi(\cdot,t)\|_{L^2}^2 + \int_0^t e^{\mu\tau}\|\psi(\cdot,\tau)\|_{H^2}^2 \leqslant C(\|\phi_0\|_{L^2}^2 + \|\psi_0\|_{H^1}^2), \tag{6.28}$$

for any $t \in [0,T]$, where $C > 0$ is a constant independent of t.

Proof. We write the second equation of (6.15), using the Cartesian coordinates, as

$$\begin{cases} \psi_t = \varepsilon\Delta\psi + 2\varepsilon\nabla V\nabla\psi + \varepsilon|\nabla\psi|^2 - (u-U), & x\in B_R,\ t>0, \\ \psi = 0, & x \in \partial B_R,\ t>0. \end{cases} \tag{6.29}$$

Multiplying (6.29) by $\Delta\psi$ and integrating the resultant equation by parts alongside Young's inequality, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{B_R}|\nabla\psi|^2 + \varepsilon\int_{B_R}|\Delta\psi|^2 \\ &= -2\varepsilon\int_{B_R}\nabla V\nabla\psi\Delta\psi - \varepsilon\int_{B_R}|\nabla\psi|^2\Delta\psi + \int_{B_R}(u-U)\Delta\psi \\ &\leqslant \frac{3\varepsilon}{4}\int_{B_R}|\Delta\psi|^2 + 4\varepsilon\int_{B_R}|\nabla V|^2|\nabla\psi|^2 + \frac{1}{\varepsilon}\int_{B_R}|u-U|^2 + \varepsilon\int_{B_R}|\nabla\psi|^4. \end{aligned} \tag{6.30}$$

By the Sobolev embedding theorem for $n = 2, 3$, we get

$$\int_{B_R}|\nabla\psi|^4 \leqslant C\|\psi\|_{H^2}^4 \leqslant CN(t)^2\|\psi\|_{H^2}^2, \tag{6.31}$$

where we have used the fact $\|\psi(\cdot,t)\|_{H^2} \leqslant N(t)$ from (6.9). Employing the standard regularity theory for the Poisson equation:

$$\begin{cases} \Delta\psi = f, & x\in B_R, \\ \psi = 0, & x\in\partial B_R, \end{cases}$$

we have

$$\|\psi\|_{H^2(B_R)} \leqslant C_1\|f\|_{L^2(B_R)} = C_1\|\Delta\psi\|_{L^2(B_R)}. \tag{6.32}$$

It then follows from (6.30)–(6.32) that

$$\frac{1}{2}\frac{d}{dt}\int_{B_R}|\nabla\psi|^2+\left(\frac{\varepsilon}{4C_1^2}-CN(t)^2\right)\|\psi(\cdot,t)\|_{H^2}^2\leqslant C\varepsilon\int_{B_R}|\nabla\psi|^2+\frac{1}{\varepsilon}\int_{B_R}|u-U|^2.$$

Thus, if $N(t)\ll 1$ such that $\frac{\varepsilon}{4C_1^2}-CN(t)^2>\frac{\varepsilon}{8C_1^2}$, we have

$$\frac{d}{dt}\int_{B_R}|\nabla\psi|^2+\frac{\varepsilon}{4C_1^2}\|\psi(\cdot,t)\|_{H^2}^2\leqslant C\int_{B_R}|\nabla\psi|^2+C\int_{B_R}|u-U|^2.$$

Now, multiplying this inequality by $e^{\mu t}$, integrating the resulting equation in t, and using Lemma 6.7, we obtain (6.28). □

We also need the H^1 estimate for $(u-U)$.

Lemma 6.10. *Let the assumptions of Proposition 6.6 hold. Assume that $N(T)\ll 1$. Then it holds that*

$$e^{\mu t}\|\nabla(u(\cdot,t)-U(\cdot))\|_{L^2}^2+\int_0^t e^{\mu\tau}\|(u-U)_\tau(\cdot,\tau)\|_{L^2}^2\leqslant C(\|\phi_0\|_{L^2}^2+\|u_0-U\|_{H^1}^2+\|\psi_0\|_{H^1}^2),$$

for any $t\in[0,T]$, where $C>0$ is a constant independent of t.

Proof. First the Equation (6.23) gives rise to

$$r^{n-1}\phi_t^2\leqslant Cr^{n-1}(|(u-U)_r|^2+V_r^2(u-U)^2+U^2\psi_r^2+(u-U)^2\psi_r^2). \tag{6.33}$$

