Inference on an Adaptive Accelerated Life Test with Application to Smart-Grid Data-Acquisition-Devices

LIJUAN SHEN

China Electric Power Research Institute, Beijing, China

DAYU SUN

The Hong Kong Polytechnic University

ZHISHENG YE

National University of Singapore

XINGQIU ZHAO

The Hong Kong Polytechnic University

An accelerated life test (ALT) is often well planned to yield the most statistical information given limited test resources. Nevertheless, ALT planning requires rough estimates of the model parameters as an input, called planning values. The discrepancy between the planning values and the true values may result in insufficient or even no failures at the low-stress level, making the subsequent data analysis difficult. Motivated by the need in the ALTs of data acquisition devices used in smart grids, an adaptive ALT scheme is proposed. The key idea is based on the observation that, when the product reliability is underestimated during the ALT design phase, it is unlikely to observe failures at the early stage of the test. Therefore, the low-stress level should be elevated to protect against insufficient failures. Under this adaptive ALT framework, order statistics techniques are used to derive the likelihood function by assuming a general log-location-scale distribution for the product lifetime. Confidence intervals for the parameters are constructed based on the large-sample approximation as well as the accelerated bootstrap method. A simulation study is conducted to demonstrate the advantages of the adaptive ALT compared with the simple constant-stress ALT. Its application is illustrated using the motivating example from smart grids.

Key Words: Adaptive Test; Log-Location-Scale Distribution; Order Statistics; Step-Stress Test; Type-I Censoring.

1. Introduction

1.1. Motivation

Accelerated life tests (ALTs) have become an important tool in the evaluation of product reliability during the design phase. During an ALT,

Dr. Shen is Reliability Engineer in the China Electric Power Research Institute, Beijing. Her email address is shenlijuan 1016@gmail.com.

Mr. Sun is a postgraduate student at the Department of Applied Mathematics, The Hong Kong Polytechnic University. His email address is dayu.sun@hotmail.com.

Dr. Ye is Assistant Professor at the Department of Industrial & Systems Engineering, National University of Singapore. He is the corresponding author. His email address is yez@nus .edu.sg.

Dr. Zhao is Associate Professor at the Department of Applied Mathematics, The Hong Kong Polytechnic University. Her email address is xingqiu.zhao@polyu.edu.hk.

testing units are exposed to harsher-than-normal use conditions, such as high temperature, voltage, and humidity, so that failure times of these units can be collected within a timely fashion. An ALT is usually stopped after a certain amount of test time, i.e., type-I censoring, or when a certain number of failures have been observed, i.e., type-II censoring. Statistical methods are then used to fit the ALT data and extrapolate the results to normal use conditions. Both constant-stress ALT and step-stress ALT can be used for the test. Although there are some advantages of step-stress ALTs (e.g., see Han and Ng (2013)), constant-stress ALTs are still widely used in industry due to their simplicity. Some comprehensive reviews of accelerated-life testing include Nelson (2009) and Bagdonavicius and Nikulin (2010).

To efficiently use the test units and the testing rigs, an ALT is often carefully designed by optimally choosing the stress levels and the proportion of test units to each level. The design usually requires a knowledge of the parametric form of the lifetime distribution, the stress-life relation, and the values of the parameters involved. Given these inputs, an ALT can be designed under a certain criterion, e.g., minimizing the (asymptotic) variance of an estimated quantile at use conditions (e.g., Liu (2012)), maximizing the determinant of the asymptotic variance covariance matrix (e.g., Guan et al. (2014)), minimizing the average variance of a predicted quantile over the entire use condition region (e.g., Pan and Yang (2014)) or optimizing the energy efficiency (e.g., Zhang and Liao (2014)). This type of design is called locally optimal design because of the prior inputs required. In practice, the parametric form of the lifetime distribution and the stress-life acceleration relationship are usually known if we already have abundant experience on this type of product. Nevertheless, planning values of the model parameters are usually subject to uncertainty. When the planning values underestimate the product reliability, which is often the case due to the conservation of the manufacturers, no sufficient failures will be generated at the low-stress level. The insufficiency in failure data poses problems in the subsequent data analysis. A motivating example of this kind is from the China Electric Power Research Institute as follows.

Advanced Metering Infrastructure (AMI) is an integrated network processing system for measurement, collection, storage, analysis, and usage of electrical data, which is the basis of a smart grid. A central hub in the AMI is the data-acquisition devices, the reliability of which is extremely important to the AMI. A schematic figure of the AMI data-acquisition devices is shown in Figure 1. The Re-

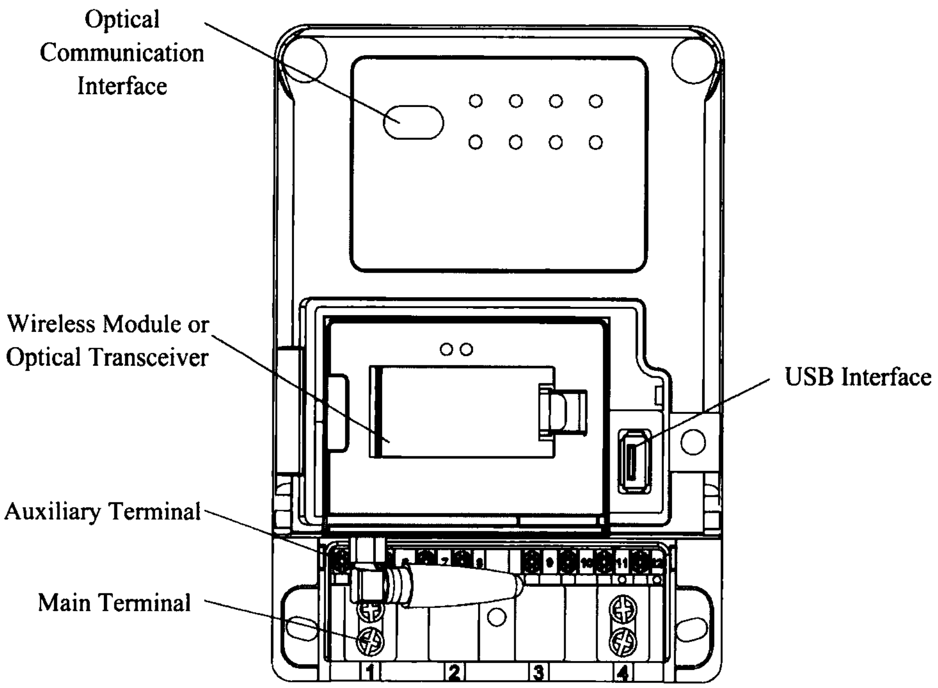

FIGURE 1. A Schematic of the AMI Data Acquisition Devices (Q/GDW 374.2, 2009).

search Institute is responsible for quality certification of AMI data acquisition devices before the devices are launched into the market. ALTs with two constant-stress levels and type-I censoring are commonly used to assess the reliability. There have been industrial standards on the lifetime distribution and the acceleration relations for the device, while distribution parameters of different brands/models of the device differ significantly. Therefore, prior reliability information required in the ALT planning is often provided by the manufacturers. Nevertheless, some manufacturers tend to over engineer the device and provide conservative reliability values much lower than the actual values. When the ALT is designed based on the conservative reliability information, ALT engineers often find that very few failures, or even no failures, are observed under the low-stress level. With few failure data at the lower stress level, extrapolation of the failure time to the use condition, if not infeasible, leads to very high variability. Such a "bad" ALT is a great waste to the institute, as the devices are often very expensive and the experiment takes a few months. The institute is thus looking for modifications to the current ALT design to mitigate the effect of conservative reliability estimation at the planning stage. Motivated by this need, this study proposes an adaptive scheme for the constant-stress ALTs.

Under the adaptive ALT, we look at the number of failures until a predetermined time under the low-stress level. The low stress is increased if the number is smaller than a threshold. The idea is based on the observation that, if the reliability is underestimated during the ALT design, then it is unlikely to observe failures at the early stage of the testing. Thus, the insufficient total number of failures before a predetermined time serves as a good indicator of the underestimation. After knowing the underestimation, an increase in the stress level induces more failures. The adaptive scheme can be regarded as an emergency control for the ALT planner to control the progress of the test. In practice, the ALT planner is usually different from the ALT experimenters. The ALT planner is usually well equipped with ALT knowledge and s/he takes care of the ALT design and the data analysis, while the experimenters who implement the test usually do not have sufficient knowledge on ALT. When there is no underestimation, the chance of increasing the low stress is slim. Therefore, the experiment goes to a regular constant-stress ALT, which can be readily handled by experimenters. In the presence of underestimation, nevertheless, the planner should step into the experiment and provide a hands-on guide to the experimenters in implementing the emergency measure. With this scheme, the risk of no failures would be greatly mitigated.

