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CAUCHY PROBLEM OF THE MAGNETOHYDRODYNAMIC BURGERS SYSTEM*

HAI-YANG JIN†, ZHI-AN WANG‡, AND LINJIE XIONG§

Abstract. In this paper, the asymptotic nonlinear stability of solutions to the Cauchy problem of a strongly coupled Burgers system arising in magnetohydrodynamic (MHD) turbulence [Fleischer and Diamond (2000), Yanase (1997)] is established. It is shown that, as time tends to infinity, the solutions of the Cauchy problem converge to constant states or rarefaction waves with large data, or viscous shock waves with arbitrarily large amplitude, where the precise asymptotic behavior depends on the relationship between the left and right end states of the initial value. Our results confirm the existence of shock waves (or turbulence) numerically found in [Fleischer and Diamond (2000), Yanase (1997)].

Key words. MHD Burgers system, rarefaction waves, viscous shock waves, nonlinear stability, weighted energy estimates.

AMS subject classifications. 35A18, 35B35, 35C06, 35C07, 35K45.

1. Introduction

To investigate the small scale structure of magnetohydrodynamic turbulence, a one-dimensional magnetohydrodynamic (MHD) Burgers system was derived in [1, 27] as follows:

$$\begin{cases} u_t+(uv)_x=Du_{xx}, & x\in\mathbb{R},\quad t>0,\\ v_t+\left(\frac{1}{2}u^2+\frac{1}{2}v^2\right)_x=\mu v_{xx}, & x\in\mathbb{R},\quad t>0, \end{cases} \tag{1.1}$$

where $u(x,t)$ is the magnetic field, $v(x,t)$ stands for the velocity field of the fluid, and D and μ are positive diffusivity. It was also shown in [1] that MHD Burgers system (1.1) is the simplest possible system which allows energy transfer between the fluid and magnetic field excitations where the turbulence is represented by an ensemble of Alfvenic shock waves on a homogeneous density background. Moreover system (1.1) may also model the opposite limit of a fluid-dominated (i.e., unmagnetized) system with arbitrary density variations reacting to an adiabatic pressure [1]. For more applications of (1.1), we refer the readers to [2, 10, 25]. Using the Elsässer variables $e^{\pm}=v\pm u$, system (1.1) is transformed into

$$\frac{\partial e^{\pm}}{\partial t}+\frac{\partial}{\partial x}\frac{(e^{\pm})^2}{2}=\frac{\mu+D}{2}\frac{\partial^2 e^{\pm}}{\partial x^2}+\frac{\mu-D}{2}\frac{\partial^2 e^{\mp}}{\partial x^2}. \tag{1.2}$$

If $D=\mu$, then e^- and e^+ do not interact with each other and system (1.1) can be reduced to two independent viscous Burgers equations for e^+ and e^-, respectively. Hence the system can be trivially solved via the Hopf-Cole transformation. The nontrivial case $D\neq\mu$ reveals more interesting interactions between the fluid and the

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†Department of Applied Mathematics, Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong (haiyang.jin@connect.polyu.hk).

‡Department of Applied Mathematics, Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong (mawza@polyu.edu.hk).

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§School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, P.R. China (xlj@whu.edu.cn).

magnetic field [1]. In the present paper, we focus on the non-trivial case $D \neq \mu$ and investigate the asymptotic behavior of solutions of (1.1) with the initial data

$$(u,v)(x,0)=(u_0,v_0)(x)\to\begin{cases}(u_-,v_-), & \text{as}\quad x\to-\infty,\\ (u_+,v_+), & \text{as}\quad x\to+\infty.\end{cases} \tag{1.3}$$

The standard hyperbolic theory (cf. [11, 22]) predicts that the time asymptotic behavior of solutions of the Cauchy problem (1.1)-(1.3) are closely related to the following Riemann problem:

$$\begin{cases}u_t+(uv)_x=0, & x\in\mathbb{R},\quad t>0,\\ v_t+\left(\frac{1}{2}u^2+\frac{1}{2}v^2\right)_x=0, & x\in\mathbb{R},\quad t>0,\\ (u,v)(x,0)=(u_0^r,v_0^r)(x)=\begin{cases}(u_-,v_-), & x<0,\\ (u_+,v_+), & x>0.\end{cases}\end{cases} \tag{1.4}$$

Writing the equations in (1.4) in the vector form

$$\begin{pmatrix}u\\ v\end{pmatrix}_t+\begin{pmatrix}v & u\\ u & v\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}_x=0, \tag{1.5}$$

we see that the Jacobian matrix $A:=\begin{pmatrix}v & u\\ u & v\end{pmatrix}$ has two real distinct eigenvalues

$$\lambda_1(u,v)=v-u,\quad \lambda_2(u,v)=v+u,$$

with corresponding eigenvectors

$$r_1(u,v)=\begin{pmatrix}1\\ -1\end{pmatrix},\quad r_2(u,v)=\begin{pmatrix}1\\ 1\end{pmatrix}.$$

Therefore it follows that $\nabla\lambda_1(u,v)\cdot r_1(u,v)=-2<0$ and $\nabla\lambda_2(u,v)\cdot r_2(u,v)=2>0$. This shows that the hyperbolic system (1.4) is genuinely nonlinear. By the hyperbolic theory [22], the solutions of the Riemann problem (1.4) are made up of three types of elementary waves (solutions): constant states, rarefaction waves, and shock waves. Moreover, we point out that the rarefaction curves of (1.5) are straight lines in the u-v plane (see Section 2), and hence the hyperbolic system (1.5) is of Temple class [24].

In this paper we shall show that as time goes to infinity, the solution of the Cauchy problem (1.1)-(1.3) will tend to a constant solution if $(u_+,v_+)=(u_-,v_-)$, or a rarefaction wave if $\lambda_i(u_-,v_-)<\lambda_i(u_+,v_+)$, or a viscous shock wave (i.e. traveling wave) if $\lambda_i(u_-,v_-)>\lambda_i(u_+,v_+)$. Specifically, the following results are proved. If the right state equals the left state, say $(u_+,v_+)=(u_-,v_-)=(\bar{u},\bar{v})$, then the solution of (1.1) with large data (1.3) will eventually approach the constant state $(\bar{u},\bar{v})$. If the right state (u_+,v_+) is connected to the left state (u_-,v_-) by a rarefaction wave, then the Cauchy problem (1.1)-(1.3) has a unique global solution which tends to the rarefaction wave of the Riemann problem (1.4) with large data. Finally if the initial value (1.3) is a small perturbation of a viscous shock wave (traveling wave), then the solution of (1.1)-(1.3) will asymptotically converge to this viscous shock wave with a proper translation, where the wave amplitude can be arbitrarily large. Our results analytically confirm the existence of shock-type waves (and hence turbulence) numerically obtained in both papers [27] and [1].

Mathematical studies on the asymptotics toward rarefaction/shock waves for viscous conservation laws have been undertaken for a long time (e.g. see [7, 17, 19]). For the general 2×2 viscous conservation laws

$$\begin{cases} u_t+[f_1(u,v)]_x=Du_{xx}, & x\in\mathbb{R},\quad t>0,\\ v_t+[f_2(u,v)]_x=\mu v_{xx}, & x\in\mathbb{R},\quad t>0 \end{cases} \tag{1.6}$$

with initial data (1.3), Xin [26] and Yang and Zhao [28] established the time asymptotic stability of weak rarefaction waves and strong rarefaction waves with small initial data, respectively. The main hypothesis on the structure of the system (1.6) is the strong coupling in the sense that

$$\frac{\partial f_1(u,v)}{\partial v}\cdot\frac{\partial f_2(u,v)}{\partial u}\neq 0, \tag{1.7}$$

which is satisfied by the system (1.1). To the best of our knowledge, for the strongly coupled system of conservation laws with large initial data, very few results are known. The asymptotic stability of viscous shock waves for the general system of conservation laws has been extensively investigated over many years. Most of results (if not all) require the wave amplitude to be small (e.g. see [3, 15, 16, 23]). The main contributions of this paper has two facets. First, exploiting the peculiar coupling structure of the MHD Burgers system (1.1), the nonlinear stability of strong rarefaction waves of (1.1)-(1.3) is established with large data. Second, the asymptotic stability of viscous shock waves of (1.1)-(1.3) is proved for large wave amplitude. Usually these results can not be proved for general hyperbolic systems as mentioned above. Finally, we mention that the asymptotic stability of viscous shock waves to (1.1) with $u_+>0$ was previously established in [6] based on the idea of [13, 14, 18] by leaving open the case $u_+=0$, which causes a singularity in the energy estimates. In this paper, we will resolve this challenging case (i.e. $u_+=0$) by invoking the weighted energy estimates inspired by the ideas of [8, 9, 12, 21].

The rest of the paper is organized as follows. In Section 2, we solve the Riemann problem (1.4) and then state the main results for the Cauchy problem (1.1)-(1.3). Then we prove the large-time behavior of solutions with constant states in Section 3. In Section 4, we show the stability of rarefaction waves. The proof of nonlinear stability of viscous shock waves is given in Section 5.

2. Preliminaries and main results

In this section, we first briefly solve the Riemann problem (1.4) in the class of functions consisting of constant states, separated by rarefaction waves or shock waves. We begin with the rarefaction waves of (1.4) by setting $\xi=x/t$. Then substituting it into the equations of (1.5), we find that (u_ξ,v_ξ) is an eigenvector of A for the eigenvalue ξ. Because the matrix A has two real and distinct eigenvalues λ_1 and λ_2, there are two families of rarefaction waves: 1-rarefaction waves and 2-rarefaction waves. The eigenvector (u_ξ,v_ξ) associated with the first eigenvalue λ_1 satisfies

$$\begin{pmatrix} u & u\\ u & u\end{pmatrix}\begin{pmatrix} u_\xi\\ v_\xi\end{pmatrix}=0, \tag{2.1}$$

which gives $u_\xi+v_\xi=0$ thanks to $u\neq 0$. This gives $\frac{du}{dv}=-1$. Integrating it, we obtain the 1-rarefaction curve $\mathrm{R}_1(u_-,v_-)$ as

$$\mathrm{R}_1(u_-,v_-)=\{(u,v)|u=-v+u_-+v_-,v>v_-\}, \tag{2.2}$$

where the entropy condition $\lambda_1(u_-,v_-)<\lambda_1(u,v)$ has been used to guarantee the uniqueness of the 1-rarefaction wave. Similarly, the 2-rarefaction curve $\mathrm{R}_2(u_-,v_-)$ can be represented as

$$\mathrm{R}_2(u_-,v_-)=\{(u,v)|u=v+u_--v_-,v>v_-\}. \tag{2.3}$$

Moreover (1.4) also has two distinct types of shock waves: 1-shock waves and 2-shock waves. To see this, we use the following jump condition (see [22, 15.11]):

$$\begin{cases} uv-u_-v_-=s(u-u_-), \\ \frac{1}{2}(u^2+v^2)-\frac{1}{2}(u_-^2+v_-^2)=s(v-v_-), \end{cases} \tag{2.4}$$

where s is the speed of the discontinuity (wave speed). Subtracting the first equation from the second equation of (2.4), we have

$$(u-v-u_-+v_-)(u-v+u_--v_-+2s)=0. \tag{2.5}$$

For 1-shock waves, the entropy condition $v-u=\lambda_1(u,v)<\lambda_1(u_-,v_-)=v_--u_-$ implies $u-v-u_-+v_->0$. Hence, from (2.5), we obtain

$$u-v+u_--v_-+2s=0. \tag{2.6}$$

Clearly $u\neq u_-$ for otherwise we have $v=v_-$ from the first equation of (2.4), and then $(u,v)=(u_-,v_-)$ is not a shock curve. Hence the first equation of (2.4) gives