Integrating this inequality multiplied by $e^{\mu t}$ and using the boundedness of ∇V and U, we obtain

$$\begin{aligned}\int_0^t e^{\mu\tau}\|\phi_t\|_{L^2}^2&\leqslant C\int_0^t e^{\mu\tau}(\|\nabla(u-U)\|_{L^2}^2+\|u-U\|_{L^2}^2+\|\nabla\psi\|_{L^2}^2+\|u-U\|_{L^\infty}^2\|\nabla\psi\|_{L^2}^2)\\&\leqslant C(\|u_0-U\|_{L^2}^2+\|\phi_0\|_{L^2}^2+\|\psi_0\|_{L^2}^2),\end{aligned}\tag{6.34}$$

where in the second inequality we have used (6.25) and Lemmas 6.7-6.8. Similarly, using the Equation (6.29), the inequality (6.31), Lemma 6.7 and Lemma 6.9, we get

$$\int_0^t e^{\mu\tau}\|\psi_t\|_{L^2}^2\leqslant C\int_0^t e^{\mu\tau}(\|\psi\|_{H^2}^2+\|\nabla\psi\|_{L^4}^4+\|u-U\|_{L^2}^2)\leqslant C(\|\phi_0\|_{L^2}^2+\|\psi_0\|_{H^1}^2). \tag{6.35}$$

We next estimate $\|\phi_t(\cdot,t)\|_{L^2}$. Differentiating Equation (6.23) with respect to t gives

$$\begin{cases}\phi_{tt}=(u-U)_{tr}-pV_r(u-U)_t-p(u-U)_t\psi_r-p[U+(u-U)]\psi_{tr}, & r\in(0,R),\\ \phi_t(0,t)=\phi_t(R,t)=0.\end{cases}\tag{6.36}$$

Note that

$$\begin{aligned}(u-U)_{tr}r^{n-1}\phi_t&=((u-U)_t r^{n-1}\phi_t)_r-(u-U)_t(r^{n-1}\phi)_{rt}\\&=((u-U)_t r^{n-1}\phi_t)_r-r^{n-1}|(u-U)_t|^2\end{aligned}$$

and

$$-p[U+(u-U)]\psi_{tr}r^{n-1}\phi_t = -p([U+(u-U)]\psi_t r^{n-1}\phi_t)_r + p[U+(u-U)]_r\psi_t r^{n-1}\phi_t + p[U+(u-U)]\psi_t r^{n-1}(u-U)_t,$$

where (6.20) has been used. Multiplying Equation (6.36) by $r^{n-1}\phi_t$ and integrating by parts along with the boundary conditions in (6.36), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^R r^{n-1}\phi_t^2 + \int_0^R r^{n-1}|(u-U)_t|^2 \\
&= -p\int_0^R V_r\phi_t r^{n-1}(u-U)_t - p\int_0^R \psi_r\phi_t r^{n-1}(u-U)_t \\
&\quad + p\int_0^R [U+(u-U)]_r\psi_t r^{n-1}\phi_t + p\int_0^R [U+(u-U)]\psi_t r^{n-1}(u-U)_t \\
&\leqslant \frac{1}{2}\int_0^R r^{n-1}|(u-U)_t|^2 + C\int_0^R r^{n-1}[\phi_t^2+\psi_r^2\phi_t^2+\psi_t^2+|(u-U)_r|^2\phi_t^2+(u-U)^2\psi_t^2],
\end{aligned} \tag{6.37}$$

where we have used the boundedness of V_r, U_r and U in the last inequality. By Hölder's inequality and the Sobolev inequality for $n=2,3$ (cf. [12, Theorem 7.10]), we get

$$\begin{aligned}
\omega_n\int_0^R r^{n-1}\psi_r^2\phi_t^2 = \int_{B_R}|\nabla\psi|^2\phi_t^2 &\leqslant \left(\int_{B_R}|\nabla\psi|^4\right)^{\frac{1}{2}}\left(\int_{B_R}|\phi_t|^4\right)^{\frac{1}{2}} \\
&\leqslant C\|\psi\|_{H^2}^2\int_{B_R}|\nabla\phi_t|^2 \\
&\leqslant CN(t)^2\int_{B_R}|\nabla\phi_t|^2,
\end{aligned} \tag{6.38}$$

where we have used $\|\psi(\cdot,t)\|_{H^2}\leqslant N(t)$ again from (6.9). Similarly, noting $\|(u-U)(\cdot,t)\|_{H^2}\leqslant N(t)$, it holds

$$\omega_n\int_0^R r^{n-1}|(u-U)_r|^2\phi_t^2 = \int_{B_R}|\nabla(u-U)|^2\phi_t^2 \leqslant C\|u-U\|_{H^2}^2\int_{B_R}|\nabla\phi_t|^2 \leqslant CN(t)^2\int_{B_R}|\nabla\phi_t|^2.$$

By (6.25), we get

$$\int_0^R r^{n-1}(u-U)^2\psi_t^2 \leqslant CN(t)^2\int_0^R r^{n-1}\psi_t^2.$$

Observe that the second term of (6.37) satisfies

$$\begin{aligned}
\int_0^R r^{n-1}|(u-U)_t|^2 &= \int_0^R r^{n-1}\left|\frac{(r^{n-1}\phi_t)_r}{r^{n-1}}\right|^2 \\
&= \int_0^R \left(r^{n-1}\phi_{tr}^2 + 2(n-1)\phi_t\phi_{tr}r^{n-2} + (n-1)^2 r^{n-3}\phi_t^2\right)
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^R r^{n-1}\phi_{tr}^2 + (n-1)\int_0^R r^{n-3}\phi_t^2 + (n-1)\phi_t^2(R,t)R^{n-2} \\
&\geqslant \frac{1}{\omega_n}\int_{B_R} |\nabla\phi_t|^2. \qquad (6.39)
\end{aligned}
$$