Another advantage of the adaptive ALT is that it may improve the estimation efficiency. ALT planners obtain the low-stress level based on the traditional optimal design for a constant-stress ALT by assuming that the planning values are equal to the true values. When there is no or mild misspecification of the parameter values, the chance of moving up the low-stress level is very small. This means that our adaptive method will not compromise the efficiency of the ALT in this case. However, when there is severe misspecification such that the reliability is highly underestimated, the true optimal value for the low stress should be higher than the design value. In this case, moving up the low-stress level to a higher level may increase the estimation efficiency.

1.2. Related Literature

The planning values of an ALT are often determined from past experience, industrial obligation, or expert knowledge. It is not uncommon for them to differ from the true values. A locally optimal design may be sensitive to the discrepancies between the planning values and the true values. There have been a slew of studies on mitigating the negative effect of the discrepancies. For instance, Ginebra et al. (1998) applied a minimax approach to minimize the determinant of the information matrix when the planning values are believed to be in a specified parameter region. Zhang and Meeker (2006) presented a general Bayesian framework for planning constant-stress ALTs. Tang and Liu (2010) proposed a sequential ALT design to plan or adjust subsequent tests by a better estimation of parameters from the first batch of samples. However, the sequential method requires heavy work for the planner (usually differs from the experimenters) to control the whole progress of the test, which is difficult for a large institute doing many ALTs simultaneously. In addition, the low-stress test has to be conducted after the high-stress test, which prolongs the total test duration.

Limited literature directly discussed the lack of failures in the low-stress level, the main problem we encountered in the motivation example. Nelson and Kielpinski (1976) suggested a compromise plan with more than two levels of stress to avoid no failures under the lowest stress level of the statistically optimum plan. Meeker and Hahn (1985) then presented some practical guidelines on designing a compromise plan. They proposed a three-level constant-stress compromised plan with a fixed 4:2:1 allocation to the low, middle, and high levels of stress, respectively. Nelson (2009) recommended a three- or four-level constant-stress test to avoid lack of failures. The key idea of those methods is to ensure enough failures from at least two stress levels by adding extra stress levels. The compromise ALT is sometimes unpractical due to equipment constrains. In the data-acquisition device example, each stress level requires an expensive thermostat. In addition, more stress levels entail more test samples. On the other hand, the adaptive ALT provides an emergency measure for engineers to flexibly change the test setting during the experiment. It uses two stress levels only and it is easy to control for the ALT planner.

1.3. Overview

The rest of the paper is organized as follows. Section 2 presents the new adaptive ALT scheme. The likelihood function is then derived by assuming a log-location-scale distribution to the lifetime. Section 3 develops a large-sample approximation and the bootstrap method to construct confidence intervals of parameters. In view of the fact that the exponential distribution provides a good fit to the lifetime of data-acquisition devices, Section 4 derives closed forms of maximum-likelihood estimators and the Fisher information matrix under the exponential distribution. Section 5 presents simulation results that compare the performances between the constant-stress ALT and the adaptive ALT. Section 6 provides a real-life example from China Electrical Power Institute. The real experiment shows its application in practice. Because the real example is in the form of a constant-stress ALT, we further generate a simulated dataset to illustrate the case where the low-stress level is switched. Section 7 includes some concluding remarks and suggestions for future research.

2. Model Description and the Log Likelihood

2.1. Notations and the Model

Consider a constant-stress ALT with two stress levels. Denote the low and the high levels as s_L and s_H, respectively. The number of test units under s_i is n_i, $i = L, H$. Let $n = n_L + n_H$. The duration of the ALT is denoted as τ_2. To avoid too few failures at s_L, suppose an adaptive scheme is implemented. The adaptive scheme sets a check-time τ_1 and monitors failures under s_L. If the number of failures until τ_1, denoted as J, is less than a threshold m, s_L is increased to a middle level s_M, $s_L < s_M < s_H$, at τ_1. Otherwise, no change is made to the test. Figure 2 gives a schematic representation of the adaptive ALT. For convenience, the stress levels are normalized as follows:

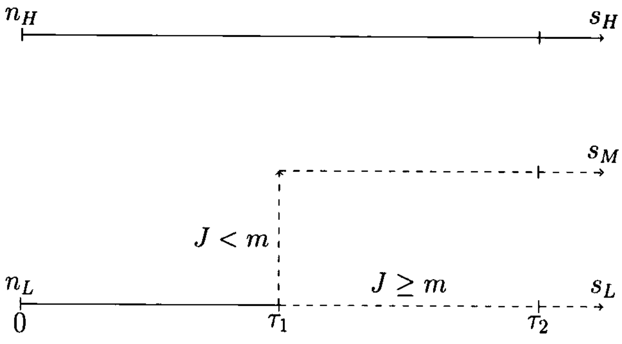

FIGURE 2. A Adaptive ALT.

$$\xi_i = \frac{\psi(s_U) - \psi(s_i)}{\psi(s_U) - \psi(s_H)}, \quad i = U, L, M, H, \tag{1}$$

where s_U is the use stress and $\psi(\cdot)$ is a function of the stress, the form of which depends on the acceleration relation. For example, $\psi(s) = \ln s$ for the inverse power law and $\psi(s) = 1/s$ for the Arrhenius law. After the normalization, $\xi_U = 0$, $\xi_H = 1$, and $0 \leq \xi_L < \xi_M < 1$.

The log-location-scale family of distributions are commonly used as lifetime distributions. See Liao and Elsayed (2010) and Chen et al. (2016a, b) to name a few. Suppose the product lifetime T under stress ξ follows a general log-location-scale distribution with cumulative distribution function (CDF) and probability density function (PDF)

$$F_i(t) = \Phi\left[\frac{\ln t - \mu(\xi)}{\sigma}\right], \quad t \geq 0 \tag{2}$$

$$f_i(t) = \frac{1}{t\sigma}\phi\left[\frac{\ln t - \mu(\xi)}{\sigma}\right], \quad t \geq 0, \tag{3}$$

respectively. $\Phi(\cdot)$ is the standard log-location-scale CDF and $\phi(\cdot)$ is the corresponding PDF. The above display has implicitly assumed that the location parameter is a function of the stress as $\mu(\xi) = \beta_0 + \beta_1\xi$, and the scale σ is constant. This is a common assumption used in the ALT literature, e.g., see Xu et al. (2015), Wang et al. (2014), and Ye et al. (2013). The exponential distribution is a special log-location-scale distribution with $\sigma = 1$ and $\Phi = 1 - \exp(-\exp(x))$.

Under the low-stress level ξ_L, the stress level increases to ξ_M at τ_1 when the number of failures J at τ_1 is less than m. To model the effect of the stress increment on product failures, the cumulative exposure model proposed by Nelson (1980) is used. This model has been widely used in ALT. See Lin et al. (2014), Liu and Qiu (2011), and Yuan et al. (2012) for some recent examples. The model assumes that, given survival up to t, the distribution of the remaining life depends only on the probability of failure up to t and the current stress level, regardless of the exposure history.

After the experiment, R_1 ordered failures, denoted as $\mathbf{X} = (X_{1:n_L}, \ldots, X_{J-1:n_L}, X_{J:n_L}, \ldots, X_{R_1:n_L})$ are observed for units allocated to ξ_L, and R_2 ordered failures, denoted as $\mathbf{Y} = (Y_{1:n_H}, \ldots, Y_{R_2:n_H})$ are observed for units allocated to ξ_H. The likelihood contributed from $\mathbf{Y}$ is easy to derive because all failures are under the same stress level ξ_H. Nevertheless, failures of the units allocated to the low stress are not independent, as the failure process after τ_1 depends on the number of failures before τ_1. This dependency motivates us to adopt order statistics. The method of order statistics is a useful tool to model censored data in reliability analysis (Arnold et al. (1992), Balakrishnan et al. (2012)). By using properties of the order statistics, the likelihood from $\mathbf{X}$ can be readily obtained.

Log-Likelihood Function

Let $\boldsymbol{\theta} = (\beta_0, \beta_1, \sigma)$ be the parameters of interest. As $\mathbf{X}$ and $\mathbf{Y}$ are independent, we decompose the log-likelihood function as $l(\boldsymbol{\theta} \mid \mathbf{X}, \mathbf{Y}) = l(\boldsymbol{\theta} \mid \mathbf{X}) + l(\boldsymbol{\theta} \mid \mathbf{Y})$. Under the adaptive scheme, $l(\boldsymbol{\theta} \mid \mathbf{X})$ cannot be simply obtained by summing the log likelihood of each individual unit as they are not mutually independent. Therefore, order statistics are used.