$$s=\frac{uv-u_-v_-}{u-u_-}. \tag{2.7}$$

Then substituting (2.7) into (2.6), one can derive the 1-shock curve $\mathrm{S}_1(u_-,v_-)$ as

$$\mathrm{S}_1(u_-,v_-)=\{(u,v)|u=-v+u_-+v_-,v<v_-\}. \tag{2.8}$$

To deduce 2-shock curves, we add the equations in (2.4) to obtain

$$(u+v-u_--v_-)(u+v+u_-+v_--2s)=0. \tag{2.9}$$

Similarly, by using the entropy condition $v+u=\lambda_2(u,v)<\lambda_2(u_-,v_-)=v_-+u_-$ and (2.7), we obtain the 2-shock curve $\mathrm{S}_2(u_-,v_-)$ as

$$\mathrm{S}_2(u_-,v_-)=\{(u,v)|u=v+u_--v_-,v<v_-\}. \tag{2.10}$$

Then curves R_1, R_2, S_1, and S_2 divide the u-v plane into four disjoint open regions I, II, III, IV defined as follows (see also figure 2.1):

$$\begin{aligned} \mathrm{I}=\mathrm{R}_1\mathrm{R}_2(u_-,v_-)&:=\{(u,v)|-v+u_-+v_-<u<v+u_--v_-\}, \\ \mathrm{II}=\mathrm{R}_1\mathrm{S}_2(u_-,v_-)&:=\{(u,v)|u<-v+u_-+v_-,u<v+u_--v_-\}, \\ \mathrm{III}=\mathrm{S}_1\mathrm{S}_2(u_-,v_-)&:=\{(u,v)|v+u_--v_-<u<-v+u_-+v_-\}, \\ \mathrm{IV}=\mathrm{S}_1\mathrm{R}_2(u_-,v_-)&:=\{(u,v)|u>-v+u_-+v_-,u>v+u_--v_-\}. \end{aligned} \tag{2.11}$$

Hence, depending on the relationship between the end states (u_+,v_+) and (u_-,v_-),

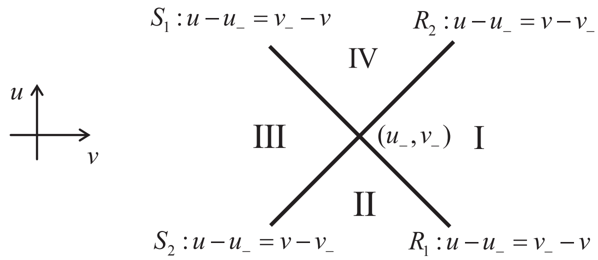

FIG. 2.1. *u-v plane*

the solutions of Riemann problem (1.4) are described as

$$
\begin{cases}
1-\text{rarefaction waves} & \text{if } u_+ + v_+ = u_- + v_- \text{ and } v_+ > v_-,\\
1-\text{shock waves} & \text{if } u_+ + v_+ = u_- + v_- \text{ and } v_+ < v_-,\\
2-\text{rarefaction waves} & \text{if } u_+ - v_+ = u_- - v_- \text{ and } v_+ > v_-,\\
2-\text{shock waves} & \text{if } u_+ - v_+ = u_- - v_- \text{ and } v_+ < v_-,\\
\text{Composite waves of two rarefaction waves} & \text{if } u_+ + v_+ > u_- + v_- \\
 & \text{and } u_+ - v_+ < u_- - v_-,\\
\text{Composite waves of two viscous shock waves} & \text{if } u_+ + v_+ < u_- + v_- \\
 & \text{and } u_+ - v_+ > u_- - v_-,\\
1-\text{rarefaction waves and } 2-\text{shock waves} & \text{if } u_+ + v_+ < u_- + v_- \\
 & \text{and } u_+ - v_+ < u_- - v_-,\\
1-\text{shock waves and } 2-\text{rarefaction waves} & \text{if } u_+ + v_+ > u_- + v_- \\
 & \text{and } u_+ - v_+ > u_- - v_-.
\end{cases}
$$

In this paper, we only consider the nonlinear stability of single waves and leave the stability of composite waves for future study. Then we are ready to state our main results in subsequent subsections. Before proceeding, we introduce some basic notations. As usual, $H^k(\mathbb{R})$ denotes the usual k-th order Sobolev space on $\mathbb{R}$ with the norm given by $\|f\|_{H^k(\mathbb{R})} := \left(\sum_{j=0}^k \|\partial_x^j f\|^2_{L^2(\mathbb{R})}\right)^{1/2}$. $H^k_w(\mathbb{R})$ denotes the weighted space of measurable functions f so that $\sqrt{w}\partial_x^j f \in L^2$ for $0 \le j \le k$ with the norm given by $\|f\|_{H^k_w(\mathbb{R})} := \left(\sum_{j=0}^k \int_{\mathbb{R}} w(x)|\partial_x^j f|^2 dx\right)^{1/2}$. Denote $\|\cdot\| := \|\cdot\|_{L^2(\mathbb{R})}$, $\|\cdot\|_k := \|\cdot\|_{H^k(\mathbb{R})}$ and $\|\cdot\|_{k,w} := \|\cdot\|_{H^k_w(\mathbb{R})}$ for simplicity. Moreover, we denote $\|(f,g)\|^2 = \|f\|^2 + \|g\|^2$ and $\|(f,g)\|_k^2 = \|f\|_k^2 + \|g\|_k^2$.

2.1. Constant states. If the end states (u_-, v_-) and (u_+, v_+) are connected by a constant, say $(u_+, v_+) = (u_-, v_-) =: (\bar{u}, \bar{v})$, and if the initial value (1.3) is a perturbation of the constant state $(\bar{u}, \bar{v})$ in $H^1(\mathbb{R})$, we have the following global asymptotic stability results.

THEOREM 2.1. *Let $(u_0 - \bar{u}, v_0 - \bar{v}) \in H^1(\mathbb{R})$. Then there exists a unique global solution $(u,v)(x,t)$ to the Cauchy problem* (1.1)-(1.3)*, which satisfies*

$$(u - \bar{u}, v - \bar{v}) \in C([0,\infty); H^1) \cap L^2((0,\infty); H^2).$$

Furthermore, the solution has the following asymptotic stability:

$$\sup_{x\in\mathbb{R}}|(u,v)(x,t)-(\bar{u},\bar{v})|\to 0, \quad \text{as} \quad t\to+\infty. \tag{2.12}$$

2.2. Stability of rarefaction waves. Without loss of generality, we consider 1-rarefaction wave solutions $(u^r,v^r)(x/t)$ of the Riemann problem (1.4) only, and our analysis can be directly applied to 2-rarefaction wave. Using (2.2), we can separate the variables u and v in (1.4) such that u satisfies the Riemann problem

$$\begin{cases} u_t+(u_-+v_--2u)u_x=0, \\ u(x,0)=u_0^r(x)=\begin{cases} u_-, & x<0, \\ u_+, & x>0, \end{cases} \end{cases} \tag{2.13}$$

and v satisfies the Riemann problem

$$\begin{cases} v_t+(2v-u_--v_-)v_x=0, \\ v(x,0)=v_0^r(x)=\begin{cases} v_-, & x<0, \\ v_+, & x>0. \end{cases} \end{cases} \tag{2.14}$$

Employing the method of characteristics, we can solve (2.13) and obtain the rarefaction wave $u^r(x/t)$ as follows:

$$u^r(x/t)=\begin{cases} u_-, & \frac{x}{t}\le v_--u_-, \\ \frac{u_-+v_-}{2}-\frac{x}{2t}, & v_--u_-\le\frac{x}{t}\le v_+-u_+, \\ u_+, & \frac{x}{t}\ge v_+-u_+. \end{cases} \tag{2.15}$$

Similarly, the rarefaction wave $v^r(x/t)$ of (2.14) can be obtained as

$$v^r(x/t)=\begin{cases} v_-, & \frac{x}{t}\le v_--u_-, \\ \frac{u_-+v_-}{2}+\frac{x}{2t}, & v_--u_-\le\frac{x}{t}\le v_+-u_+, \\ v_+, & \frac{x}{t}\ge v_+-u_+. \end{cases} \tag{2.16}$$

Then the result on asymptotic stability of the 1-rarefaction waves $(u^r,v^r)(x/t)$ is as follows.

THEOREM 2.2. *Let $(u_+,v_+)\in \mathrm{R}_1(u_-,v_-)$ and $v_+>v_-$. If $(u_0-u_0^r,v_0-v_0^r)\in L^2(\mathbb{R})$ and $(u_{0x},v_{0x})\in L^2(\mathbb{R})$, then the Cauchy problem* (1.1)-(1.3) *has a unique global solution (u,v) satisfying*

$$\begin{cases} (u-u^r,v-v^r)\in C([0,\infty);L^2)\cap L^\infty((0,\infty);L^2), \\ (u_x,v_x)\in C([0,\infty);L^2)\cap L^\infty((0,\infty);L^2)\cap L^2((0,\infty);H^1), \end{cases}$$

and

$$\sup_{x\in\mathbb{R}}|(u,v)(x,t)-(u^r,v^r)(x/t)|\to 0, \quad \text{as} \quad t\to+\infty. \tag{2.17}$$

REMARK 2.1. If $(u_+,v_+)\in \mathrm{R}_2(u_-,v_-)$ and $v_+>v_-$, then a similar stability result can be obtained.

2.3. Stability of viscous shock waves. The existence of traveling wave solutions of (1.1) with $0\le u_+<u_-$ and $0\le v_+<v_-$ was established in [6] by the phase plane analysis and the nonlinear stability of traveling wave solutions was prove only for $u_+>0$ by the method of energy estimates, whereas the stability for $u_+=0$ remains open. In this paper, we shall solve this open question by using the weighted energy estimates. Toward this end, we identify the decay rates of traveling wave solutions as $z\to\pm\infty$ and choose appropriate exponential weight functions. For completeness, we shall briefly recall the existence of traveling wave solutions for $u_+=0$ and derive the asymptotic decay rates of traveling wave solutions.

The traveling wave solution of (1.1) with (1.3) is a special solution in the form

$$(u,v)(x,t)=(U,V)(z),\quad z=x-st,$$

where $(U,V)\in C^\infty(\mathbb{R})$ satisfies

$$\begin{cases}-sU'+(UV)'=DU'',\\ -sV'+\frac{1}{2}\left(U^2+V^2\right)'=\mu V'',\end{cases}\tag{2.18}$$

with boundary condition

$$U(\pm\infty)=u_\pm,\quad V(\pm\infty)=v_\pm,\quad U'(\pm\infty)=V'(\pm\infty)=0,\tag{2.19}$$

where $'=\frac{d}{dz}$. Integrating (2.18) once yields that

$$\begin{cases}DU'=-sU+UV+\varrho_1,\\ \mu V'=-sV+\frac{1}{2}(U^2+V^2)+\varrho_2,\end{cases}\tag{2.20}$$

where ϱ_1 and ϱ_2 are constants satisfying

$$\begin{cases}\varrho_1=su_+-u_+v_+=su_--u_-v_-,\\ \varrho_2=sv_+-\frac{1}{2}(u_+^2+v_+^2)=sv_--\frac{1}{2}(u_-^2+v_-^2),\end{cases}$$

which gives

$$\begin{cases}s(u_+-u_-)=u_+v_+-u_-v_-,\\ s(v_+-v_-)=\frac{1}{2}(u_+^2+v_+^2)-\frac{1}{2}(u_-^2+v_-^2).\end{cases}\tag{2.21}$$

Then (2.21) with $u_+=0$ yields

$$s^2-v_-s=0,\tag{2.22}$$

and hence $s=0$ or $s=v_-$, which corresponds to the wave speed of the 1st and 2nd characteristic family of shock waves of (1.1). If $(u_+,v_+)\in \mathrm{S}_1(u_-,v_-)$, using (2.8), we have $u_+=-v_++u_-+v_-$ and $v_+<v_-$, which yield $0=u_+>u_-$ and $v_+<v_-$. Similarly, when $(u_+,v_+)\in \mathrm{S}_2(u_-,v_-)$, from (2.10), we obtain $u_+-u_-=v_+-v_-$ and $v_+<v_-$, which imply that

$$0=u_+<u_-\quad\text{and}\quad v_+<v_-.\tag{2.23}$$

In this paper, we only consider the case $s=v_-$, for the analysis for $s=0$ is similar. We first have the following existence results for the 2-shock profile $(U,V)(x-st)$.