It then follows from (6.37) that

$$
\frac{d}{dt}\int_0^R r^{n-1}\phi_t^2 + \frac{1}{2}\int_0^R r^{n-1}\left|(u-U)_t\right|^2 + \left(\frac{1}{2\omega_n} - CN(t)^2\right)\int_{B_R}|\nabla\phi_t|^2 \leqslant C\int_0^R r^{n-1}(\phi_t^2+\psi_t^2).
$$

If $N(t) \ll 1$ such that $\frac{1}{2\omega_n} > CN(t)^2$, then

$$
\frac{d}{dt}\int_0^R r^{n-1}\phi_t^2 + \frac{1}{2}\int_0^R r^{n-1}\left|(u-U)_t\right|^2 \leqslant C\int_0^R r^{n-1}(\phi_t^2+\psi_t^2). \qquad (6.40)
$$

Now multiplying (6.40) by $e^{\mu t}$ and integrating the resulting equation in t, by the estimates (6.34)–(6.35), we have

$$
e^{\mu t}\int_{B_R}\phi_t^2 + \int_0^t e^{\mu\tau}\int_{B_R}\left|(u-U)_t\right|^2 \leqslant C(\|\phi_0\|_{L^2}^2 + \|u_0-U\|_{H^1}^2 + \|\psi_0\|_{H^1}^2), \qquad (6.41)
$$

where we have used $\phi_t^2(\cdot,0) \leqslant C(|(u_0-U)_r|^2 + (u_0-U)^2 + \psi_{0r}^2 + \psi_{0r}^2)$ owing to (6.33). Using Equation (6.23) again, by (6.41) and Lemmas 6.8–6.9, we have

$$
\begin{aligned}
e^{\mu t}\int_{B_R}|\nabla(u-U)|^2 &\leqslant Ce^{\mu t}\int_{B_R}(\phi_t^2 + (u-U)^2 + |\nabla\psi|^2) \\
&\leqslant C(\|\phi_0\|_{L^2}^2 + \|u_0-U\|_{H^1}^2 + \|\psi_0\|_{H^1}^2).
\end{aligned} \qquad (6.42)
$$

The desired estimate follows from (6.41) and (6.42). □

The H^2 estimate for ψ is as follows.

Lemma 6.11. *Let the assumptions of Proposition 6.6 hold. Assume that $N(T) \ll 1$, then it holds*

$$
e^{\mu t}\|\psi(\cdot,t)\|_{H^2}^2 + \int_0^t e^{\mu\tau}\|\psi(\cdot,\tau)\|_{H^3}^2 \leqslant C(\|\phi_0\|_{L^2}^2 + \|u_0-U\|_{H^1}^2 + \|\psi_0\|_{H^2}^2), \qquad (6.43)
$$

for any $t \in [0,T]$, where $C > 0$ is a constant independent of t.

Proof. Differentiating Equation (6.29) in t yields

$$
\begin{cases}
\psi_{tt} = \varepsilon\Delta\psi_t + 2\varepsilon\nabla V\nabla\psi_t + 2\varepsilon\nabla\psi\nabla\psi_t - (u-U)_t, & x \in B_R,\ t > 0 \\
\psi_t = 0, & x \in \partial B_R,\ t > 0.
\end{cases} \qquad (6.44)
$$

Multiplying (6.44) by ψ_t and integrating by parts, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{B_R}\psi_t^2+\varepsilon\int_{B_R}|\nabla\psi_t|^2\\
&=2\varepsilon\int_{B_R}\nabla V\nabla\psi_t\psi_t+2\varepsilon\int_{B_R}\nabla\psi\nabla\psi_t\psi_t-\int_{B_R}(u-U)_t\psi_t \qquad (6.45)\\
&\leqslant\frac{\varepsilon}{4}\int_{B_R}|\nabla\psi_t|^2+C\int_{B_R}(|\nabla V|^2|\psi_t|^2+|(u-U)_t|^2+|\psi_t|^2)+C\int_{B_R}|\nabla\psi|^2|\psi_t|^2.
\end{aligned}$$

As in (6.38), by the Hölder and Sobolev inequalities, we get

$$\int_{B_R}|\nabla\psi|^2|\psi_t|^2\leqslant\left(\int_{B_R}|\nabla\psi|^4\right)^{\frac{1}{2}}\left(\int_{B_R}|\psi_t|^4\right)^{\frac{1}{2}}\leqslant C\|\psi\|_{H^2}^2\int_{B_R}|\nabla\psi_t|^2\leqslant CN(t)^2\int_{B_R}|\nabla\psi_t|^2.$$