Using the order statistics, we have

$$l(\boldsymbol{\theta} \mid \mathbf{X}) = \ln g_{X_{1:n_L}}(x_{1:n_L}) + \sum_{i=2}^{R_1} \ln g_{X_{i:n_L}}(x_{i:n_L} \mid x_{i-1:n_L}, \ldots, x_{1:n_L}),$$

and

$$l(\boldsymbol{\theta} \mid \mathbf{Y}) = \ln g_{Y_{1:n_H}}(y_{1:n_H}) + \sum_{i=2}^{R_2} \ln g_{y_{i:n_H}}(y_{i:n_H} \mid y_{i-1:n_H}, \ldots, y_{1:n_H}).$$

Here, $g_{X_{1:n_L}}(\cdot)$ and $g_{Y_{1:n_H}}(\cdot)$ are the respective PDFs of $X_{1:n_L}$ and $Y_{1:n_H}$, while $g_{X_{i:n_L}}(\cdot|x_{i-1:n_L}, \ldots, x_{1:n_L})$ and $g_{y_{i:n_H}}(\cdot \mid y_{i-1:n_H}, \ldots, y_{1:n_H})$ are the conditional PDFs of $X_{i:n_L}$ and $Y_{i:n_H}$ given all prior failure times, respectively. Because all units allocated to the high-stress level operate under the constant stress ξ_H, it is easy to see that

$$g_{Y_{1:n_H}}(t) = n_H f_H(t)(1 - F_H(t))^{n_H - 1}$$

and

$$g_{Y_{i:n_H}}(t \mid y_{i-1:n_H}, \ldots, y_{1:n_H}) = \frac{(n_H - i + 1) f_H(t)(1 - F_H(t))^{n_H - i}}{(1 - F_H(y_{i-1:n_L}))^{n_H - i + 1}}.$$

On the other hand, the distribution for $X_{i:n_L}$ is much more complicated. For $X_{1:n_L}$, it is readily seen that

$$g_{X_{1:n_L}}(t) = \begin{cases} n_L f_L(t)(1 - F_L(t))^{n_L - 1}, & \text{if } x_{1:n_L} \le \tau_1; \\ n_L f_M(t - \tau_1 + \tau') \\ \times (1 - F_M(t - \tau_1 + \tau'))^{n_L - 1}, & \text{if } x_{1:n_L} > \tau_1, \end{cases}$$

where $\tau' = F_M^{-1}(F_L(\tau_1)) = \tau_1 \exp(\beta_1(\xi_M - \xi_L))$ is the equivalent operating time under s_M for τ_1 under s_L. It is obtained from the cumulative exposure model. The distribution of $X_{i:n_L}$ given all $i-1$ failures before it is more involved. After some tedious calculation, it can be shown that

$$g_{X_{i:n_L}}(t \mid x_{i-1:n_L}, \ldots, x_{1:n_L}) = \begin{cases} \dfrac{(n_L - i + 1)\, f_L(t)\, (1 - F_L(t))^{n_L - i}}{(1 - F_L(x_{i-1:n_L}))^{n_L - i + 1}}, & \text{if } x_{i:n_L} \le \tau_1 \text{ or if } x_{i:n_L} > \tau_1 \text{ and } J \ge m; \\ \dfrac{(n_L - i + 1) f_M(t - \tau_1 + \tau')(1 - F_M(t - \tau_1 + \tau'))^{n_L - i}}{(1 - F_L(x_{i-1:n_L}))^{n_L - i + 1}}, & \text{if } x_{i:n_L} > \tau_1,\ x_{i-1:n_L} < \tau_1 \text{ and } J < m; \\ \dfrac{(n_L - i + 1) f_M(t - \tau_1 + \tau')(1 - F_M(t - \tau_1 + \tau'))^{n_L - i}}{(1 - F_M(x_{i-1:n_L} - \tau_1 + \tau'))^{n_L - i + 1}}, & \text{if } x_{i:n_L} > \tau_1,\ x_{i-1:n_L} > \tau_1 \text{ and } J < m. \end{cases}$$

Combining the above results, the log-likelihood function when $m \leq J \leq n_L$ is

$$\begin{aligned} l(\boldsymbol{\theta} \mid \mathbf{X}, \mathbf{Y}) &= D_0 + \sum_{i=1}^{R_1} \ln(f_L(x_{i:n_L})) \\ &\quad + (n - R_1)\ln(1 - F_L(\tau_2)) \\ &\quad + \sum_{i=1}^{R_2} \ln(f_H(y_{i:n_H})) \\ &\quad + (n_H - R_2)\ln(1 - F_H(\tau_2)). \quad (4) \end{aligned}$$

When $0 \leq J < m$,

$$\begin{aligned} l(\boldsymbol{\theta} \mid \mathbf{X}, \mathbf{Y}) &= D_0 + \sum_{i=1}^{J} \ln(f_L(x_{i:n_L})) \\ &\quad + \sum_{i=J+1}^{R_1} \ln(f_M(x_{i:n_L} - \tau_1 + \tau')) \\ &\quad + (n - R_1)\ln(1 - F_M(t - \tau_1 + \tau')) \\ &\quad + \sum_{i=1}^{R_2} \ln(f_H(y_{i:n_H})) \\ &\quad + (n_H - R_2)\ln(1 - F_H(\tau_2)), \end{aligned}$$

where

$$D_0 = \ln\left(\frac{n_L!}{(n_L - R_1)!}\frac{n_H!}{(n_H - R_2)!}\right)$$

is a constant. The second term of Equation (5) will vanish if $J = 0$ and so will the third term if $J = R_1$. Detailed expressions of the log-likelihood functions for different log-location-scale distributions are provided in the Appendix.

Maximum-likelihood estimators (MLEs) of β_1, β_0, and σ can be obtained by maximizing Equation (4) or Equation (5). The MLEs do not exist when no failures are observed on either stress level, i.e., either R_1 or R_2 is zero. Therefore, a rational ALT should minimize the chance of no failures, which is the case for our adaptive scheme. When $J \geq m$, the log-likelihood function is the same as that from a simple constant-stress ALT. In which case, the MLEs are the same as those from the constant-stress. However, we will show later that the interval estimations differ. Usually, the MLEs do not have closed forms under a general log-location-scale distribution. Numerical methods can be used to do the optimization. Nevertheless, closed forms of the MLEs exist for the exponential distribution when $m \leq J \leq n_L$. See Section 4 for more details.

3. Interval Estimation

In this section, we present two different methods to construct confidence intervals (CIs) for the parameters β_0, β_1, and σ. First, we present approximate CIs for β_0, β_1, and σ by using large-sample approximation. Then the parametric bootstrap method is used.

3.1. Large-Sample Approximate Confidence Intervals

When the sample size is not very small, normal approximation to the MLEs is often an accurate method for confidence intervals. The large-sample approximate covariance–variance matrix of $\hat{\beta}_1$, $\hat{\beta}_0$ and $\hat{\sigma}$, denoted by $\boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}}$, is the inverse of the Fisher information matrix (FIM) $\mathbf{I}(\boldsymbol{\theta})$. It is the expectation of the observed information matrix $I(\boldsymbol{\theta})$, which is the negative of the second partial derivatives of the log-likelihood function with respect to parameters. When the expectation is difficult to compute, which is the case in our problem, $\boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}}$ can be alternatively obtained by inverting the observed information matrix, which is given by

$$\begin{aligned} \boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}} &= \begin{pmatrix} \mathrm{Var}(\hat{\beta}_0) & \mathrm{Cov}(\hat{\beta}_0, \hat{\beta}_1) & \mathrm{Cov}(\hat{\beta}_0, \hat{\sigma}) \\ \mathrm{Cov}(\hat{\beta}_0, \hat{\beta}_1) & \mathrm{Var}(\hat{\beta}_1) & \mathrm{Cov}(\hat{\beta}_1, \hat{\sigma}) \\ \mathrm{Cov}(\hat{\beta}_0, \sigma) & \mathrm{Cov}(\hat{\beta}_1, \sigma) & \mathrm{Var}(\hat{\sigma}) \end{pmatrix} \\ &= I^{-1}(\boldsymbol{\theta}). \end{aligned}$$

Then the two-sided $100(1-a)\%$ confidence intervals of β_0, β_1, σ are

$$\begin{aligned} &\hat{\beta}_0 \pm z_{a/2}\sqrt{\mathrm{Var}(\hat{\beta}_0)} \\ &\hat{\beta}_1 \pm z_{a/2}\sqrt{\mathrm{Var}(\hat{\beta}_1)} \\ &\hat{\sigma} \pm z_{a/2}\sqrt{\mathrm{Var}(\hat{\sigma})}. \end{aligned}$$

Details of the observed information matrix $I(\boldsymbol{\theta})$ for some common log-location-scale distributions are given in the Appendix. Under the special case of exponential distributions with $\boldsymbol{\theta} = (\beta_0, \beta_1)$, the explicit expression of the Fisher information matrix is available, as shown in the next section. The closed-form expressions enable us to compare the difference between our adaptive scheme and the simple constant-stress ALT.