LEMMA 2.3. *Let $u_\pm$ and $v_\pm$ satisfy* (2.23)*. Then there exists a monotone shock profile $(U,V)(x-st)$ to the system* (2.18)*-*(2.19)*, with wave speed $s=v_-$, which is unique up to a translation and satisfies $U_z<0$, $V_z<0$. Furthermore, the solution profile $(U,V)(x-st)$ decays exponentially at $\pm\infty$ with rates*

$$\begin{aligned} U-u_\pm &\sim e^{\sigma_\pm z}, \text{ as } z\to\pm\infty,\\ V-v_\pm &\sim e^{\sigma_\pm z}, \text{ as } z\to\pm\infty, \end{aligned} \tag{2.24}$$

where

$$\sigma_-=\frac{u_-}{\sqrt{D\mu}},\quad \sigma_+=\begin{cases}\frac{v_+-s}{D}, & D>\mu,\\ \frac{v_+-s}{\mu}, & D<\mu.\end{cases} \tag{2.25}$$

Proof. The existence of monotone shock profiles $(U,V)(x-st)$ to system (2.18)-(2.19) has been proved in [6] by phase plane analysis. It remains only to derive the asymptotic decay rates, which are eigenvalues of the linearized system at equilibria $(u_\pm,v_\pm)$. To see this, we linearize the system (2.20) at $(u_\pm,v_\pm)$ and obtain the corresponding Jacobian matrix

$$J(u_\pm,v_\pm)=\begin{bmatrix}\frac{v_\pm-s}{D} & \frac{u_\pm}{D}\\ \frac{u_\pm}{\mu} & \frac{v_\pm-s}{\mu}\end{bmatrix}, \tag{2.26}$$

whose eigenvalue σ satisfies

$$\sigma^2+\frac{D+\mu}{D\mu}(s-v_\pm)\sigma+\frac{(s-v_\pm)^2-u_\pm^2}{D\mu}=0. \tag{2.27}$$

By (2.23) and $s=v_-$, we can readily check that the equilibrium (u_-,v_-) is a saddle and (u_+,v_+) is a stable node. Then solving the equation (2.27), we obtain the decay rates as announced. ∎

Then we proceed to consider the asymptotic stability of traveling wave solutions obtained in Lemma 2.3 under the small initial perturbation of the form

$$\int_{-\infty}^{+\infty}\begin{pmatrix}u_0(x)-U(x)\\ v_0(x)-V(x)\end{pmatrix}dx=x_0\begin{pmatrix}u_+-u_-\\ v_+-v_-\end{pmatrix}+\beta r_1(u_-,v_-). \tag{2.28}$$

The coefficients x_0 and β are uniquely determined by the initial data $(u_0(x),v_0(x))$. When $\beta\neq 0$, the diffusion wave will appear. The stability of viscous shock waves with a diffusion wave for small wave strength have been investigated previously (e.g. see [15, 23]). The stability of shock waves with a diffusion wave and large wave strength still remains open up to present. In this paper we do not consider the diffusion wave (i.e. assuming $\beta=0$) but consider large wave strength. Then by conservation law (1.1), we can derive that

$$\begin{aligned} &\int_{-\infty}^{+\infty}\begin{pmatrix}u(x,t)-U(x+x_0-st)\\ v(x,t)-V(x+x_0-st)\end{pmatrix}dx=\int_{-\infty}^{+\infty}\begin{pmatrix}u_0(x)-U(x+x_0)\\ v_0(x)-V(x+x_0)\end{pmatrix}dx\\ =&\int_{-\infty}^{+\infty}\begin{pmatrix}u_0(x)-U(x)\\ v_0(x)-V(x)\end{pmatrix}dx+\int_{-\infty}^{+\infty}\begin{pmatrix}U(x)-U(x+x_0)\\ V(x)-V(x+x_0)\end{pmatrix}dx\\ =&\int_{-\infty}^{+\infty}\begin{pmatrix}u_0(x)-U(x)\\ v_0(x)-V(x)\end{pmatrix}dx-x_0\begin{pmatrix}u_+-u_-\\ v_+-v_-\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}. \end{aligned} \tag{2.29}$$

Thus we decompose the solution of (1.1) into the form

$$(u,v)(x,t)=(U,V)(x+x_0-st)+(\phi_z,\psi_z)(z,t), \tag{2.30}$$

where

$$(\phi(z,t),\psi(z,t))=\int_{-\infty}^{z}(u(y,t)-U(y+x_0-st),v(y,t)-V(y+x_0-st))\,dy.$$

Clearly, for all $t>0$, if follows from (2.29) that

$$\phi(\pm\infty,t)=\psi(\pm\infty,t)=0.$$

Without loss of generality we may assume that $x_0=0$, for otherwise we make a translation of the traveling wave solutions. Hence, the initial value of the perturbation (ϕ,ψ) is given by

$$(\phi_0,\psi_0)(z)=\int_{-\infty}^{z}(u_0-U,v_0-V)(y)dy. \tag{2.31}$$

Then we have the following stability results on the traveling wave solutions.

THEOREM 2.4. *Let* (2.23) *hold, and let* $(U,V)(x-st)$ *be a traveling wave solution obtained in Lemma* 2.3. *If* $D\geq\mu$, *there exists a constant* $\varepsilon_0>0$ *such that if* $\|u_0-U\|_{1,w}+\|v_0-V\|_{1,w}+\|(\phi_0,\psi_0)\|_w\leq\varepsilon_0$, *then the Cauchy problem* (1.1)-(1.3) *has a unique global solution* $(u,v)(x,t)$ *satisfying*

$$(u-U,v-V)\in C([0,\infty);H_w^1)\cap L^2((0,\infty);H_w^2), \tag{2.32}$$

where the weight function w *is defined as*

$$w(z):=1+e^{\eta z},\ \eta=\frac{s-v_+}{D}>0. \tag{2.33}$$

Furthermore, the solution has the following asymptotic stability:

$$\sup_{x\in\mathbb{R}}|(u,v)(x,t)-(U,V)(x-st)|\to 0,\quad \text{as}\quad t\to+\infty. \tag{2.34}$$

REMARK 2.2. To establish the L^2-energy estimates, the conditions $D\geq\mu$ is needed, see (5.9). The nonlinear stability result for the case $D<\mu$ still remains unknown.

REMARK 2.3. When $u_+=0$ and $D\geq\mu$, it can be easily verified that there exist two constants $C_2>C_1>0$ such that the traveling wave solution (U,V) obtained in Lemma 2.3 satisfies

$$C_1w(z)\leq\frac{1}{U(z)}\leq C_2w(z)\quad\text{for all } z\in\mathbb{R}. \tag{2.35}$$

3. Proof of Theorem 2.1

In this section, we shall prove Theorem 2.1. For the case $(u_+,v_+)=(u_-,v_-)=(\bar{u},\bar{v})$, we seek the solution of (1.1)-(1.3) in the following solution space:

$$X_1(0,T)=\{(u,v):(u-\bar{u},v-\bar{v})\in C([0,T];H^1);(u_x,v_x)\in L^2((0,T);H^1)\}.$$

Then Theorem 2.1 is a consequence of the following proposition.

PROPOSITION 3.1. *There exists a unique global solution* $(u,v)\in X_1(0,\infty)$ *to* (1.1)-(1.3) *such that*

$$\|(u-\bar{u},v-\bar{v})\|_1^2+D\int_0^\infty \|u_x(\cdot,t)\|_1^2dt+\mu\int_0^\infty \|v_x(\cdot,t)\|_1^2dt\le C\|(u_0-\bar{u},v_0-\bar{v})\|_1^2. \tag{3.1}$$

Next, we prove Proposition 3.1 by continuing a unique local solution with the a *priori* estimates. The construction on the local existence of solutions is standard, and is based on an iteration argument and fixed point Theorem (cf. [5]). We omit the details for brevity. Hereafter, we denote $\int_{\mathbb{R}} f dx =: \int f dx$ for convenience and use C to denote a generic positive constant which may vary in the context.

LEMMA 3.2 (Local existence). *If* $(u_0-\bar{u},v_0-\bar{v})\in H^1(\mathbb{R})$, *then there exists a positive constant* T_0 *such that the Cauchy problem* (1.1)-(1.3) *admits a unique smooth solution* $(u,v)\in X_1(0,T_0)$ *satisfying*

$$\|(u(\cdot,t)-\bar{u},v(\cdot,t)-\bar{v})\|_1\le 2\|(u_0-\bar{u},v_0-\bar{v})\|_1, \quad \textit{for all} \quad 0\le t\le T_0. \tag{3.2}$$

PROPOSITION 3.3 (A *priori* estimates). *Suppose the Cauchy problem* (1.1)-(1.3) *has a solution* $(u,v)\in X_1(0,T)$ *for some* $T>0$. *Then there exists a constant* C *independent of* T *such that*

$$\begin{aligned}&\|(u(\cdot,t)-\bar{u},v(\cdot,t)-\bar{v})\|_1^2+D\int_0^t \|u_x(\cdot,\tau)\|_1^2d\tau+\mu\int_0^t \|v_x(\cdot,\tau)\|_1^2d\tau\\ \le& C\|(u_0-\bar{u},v_0-\bar{v})\|_1^2.\end{aligned} \tag{3.3}$$

Proof. Letting $\phi=u-\bar{u}$ and $\psi=v-\bar{v}$, and substituting (ϕ,ψ) into (1.1), we have

$$\begin{cases}\phi_t+(\bar{u}\psi+\bar{v}\phi+\phi\psi)_x=D\phi_{xx},\\ \psi_t+\left(\frac{1}{2}\phi^2+\frac{1}{2}\psi^2+\bar{u}\phi+\bar{v}\psi\right)_x=\mu\psi_{xx}.\end{cases} \tag{3.4}$$

Step 1 (L^2-estimates). Multiplying the first equation of (3.4) by ϕ and second equation by ψ, adding them and integrating the resulting equation with respect to x, we end up with

$$\begin{aligned}&\frac{d}{dt}\int\left(\frac{\phi^2}{2}+\frac{\psi^2}{2}\right)dx+D\int\phi_x^2dx+\mu\int\psi_x^2dx\\ =&-\int(\bar{u}\psi+\bar{v}\phi+\phi\psi)_x\phi dx-\int\left(\frac{\phi^2}{2}+\frac{\psi^2}{2}+\bar{u}\phi+\bar{v}\psi\right)_x\psi dx\\ =&-\int\left(\bar{v}\frac{\phi^2}{2}+\bar{v}\frac{\psi^2}{2}+\frac{\psi^3}{3}+\bar{u}\phi\psi+\phi^2\psi\right)_x dx=0.\end{aligned} \tag{3.5}$$

Hence

$$\int\left(\frac{\phi^2}{2}+\frac{\psi^2}{2}\right)dx+D\int_0^t\int\phi_x^2dxd\tau+\mu\int_0^t\int\psi_x^2dxd\tau=\int\left(\frac{\phi_0^2}{2}+\frac{\psi_0^2}{2}\right)dx,$$

which yields

$$\begin{aligned}&\|(u(\cdot,t)-\bar{u},v(\cdot,t)-\bar{v})\|^2+2D\int_0^t\|u_x(\cdot,\tau)\|^2d\tau+2\mu\int_0^t\|v_x(\cdot,\tau)\|^2d\tau\\=&\|(u_0-\bar{u},v_0-\bar{v})\|^2.\end{aligned} \tag{3.6}$$