If $N(t)\ll 1$, it then follows from (6.45) that

$$\frac{d}{dt}\int_{B_R}\psi_t^2+\varepsilon\int_{B_R}|\nabla\psi_t|^2\leqslant C\int_{B_R}(|\psi_t|^2+|(u-U)_t|^2).$$

Multiplying this inequality by $e^{\mu t}$ and integrating in t, by (6.35) and Lemma 6.10, we have

$$e^{\mu t}\int_{B_R}\psi_t^2+\varepsilon\int_0^t e^{\mu\tau}\int_{B_R}|\nabla\psi_t|^2\leqslant C(\|\phi_0\|_{L^2}^2+\|u_0-U\|_{H^1}^2+\|\psi_0\|_{H^2}^2), \qquad (6.46)$$

where we have used $\|\psi_t(\cdot,0)\|_{L^2}^2\leqslant C(\|\psi_0\|_{H^2}^2+\|u_0-U\|_{L^2}^2)$. Write (6.29) as an elliptic equation

$$\begin{cases}-\varepsilon\Delta\psi=-\psi_t+2\varepsilon\nabla V\nabla\psi+\varepsilon|\nabla\psi|^2-(u-U), & x\in B_R,\\ \psi=0, & x\in\partial B_R.\end{cases} \qquad (6.47)$$

Thanks to (6.31) and (6.32), it follows that

$$\begin{aligned}
e^{\mu t}\|\psi\|_{H^2}^2&\leqslant Ce^{\mu t}(\|\psi_t\|_{L^2}^2+\|\psi\|_{H^1}^2+\|u-U\|_{L^2}^2)\\
&\leqslant C(\|\phi_0\|_{L^2}^2+\|u_0-U\|_{H^1}^2+\|\psi_0\|_{H^2}^2),
\end{aligned} \qquad (6.48)$$

where we have used (6.46) and Lemmas 6.8–6.9. Applying the regularity theory of elliptic equation for (6.47), (cf. [12, Theorem 8.13]), one has

$$\|\psi\|_{H^3}^2\leqslant C(\|\psi_t\|_{H^1}^2+\|\psi\|_{H^2}^2+\||\nabla\psi|^2\|_{H^1}^2+\|u-U\|_{H^1}^2). \qquad (6.49)$$

By (6.31) and the Sobolev embedding theorem, we get

$$\begin{aligned}\||\nabla\psi|^2\|_{H^1}^2 &\leqslant C\int_{B_R}|\nabla\psi|^4 + C\sum_{i,j=1}^n\int_{B_R}|\nabla\psi|^2|\partial^2_{x_ix_j}\psi|^2\\ &\leqslant CN(t)^2\|\psi\|_{H^2}^2 + C\sum_{i,j=1}^n\left(\int_{B_R}|\nabla\psi|^4\right)^{\frac12}\left(\int_{B_R}|\partial^2_{x_ix_j}\psi|^4\right)^{\frac12}\\ &\leqslant C\|\psi\|_{H^2}^2 + C\|\psi\|_{H^2}^2\|\psi\|_{H^3}^2\\ &\leqslant C\|\psi\|_{H^2}^2 + CN(t)^2\|\psi\|_{H^3}^2.\end{aligned}$$

Substituting this inequality into (6.49), when $N(t) \ll 1$, we get

$$\|\psi\|_{H^3}^2 \leqslant C(\|\psi_t\|_{H^1}^2 + \|\psi\|_{H^2}^2 + \|u-U\|_{H^1}^2). \tag{6.50}$$

Multiplying (6.50) by $e^{\mu t}$ and integrating in t, by (6.35), (6.46) and Lemmas 6.7–6.9, we obtain

$$\int_0^t e^{\mu\tau}\|\psi(\cdot,\tau)\|_{H^3}^2 \leqslant C(\|\phi_0\|_{L^2}^2 + \|u_0-U\|_{H^1}^2 + \|\psi_0\|_{H^2}^2). \tag{6.51}$$

The desired estimate (6.43) follows from (6.48) and (6.51). □

Finally, we establish the H^2 estimate for $(u-U)$.

Lemma 6.12. *Let the assumptions of Proposition 6.6 hold. If $N(T) \ll 1$, then it holds*

$$e^{\mu t}\|(u(\cdot,t)-U(\cdot))\|_{H^2}^2 + \int_0^t e^{\mu\tau}\|(u-U)(\cdot,\tau)\|_{H^3}^2 \leqslant C(\|u_0-U\|_{H^2}^2 + \|\psi_0\|_{H^2}^2), \tag{6.52}$$

for any $t \in [0,T]$, where $C > 0$ is a constant independent of t.