3.2. Parametric Bootstrap Confidence Intervals

The confidence intervals can also be constructed based on the parametric bootstrap. The biased-corrected and accelerated (BCa) percentile introduced in Efron and Tibshirani (1994) is adopted. The algorithm, when applied to our inference problem, is as follows:

1. Obtain the MLEs $\hat{\boldsymbol{\theta}} = (\hat{\beta}_0, \hat{\beta}_1, \hat{\sigma})$ based on the observed data $\mathbf{X}$ and $\mathbf{Y}$. Let $\hat{\tau} = F_M^{-1}(F_L(\tau_1; \hat{\boldsymbol{\theta}}); \hat{\boldsymbol{\theta}})$.
2. Generate two sets of standard uniform random samples of size n_L and n_H, respectively. Obtain the two samples of order statistics $(U_{1:n_L}, \ldots, U_{n_L:n_L})$ and $(V_{1:n_H}, \ldots, V_{n_H:n_H})$.
3. Find J^* such that $U_{J^*:n_L} < F_L(\tau_1; \hat{\boldsymbol{\theta}}) \leq U_{J^*+1:n_L}$. Generate a pseudo-random sample for units allocated to the low stress as
   - If $0 \leq J^* < m$, find R_1^* such that $U_{R_1^*:n_L} < F_M(\tau_2 - \tau_1 + \hat{\tau}; \hat{\boldsymbol{\theta}}) \leq U_{R_1^*+1:n_L}$. Then, we set $x_{i:n}^* = F_L^{-1}(U_{i:n_L}; \hat{\boldsymbol{\theta}})$ for $1 \leq i \leq J^*$ and $x_{i:n}^* = F_M^{-1}(U_{i:n_L}; \hat{\boldsymbol{\theta}}) + \tau_1 - \hat{\tau}$ for $J+1 \leq i \leq R_1^*$.
   - If $m \leq J^* \leq n_L$, find R_1^* such that $U_{R_1^*:n_L} < F_L(\tau_2; \hat{\boldsymbol{\theta}}) \leq U_{R_1^*+1:n_L}$. Then set $x_{i:n}^* = F_L^{-1}(U_{i:n_L}; \hat{\boldsymbol{\theta}})$ for $1 \leq i \leq R_1^*$.
4. Find R_2^* such that $V_{R_2^*:n_H} < F_H(\tau_2; \hat{\boldsymbol{\theta}}) \leq V_{R_2^*+1:n_H}$. Generate a pseudo-random sample for units allocated to the high stress as $y_{i:n_H}^* = F_H^{-1}(V_{i:n_H}; \hat{\boldsymbol{\theta}})$, $1 \leq i \leq R_2^*$.
5. If $R_1^*, R_2^* > 0$, obtain a bootstrap estimate $\hat{\boldsymbol{\theta}}^* = (\hat{\beta}_0^*, \hat{\beta}_1^*, \hat{\sigma}^*)$ based on the data generated.
6. Repeat steps 2–5 to obtain B bootstrap estimates $\hat{\boldsymbol{\theta}}^{*(k)}$, $k = 1, 2, \ldots, B$.

Let θ_i be the ith element of $\boldsymbol{\theta}$ for convenience. Sort $\hat{\theta}_i^{*(k)}$ in ascending order. Then a two-sided $100(1-\alpha)\%$ BCa bootstrap confidence interval of θ_i is given by

$$(\hat{\theta}_i^{*[\alpha_{1i}B]}, \hat{\theta}_i^{*[\alpha_{2i}B]}), \quad i = 0, 1, 2,$$

where

$$\alpha_{1i} = \Phi\left\{\hat{z}_{0i} + \frac{\hat{z}_{0i} + z_{\alpha/2}}{1 - \hat{\alpha}_i(\hat{z}_{0i} + z_{\alpha/2})}\right\}$$

and

$$\alpha_{2i} = \Phi\left\{\hat{z}_{0i} + \frac{\hat{z}_{0i} + z_{1-\alpha/2}}{1 - \hat{\alpha}_i(\hat{z}_{0i} + z_{1-\alpha/2})}\right\}.$$

In the above display, $\Phi(\cdot)$ is the standard normal CDF and

$$\hat{z}_{0i} = \Phi^{-1}\left\{\frac{\#\text{ of } \hat{\beta}_i^{*[k]} < \hat{\beta}_i}{B}\right\}, \quad k = 1, \ldots, B.$$

A good estimate of the acceleration factor α_i is

$$\hat{\alpha}_i = \frac{\sum_{k=1}^{R_1+R_2}(\hat{\beta}_i^{(\cdot)} - \hat{\beta}_i^{(k)})^3}{6\left[\sum_{k=1}^{R_1+R_2}(\hat{\beta}_i^{(\text{dot})} - \hat{\beta}_i^{(k)})^2\right]^{3/2}}, \quad k = 1, 2,$$

where $\hat{\beta}_i^{(k)}$ is the MLE of β_i, based on the original sample combined $\mathbf{X}$ and $\mathbf{Y}$ ascending with the ith observation deleted (i.e., the jackknife estimate), and

$$\hat{\beta}_i^{(\cdot)} = \sum_{k=1}^{R_1+R_2} \hat{\beta}_i^{(k)}/(R_1 + R_2), \quad k = 1, 2.$$

4. Inference Under Exponential Distribution

Under the special case of the exponential distribution with $\boldsymbol{\theta} = (\beta_0, \beta_1)$, the closed form of the Fisher information matrix is available. The log-likelihood function given the exponential lifetime is

$$l(\boldsymbol{\theta} \mid \mathbf{X}, \mathbf{Y}) = \begin{cases} l_1 = D_0 - (R_1 + R_2)\beta_0 \\ \quad -\beta_1(J\xi_L + (R_1 - J)\xi_M \\ \quad +R_2\xi_H) \\ \quad -D_1 - D_2 - D_3, & \text{if } J < m; \\ l_2 = D_0 - (R_1 + R_2)\beta_0 \\ \quad -\beta_1(R_1\xi_L + R_2\xi_H) \\ \quad -D_1 - D_4 - D_3, & \text{if } J \geq m, \end{cases} \tag{6}$$

where

$$D_1 = \begin{cases} e^{-(\beta_0+\beta_1\xi_L)}\left(\sum_{i=1}^{J} x_{i:n_L} + \tau_1(n_L - J)\right), & \text{if } J > 0; \\ e^{-(\beta_0+\beta_1\xi_L)}\tau_1 n_L, & \text{if } J = 0, \end{cases}$$

$$D_2 = \begin{cases} e^{-(\beta_0+\beta_1\xi_M)}\left(\sum_{i=J+1}^{R_1} x_{i:n_L} - \tau_1(n_L - J) + (n_L - R_1)\tau_2\right), & \text{if } R_1 > J; \\ e^{-(\beta_0+\beta_1\xi_M)}(\tau_2 - \tau_1)n_L, & \text{if } R_1 = J, \end{cases}$$

$$D_3 = e^{-(\beta_0+\beta_1\xi_H)}\left(\sum_{i=1}^{R_2} y_{i:n_H} + \tau_2(n_H - R_2)\right),$$

and

$$D_4 = \begin{cases} e^{-(\beta_0+\beta_1\xi_L)}\left(\sum_{i=J+1}^{R_1} x_{i:n_L} - \tau_1(n_L - J) + (n_L - R_1)\tau_2\right), & \text{if } R_1 > J; \\ e^{-(\beta_0+\beta_1\xi_L)}(\tau_2 - \tau_1)n_L, & \text{if } R_1 = J. \end{cases}$$

When $J \geq m$, the closed forms of MLEs of $\boldsymbol{\theta}$ are as follows:

$$\hat{\beta}_0 = \frac{\xi_H(\beta_0 + \beta_1\xi_L)\ln\left(\frac{D_1+D_4}{R_1}\right)}{\xi_H - \xi_L} - \frac{\xi_L(\beta_0 + \beta_1\xi_H)\ln\left(\frac{D_3}{R_2}\right)}{\xi_H - \xi_L}$$

$$\hat{\beta}_1 = \frac{(\beta_0 + \beta_1\xi_H)\ln\left(\frac{D_3}{R_2}\right)}{\xi_H - \xi_L} - \frac{(\beta_0 + \beta_1\xi_L)\ln\left(\frac{D_1+D_4}{R_1}\right)}{\xi_H - \xi_L}.$$

When $J < m$, the MLEs have to be obtained by numerical methods as the closed forms are not available.