Step 2 (H^1-estimates). Multiplying the first equation of (3.4) by $-\phi_{xx}$ and second equation by $-\psi_{xx}$, adding them, and integrating the results with respect to x yields that

$$\begin{aligned}&\frac{d}{dt}\int\left(\frac{\phi_x^2}{2}+\frac{\psi_x^2}{2}\right)dx+D\int\phi_{xx}^2dx+\mu\int\psi_{xx}^2dx\\=&\int(\bar{u}\psi+\bar{v}\phi+\phi\psi)_x\phi_{xx}dx+\int\left(\frac{\phi^2}{2}+\frac{\psi^2}{2}+\bar{u}\phi+\bar{v}\psi\right)_x\psi_{xx}dx\\\le&\int\phi\psi_x\phi_{xx}dx+\int\psi\phi_x\phi_{xx}dx+\int\phi\phi_x\psi_{xx}dx+\int\psi\psi_x\psi_{xx}dx\\&+\frac{\bar{u}^2}{D}\|\psi_x\|^2+\frac{D}{4}\|\phi_{xx}\|^2+\frac{\bar{u}^2}{\mu}\|\phi_x\|^2+\frac{\mu}{4}\|\psi_{xx}\|^2.\end{aligned} \tag{3.7}$$

Using (3.6), we have $\|(\phi,\psi)\|\le C$, and hence

$$\begin{aligned}&\int\phi\psi_x\phi_{xx}dx+\int\psi\phi_x\phi_{xx}dx\\\le&\|\psi_x\|_{L^\infty}\|\phi\|\|\phi_{xx}\|+\|\phi_x\|_{L^\infty}\|\psi\|\|\phi_{xx}\|\\\le&\frac{4}{D}(\|\psi_x\|_{L^\infty}^2\|\phi\|^2+\|\phi_x\|_{L^\infty}^2\|\psi\|^2)+\frac{D}{8}\|\phi_{xx}\|^2\\\le&C\|\psi_x\|\|\psi_{xx}\|+C\|\phi_x\|\|\phi_{xx}\|+\frac{D}{8}\|\phi_{xx}\|^2\\\le&C(\|\phi_x\|^2+\|\psi_x\|^2)+\frac{3D}{16}\|\phi_{xx}\|^2+\frac{\mu}{16}\|\psi_{xx}\|^2.\end{aligned} \tag{3.8}$$

Similarly, we have

$$\begin{aligned}&\int\phi\phi_x\psi_{xx}dx+\int\psi\psi_x\psi_{xx}dx\\\le&\frac{8}{\mu}\|\phi_x\|_{L^\infty}^2\|\phi\|^2+\frac{\mu}{32}\|\psi_{xx}\|^2+\frac{8}{\mu}\|\psi_x\|_{L^\infty}^2\|\psi\|^2+\frac{\mu}{32}\|\psi_{xx}\|^2\\\le&C\|\phi_x\|\|\phi_{xx}\|+C\|\psi_x\|\|\psi_{xx}\|+\frac{\mu}{16}\|\psi_{xx}\|^2\\\le&C(\|\phi_x\|^2+\|\psi_x\|^2)+\frac{D}{16}\|\phi_{xx}\|^2+\frac{3\mu}{16}\|\psi_{xx}\|^2.\end{aligned} \tag{3.9}$$

Substituting (3.8) and (3.9) into (3.7), we have

$$\frac{d}{dt}\int\left(\frac{\phi_x^2}{2}+\frac{\psi_x^2}{2}\right)dx+\frac{D}{2}\int\phi_{xx}^2dx+\frac{\mu}{2}\int\psi_{xx}^2dx\le C(\|\phi_x\|^2+\|\psi_x\|^2). \tag{3.10}$$

Integrating (3.10) over $[0,t]$ and using (3.6), we get

$$\|(u_x,v_x)(\cdot,t)\|^2+D\int_0^t\|u_{xx}(\cdot,\tau)\|^2d\tau+\mu\int_0^t\|v_{xx}(\cdot,\tau)\|^2d\tau\le C\|(u_0-\bar{u},v_0-\bar{v})\|_1^2. \tag{3.11}$$

The combination of (3.6) and (3.11) yields (3.3). Then the proof of Proposition 3.3 is completed. ∎

With the above results, we are now in a position to prove Theorem 2.1.

Proof. Proposition 3.1 can be obtained by combining the existence of local solutions and the a *priori* estimates. This completes the proof of global existence in Theorem 2.1. Next we derive (2.12). From (3.1), one has $\|(u(\cdot,t)-\bar{u},v(\cdot,t)-\bar{v})\| \le C$ and

$$\|(u_x(\cdot,t),v_x(\cdot,t))\| \to 0 \quad \text{as} \quad t \to \infty.$$

Consequently, for all $x \in \mathbb{R}$, it follows that

$$\begin{aligned}(u(x,t)-\bar{u})^2 &= 2\int_{-\infty}^{x} (u(y,t)-\bar{u})(u(y,t)-\bar{u})_y dy \\ &\le 2\left(\int (u(y,t)-\bar{u})^2 dy\right)^{\frac{1}{2}} \left(\int u_y^2 dy\right)^{\frac{1}{2}} \\ &\le 2C\|u_x(\cdot,t)\| \to 0 \quad \text{as} \quad t\to\infty. \end{aligned} \tag{3.12}$$

This implies $\sup_{x\in\mathbb{R}} |u(x,t)-\bar{u}| \to 0$ as $t\to\infty$. Similarly, we can prove $\sup_{x\in\mathbb{R}} |v(x,t)-\bar{v}| \to 0$ as $t\to\infty$. Hence (2.12) is proved and the proof of Theorem 2.1 is completed. ∎

4. Proof of Theorem 2.2

To study the nonlinear stability of rarefaction waves, we first construct a smooth approximation of solutions $(u^r,v^r)(x/t)$ of the Riemann problem (1.4).

4.1. Smooth approximate solution of the Riemann problem. It is well-known (e.g. see [17]) that the Riemann problem of the Burgers equation,

$$\begin{cases} w_t + ww_x = 0, & x\in\mathbb{R}, \quad t>0, \\ w(x,0) = w_0^r(x) = \begin{cases} v_- - u_-, & x<0, \\ v_+ - u_+, & x>0, \end{cases} \end{cases} \tag{4.1}$$

where $v_- - u_- \le v_+ - u_+$, has a continuous weak solution $w^r(x/t)$ of the form

$$w^r(x/t) = \begin{cases} v_- - u_-, & \frac{x}{t} \le v_- - u_-, \\ \frac{x}{t}, & v_- - u_- \le \frac{x}{t} \le v_+ - u_+, \\ v_+ - u_+, & \frac{x}{t} \ge v_+ - u_+. \end{cases} \tag{4.2}$$

Then the 1-rarefaction wave solutions $(u^r,v^r)(x/t)$ given by (2.15) and (2.16) can be written as

$$u^r(x/t) = \frac{u_- + v_- - w^r(x/t)}{2}, \qquad v^r(x/t) = \frac{u_- + v_- + w^r(x/t)}{2}. \tag{4.3}$$

We approximate $w^r(x/t)$ by the solution $w(x,t)$ of the following initial value problem:

$$\begin{cases} w_t + ww_x = 0, & x\in\mathbb{R}, \quad t>0, \\ w(x,0) = w_0(x) := \frac{v_+ - u_+ + v_- - u_-}{2} + \frac{v_+ - u_+ - v_- + u_-}{2} k_q \int_0^{\varepsilon x} (1+y^2)^{-q} dy, \end{cases} \tag{4.4}$$

where $\varepsilon>0$ is a constant to be determined later and k_q is a constant such that $k_q\int_0^\infty(1+y^2)^{-q}dy=1$ for each $q>\frac{3}{2}$. Then the solution of the Cauchy problem (4.4) has the following properties.

Lemma 4.1 ([17]). *If $v_- - u_- < v_+ - u_+$, then the Cauchy problem (4.4) has a unique smooth global solution $w(x,t)$ satisfying the following:*

(i) $v_- - u_- < w(x,t) < v_+ - u_+, \quad w_x(x,t)>0, \quad for\ (x,t)\in\mathbb{R}\times\mathbb{R}_+.$

(ii) *For any $p\in[1,\infty]$, there exists a constant $C_{p,q}$ such that for any $t\in\mathbb{R}_+$,*

$$\begin{aligned}&\|w_x(\cdot,t)\|_{L^p}\le C_{p,q}\min\left\{\varepsilon^{1-\frac{1}{p}},\quad(1+t)^{-1+\frac{1}{p}}\right\},\\&\|w_{xx}(\cdot,t)\|_{L^p}\le C_{p,q}\min\left\{\varepsilon^{2-\frac{1}{p}},\quad\varepsilon^{(1-\frac{1}{2q})(1-\frac{1}{p})}(1+t)^{-1-\frac{p-1}{2pq}}\right\}.\end{aligned}\tag{4.5}$$

(iii) $\lim_{t\to\infty}\sup_{x\in\mathbb{R}}|w(x,t)-w^r(x/t)|=0.$

Using (4.3) and Lemma 4.1, the smooth approximation of the rarefaction wave profile $(u^r,v^r)(x/t)$ can be constructed via

$$\widetilde{U}=\frac{u_-+v_--w}{2},\qquad\widetilde{V}=\frac{u_-+v_-+w}{2},\tag{4.6}$$

which satisfy

$$\begin{cases}\widetilde{U}_t+(\widetilde{U}\widetilde{V})_x=0,\\\widetilde{V}_t+(\frac{1}{2}\widetilde{U}^2+\frac{1}{2}\widetilde{V}^2)_x=0,\\(\widetilde{U},\widetilde{V})(x,0)=(\widetilde{U}_0,\widetilde{V}_0)(x)=\left(\frac{u_-+v_--w_0(x)}{2},\frac{u_-+v_-+w_0(x)}{2}\right),\end{cases}\tag{4.7}$$

where w_0 is defined in (4.4). Moreover the following properties can be readily verified.

Lemma 4.2. *The smooth function $(\widetilde{U},\widetilde{V})(x,t)$ given in (4.6) has the following properties:*

(i) $\widetilde{V}_x=-\widetilde{U}_x>0.$

(ii) *For any $p\in[1,+\infty]$, there exists a positive constant $C_{p,q}$ such that*

$$\begin{aligned}&\left\|(\widetilde{U}_x,\widetilde{V}_x)(\cdot,t)\right\|_{L^p}\le C_{p,q}\min\{\varepsilon^{1-\frac{1}{p}},\quad(1+t)^{-1+\frac{1}{p}}\},\\&\left\|(\widetilde{U}_{xx},\widetilde{V}_{xx})(\cdot,t)\right\|_{L^p}\le C_{p,q}\min\left\{\varepsilon^{2-\frac{1}{p}},\quad\varepsilon^{(1-\frac{1}{2q})(1-\frac{1}{p})}(1+t)^{-1-\frac{p-1}{2pq}}\right\}.\end{aligned}$$

In particular, for $p>1$, it holds that

$$\int_0^\infty\left\|(\widetilde{U}_{xx},\widetilde{V}_{xx})\right\|_{L^p}dt\le C_{p,q}.\tag{4.8}$$

(iii) $\lim_{t\to\infty}\sup_{x\in\mathbb{R}}\left|(u^r-\widetilde{U},v^r-\widetilde{V})(x,t)\right|=0.$

Proof. The properties (ii) and (iii) can de derived from Lemma 4.1 and (4.6) directly. We only need to prove (i). Indeed, using (4.6) and Lemma 4.1 (i), we have

$$\widetilde{V}_x-\widetilde{U}_x=w_x(x,t)>0,\qquad\widetilde{U}_x=-\widetilde{V}_x,\tag{4.9}$$

which implies $\widetilde{V}_x=-\widetilde{U}_x>0$. ∎

4.2. Reformulated problem. By the approximate smooth solution $(\widetilde{U},\widetilde{V})$ constructed in (4.6), we define $(\phi,\psi)=(u-\widetilde{U},v-\widetilde{V})$ and rewrite the Cauchy problem (1.1)-(1.3) as

$$\begin{cases}\phi_t+(\widetilde{U}\psi+\widetilde{V}\phi+\phi\psi)_x=D\phi_{xx}+D\widetilde{U}_{xx},\\ \psi_t+\frac{1}{2}(\phi^2+\psi^2)_x+(\widetilde{V}\psi+\widetilde{U}\phi)_x=\mu\psi_{xx}+\mu\widetilde{V}_{xx},\end{cases}\tag{4.10}$$

with initial data

$$(\phi,\psi)(x,0)=(\phi_0,\psi_0)(x)=(u_0(x)-\widetilde{U}_0(x),v_0(x)-\widetilde{V}_0(x)),\tag{4.11}$$

where (4.7) has been used.