Proof. To simplify the notation, we set $j := \nabla u - pu\nabla v$ and $J := \nabla U - pU\nabla V$. Then $(u-U)$ satisfies

$$\begin{cases}(u-U)_t = \nabla\cdot(j-J), & x\in B_R, t>0,\\ (j-J)\cdot\nu = 0, & x\in\partial B_R, t>0.\end{cases}$$

Differentiating the above equations with respect to t yields

$$\begin{cases}(u-U)_{tt} = \nabla\cdot(j-J)_t, & x\in B_R, t>0,\\ (j-J)_t\cdot\nu = 0, & x\in\partial B_R, t>0.\end{cases} \tag{6.53}$$

Multiplying (6.53) by $(u-U)_t$ and integrating the result by parts, we have

$$\frac12\frac{d}{dt}\int_{B_R}(u-U)_t^2 = -\int_{B_R}(j-J)_t\nabla(u-U)_t. \tag{6.54}$$

A direct calculation gives

$$(j-J)_t = \nabla(u-U)_t - p(u-U)_t\nabla V - p(u-U)_t\nabla\psi - p(u-U)\nabla\psi_t - pU\nabla\psi_t.$$

It then follows from (6.54) that

$$\begin{aligned}
&\frac{d}{dt}\int_{B_R}(u-U)_t^2+2\int_{B_R}|\nabla(u-U)_t|^2\\
&\leqslant C\int_{B_R}(|(u-U)_t|^2|\nabla V|^2+|u-U|^2|\nabla\psi_t|^2+U^2|\nabla\psi_t|^2)\\
&\quad+C\int_{B_R}|\nabla\psi|^2|(u-U)_t|^2+\frac{1}{2}\int_{B_R}|\nabla(u-U)_t|^2.
\end{aligned}\tag{6.55}$$

Noting $\int_{B_R}(u-U)_t dx = \frac{d}{dt}\int_{B_R}(u-U)dx = 0$, when $n = 2, 3$, it follows from Poincaré's inequality that

$$\|(u-U)_t\|_{L^4}\leqslant C\|\nabla(u-U)_t\|_{L^2},$$

which implies

$$\int_{B_R}|\nabla\psi|^2|(u-U)_t|^2\leqslant\left(\int_{B_R}|\nabla\psi|^4\right)^{\frac{1}{2}}\left(\int_{B_R}|(u-U)_t|^4\right)^{\frac{1}{2}}\leqslant CN(t)^2\int_{B_R}|\nabla(u-U)_t|^2,$$

where we have used $\|\nabla\psi(\cdot,t)\|_{L^4}\leqslant C\|\psi(\cdot,t)\|_{H^2}\leqslant CN(t)$. Thus if $N(t)\ll 1$, using the boundedness of ∇V and U, by (6.10), we get from (6.55) that

$$\frac{d}{dt}\int_{B_R}(u-U)_t^2+\int_{B_R}|\nabla(u-U)_t|^2\leqslant C\int_{B_R}(|(u-U)_t|^2+|\nabla\psi_t|^2).$$

Multiplying this inequality by $e^{\mu t}$ and integrating the result in t, by (6.46) and Lemma 6.10, we have

$$\begin{aligned}
&e^{\mu t}\int_{B_R}(u-U)_t^2+\int_0^t e^{\mu\tau}\int_{B_R}|\nabla(u-U)_t|^2\\
&\leqslant C(\|\phi_0\|_{L^2}^2+\|u_0-U\|_{H^1}^2+\|\psi_0\|_{H^2}^2+\|(u-U)_t(\cdot,0)\|_{L^2}^2)\\
&\leqslant C(\|u_0-U\|_{H^2}^2+\|\psi_0\|_{H^2}^2),
\end{aligned}\tag{6.56}$$

where we have used the following inequalities:

$$\begin{aligned}
&\|\phi_0\|_{L^2}^2\leqslant C\|u_0-U\|_{L^\infty}^2\leqslant C\|u_0-U\|_{H^2}^2,\\
&|(u-U)_t(\cdot,0)\|_{L^2}^2\leqslant C(\|u_0-U\|_{H^2}^2+\|\psi_0\|_{H^2}^2).
\end{aligned}$$

We next estimate $\|(u-U)(\cdot,t)\|_{H^2}$. Set $h := ue^{-pv}$ and $H := Ue^{-pV}$, then by the first equation of (6.6), h satisfies

$$\begin{cases}
h_t=\Delta h-phv_t+p\nabla h\nabla v, & x\in B_R,\ t>0,\\
\frac{\partial h}{\partial\nu}=0, & x\in\partial B_R,\ t>0,\\
h(x,0)=u_0(x)e^{-pv_0(x)}, & x\in B_R,
\end{cases}$$

and $(h-H)$ satisfies

$$\begin{cases} -\Delta(h-H) = -h_t - ph\psi_t + p\nabla(h-H)\nabla V + p\nabla(h-H)\nabla\psi + p\nabla H\nabla\psi, & x\in B_R,\\ \frac{\partial(h-H)}{\partial\nu} = 0, & x\in\partial B_R. \end{cases} \tag{6.57}$$

Thus, it holds

$$\int_{B_R} |\Delta(h-H)|^2 \leqslant C\int_{B_R} (h_t^2 + h^2\psi_t^2 + (|\nabla V|^2 + |\nabla\psi|^2)|\nabla(h-H)|^2 + |\nabla H|^2|\nabla\psi|^2). \tag{6.58}$$