The Fisher information matrix can be obtained by the observed information conditioned on J as

$$\begin{aligned}\mathcal{I}(\boldsymbol{\theta}) &= \mathbb{E}[\mathbb{E}[I(\boldsymbol{\theta}) \mid J]] \\ &= \sum_{j=0}^{m-1} \mathbb{E}[I_1(\boldsymbol{\theta}) \mid J = j]\Pr(J = j) \\ &\quad + \sum_{j=m}^{n_L} \mathbb{E}[I_2(\boldsymbol{\theta}) \mid J = j]\Pr(J = j),\end{aligned}$$

where $I_i(\boldsymbol{\theta})$ is the corresponding observed information matrix of $l_i(\boldsymbol{\theta})$. Therefore, the entries of the Fisher information matrix are given by

$$\begin{aligned}\mathcal{I}_{ik} &= -\xi_L^{i+k-2}\left(\mathbb{E}[D_1] + \sum_{j=m}^{n_L} \mathbb{E}[D_4 \mid J = j]\Pr(J = j)\right) \\ &\quad - \xi_M^{i+k-2}\sum_{j=0}^{m-1} \mathbb{E}[D_2 \mid J = j]\Pr(J = j)) \\ &\quad - \xi_H^{i+k-2}\mathbb{E}[D_3],\end{aligned}$$

where $i, k = 1, 2$. With the aid of the zero expectation of the first-order derivatives to the log-likelihood functions, we have

$$\begin{aligned}\mathbb{E}[D_1] &+ \sum_{j=m}^{n_L} \mathbb{E}[D_4 \mid J = j]\Pr(J = j) \\ &= \mathbb{E}[J] + \sum_{j=m}^{n_L} \mathbb{E}[R_1 - J \mid J = j]\Pr(J = j),\end{aligned}$$

$$\begin{aligned}\sum_{j=0}^{m-1} \mathbb{E}[D_2 \mid J = j]\Pr(J = j) &= \sum_{j=0}^{m-1} \mathbb{E}[R_1 - J \mid J = j]\Pr(J = j), \\ \mathbb{E}[D_3] &= E[R_2].\end{aligned}$$

Based on the above results, $\mathbf{I}_{ik}$ can be simplified as

$$\begin{aligned}\mathcal{I}_{ik} &= \xi_L^{i+k-2}\left(\mathbb{E}[J] + \sum_{j=m}^{n_L} \mathbb{E}[R_1 - J \mid J = j]\Pr(J = j)\right) \\ &\quad + \xi_M^{i+k-2}\sum_{j=0}^{m-1} \mathbb{E}[R_1 - J \mid J = j]\Pr(J = j) \\ &\quad + \xi_H^{i+k-2}\mathbb{E}[R_2].\end{aligned}$$

Details of the derivation of $\mathcal{I}_{ik}$ are shown in the Appendix. Under the simple constant-stress ALT, the second term of the entries of the Fisher information matrix vanishes and m is 0. This implies that confidence intervals constructed by the Fisher information matrix of the adaptive ALT differ from those of the constant-stress one.

5. Simulation Study

Comprehensive Monte Carlo simulations are conducted to compare the performances of the adaptive ALT with the simple constant-stress ALT under different lifetime distributions. The simulations are based on three commonly used log-location-scale distributions, namely, the Weibull, log normal, and exponential distributions. The simulation settings mimic the experiment reported in Section 6. In all simulations, we let $\xi_L = 0.587$, $n_L = 30$, and $n_H = 13$. The nominal confidence level is set to be 90%. The duration of the test is $\tau_2 = 160$.

For each distribution, we assume that planning values of $\boldsymbol{\theta}^{\square} = (\beta_0^{\square}, \beta_1^{\square}, \sigma^{\square})$ are provided by the manufacturer, as shown in Table 1. These values are similar to MLEs of parameters for each distribution based on the real data in Section 6. In our simulations, we first consider the scenario that the true values are the same as the planning values. Then we consider underestimation of the reliability by letting the true value of one parameter differ from its planning value by 3% or −3%, whichever leads to underestimation in reliability. For each scenario, we simulate both the simple constant-stress ALT and six adaptive schemes by varying m, ξ_M. The values of m are from 1 to 2 and $\xi_M = 0.618, 0.647, 0.676$.

Regarding the choice of τ_1, if we know the true values of the model parameters when there is misspecification, we can determine the optimal switch time τ_1 using some statistical criterion. This is similar to

TABLE 1. Planning Values of the Model Parameters for the Three Distributions

Distribution	$\beta_0^\square$	$\beta_1^\square$	$\sigma^\square$
Weibull	10.733	−6.713	1.022
Lognormal	12.400	−8.349	2.500
Exponential	11.560	−8.293	

the determination of control limits in the design of a control chart. The challenge is that we do not know the true values of the model parameters when there is misspecification. Therefore, it might be difficult to quantitatively determine the optimal switch setting. However, we can still choose τ_1 by some heuristic criteria as follows. (a) If the model parameters $\boldsymbol{\theta}$ are close to the planning values $\boldsymbol{\theta}^\square$, the misspecification is mild and we may not need to change the stress. Therefore, we should choose τ_1 such that $\Pr(J(\tau_1) < m \mid \boldsymbol{\theta}^\square)$ is small, say, less than 0.1. For a fixed m, this implies that a small τ_1 is preferred. (b) On the other hand, if the model parameters $\boldsymbol{\theta}$ are far from the planning values $\boldsymbol{\theta}^\square$, we should choose τ_1 such that $\Pr(J(\tau_1) < m \mid \boldsymbol{\theta})$ is large enough. This entails a large τ_1 when m is fixed. (c) To have an effective adaptive plan, we also want to choose τ_1 such that the discrepancy between $\Pr(J(\tau_1) < m \mid \boldsymbol{\theta}^\square)$ and $\Pr(J(\tau_1) < m \mid \boldsymbol{\theta})$ is as large as possible. In order to achieve a balance between (a) and (b), τ_1 should neither be too large nor too small. In the simulation, the choice of τ_1 is mainly based on the Weibull model and a conservative $m = 1$. In the settings for the Weibull distribution, the most significant misspecification (in terms of MTTF) happens when β_0 is 3% larger than the plan value. In this scenario, one finds that, when $\tau_1 = 60, 61, 62$, criteria (a) and (b) are satisfied. Meanwhile, the difference between $\Pr(J(60) < m \mid \boldsymbol{\theta}^\square)$ and $\Pr(J(60) < m \mid \boldsymbol{\theta})$ is the largest when $\tau_1 = 60$. Therefore, a good choice of τ_1 is 60 for this scenario. When the same analysis is applied to the exponential model, the choice of τ_1 is similar. On the other hand, the best τ_1 for the lognormal model is circa 45. Because we want to compare the three models under the same testing scheme, we choose $\tau_1 = 60$ in our simulation.

In each scenario, 10,000 Monte Carlo replications are used to obtain the root-mean squared errors (RMSEs) of the MLEs as well as the coverage probabilities of the confidence intervals. In the bootstrap, the bootstrap sample size is $B = 2{,}000$. We also report the percentage of replications with $R_1 \leq 1$, under which the MLEs are erratic. Tables 2 4 present the simulation results on the log-normal, Weibull, and exponential distributions.

Several important observations are readily made. The percentage of experiments with number of failures no more than one at the low stress can be very high (nearly 10%) under the simple constant-stress ALTs when there is underestimation in reliability. The adaptive scheme effectively mitigates, though not eradicates, the problem of lack of failures under the low stress. This observation provides a strong motivation in adopting the adaptive ALT to mitigate the lack of failures. An interesting observation from Tables 2 4 is that the RMSEs of $\hat{\beta}_0$ and $\hat{\beta}_1$ of all the adaptive tests are smaller than those from the simple ALTs. This might be explained by the larger number of failures under the adaptive scheme. Nevertheless, the RMSEs of $\hat{\sigma}$ tend to be slightly larger under some cases of the adaptive scheme. This might be explained by the fact that the increase from ξ_L to ξ_M reduces the spacing between the stresses and thus increases the difficulty in the failure-time regression. Therefore, there seems to be a trade-off between the estimation accuracies of σ and β_i's. However, our simulation experiments with other parameter settings suggest that, when the reliability is seriously underestimated, the RMSEs of all the three estimators $\hat{\beta}_0, \hat{\beta}_1, \hat{\sigma}$ under the adaptive scheme tend be smaller than those under the simple constant-stress ALT.

The adaptive ALT also provides better interval estimation. It is clear that the bootstrap confidence intervals under the adaptive ALT have higher coverage probabilities than those of the simple ALT if ξ_M is not very large. Moreover, the bootstrap confidence intervals of the adaptive ALT maintain satisfactory coverage probabilities close to the nominal level when ξ_M is relatively small. This might be because the adaptive ALT tends to yield more failures in each bootstrap replication and thus more accurate parameter estimation when ξ_M is not too high. Meanwhile, under Weibull and log-normal distributions, confidence intervals constructed by the observed information matrix have worse performances under the adaptive test than the simple ALT. Under the exponential distribution, coverage probabilities of all three methods are similar or slightly larger than the nominal levels. Considering heavy computation effort of the bootstrap method, the large sample approximation is preferred if the model is exponential. In conclusion, we recommend the bootstrap method for the Weibull and log-normal lifetimes and large sample approximation for the exponential distribution.