We seek the solution of (4.10)-(4.11) in the space $X_2(0,T)$ defined by

$$X_2(0,T)=\{(\phi,\psi):(\phi,\psi)\in C([0,T];H^1);(\phi_x,\psi_x)\in L^2((0,T);H^1)\}.$$

For the proof of Theorem 2.2, it suffices to show the following results.

Proposition 4.3. *There exists a unique global solution* $(\phi,\psi)\in X_2(0,\infty)$ *to* (4.10)-(4.11) *such that*

$$\|(\phi,\psi)(\cdot,t)\|_1^2+D\int_0^\infty\|\phi_x(\cdot,t)\|_1^2dt+\mu\int_0^\infty\|\psi_x(\cdot,t)\|_1^2dt\le C\|(\phi_0,\psi_0)\|_1^2.\tag{4.12}$$

Proposition 4.3 is obtained by the combination of local existence of solutions and the a *priori* estimates. The proof of the local existence of solutions is standard, and is based on an iteration argument and a fixed point theorem (cf. [26]). We state the local existence theorem without proof.

Proposition 4.4 (Local existence). *If* $(\phi_0,\psi_0)\in H^1$, *then there exists a positive constant* T_0 *such that the Cauchy problem* (4.10)-(4.11) *admits a unique solution* $(\phi,\psi)\in X_2(0,T_0)$ *satisfying*

$$\|(\phi,\psi)(\cdot,t))\|_1\le 2\|(\phi_0,\psi_0)\|_1,\quad for\ all\quad 0\le t\le T_0.\tag{4.13}$$

Proposition 4.5 (A *priori* estimates). *Suppose the Cauchy problem* (4.10)-(4.11) *has a solution* $(\phi,\psi)\in X_2(0,T)$ *for some* $T>0$. *Then there exists a constant* C *independent of* T *such that*

$$\begin{aligned}&\|(\phi,\psi)(\cdot,t)\|_1^2+D\int_0^t\|\phi_x(\cdot,\tau)\|_1^2d\tau+\mu\int_0^t\|\psi_x(\cdot,\tau)\|_1^2d\tau+\int_0^t\|\sqrt{\widetilde{V}_x}(\phi-\psi)(\cdot,\tau)\|^2d\tau\\&\qquad+\int_0^t\|\sqrt{\widetilde{V}_x}(\phi_x,\psi_x)(\cdot,\tau)\|^2d\tau\\&\le C(\|(\phi_0,\psi_0)\|_1^2+1),\quad for\quad all\quad t\in[0,T].\end{aligned}\tag{4.14}$$

To prove Proposition 4.5, we first derive the L^2-estimates of (ϕ,ψ).

Lemma 4.6 (L^2-estimates). *Let* $(\phi,\psi)\in X_2(0,T)$ *be a solution of* (4.10)-(4.11) *for some* $T>0$. *Then it holds that*

$$\|(\phi,\psi)(\cdot,t)\|^2+D\int_0^t\|\phi_x(\cdot,\tau)\|^2d\tau+\mu\int_0^t\|\psi_x(\cdot,\tau)\|^2d\tau+\int_0^t\|\sqrt{\widetilde{V}_x}(\phi-\psi)(\cdot,\tau)\|^2d\tau$$

$$\leq C(\|(\phi_0,\psi_0)\|^2+1), \tag{4.15}$$

where C is a constant independent of T.

Proof. We multiply the first equation of (4.10) by ϕ and the second by ψ, then integrate the results with respect to x to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int(\phi^2+\psi^2)dx+D\int\phi_x^2dx+\mu\int\psi_x^2dx\\ =&-\int(\widetilde{U}\psi+\widetilde{V}\phi+\psi\phi)_x\phi dx-\int(\widetilde{V}\psi+\widetilde{U}\phi)_x\psi dx-\int\phi\phi_x\psi dx\\ &\quad+D\int\widetilde{U}_{xx}\phi dx+\mu\int\widetilde{V}_{xx}\psi dx.\end{aligned} \tag{4.16}$$

Notice that

$$\begin{cases}(\widetilde{U}\psi+\widetilde{V}\phi+\psi\phi)_x\phi=\left(\widetilde{U}\phi\psi+\frac{\widetilde{V}\phi^2}{2}+\phi^2\psi\right)_x+\frac{1}{2}\widetilde{V}_x\phi^2-\widetilde{U}\psi\phi_x-\phi\phi_x\psi,\\ (\widetilde{V}\psi+\widetilde{U}\phi)_x\psi=\left(\frac{\widetilde{V}}{2}\psi^2+\widetilde{U}\phi\psi\right)_x+\frac{1}{2}\widetilde{V}_x\psi^2-\widetilde{U}\phi\psi_x.\end{cases} \tag{4.17}$$

Substituting (4.17) into (4.16) and using Lemma 4.2 (i), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int(\phi^2+\psi^2)dx+D\int\phi_x^2dx+\mu\int\psi_x^2dx+\frac{1}{2}\int\widetilde{V}_x(\phi^2+\psi^2)dx\\ =&\int\widetilde{U}(\phi\psi)_xdx+D\int\widetilde{U}_{xx}\phi dx+\mu\int\widetilde{V}_{xx}\psi dx\\ =&\int\widetilde{V}_x\phi\psi dx+D\int\widetilde{U}_{xx}\phi dx+\mu\int\widetilde{V}_{xx}\psi dx,\end{aligned}$$

which yields

$$\begin{aligned}&\frac{d}{dt}\int(\phi^2+\psi^2)dx+2D\int\phi_x^2dx+2\mu\int\psi_x^2dx+\int\widetilde{V}_x(\phi-\psi)^2dx\\ =&2D\int\widetilde{U}_{xx}\phi dx+2\mu\int\widetilde{V}_{xx}\psi dx\\ \leq&2D\|\widetilde{U}_{xx}\|\,\|\phi\|+2\mu\|\widetilde{V}_{xx}\|\,\|\psi\|\\ \leq&D^2\|\widetilde{U}_{xx}\|+\|\widetilde{U}_{xx}\|\,\|\phi\|^2+\mu^2\|\widetilde{V}_{xx}\|+\|\widetilde{V}_{xx}\|\|\psi\|^2,\end{aligned} \tag{4.18}$$

where we have used the Hölder and Cauchy-Schwarz inequalities. Applying Gronwall's inequality to (4.18), we obtain (4.15) by using (4.8) and the fact $\widetilde{V}_x>0$ in Lemma 4.2 (i). ∎

LEMMA 4.7 (H^1-estimates). *Suppose the Cauchy problem (4.10)-(4.11) has a solution $(\phi,\psi)\in X_2(0,T)$ for some $T>0$. Then there exists a constant C independent of T such that*

$$\begin{aligned}&\|(\phi_x,\psi_x)(\cdot,t)\|^2+D\int_0^t\|\phi_{xx}(\cdot,\tau)\|^2d\tau+\mu\int_0^t\|\psi_{xx}(\cdot,\tau)\|^2d\tau\\ &\qquad+\int_0^t\|\sqrt{\widetilde{V}_x}(\phi_x,\psi_x)(\cdot,\tau)\|^2d\tau\leq C(\|(\phi_0,\psi_0)\|_1^2+1).\end{aligned} \tag{4.19}$$

Proof. Multiplying the first equation of (4.10) by $-\phi_{xx}$ and the second by $-\psi_{xx}$, and integrating them with respect to x, we end up with

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int(\phi_x^2+\psi_x^2)dx+D\int\phi_{xx}^2dx+\mu\int\psi_{xx}^2dx+\frac{3}{2}\int\widetilde{V}_x(\phi_x^2+\psi_x^2)dx\\
=&\int\phi\phi_x\psi_{xx}dx+\int\psi\psi_x\psi_{xx}dx+\int\phi\psi_x\phi_{xx}dx+\int\phi_x\psi\phi_{xx}dx\\
&-\int\widetilde{V}_{xx}(\phi\phi_x+\psi\psi_x)dx-\int\widetilde{U}_{xx}(\psi\phi_x+\phi\psi_x)dx\\
&-3\int\widetilde{U}_x\phi_x\psi_xdx-D\int\widetilde{U}_{xx}\phi_{xx}dx-\mu\int\widetilde{V}_{xx}\psi_{xx}dx. \qquad (4.20)
\end{aligned}$$

Integrating (4.20) with respect to t leads to

$$\begin{aligned}
&\frac{1}{2}\int(\phi_x^2+\psi_x^2)dx+D\int_0^t\int\phi_{xx}^2dxd\tau+\mu\int_0^t\int\psi_{xx}^2dxd\tau+\frac{3}{2}\int_0^t\int\widetilde{V}_x(\phi_x^2+\psi_x^2)dxd\tau\\
=&\frac{1}{2}\int(\phi_{0x}^2+\psi_{0x}^2)dx+\int_0^t\int\phi\phi_x\psi_{xx}dxd\tau+\int_0^t\int\psi\psi_x\psi_{xx}dxd\tau+\int_0^t\int\phi\psi_x\phi_{xx}dxd\tau\\
&+\int_0^t\int\psi\phi_x\phi_{xx}dxd\tau-\int_0^t\int\widetilde{V}_{xx}(\phi\phi_x+\psi\psi_x)dxd\tau-\int_0^t\int\widetilde{U}_{xx}(\psi\phi_x+\phi\psi_x)dxd\tau\\
&-3\int_0^t\int\widetilde{U}_x\phi_x\psi_xdxd\tau-D\int_0^t\int\widetilde{U}_{xx}\phi_{xx}dxd\tau-\mu\int_0^t\int\widetilde{V}_{xx}\psi_{xx}dxd\tau\\
=&\frac{1}{2}\|(\phi_{0x},\psi_{0x})\|^2+\sum_{j=1}^{9}I_j. \qquad (4.21)
\end{aligned}$$

Using Lemma 4.6, one has $\|(\phi,\psi)(\cdot,t)\|^2+\int_0^t\|(\phi_x,\psi_x)(\cdot,\tau)\|^2d\tau\le C(\|(\phi_0,\psi_0)\|^2+1)$. Then

$$\begin{aligned}
I_1\le\int_0^t\int|\phi\phi_x\psi_{xx}|dxd\tau&\le\frac{2}{\mu}\int_0^t\int\phi^2\phi_x^2dxd\tau+\frac{\mu}{8}\int_0^t\|\psi_{xx}\|^2d\tau\\
&\le\frac{2}{\mu}\int_0^t\|\phi_x\|_{L^\infty}^2\|\phi\|^2d\tau+\frac{\mu}{8}\int_0^t\|\psi_{xx}\|^2d\tau\\
&\le C\int_0^t\|\phi_{xx}\|\|\phi_x\|d\tau+\frac{\mu}{8}\int_0^t\|\psi_{xx}\|^2d\tau\\
&\le C\int_0^t\|\phi_x\|^2d\tau+\frac{D}{8}\int_0^t\|\phi_{xx}\|^2d\tau+\frac{\mu}{8}\int_0^t\|\psi_{xx}\|^2d\tau\\
&\le C(\|(\phi_0,\psi_0)\|^2+1)+\frac{D}{8}\int_0^t\|\phi_{xx}\|^2d\tau+\frac{\mu}{8}\int_0^t\|\psi_{xx}\|^2d\tau. \qquad (4.22)
\end{aligned}$$