A direct calculation gives

$$h-H = (u-U)e^{-p(V+\psi)} + Ue^{-pV}(e^{-p\psi}-1), \tag{6.59}$$

which implies

$$\|h\|_{L^\infty} \leqslant \|h-H\|_{L^\infty} + \|H\|_{L^\infty} \leqslant C,\ \|\nabla(h-H)\|_{L^2}^2 \leqslant C(\|u-U\|_{H^1}^2 + \|\psi\|_{H^1}^2),$$

and

$$\|h_t\|_{L^2}^2 \leqslant C(\|(u-U)_t\|_{L^2}^2 + \|\psi_t\|_{L^2}^2).$$

By the Hölder's inequality and the Sobolev embedding theorem for $n=2,3$, we have

$$\int_{B_R} |\nabla\psi|^2|\nabla(h-H)|^2 \leqslant \|\nabla\psi\|_{L^4}^2\|\nabla(h-H)\|_{L^4}^2 \leqslant C\|\psi\|_{H^2}^2\|h-H\|_{H^2}^2 \leqslant CN(t)^2\|h-H\|_{H^2}^2.$$

It then follows from (6.58) that

$$\|\Delta(h-H)\|_{L^2}^2 \leqslant C(\|(u-U)_t\|_{L^2}^2 + \|\psi_t\|_{L^2}^2 + \|u-U\|_{H^1}^2 + \|\psi\|_{H^1}^2) + CN(t)^2\|h-H\|_{H^2}^2. \tag{6.60}$$

As in (6.32), by the standard regularity theory for the Poisson equation, we get from (6.60) and (6.59) that

$$\begin{aligned} \|h-H\|_{H^2}^2 &\leqslant C(\|\Delta(h-H)\|_{L^2}^2 + \|h-H\|_{L^2}^2)\\ &\leqslant C(\|(u-U)_t\|_{L^2}^2 + \|\psi_t\|_{L^2}^2 + \|u-U\|_{H^1}^2 + \|\psi\|_{H^1}^2)\\ &\quad + CN(t)^2\|h-H\|_{H^2}^2 + C\|h-H\|_{L^2}^2\\ &\leqslant C(\|(u-U)_t\|_{L^2}^2 + \|\psi_t\|_{L^2}^2 + \|u-U\|_{H^1}^2 + \|\psi\|_{H^1}^2 + N(t)^2\|h-H\|_{H^2}^2). \end{aligned}$$

Thus, if $N(t) \ll 1$, then we have

$$\|h-H\|_{H^2}^2 \leqslant C(\|(u-U)_t\|_{L^2}^2 + \|\psi_t\|_{L^2}^2 + \|u-U\|_{H^1}^2 + \|\psi\|_{H^1}^2). \tag{6.61}$$

By (6.59), we also get

$$u-U = e^{p(V+\psi)}(h-H) - Ue^{p\psi}(e^{-p\psi}-1), \tag{6.62}$$

which in combination with (6.61) leads to

$$\|u-U\|_{H^2}^2 \leqslant C(\|h-H\|_{H^2}^2 + \|\psi\|_{H^2}^2) \leqslant C(\|(u-U)_t\|_{L^2}^2 + \|\psi_t\|_{L^2}^2 + \|u-U\|_{H^1}^2 + \|\psi\|_{H^2}^2).$$

It then follows from (6.56), (6.46) and Lemmas 6.8 and 6.10–6.11 that

$$e^{\mu t}\|u-U\|_{H^2}^2 \leqslant C(\|u_0-U\|_{H^2}^2+\|\psi_0\|_{H^2}^2). \tag{6.63}$$

As in (6.50), by the regularity theory of elliptic equation for (6.57), we are led to

$$\int_0^t e^{\mu\tau}\|(h-H)(\cdot,\tau)\|_{H^3}^2 \leqslant C\int_0^t e^{\mu\tau}(\|h_t\|_{H^1}^2+\|\psi_t\|_{H^1}^2+\|h-H\|_{H^2}^2+\|\psi\|_{H^3}^2).$$

Thanks to (6.62), we further have

$$\begin{aligned}\int_0^t e^{\mu\tau}\|(u-U)(\cdot,\tau)\|_{H^3}^2 &\leqslant C\int_0^t e^{\mu\tau}(\|(u-U)_t\|_{H^1}^2+\|\psi_t\|_{H^1}^2+\|u-U\|_{H^2}^2+\|\psi\|_{H^3}^2)\\ &\leqslant C(\|u_0-U\|_{H^2}^2+\|\psi_0\|_{H^2}^2),\end{aligned} \tag{6.64}$$

where we have used (6.35), (6.46), (6.56) and Lemmas 6.7–6.11. The desired estimate (6.52) follows from (6.63) and (6.64). □