TABLE 2. RMSEs of the Log-Normal MLEs and the Coverages Probabilities with Nominal Confidence Level $1-\alpha=90\%$ of the Normal Approximation Method (App.) and the Bootstrap (BS) Under the Simple Constant-Stress ALT and the Adaptive ALTs. The percentage of number of failures at the low stress no more than one and the percentage of changing stress for each scenario are also presented, respectively

Deviation (in %) of			% Number of failures	% Stress	RMSE			Coverage probability (in %)					
								App.			BS		
$\beta_0/\beta_1/\sigma$	ξ_M	m	≤ 1	change	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\sigma}$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\sigma}$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\sigma}$
0/0/0	Simple		3.07	0.0	2.41	2.74	0.557	87.6	87.5	84.3	87.5	87.3	89.8
	0.618	1	2.45	6.3	2.41	2.74	0.557	87.1	87.3	83.0	87.7	87.2	89.7
		2	1.76	25.3	2.38	2.72	0.557	84.6	86.3	77.7	87.6	87.1	89.7
	0.647	1	1.98	6.3	2.39	2.73	0.557	87.1	87.3	82.8	88.3	87.8	89.4
		2	0.92	25.3	2.34	2.69	0.557	84.3	86.1	77.3	88.3	87.6	89.5
	0.676	1	1.75	6.3	2.39	2.73	0.558	87.1	87.3	82.7	88.8	88.2	89.4
		2	0.44	25.3	2.33	2.69	0.559	83.8	86.0	76.8	88.3	87.9	89.5
3/0/0	Simple		8.49	0.0	2.63	2.91	0.602	86.9	87.1	83.8	86.2	85.9	90.1
	0.618	1	6.89	12.8	2.62	2.90	0.603	85.4	86.8	80.9	86.9	86.2	90.0
		2	5.49	40.7	2.60	2.89	0.601	77.9	83.2	71.9	87.2	86.5	90.0
	0.647	1	5.67	12.8	2.61	2.90	0.603	85.3	86.7	80.9	87.9	87.1	89.8
		2	3.39	40.7	2.58	2.89	0.600	76.8	82.8	71.5	88.3	87.3	89.8
	0.676	1	4.72	12.8	2.61	2.90	0.604	85.1	86.7	80.5	88.6	87.7	89.8
		2	1.86	40.7	2.58	2.91	0.600	75.4	82.4	70.6	88.7	87.5	89.9
0/−3/0	Simple		4.71	0.0	2.50	2.80	0.580	87.1	87.6	84.0	87.0	86.9	89.8
	0.618	1	3.77	8.6	2.48	2.79	0.581	86.2	87.4	82.0	87.7	87.0	89.7
		2	2.96	30.7	2.46	2.78	0.579	82.0	85.5	75.8	87.8	87.1	89.6
	0.647	1	3.06	8.6	2.47	2.79	0.581	86.2	87.4	81.8	88.4	87.5	89.5
		2	1.67	30.7	2.44	2.77	0.580	81.5	85.3	75.1	88.2	87.5	89.5
	0.676	1	2.53	8.6	2.46	2.78	0.582	86.1	87.4	81.6	88.9	88.0	89.5
		2	0.83	30.7	2.42	2.76	0.581	80.6	85.0	74.5	88.9	87.8	89.6
0/0/−3	Simple		3.82	0.0	2.37	2.69	0.541	87.4	87.7	84.3	87.2	86.9	89.8
	0.618	1	3.02	7.9	2.36	2.68	0.542	86.8	87.6	82.6	87.5	87.0	89.5
		2	2.28	29.0	2.34	2.66	0.541	84.1	86.6	76.9	87.6	87.1	89.7
	0.647	1	2.38	7.9	2.35	2.67	0.542	86.8	87.6	82.4	88.1	87.8	89.7
		2	1.21	29.0	2.31	2.65	0.542	83.8	86.4	76.4	88.1	87.5	89.7
	0.676	1	2.05	7.9	2.34	2.67	0.543	86.7	87.6	82.2	88.8	88.3	89.5
		2	0.54	29.0	2.29	2.64	0.542	83.3	86.2	76.0	88.7	88.1	89.6

The simulation results imply that the heuristic method used to determine τ_1 is reasonable. Compared with the Weibull model, the adaptive scheme has less RMSE reduction in $\hat{\beta}_1$ for the log-normal model, especially when β_0 is 3% larger than the planning value. This is because a good choice of τ_1 for the log-normal model is 45, which is less than 60. When the emergency measure is initiated and the low-stress level is increased, the remaining test time is not sufficient to yield enough failures. With insufficient failures, the adaptive ALT for the log-normal model is not as effective as the Weibull model. The above observation reveals the benefits of the heuristic method in choosing τ_1.

The simulations also provide some suggestions on how to choose m and ξ_M. It is immediately seen that small or moderate m and ξ_M are preferred as large m and ξ_M could lead to worse interval estimation and larger RMSEs of σ. Some ALTs aim to esti-

TABLE 3. RMSEs of the Weibull MLEs and the Coverages Probabilities with Nominal Confidence Level $1 - \alpha = 90\%$ of the Normal Approximation Method (App.) and the Bootstrap (BS) Under the Simple Constant-Stress ALT and the Adaptive ALTs. The percentage of number of failures at the low stress no more than one and the percentage of changing stress for each scenario are also presented, respectively

Deviation (in %) of			% Number of failures	% Stress	RMSE			Coverage probability (in %)					
								App.			BS		
$\beta_0/\beta_1/\sigma$	ξ_M	m	≤ 1	change	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\sigma}$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\sigma}$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\sigma}$
0/0/0	Simple		2.75	0.0	1.65	1.78	0.217	88.5	88.7	84.2	88.6	88.2	89.6
	0.618	1	1.90	11.6	1.63	1.77	0.217	88.1	88.4	83.9	88.8	88.4	89.6
		2	1.39	37.8	1.61	1.75	0.218	87.1	87.7	83.3	88.9	88.3	89.6
	0.647	1	1.47	11.6	1.61	1.75	0.218	87.9	88.3	83.7	88.8	88.3	89.5
		2	0.61	37.8	1.56	1.71	0.218	86.3	87.2	82.7	88.6	88.0	89.6
	0.676	1	1.19	11.6	1.60	1.74	0.218	87.6	88.0	83.4	88.8	88.5	89.9
		2	0.23	37.8	1.54	1.69	0.219	85.3	86.2	81.9	88.4	88.0	89.9
3/0/0	Simple		9.35	0.0	1.92	2.04	0.235	88.1	88.5	84.9	87.2	87.1	89.9
	0.618	1	7.22	20.7	1.91	2.03	0.235	87.0	87.8	84.6	88.0	87.5	89.8
		2	6.05	54.1	1.89	2.01	0.235	85.2	87.0	83.8	88.1	87.6	89.8
	0.647	1	5.56	20.7	1.90	2.02	0.236	86.9	87.8	84.3	88.7	88.3	89.8
		2	3.47	54.1	1.87	1.99	0.234	84.8	86.9	83.3	88.7	88.4	89.9
	0.676	1	4.37	20.7	1.89	2.01	0.236	86.5	87.5	84.1	88.9	88.6	89.8
		2	1.71	54.1	1.84	1.98	0.235	83.7	86.3	83.0	88.9	88.1	89.9
0/−3/0	Simple		4.56	0.0	1.76	1.88	0.226	88.4	88.8	84.6	88.5	88.1	89.8
	0.618	1	3.39	14.5	1.74	1.87	0.226	87.7	88.2	84.1	89.0	88.4	89.6
		2	2.68	44.0	1.72	1.85	0.226	86.5	87.5	83.5	88.9	88.2	89.6
	0.647	1	2.48	14.5	1.73	1.86	0.227	87.4	88.1	84.0	89.1	88.6	89.6
		2	1.30	44.0	1.69	1.83	0.227	85.8	87.1	83.2	89.0	88.2	89.8
	0.676	1	2.01	14.5	1.71	1.84	0.227	87.4	88.1	83.8	89.3	88.7	89.7
		2	0.52	44.0	1.65	1.80	0.227	85.3	86.7	82.6	88.9	88.1	90.1
0/0/−3	Simple		3.49	0.0	1.65	1.77	0.212	88.4	88.7	84.3	88.3	87.9	89.4
	0.618	1	2.43	13.6	1.63	1.76	0.212	87.7	88.2	83.9	88.5	88.1	89.3
		2	1.76	42.1	1.60	1.73	0.212	86.8	87.5	83.1	88.8	88.1	89.4
	0.647	1	1.73	13.6	1.61	1.74	0.212	87.7	88.0	83.6	88.6	88.1	89.4
		2	0.74	42.1	1.55	1.69	0.212	86.2	86.9	82.5	88.4	88.0	89.7
	0.676	1	1.47	13.6	1.60	1.73	0.213	87.2	87.7	83.5	88.8	88.4	89.8
		2	0.31	42.1	1.53	1.68	0.213	85.0	85.9	81.8	88.3	87.6	90.0

mate a certain percentile of the failure time at the use stress level. Too small spacing between the stress levels causes more extrapolation of the use conditions and hence may lead to less accurate prediction of the lifetime percentile. Moreover, the moderate level of $\xi_M = 0.647$ sufficiently diminishes the probability of no failures when the parameter values are misspecified. This achieves the main purpose of the adaptive ALT and hence there is no need to increase the low-stress level too much. To sum up, for practitioners, we recommend that m be small, say 1 or 2, and ξ_M be less than the middle point of ξ_L and ξ_H.