Applying the same procedure to I_2, I_3, and I_4, we have

$$\begin{aligned}
&I_2+I_3+I_4\\
\le&\int_0^t\int|\psi\psi_x\psi_{xx}|dxd\tau+\int_0^t\int|\phi\psi_x\phi_{xx}|dxd\tau+\int_0^t\int|\psi\phi_x\phi_{xx}|dxd\tau
\end{aligned}$$

$$
\begin{aligned}
&\quad +\frac{\mu}{16}\int_0^t \|\psi_{xx}\|^2 d\tau+\frac{D}{16}\int_0^t \|\phi_{xx}\|^2 d\tau\\
&\le C\int_0^t \|\psi_{xx}\|\|\psi_x\| d\tau + C\int_0^t \|\phi_{xx}\|\|\phi_x\| d\tau+\frac{\mu}{16}\int_0^t \|\psi_{xx}\|^2 d\tau+\frac{D}{16}\int_0^t \|\phi_{xx}\|^2 d\tau\\
&\le C\int_0^t (\|\phi_x\|^2+\|\psi_x\|^2)d\tau+\frac{\mu}{8}\int_0^t \|\psi_{xx}\|^2 d\tau+\frac{D}{8}\int_0^t \|\phi_{xx}\|^2 d\tau\\
&\le C(\|(\phi_0,\psi_0)\|^2+1)+\frac{\mu}{8}\int_0^t \|\psi_{xx}\|^2 d\tau+\frac{D}{8}\int_0^t \|\phi_{xx}\|^2 d\tau.
\end{aligned}
\tag{4.23}
$$

Using the Hölder and Cauchy-Schwarz inequalities, we can estimate the terms I_5, I_6, and I_7 as follows:

$$
\begin{aligned}
&I_5+I_6+I_7\\
&\le \int_0^t\int |\widetilde{V}_{xx}(\phi\phi_x+\psi\psi_x)|dxd\tau+\int_0^t\int |\widetilde{U}_{xx}(\psi\phi_x+\phi\psi_x)|dxd\tau+3\int_0^t\int |\widetilde{U}_x\phi_x\psi_x|dxd\tau\\
&\le \int_0^t \|\widetilde{V}_{xx}\|_{L^\infty}(\|\phi\|\|\phi_x\|+\|\psi\|\|\psi_x\|)d\tau+\int_0^t \|\widetilde{U}_{xx}\|_{L^\infty}(\|\psi\|\|\phi_x\|+\|\phi\|\|\psi_x\|)d\tau\\
&\quad +\frac{3}{2}\int_0^t \|\widetilde{U}_x\|_{L^\infty}(\|\phi_x\|^2+\|\psi_x\|^2)d\tau\\
&\le \frac{1}{2}\int_0^t (\|\widetilde{U}_{xx}\|_{L^\infty}+\|\widetilde{V}_{xx}\|_{L^\infty})\|(\phi,\psi)\|^2 d\tau+\frac{1}{2}\int_0^t (\|\widetilde{U}_{xx}\|_{L^\infty}+\|\widetilde{V}_{xx}\|_{L^\infty})\|(\phi_x,\psi_x)\|^2 d\tau\\
&\quad +\frac{3}{2}\int_0^t \|\widetilde{U}_x\|_{L^\infty}(\|\phi_x\|^2+\|\psi_x\|^2)d\tau.
\end{aligned}
\tag{4.24}
$$

From Lemma 4.2 (ii), we have

$$
\|(\widetilde{U}_x,\widetilde{U}_{xx},\widetilde{V}_{xx})\|_{L^\infty}\le C \quad\text{and}\quad \int_0^t \|(\widetilde{U}_{xx},\widetilde{V}_{xx})\|_{L^\infty}d\tau\le C.
\tag{4.25}
$$

Substituting (4.25) into (4.24), and using Lemma 4.6, one has

$$
\begin{aligned}
I_5+I_6+I_7&\le C\int_0^t (\|\widetilde{U}_{xx}\|_{L^\infty}+\|\widetilde{V}_{xx}\|_{L^\infty})d\tau+C\int_0^t (\|\phi_x\|^2+\|\psi_x\|^2)d\tau\\
&\le C(\|(\phi_0,\psi_0)\|^2+1).
\end{aligned}
\tag{4.26}
$$

Finally, we use the Hölder inequality, the Cauchy-Schwarz inequality, and Lemma 4.2 (ii) to estimate the last two terms I_8 and I_9 as follows:

$$
\begin{aligned}
I_8+I_9&\le D\int_0^t\int |\widetilde{U}_{xx}\phi_{xx}|dxd\tau+\mu\int_0^t\int |\widetilde{V}_{xx}\psi_{xx}|dxd\tau\\
&\le D\int_0^t \|\widetilde{U}_{xx}\|^2 d\tau+\mu\int_0^t \|\widetilde{V}_{xx}\|^2 d\tau+\frac{D}{4}\int_0^t \|\phi_{xx}\|^2 d\tau+\frac{\mu}{4}\int_0^t \|\psi_{xx}\|^2 d\tau\\
&\le C+\frac{D}{4}\int_0^t \|\phi_{xx}\|^2 d\tau+\frac{\mu}{4}\int_0^t \|\psi_{xx}\|^2 d\tau.
\end{aligned}
\tag{4.27}
$$

Substituting (4.22), (4.23), (4.26), and (4.27) into (4.21), we obtain

$$
\begin{aligned}
&\frac{1}{2}\int(\phi_x^2+\psi_x^2)dx+\frac{D}{2}\int_0^t\int\phi_{xx}^2dxd\tau+\frac{\mu}{2}\int_0^t\int\psi_{xx}^2dxd\tau+\frac{3}{2}\int_0^t\int\widetilde{V}_x(\phi_x^2+\psi_x^2)dxd\tau\\
\le&\frac{1}{2}\|(\phi_{0x},\psi_{0x})\|^2+C(\|(\phi_0,\psi_0)\|^2+1)\\
\le&C(\|(\phi_0,\psi_0)\|_1^2+1),
\end{aligned}
$$

which implies (4.19). Then we complete the proof of Lemma 4.7. □

With the above lemmas in hand, we now prove Theorem 2.2.

Proof. Proposition 4.5 follows from Lemmas 4.6 and 4.7. Then using Proposition 4.4 and Proposition 4.5, we can derive Proposition 4.3 by the continuity argument. From Proposition 4.3, one has $\|(\phi,\psi)(\cdot,t)\|\le C$ and $\|(\phi_x,\psi_x)(\cdot,t)\|\to 0$ as $t\to\infty$. Hence, the same argument as in the proof of Theorem 2.1 leads to

$$\sup_{x\in\mathbb{R}}|u(x,t)-\widetilde{U}(x,t)|\to 0 \quad \text{as} \quad t\to\infty \tag{4.28}$$

and

$$\sup_{x\in\mathbb{R}}|v(x,t)-\widetilde{V}(x,t)|\to 0 \quad \text{as} \quad t\to\infty. \tag{4.29}$$

The combination of (4.28) and Lemma 4.2 (iii) gives

$$\sup_{x\in\mathbb{R}}|u(x,t)-u^r(x/t)|\le\sup_{x\in\mathbb{R}}|u(x,t)-\widetilde{U}(x,t)|+\sup_{x\in\mathbb{R}}|u^r(x/t)-\widetilde{U}(x,t)|\to 0 \;\; \text{as} \;\; t\to\infty,$$

Similarly, the combination of (4.29) and Lemma 4.2 (iii) gives

$$\sup_{x\in\mathbb{R}}|v(x,t)-v^r(x/t)|\to 0 \quad \text{as} \quad t\to\infty.$$

Then the proof of Theorem 2.2 is completed. □

5. Proof of Theorem 2.4

5.1. Reformulation of the problem. Substituting (2.30) into (1.1), using (2.18) and integrating the system with respect to z, we obtain the equations for the perturbation (ϕ,ψ):

$$\begin{cases}\phi_t=D\phi_{zz}+(s-V)\phi_z-U\psi_z-\phi_z\psi_z,\\ \psi_t=\mu\psi_{zz}+(s-V)\psi_z-U\phi_z-\frac{1}{2}(\phi_z^2+\psi_z^2),\end{cases} \tag{5.1}$$

with initial data

$$(\phi,\psi)(z,0)=(\phi_0,\psi_0)(z), \quad z\in\mathbb{R}, \tag{5.2}$$

where (ϕ_0,ψ_0) is defined in (2.31). We look for solutions of the reformulated system (5.1) in the following solution space:

$$X_3(0,T)=\left\{(\phi,\psi):(\phi,\psi)\in C([0,T];H_w^2),(\phi_z,\psi_z)\in L^2((0,T);H_w^2)\right\},$$

where the weight function w is defined by (2.33).

Clearly, if $\phi \in H_w^2$, then $\phi \in H^2$ because $w \geq 1$. Define

$$N(t) := \sup_{\tau \in [0,t]} (\|\phi(\cdot,\tau)\|_{2,w} + \|\psi(\cdot,\tau)\|_{2,w}).$$

By the Sobolev embedding inequality, one has

$$\sup_{\tau \in [0,t]} \{\|\phi(\cdot,\tau)\|_{L^\infty}, \|\phi_z(\cdot,\tau)\|_{L^\infty}, \|\psi(\cdot,\tau)\|_{L^\infty}, \|\psi_z(\cdot,\tau)\|_{L^\infty}\} \leq N(t). \tag{5.3}$$

Then Theorem 2.4 is a consequence of the following theorem.

THEOREM 5.1. *Let* (2.23) *hold, and let $D \geq \mu$. Then there exists a positive constant ε_1, such that if $N(0) \leq \varepsilon_1$, then the Cauchy problem* (5.1)-(5.2) *has a unique global solution $(\phi,\psi) \in X_3(0,\infty)$ satisfying*

$$\begin{aligned} &\|\phi(\cdot,t)\|_{2,w}^2 + \|\psi(\cdot,t)\|_{2,w}^2 + \int_0^t (\|\phi_z(\cdot,\tau)\|_{2,w}^2 + \|\psi_z(\cdot,\tau)\|_{2,w}^2) d\tau \\ \leq & C\Big(\|\phi_0\|_{2,w}^2 + \|\psi_0\|_{2,w}^2\Big) \leq CN^2(0) \end{aligned} \tag{5.4}$$

for any $t \in [0,+\infty)$. Moreover, it follows that

$$\sup_{z \in \mathbb{R}} |(\phi_z,\psi_z)(z,t)| \to 0 \quad as \quad t \to \infty. \tag{5.5}$$

The global existence of (ϕ,ψ) announced in Theorem 5.1 follows from the local existence of solutions and the *a priori* estimates, which are given below.