Proof of Theorem 6.5.. First note that Proposition 6.6 is a direct consequence of Lemmas 6.11–6.12. The a priori estimate (6.11) guarantees that if $\|(u_0-U,v_0-V)\|_{H^2}$ is small enough, then $N(t)$ is small for all $t>0$. Therefore, applying the standard extension argument, we obtain the global well-posedness of the system (6.6) in $C([0,+\infty);H^2(B_R))$. Moreover, the estimate (6.11) implies the exponential stability of the steady-state (U,V) in $H^2(B_R)$. By the Sobolev embedding theorem, we further have

$$\|(u-U)(\cdot,t)\|_{L^\infty} \leqslant C\|(u-U)(\cdot,t)\|_{H^2} \leqslant C\|u_0-U\|_{H^2}.$$

Thus,

$$u(x,t) \geqslant U(x)-C\|u_0-U\|_{H^2} \geqslant U_{\min}-C\|u_0-U\|_{H^2} > 0,$$

provided that $\|u_0-U\|_{H^2} < \delta_2 := \frac{U_{\min}}{C}$. This completes the proof. □

Proof of Theorem 6.2.. We complete the proof in two steps.

Step 1. We first show the uniqueness of solutions for the system (1.2) in the space Y_T defined by

$$Y_T := \{(u,w)(x,t) \mid u(x,t)>0, w(x,t)>0, (u,w)\in C([0,T];H^2(B_R))\}.$$

Without loss of generality, we only consider the case of $n=3$. Suppose that in Y_T the system (1.2) has two solutions (u_1,w_1) and (u_2,w_2) with the same initial data $(u_0,w_0)(x)$. Set $\Phi := u_1-u_2$ and $\Psi := w_1-w_2$. Then a direct calculation yields the equations of (Φ,Ψ)

$$\begin{cases}\Phi_t = \Delta\Phi - p\nabla\cdot\left(\frac{u_2}{w_1}\nabla\Psi - \frac{u_2\nabla w_2}{w_1w_2}\Psi + \frac{\nabla w_1}{w_1}\Phi\right), & x\in B_R,\ t>0,\\ \Psi_t = \varepsilon\Delta\Psi - w_1\Phi - u_2\Psi, & x\in B_R,\ t>0,\\ \Phi(x,0)=\Psi(x,0)=0, & x\in B_R,\\ \frac{\partial\Phi}{\partial\nu} - p(\frac{u_2}{w_1}\frac{\partial\Psi}{\partial\nu} - \frac{u_2\Psi}{w_1w_2}\frac{\partial w_2}{\partial\nu} + \frac{\Phi}{w_1}\frac{\partial w_1}{\partial\nu}) = 0,\ \Psi=0, & x\in\partial B_R,\ t>0.\end{cases} \tag{6.65}$$

Multiplying the first equation of (6.65) by Φ and integrating the equation on B_R, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{B_R}|\Phi|^2+\int_{B_R}|\nabla\Phi|^2\\
&=p\int_{B_R}\frac{u_2}{w_1}\nabla\Psi\nabla\Phi-p\int_{B_R}\frac{u_2\nabla w_2}{w_1w_2}\Psi\nabla\Phi+p\int_{B_R}\frac{\nabla w_1}{w_1}\Phi\nabla\Phi \qquad (6.66)\\
&\leqslant\frac{1}{8}\int_{B_R}|\nabla\Phi|^2+Cp^2\int_{B_R}\frac{u_2^2}{w_1^2}|\nabla\Psi|^2+Cp^2\int_{B_R}\frac{u_2^2|\nabla w_2|^2}{w_1^2w_2^2}\Psi^2+Cp^2\int_{B_R}\frac{|\nabla w_1|^2}{w_1^2}\Phi^2.
\end{aligned}$$

For convenience, set $Q_T:=B_R\times(0,T)$, $A_i(T):=\min_{(x,t)\in Q_T}w_i^2$ and $C_i(T):=\|(u_i,w_i)\|^2_{C([0,T];H^2)}$ for $i=1,2$. By the Hölder's inequality and the Sobolev inequality, we get

$$\int_{B_R}\frac{u_2^2}{w_1^2}|\nabla\Psi|^2\leqslant\frac{\|u_2\|^2_{L^\infty(Q_T)}}{\min_{(x,t)\in Q_T}w_1^2}\int_{B_R}|\nabla\Psi|^2\leqslant\frac{CC_2(T)}{A_1(T)}\int_{B_R}|\nabla\Psi|^2.$$

Similarly,

$$\begin{aligned}
\int_{B_R}\frac{u_2^2|\nabla w_2|^2}{w_1^2w_2^2}\Psi^2&\leqslant\frac{CC_2(T)}{A_1(T)A_2(T)}\left(\int_{B_R}|\nabla w_2|^4\right)^{\frac12}\left(\int_{B_R}|\Psi|^4\right)^{\frac12}\\
&\leqslant\frac{CC_2(T)}{A_1(T)A_2(T)}\|w_2\|^2_{H^2}\int_{B_R}|\nabla\Psi|^2\\
&\leqslant\frac{CC_2^2(T)}{A_1(T)A_2(T)}\int_{B_R}|\nabla\Psi|^2.
\end{aligned}$$