6. Illustrative Example

6.1. A Real Example

This section uses a real experiment from China Electrical Power Research Institute to demonstrate the applicability of the proposed adaptive schemes and the inference procedures. Recently, smart-grid technology has been a very hot topic in the energy area. One of the main objectives of a smart grid is to achieve an interactive grid. The crux of an interactive smart grid is to handle energy data in an intelligent way via AMI data-acquisition devices. In order

TABLE 4. RMSEs of the Exponential MLEs and the Coverages Probabilities with Nominal Confidence Level $1-\alpha=90\%$ of the Normal Approximation Method Constructed by Fisher Information Matrix (FIM) and Observed Information Matrix (OIM) as well as the Bootstrap (BS) Under the Simple Constant-Stress ALT and the Adaptive ALTs. The percentage of number of failures at the low stress no more than one and the percentage of changing stress for each scenario are also presented, respectively

Deviation (in %) of			% Number of failures	% Stress	Coverage probability (in %)							
					RMSE		FIM		OIM		BS	
$\beta_0/\beta_1/\sigma$	ξ_M	m	≤ 1	change	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_0$	$\hat{\beta}_1$
0/0	Simple		1.88	0.0	1.26	1.38	91.9	91.0	91.9	91.0	89.6	90.0
	0.618	1	1.18	10.5	1.23	1.36	91.6	90.8	91.7	90.9	90.1	90.3
		2	0.69	35.2	1.22	1.34	91.5	90.7	91.4	90.7	90.3	90.2
	0.647	1	0.89	10.5	1.22	1.35	91.3	90.6	91.6	90.8	90.6	90.5
		2	0.24	35.2	1.19	1.32	91.2	90.6	91.0	90.4	90.6	90.4
	0.676	1	0.75	10.5	1.21	1.34	91.1	90.5	91.4	90.7	90.6	90.5
		2	0.03	35.2	1.16	1.30	90.9	90.5	90.6	90.3	90.7	90.3
3/0	Simple		8.54	0.0	1.44	1.55	93.4	92.8	93.4	92.8	88.0	87.3
	0.618	1	6.02	20.2	1.42	1.53	92.8	92.2	92.8	92.4	88.8	88.8
		2	4.61	53.3	1.40	1.52	92.6	92.0	92.6	92.0	88.9	89.0
	0.647	1	4.35	20.2	1.40	1.52	92.0	91.7	92.1	91.8	89.0	89.3
		2	2.06	53.3	1.37	1.49	91.7	91.5	91.6	91.3	89.4	89.6
	0.676	1	3.34	20.2	1.38	1.50	91.4	91.3	91.8	91.6	89.3	89.5
		2	0.68	53.3	1.34	1.47	91.2	91.1	91.0	90.9	90.3	90.2
0/−3	Simple		3.89	0.0	1.34	1.45	92.2	91.7	92.2	91.8	89.2	89.5
	0.618	1	2.59	14.0	1.32	1.44	91.6	91.3	91.6	91.4	89.6	89.9
		2	1.82	43.1	1.30	1.42	91.6	91.2	91.4	91.2	89.9	90.1
	0.647	1	1.84	14.0	1.30	1.42	91.3	91.1	91.5	91.2	90.0	90.3
		2	0.67	43.1	1.27	1.40	91.2	90.9	91.0	90.9	90.4	90.2
	0.676	1	1.53	14.0	1.29	1.41	91.0	90.9	91.4	91.1	90.2	90.3
		2	0.22	43.1	1.24	1.38	90.7	90.6	90.7	90.6	90.8	90.3

to ensure accurate acquisition of electrical data, the data-acquisition devices in the AMI are required to be highly reliable and robust under different operating environments. However, the high reliability of the devices under normal operating posts a challenge on reliability assessment during the design phases. ALTs are effective in shortening the test time. The temperature as an important environmental factor is selected as the accelerating stress in the test. The standard approach to the ALT planning of data-acquisition devices is mainly based on IEC 62059-31-1-2008. The standard requires a guess of product reliability as an input, which is often provided by manufacturers.

After the adaptive ALT scheme discussed in this study was proposed to the Research Institute in early 2014, an ALT has been conducted based on the scheme. For the device in the ALT, the reliability estimate provided by the manufacturer was 10% failures in 7.5 years under room temperature $s_U = 20°C$. Based on the reliability information, the duration of the test was scheduled to be 3 months ($\tau_2 = 90$ days) by the Institute, and the low- and high-temperature levels were set as $s_L = 55°C$ and $s_H = 85°C$. The number of samples allocated to the low- and the high-stress levels were 30 and 13, respectively. To hedge against any underestimation in the reliability, the adaptive scheme was adopted with check-time $\tau_1 = 45$ days and $m = 1$. That is, if no failures in the low-stress test are observed in one month and a half, the temperature will be increased by 5°C.

The Arrhenius relation is a common model used to

TABLE 5. Settings of the Adaptive ALT Scheme for an AMI Data-Acquisition Device

Stress level	Temperature	Normalized stress	Sample size
Low	338K (55°C)	0.587	30
Medium	348K (60°C)	0.662	
High	358K (85°C)	1	13

describe the temperature acceleration to the lifetime. When this model is applied to the log-location-scale family, it is equivalent to requiring that the scale parameter μ be a linear function of the inverse temperature $\psi(s) = 1/s$, where the temperature is in the Kelvin scale. After normalizing the stress using Equation (1), we have

$$\mu = \beta_0 + \beta_1 \xi, \qquad \text{where } \xi = \frac{1/s_U - 1/s}{1/s_U - 1/s_H}.$$

A summary of the test is given in Table 5.

The failure data are reported in Table 6. During the test, two failures in the low-stress test were observed in 45 days, and so the low-stress level was unchanged throughout the test.

The Weibull, log-normal, and exponential distributions are used to fit the data. For each distribution, the maximum log-likelihood $l(\hat{\boldsymbol{\theta}})$ and the corresponding Akaike's Information Criterion (AIC) value $2k - 2l(\hat{\boldsymbol{\theta}})$ with k being the number of model parameters are presented in Table 7. The exponential distribution has the smallest AIC value and thus is chosen for the data. We also plot an exponential-probability plot of failure times from high-stress level in Figure 3 to verify the goodness of fit of the exponential model.

Using the inference methods developed above, parameters in the exponential ALT model can be estimated, as shown in Table 8. The confidence level used here is 80%. Even if the test is operated under con-

TABLE 6. Failure Data of the AMI Data Acquisition Devices Under the Adaptive ALT Scheme: 27 Units Were Censored Under s_L and 3 Units Were Censored Under s_H, with Censoring Time $\tau_2 = 90$ Days

	The failure time (days)									
Low level	8	31	60							
High level	7	9	12	15	23	23	28	31	66	73

stant stresses, the inferential conclusions are still diferent from those from a simple constant-stress ALT. Assuming the data is from a simple constant-stress ALT, the estimation for the parameters under the exponential model is shown in Table 9. It is seen that, even though the point estimation is exactly the same, the confidence intervals are slightly different. It can be seen that this quantile is higher than that provided by the manufacturer, which supports our argument that the manufacturer tends to report a lower reliability. We can also obtain the pointwise confidence band for the quantile function. The confidence intervals are computed by substituting the FIM confidence interval boundaries of $\hat{\beta}_0$ into the quantile function. The results are presented in Figure 4.