PROPOSITION 5.2 (Local existence). *For any $\varepsilon_2 > 0$, there exists a positive constant T_0 depending on ε_2 such that if $(\phi_0,\psi_0) \in H_w^2$ with $N(0) \leq \varepsilon_2$, then the problem* (5.1)-(5.2) *has a unique solution $(\phi,\psi) \in X_3(0,T_0)$ satisfying $N(t) \leq 2N(0)$ for any $0 \leq t \leq T_0$.*

PROPOSITION 5.3 (A *priori* estimates). *Assume that $(\phi,\psi) \in X_3(0,T)$ is a solution obtained in Proposition* 5.2 *for a positive constant T. Then there is a positive constant $\varepsilon_3 > 0$, independent of T, such that if*

$$N(t) \leq \varepsilon_3$$

for any $0 \leq t \leq T$, then the solution (ϕ,ψ) of (5.1)-(5.2) *satisfies* (5.4) *for any $0 \leq t \leq T$.*

The local existence in Proposition 5.2 can be proved by the standard argument (cf. [20]), so we omit the details for brevity. Next, we shall prove Proposition 5.3 by using the weighted energy estimates. In the following, we assume $N(t) < \min\{\mu, D\}$ without loss of generality.

5.2. Weighted energy estimates.

LEMMA 5.4 (L^2-estimates). *Let the assumptions of Theorem* 5.1 *hold and assume that $(\phi,\psi) \in X_3(0,T)$ is a solution obtained in Proposition* 5.2. *Then there exists a constant $C > 0$ such that*

$$\|\phi(\cdot,t)\|_w^2 + \|\psi(\cdot,t)\|_w^2 + D\int_0^t \|\phi_z(\cdot,\tau)\|_w^2 d\tau + \mu\int_0^t \|\psi_z(\cdot,\tau)\|_w^2 d\tau \leq C(\|\phi_0\|_w^2 + \|\psi_0\|_w^2). \tag{5.6}$$

Proof. Multiplying the first equation of (5.1) by ϕ/U and the second by ψ/U, integrating the resultant equations with respect to z and adding them, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\frac{\phi^2+\psi^2}{U}dz+D\int\frac{\phi_z^2}{U}dz+\mu\int\frac{\psi_z^2}{U}dz\\
=&\frac{1}{2}\int\left[\left(\frac{D}{U}\right)_{zz}-\left(\frac{s-V}{U}\right)_z\right]\phi^2dz+\frac{1}{2}\int\left[\left(\frac{\mu}{U}\right)_{zz}-\left(\frac{s-V}{U}\right)_z\right]\psi^2dz\\
&-\int\frac{\phi\phi_z\psi_z}{U}dz-\frac{1}{2}\int\frac{\psi(\phi_z^2+\psi_z^2)}{U}dz.
\end{aligned}\tag{5.7}$$

Using (2.18) and the fact that $u_+=0$, it can be checked that

$$\left(\frac{D}{U}\right)_{zz}-\left(\frac{s-V}{U}\right)_z=\frac{2U_z}{U^3}(s-v_+)u_+=0.\tag{5.8}$$

The combination of (2.18) and the facts $U_z<0,\ V_z<0,\ s-V>0,\ U>0$, and $D\geq\mu$ gives

$$\left(\frac{\mu}{U}\right)_{zz}-\left(\frac{s-V}{U}\right)_z=\frac{D-\mu}{DU}\left(V_z+\frac{(s-V)U_z}{U}\right)\leq 0.\tag{5.9}$$

Substituting (5.8) and (5.9) into (5.7) and integrating the equation with respect to t, with the fact $\|(\phi,\psi)(\cdot,t)\|_{L^\infty}\leq N(t)$, we derive

$$\begin{aligned}
&\frac{1}{2}\int\frac{\phi^2+\psi^2}{U}dz+D\int_0^t\int\frac{\phi_z^2}{U}dzd\tau+\mu\int_0^t\int\frac{\psi_z^2}{U}dzd\tau\\
&\quad+\frac{D-\mu}{2D}\int_0^t\int\left[-V_z+\frac{s-V}{U}(-U_z)\right]\frac{\psi^2}{U}dzd\tau\\
=&\frac{1}{2}\int\frac{\phi_0^2+\psi_0^2}{U}dz-\int_0^t\int\frac{\phi\phi_z\psi_z}{U}dzd\tau-\frac{1}{2}\int_0^t\int\frac{\psi(\phi_z^2+\psi_z^2)}{U}dzd\tau\\
\leq&\frac{1}{2}\int\frac{\phi_0^2+\psi_0^2}{U}dz+\frac{N(t)}{2}\int_0^t\int\left(\frac{\phi_z^2}{U}+\frac{\psi_z^2}{U}\right)dzd\tau+\frac{N(t)}{2}\int_0^t\int\frac{\phi_z^2+\psi_z^2}{U}dzd\tau\\
\leq&\frac{1}{2}\int\frac{\phi_0^2+\psi_0^2}{U}dz+N(t)\int_0^t\int\frac{\phi_z^2}{U}dzd\tau+N(t)\int_0^t\int\frac{\psi_z^2}{U}dzd\tau,
\end{aligned}$$

which yields that

$$\int\frac{\phi^2+\psi^2}{U}dz+2(D-N(t))\int_0^t\int\frac{\phi_z^2}{U}dzd\tau+2(\mu-N(t))\int_0^t\int\frac{\psi_z^2}{U}dzd\tau\leq\int\frac{\phi_0^2+\psi_0^2}{U}dz.\tag{5.10}$$

Then using the assumption $N(t)<\min\{\mu,D\}$ and Remark 2.3, we obtain (5.6) from (5.10). ∎

LEMMA 5.5 (H^1-estimates). *Let the assumptions of Lemma 5.4 hold. Then it follows that*

$$\begin{aligned}
&\|\phi(\cdot,t)\|_{1,w}^2+\|\psi(\cdot,t)\|_{1,w}^2+D\int_0^t\|\phi_z(\cdot,\tau)\|_{1,w}^2d\tau+\mu\int_0^t\|\psi_z(\cdot,\tau)\|_{1,w}^2d\tau\\
\leq&C\left(\|\phi_0\|_{1,w}^2+\|\psi_0\|_{1,w}^2\right),
\end{aligned}\tag{5.11}$$

where $C>0$ is a constant.

Proof. We differentiate (5.1) with respect to z to get

$$\begin{cases}\phi_{zt}=D\phi_{zzz}-V_z\phi_z+(s-V)\phi_{zz}-U_z\psi_z-U\psi_{zz}-(\phi_z\psi_z)_z,\\ \psi_{zt}=\mu\psi_{zzz}-V_z\psi_z+(s-V)\psi_{zz}-U_z\phi_z-U\phi_{zz}-\frac{1}{2}(\phi_z^2+\psi_z^2)_z.\end{cases} \tag{5.12}$$

Multiplying the first equation of (5.12) by ϕ_z/U and the second by ψ_z/U, after some algebra, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\frac{\phi_z^2+\phi_z^2}{U}dz+D\int\frac{\phi_{zz}^2}{U}dz+\mu\int\frac{\psi_{zz}^2}{U}dz\\ =&\frac{1}{2}\int\left[\left(\frac{D}{U}\right)_{zz}-\left(\frac{s-V}{U}\right)_z\right]\phi_z^2dz+\frac{1}{2}\int\left[\left(\frac{\mu}{U}\right)_{zz}-\left(\frac{s-V}{U}\right)_z\right]\psi_z^2dz\\ &\quad-2\int\frac{U_z}{U}\phi_z\psi_zdz-\int\frac{V_z}{U}(\phi_z^2+\psi_z^2)dz-\int\frac{\psi_z^2\psi_{zz}+2\psi_z\phi_z\phi_{zz}+\psi_{zz}\phi_z^2}{U}dz\\ \le&-2\int\frac{U_z}{U}\phi_z\psi_zdz-\int\frac{V_z}{U}(\phi_z^2+\psi_z^2)dz-\int\frac{\psi_z^2\psi_{zz}+2\psi_z\phi_z\phi_{zz}+\psi_{zz}\phi_z^2}{U}dz,\end{aligned} \tag{5.13}$$

where (5.8) and (5.9) have been used. Using (2.20) and the facts $s=v_-$, $0=u_+<U<u_-$, and $v_+<V<v_-$, it is easy to check that

$$\left|\frac{U_z}{U}\right|=\left|\frac{V-s}{D}\right|\le\frac{v_--v_+}{D},\quad |V_z|\le\left|-\frac{s}{\mu}V+\frac{1}{2\mu}(U^2+V^2)+\frac{\rho_2}{\mu}\right|\le C. \tag{5.14}$$

Integrating (5.13) in t and using the fact $\psi_z^2\le\frac{C\psi_z^2}{U}$ and (5.3), we obtain from (5.13) and (5.14) that

$$\begin{aligned}&\int\frac{\phi_z^2+\phi_z^2}{U}dz+2D\int_0^t\int\frac{\phi_{zz}^2}{U}dzd\tau+2\mu\int_0^t\int\frac{\psi_{zz}^2}{U}dzd\tau\\ \le&\int\frac{\phi_{0z}^2+\phi_{0z}^2}{U}dz+\frac{v_--v_+}{D}\int_0^t\int U\psi_z^2dzd\tau\\ &\quad+\frac{v_--v_+}{D}\int_0^t\int\frac{\phi_z^2}{U}dzd\tau+C\int_0^t\int\frac{\phi_z^2+\psi_z^2}{U}dzd\tau\\ &\quad+2N(t)\left(\int_0^t\int\frac{\psi_z^2+\psi_{zz}^2}{2U}dzd\tau+\int_0^t\int\frac{\phi_z^2+\phi_{zz}^2}{U}dzd\tau+\int_0^t\int\frac{\phi_z^2+\psi_{zz}^2}{2U}dzd\tau\right)\\ \le&\int\frac{\phi_{0z}^2+\phi_{0z}^2}{U}dz+C(1+N(t))\int_0^t\int\frac{\phi_z^2+\psi_z^2}{U}dzd\tau+2N(t)\int_0^t\int\frac{\phi_{zz}^2+\psi_{zz}^2}{U}dzd\tau,\end{aligned}$$

which entails that

$$\begin{aligned}&\int\frac{\phi_z^2+\phi_z^2}{U}dz+2(D-N(t))\int_0^t\int\frac{\phi_{zz}^2}{U}dzd\tau+2(\mu-N(t))\int_0^t\int\frac{\psi_{zz}^2}{U}dzd\tau\\ \le&\int\frac{\phi_{0z}^2+\phi_{0z}^2}{U}dz+C(1+N(t))\int_0^t\int\frac{\phi_z^2+\psi_z^2}{U}dzd\tau\\ \le&C(\|\phi_{0z}\|_w^2+\|\psi_{0z}\|_w^2)+C(1+N(t))\int_0^t(\|\phi_z(\cdot,\tau)\|_w^2+\|\psi_z(\cdot,\tau)\|_w^2)d\tau,\end{aligned} \tag{5.15}$$

where we have used the fact $\frac{1}{U(z)} \le C_2 w(z)$ for all $z \in \mathbb{R}$ (see Remark 2.3). The combination (5.6) and (5.15) gives that

$$\begin{aligned}&\int \frac{\phi_z^2+\phi_z^2}{U}dz+2(D-N(t))\int\int_0^t \frac{\phi_{zz}^2}{U}dzd\tau+2(\mu-N(t))\int\int_0^t \frac{\psi_{zz}^2}{U}dzd\tau \\ &\le C\left(\|\phi_0\|_{1,w}^2+\|\psi_0\|_{1,w}^2\right).\end{aligned} \tag{5.16}$$

Using the fact $C_1 w(z) \le \frac{1}{U(z)}$ for all $z \in \mathbb{R}$ (see Remark 2.3) and the assumption $N(t) < \min\{\mu, D\}$, we obtain (5.11) from (5.16). ∎

Next, we give the estimates of the second order derivative of (ϕ,ψ).