By the interpolation inequality, one has

$$\begin{aligned}
Cp^2\int_{B_R}\frac{|\nabla w_1|^2}{w_1^2}\Phi^2&\leqslant\frac{Cp^2}{A_1(T)}\left(\int_{B_R}|\nabla w_1|^4\right)^{\frac12}\left(\int_{B_R}|\Phi|^4\right)^{\frac12}\\
&\leqslant\frac{Cp^2}{A_1(T)}\|w_1\|^2_{H^2}\|\Phi\|^{\frac12}_{L^2}\|\Phi\|^{\frac32}_{L^6}\\
&\leqslant\frac{Cp^2C_1(T)}{A_1(T)}\|\Phi\|^{\frac12}_{L^2}\|\Phi\|^{\frac32}_{H^1}\\
&\leqslant\frac34\int_{B_R}|\nabla\Phi|^2+\left(\frac{Cp^8C_1^4(T)}{A_1^4(T)}+\frac{Cp^2C_1(T)}{A_1(T)}\right)\int_{B_R}|\Phi|^2.
\end{aligned}$$

It hence follows from (6.66) that

$$\frac{d}{dt}\int_{B_R}|\Phi|^2+\frac14\int_{B_R}|\nabla\Phi|^2\leqslant C(T)\int_{B_R}|\nabla\Psi|^2+C(T)\int_{B_R}\Phi^2. \qquad (6.67)$$

Multiplying the second equation of (6.65) by 2Ψ and integrating the equation on B_R, we get

$$\begin{aligned}
&\frac{d}{dt}\int_{B_R}|\Psi|^2+2\varepsilon\int_{B_R}|\nabla\Psi|^2\\
&=-2\int_{B_R}w_1\Phi\Psi-2\int_{B_R}u_2\Psi^2\\
&\leqslant\|w_1\|_{L^\infty(Q_T)}\int_{B_R}|\Phi|^2+\|w_1\|_{L^\infty(Q_T)}\int_{B_R}|\Psi|^2+2\|u_2\|_{L^\infty(Q_T)}\int_{B_R}|\Psi|^2\\
&\leqslant CC_1(T)\int_{B_R}(|\Phi|^2+|\Psi|^2).
\end{aligned}\tag{6.68}$$

Multiplying (6.68) by a large constant K and combining the equation with (6.67), we obtain

$$\frac{d}{dt}\int_{B_R}(|\Phi|^2+K|\Psi|^2)\leqslant C(T)\int_{B_R}(|\Phi|^2+K|\Psi|^2).$$

Noting $(\Phi,\Psi)(x,0)=(0,0)$, it follows from Gronwall's inequality that

$$\int_{B_R}(|\Phi|^2+K|\Psi|^2)=0 \text{ for any } t\in[0,T],$$

which implies

$$(\Phi,\Psi)(x,t)\equiv(0,0) \text{ on } Q_T,$$

and hence $(u_1,w_1)(x,t)\equiv(u_2,w_2)(x,t)$ on Q_T.

Step 2. We next construct a global solution for the system (1.2) in Y_T. Recall that v_0 and w_0 satisfy

$$v_0-V=\log\left(1+\frac{w_0-W}{W}\right).\tag{6.69}$$

By the Sobolev embedding theorem, it holds that $\|w_0-W\|_{L^\infty}\leqslant C\|w_0-W\|_{H^2}$. Then with $\min\limits_{x\in\overline{B_R}}W(x)>0$, when $\|w_0-W\|_{H^2}\ll 1$, by Taylor's formula, we get from (6.69) that

$$\|v_0-V\|_{H^2}\leqslant C_0\|w_0-W\|_{H^2}.$$

Now we take $\delta_1=\frac{\delta_2}{1+C_0}$, where δ_2 is the constant obtained in Theorem 6.5. If (u_0,w_0) satisfies

$$\|(u_0-U,w_0-W)\|_{H^2}\leqslant\delta_1,$$

then the initial value of the system (6.6) satisfies

$$\|(u_0-U,v_0-V)\|_{H^2}\leqslant(1+C_0)\|(u_0-U,w_0-W)\|_{H^2}\leqslant(1+C_0)\delta_1=\delta_2.$$

Thus, by Theorem 6.5, the system (6.6) has a unique global radial solution (u,v) satisfying $u(x,t)>0$, $(u,v)\in C([0,+\infty);H^2(B_R))$ and the exponential convergence (6.8). Now we define $w(x,t)=e^{v(x,t)}$. Then it is easy to verify that $(u,w)\in C([0,+\infty);H^2(B_R))$ is the unique global solution of the original system (1.2). Moreover, since $w - W = e^v - e^V = e^V(e^\psi - 1)$ due to (6.14), we further have by the Taylor's formula again

$$\|w - W\|_{H^2}^2 \leqslant C\|\psi\|_{H^2}^2 \leqslant Ce^{-\mu t}(\|u_0 - U\|_{H^2}^2 + \|\psi_0\|_{H^2}^2),$$

which, along with the convergence of $(u - U)$ in Theorem 6.5, gives the estimate (6.5). We complete the proof. □

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