LEMMA 5.6 (H^2-estimates). *Let the assumptions of Lemma* 5.4 *hold. Then there exists a constant* $C>0$ *such that*

$$\begin{aligned}&\|\phi(\cdot,t)\|_{2,w}^2+\|\psi(\cdot,t)\|_{2,w}^2+D\int_0^t \|\phi_z(\cdot,\tau)\|_{2,w}^2 d\tau+\mu\int_0^t \|\psi_z(\cdot,\tau)\|_{2,w}^2 d\tau \\ &\le C\left(\|\phi_0\|_{2,w}^2+\|\psi_0\|_{2,w}^2\right).\end{aligned} \tag{5.17}$$

Proof. We differentiate (5.1) with respect to z twice to get

$$\begin{cases}\phi_{zzt}=D\phi_{zzzz}-V_{zz}\phi_z-2V_z\phi_{zz}+(s-V)\phi_{zzz}-U_{zz}\psi_z-2U_z\psi_{zz} \\ \qquad\qquad -U\psi_{zzz}-(\phi_z\psi_z)_{zz}, \\ \psi_{zzt}=\mu\psi_{zzzz}-V_{zz}\psi_z-2V_z\psi_{zz}+(s-V)\psi_{zzz}-U_{zz}\phi_z-2U_z\phi_{zz} \\ \qquad\qquad -U\phi_{zzz}-\dfrac{1}{2}(\phi_z^2+\psi_z^2)_{zz}.\end{cases} \tag{5.18}$$

Multiplying the first equation of (5.18) by ϕ_{zz}/U and the second equation by ψ_{zz}/U, using the facts

$$\begin{cases}\phi_{zzzz}\cdot\frac{\phi_{zz}}{U}=\left(\phi_{zzz}\cdot\frac{\phi_{zz}}{U}\right)_z-\frac{\phi_{zzz}^2}{U}-\frac{1}{2}\left(\phi_{zz}^2\left(\frac{1}{U}\right)_z\right)_z+\frac{1}{2}\left(\frac{1}{U}\right)_{zz}\phi_{zz}^2, \\ (s-V)\phi_{zzz}\cdot\frac{\phi_{zz}}{U}=\frac{1}{2}\left(\phi_{zz}^2\frac{(s-V)}{U}\right)_z-\frac{1}{2}\left(\frac{s-V}{U}\right)_z\phi_{zz}^2, \\ \psi_{zzzz}\cdot\frac{\psi_{zz}}{U}=\left(\psi_{zzz}\cdot\frac{\psi_{zz}}{U}\right)_z-\frac{\psi_{zzz}^2}{U}-\frac{1}{2}\left(\psi_{zz}^2\left(\frac{1}{U}\right)_z\right)_z+\frac{1}{2}\left(\frac{1}{U}\right)_{zz}\psi_{zz}^2, \\ (s-V)\psi_{zzz}\cdot\frac{\psi_{zz}}{U}=\frac{1}{2}\left(\psi_{zz}^2\frac{(s-V)}{U}\right)_z-\frac{1}{2}\left(\frac{s-V}{U}\right)_z\psi_{zz}^2,\end{cases}$$

we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\frac{\phi_{zz}^2+\psi_{zz}^2}{U}dz+D\int\frac{\phi_{zzz}^2}{U}dz+\mu\int\frac{\psi_{zzz}^2}{U}dz \\ =&\frac{1}{2}\int\left[\left(\frac{D}{U}\right)_{zz}-\left(\frac{s-V}{U}\right)_z\right]\phi_{zz}^2dz+\frac{1}{2}\int\left[\left(\frac{\mu}{U}\right)_{zz}-\left(\frac{s-V}{U}\right)_z\right]\psi_{zz}^2dz \\ &-\int\frac{V_{zz}}{U}(\phi_z\phi_{zz}+\psi_z\psi_{zz})dz-2\int\frac{V_z}{U}(\phi_{zz}^2+\psi_{zz}^2)dz-\int\frac{U_{zz}}{U}(\psi_z\phi_{zz}+\phi_z\psi_{zz})dz \\ &-4\int\frac{U_z}{U}\phi_{zz}\psi_{zz}dz-\int\frac{(\phi_z\psi_z)_{zz}\phi_{zz}}{U}dz-\frac{1}{2}\int\frac{(\phi_z^2+\psi_z^2)_{zz}\psi_{zz}}{U}dz.\end{aligned} \tag{5.19}$$

Using (5.8) and (5.9), one has

$$\frac{1}{2}\int\left[\left(\frac{D}{U}\right)_{zz}-\left(\frac{s-V}{U}\right)_z\right]\phi_{zz}^2dz+\frac{1}{2}\int\left[\left(\frac{\mu}{U}\right)_{zz}-\left(\frac{s-V}{U}\right)_z\right]\psi_{zz}^2dz\le 0. \tag{5.20}$$

The combination of (5.19) and (5.20) yields that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \frac{\phi_{zz}^2+\psi_{zz}^2}{U}dz + D\int \frac{\phi_{zzz}^2}{U}dz + \mu\int \frac{\psi_{zzz}^2}{U}dz \\
\le & -\int \frac{V_{zz}}{U}(\phi_z\phi_{zz}+\psi_z\psi_{zz})dz - 2\int \frac{V_z}{U}(\phi_{zz}^2+\psi_{zz}^2)dz - \int \frac{U_{zz}}{U}(\psi_z\phi_{zz}+\phi_z\psi_{zz})dz \\
&-4\int \frac{U_z}{U}\phi_{zz}\psi_{zz}dz - \int \frac{(\phi_z\psi_z)_{zz}\phi_{zz}}{U}dz - \frac{1}{2}\int \frac{(\phi_z^2+\psi_z^2)_{zz}\psi_{zz}}{U}dz. \qquad (5.21)
\end{aligned}$$

Using (2.18), (5.14), and the facts $0=u_+<U<u_-$ and $v_+<V<v_-$, one can derive that

$$\begin{aligned}
|U_z| &= \left|\frac{(V-s)U}{D}\right| \le \frac{(v_--v_+)u_-}{D}, \\
|U_{zz}| &= \left|\frac{(V-s)U_z+UV_z}{D}\right| \le \frac{(v_--v_+)^2u_-}{D^2} + C\cdot\frac{u_-}{D} \le C, \qquad (5.22) \\
|V_{zz}| &= \left|\frac{(V-s)V_z+UU_z}{\mu}\right| \le C.
\end{aligned}$$

Then we have the following estimates by using (5.22) and the Cauchy-Schwarz inequality:

$$\begin{aligned}
-\int \frac{V_{zz}}{U}(\phi_z\phi_{zz}+\psi_z\psi_{zz})dz &\le C\int \frac{|\phi_z\phi_{zz}+\psi_z\psi_{zz}|}{U}dz \\
&\le C\int \frac{\phi_z^2+\psi_z^2+\phi_{zz}^2+\psi_{zz}^2}{U}dz, \\
-2\int \frac{V_z}{U}(\phi_{zz}^2+\psi_{zz}^2)dz &\le C\int \frac{\phi_{zz}^2+\psi_{zz}^2}{U}dz, \qquad (5.23) \\
-\int \frac{U_{zz}}{U}(\psi_z\phi_{zz}+\phi_z\psi_{zz})dz &\le C\int \frac{\phi_z^2+\psi_z^2+\phi_{zz}^2+\psi_{zz}^2}{U}dz, \\
-4\int \frac{U_z}{U}\phi_{zz}\psi_{zz}dz &\le C\int \frac{\phi_{zz}^2+\psi_{zz}^2}{U}dz.
\end{aligned}$$

Using (5.3), (5.14), and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&-\int \frac{(\phi_z\psi_z)_{zz}\phi_{zz}}{U}dz \\
=& \int \frac{(\phi_z\psi_z)_z\phi_{zzz}}{U}dz - \int \frac{(\phi_z\psi_z)_z\phi_{zz}U_z}{U^2}dz \\
\le& N(t)\int \frac{|\psi_{zz}\phi_{zzz}+\phi_{zz}\phi_{zzz}|}{U} + \frac{v_--v_+}{D}N(t)\int \frac{\phi_{zz}^2+|\psi_{zz}\phi_{zz}|}{U}dz \\
\le& \frac{3v_--3v_++D}{2D}N(t)\int \frac{\phi_{zz}^2}{U}dz + \frac{v_--v_++D}{2D}N(t)\int \frac{\psi_{zz}^2}{U}dz + N(t)\int \frac{\phi_{zzz}^2}{U}dz, \qquad (5.24)
\end{aligned}$$

and

$$-\frac{1}{2}\int \frac{(\phi_z^2+\psi_z^2)_{zz}\psi_{zz}}{U}dz$$

$$
\begin{aligned}
&= -\frac{1}{2}\int \frac{(\phi_z^2+\psi_z^2)_z\psi_{zz}U_z}{U^2}dz + \frac{1}{2}\int \frac{(\phi_z^2+\psi_z^2)_z\psi_{zzz}}{U}dz \\
&\le \frac{(v_- - v_+)}{D}\int \frac{|\phi_z\phi_{zz}\psi_{zz}+\psi_z\psi_{zz}^2|}{U}dz + \int \frac{|\phi_z\phi_{zz}\psi_{zzz}+\psi_z\psi_{zz}\psi_{zzz}|}{U}dz \\
&\le \frac{(v_- - v_+)}{2D}N(t)\int \frac{\phi_{zz}^2}{U}dx + \frac{3(v_- - v_+)}{2D}N(t)\int \frac{\psi_{zz}^2}{U}dx + \frac{N(t)}{2}\int \frac{\phi_{zz}^2+\psi_{zz}^2}{U}dz \\
&\quad + N(t)\int \frac{\psi_{zzz}^2}{U}dz \\
&\le \frac{v_- - v_+ + D}{2D}N(t)\int \frac{\phi_{zz}^2}{U}dz + \frac{3v_- - 3v_+ + D}{2D}N(t)\int \frac{\psi_{zz}^2}{U}dz + N(t)\int \frac{\psi_{zzz}^2}{U}dz.
\end{aligned}
\tag{5.25}
$$

Inserting (5.23), (5.24), and (5.25) into (5.21), one has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \frac{\phi_{zz}^2+\psi_{zz}^2}{U}dz + (D-N(t))\int \frac{\phi_{zzz}^2}{U}dz + (\mu - N(t))\int \frac{\psi_{zzz}^2}{U}dz \\
&\le \frac{2v_- - 2v_+ + D}{D}N(t)\int \left(\frac{\phi_{zz}^2}{U}+\frac{\psi_{zz}^2}{U}\right)dz + C\int \frac{\phi_z^2+\psi_z^2+\phi_{zz}^2+\psi_{zz}^2}{U}dz.
\end{aligned}
\tag{5.26}
$$

Integrating (5.26) with respect to t, then using (2.35), Lemma 5.5, and the assumption $N(t) < \min\{\mu, D\}$, we obtain (5.17). Then the proof of Lemma 5.6 is completed. □

5.3. Proof of Theorem 5.1. Now we are in a position to prove Theorem 5.1. In fact we only need to prove (5.5). From the global estimate (5.4) which has been indicated by lemmas 5.4-5.6, we have

$$
\|(\phi_z(\cdot,t), \psi_z(\cdot,t))\|_{1,w} \to 0 \quad \text{as} \quad t \to +\infty. \tag{5.27}
$$

Hence, for all $z \in \mathbb{R}$, we have

$$
\begin{aligned}
\phi_z^2(z,t) &= 2\int_{-\infty}^{z} \phi_z\phi_{zz}(y,t)dy \\
&\le 2\left(\int_{-\infty}^{\infty}\phi_z^2 dy\right)^{1/2}\left(\int_{-\infty}^{\infty}\phi_{zz}^2 dy\right)^{1/2} \to 0 \quad \text{as} \quad t \to +\infty.
\end{aligned}
\tag{5.28}
$$

Applying the same procedure to ψ_z leads to

$$
\psi_z(z,t) \to 0 \quad \text{as} \quad t \to +\infty \quad \text{for all} \quad z \in \mathbb{R}. \tag{5.29}
$$

Thus (5.5) is proved.

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