6.2. A Simulated Dataset

To demonstrate the emergency measure, we generate a simulated dataset under the exponential model such that J is smaller than m. The MLEs in the previous section are used as the planning values, while the true values are assumed to be $\beta_0 = 11.0$ and $\beta_1 = -6.590$. This setting mimics the situation that the true values of the model parameters are very different from the planning values. The true 0.1 quantile $t_{0.1}$ is 17.283 years, which is much larger than 11.702 years, the 0.1-quantile under the planning values. All

TABLE 7. MLEs, Values of Log-Likelihood Function and AIC Under Different Models

	Exponential		Weibull			Log normal		
	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\sigma}$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\sigma}$
MLE	10.610	−6.590	10.733	−6.713	1.022	9.803	−6.190	1.429
Log likelihood	−42.550			−42.545			−42.634	
AIC	89.099			91.091			91.268	

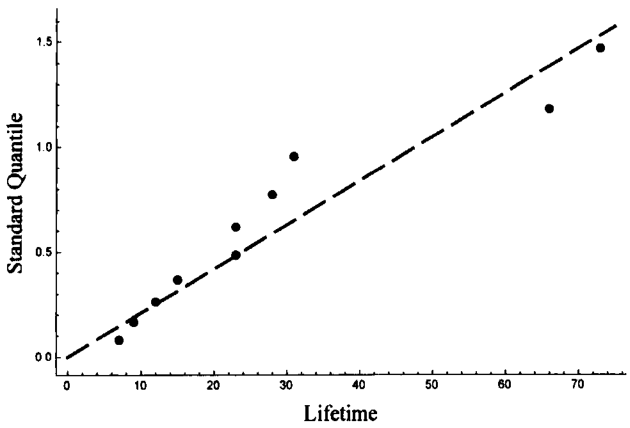

FIGURE 3. The Exponential Probability Plot of 10 Failure Times from the High-Stress Level with Three Test Units Censored. The R square of fitted line is 0.966.

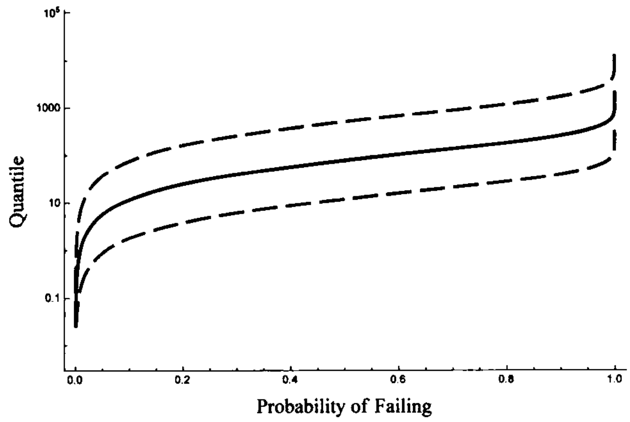

FIGURE 4. The Estimated Quantile Function of Lifetime and the 80% Confidence Band of the Quantile Function of Lifetime Under 20°C.

other experiment settings are the same as the real test. A simulated dataset is presented in Table 10. No failures are observed before $\tau_1 = 45$ and hence the low-stress level is elevated to s_M.

From Table 11, the exponential distribution has the smallest AIC and is chosen again. Table 12 shows the MLEs and confidence intervals for the adaptive ALT when the low-stress level is increased. The con-

TABLE 8. MLEs Under the Exponential Distribution and the 80% Confidence Intervals Constructed by Fisher Information Matrix (FIM), Observed Information Matrix (OIM), and the Bootstrap Method, Respectively

		Confidence interval		
	MLE	BS	FIM	OIM
$\hat{\beta}_0$	10.610	(8.753, 12.620)	(8.739, 12.481)	(8.725, 12.495)
$\hat{\beta}_1$	−6.590	(−8.621, −4.559)	(−8.636, −4.544)	(−8.668, −4.451)

TABLE 9. Assuming the Data Is Obtained by a Simple Constant-Stress ALT, MLEs Under the Exponential Distribution and the 80% Confidence Intervals Constructed by Fisher Information Matrix (FIM), Observed Information Matrix (OIM), and the Bootstrap Method, Respectively

		Confidence interval		
	MLE	BS	FIM	OIM
$\hat{\beta}_0$	10.610	(8.904, 12.853)	(8.739, 12.481)	(8.725, 12.495)
$\hat{\beta}_1$	−6.590	(−8.657, −4.567)	(−8.617, −4.563)	(−8.636, −4.544)

TABLE 10. Simulated Failure Data of the AMI Data-Acquisition Devices Under the Adaptive ALT Scheme: 27 Units Were Censored Under s_L and 4 Units Were Censored Unders_H, with Censoring Time $\tau_2 = 90$ Days

Low level	54.509	69.441	72.194						
High level	6.970	15.220	17.418	21.718	54.945	59.930	70.054	79.997	85.963

TABLE 11. MLEs, Values of Log-Likelihood Function and AIC Under Different Models Based on the Simulated Sample

	Exponential		Weibull			Lognormal		
	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\sigma}$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\sigma}$
MLE	10.962	−6.5370	8.704	−4.291	0.680	8.812	−4.755	1.032
log-likelihood	−42.273			−42.249			−41.880	
AIC	88.546			90.499			89.761	

TABLE 12. MLEs Under the Exponential Distribution and the 80% Confidence Intervals Constructed by Fisher Information Matrix (FIM), Observed Information Matrix (OIM), and the Bootstrap Method Based on the Simulated Sample

		Confidence interval		
	MLE	BS	FIM	OIM
$\hat{\beta}_0$	10.962	(9.004, 13.257)	(8.732, 13.193)	(8.756, 13.169)
$\hat{\beta}_1$	−6.537	(−8.956, −4.234)	(−8.934, −4.139)	(−8.912, −4.161)

fidence level used here is also 80%. Based on the exponential model, the estimated 0.1 quantile $\hat{t}_{0.1}$ is 16.63 years.

Likewise, we plot the pointwise confidence band for the quantile function in Figure 5. The confidence intervals are also calculated by substituting the FIM confidence interval boundaries of $\hat{\beta}_0$ into the quantile function.

7. Conclusion

In this study, we have successfully proposed an adaptive ALT scheme and implemented it in an accelerated test on data acquisition devices used in smart grids. When the manufacturer underestimates the product reliability or when the planning values of the test are significantly different from the true values, there is a high chance to observe no failures under the low-stress level. The adaptive scheme monitors the failure process at the low-stress level and it effectively mitigates this possibility. The adaptive scheme serves like an emergency control for the ALT planner. When there is no serious underestimation of the product reliability, the low-stress level remains unchanged with high probability. Therefore, the test is the same as the simple constant-stress ALT, which can be easily implemented by the experimenters. When there is serious underestimation, the ALT planner can step into the test midway and guide the experimenters to make necessary change to the low-stress level.

Based on the adaptive scheme, we have developed inference techniques to analyze ALT data collected from the test. The log-location-scale distributions were assumed for the product lifetime and the log-likelihood function was derived. Large-sample normal approximation and the bootstrap were used to construct confidence intervals. The simulation revealed that the adaptive scheme greatly mitigates the problem of lack of failures and improves the estimation accuracy in the presence of reliability underestimation. The developed methods were successfully applied to the ALT dataset from the data-acquisition devices.

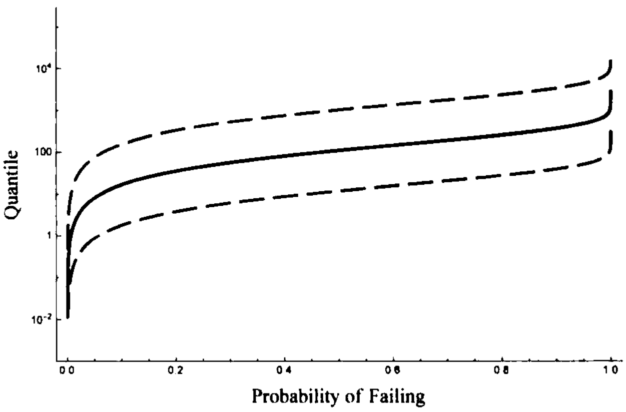

FIGURE 5. The Estimated Quantile Function of Lifetime from the Simulated Data and the 80% Confidence Band of the Quantile Function of Lifetime Under 20°C.

In an adaptive ALT, m, τ_1, and ξ_M are main factors to influence estimation efficiency. We have provided a useful heuristic method to determine τ_1. It would be also useful to rigorously determine the optimal values of those parameters for an adaptive ALT in future research.

Appendix

This Appendix provides the first-order and second-order derivatives of the log likelihood under the general log-location-scale distributions and the exponential distribution. Based on the second-order derivatives of the exponential distribution, the closed form of Fisher information matrix under exponential distribution is also derived.

A. Derivatives of Log Likelihood Under Log-Location-Scale Distributions

For convenience, we define

$$\omega_i(x) = \frac{\phi'\left(\frac{\ln(x)-\beta_0-\beta_1\xi_i}{\sigma}\right)}{\phi\left(\frac{\ln(x)-\beta_0-\beta_1\xi_i}{\sigma}\right)},$$

$$\psi_i(x) = \frac{\phi\left(\frac{\ln(x)-\beta_0-\beta_1\xi_i}{\sigma}\right)}{1-\Phi\left(\frac{\ln(x)-\beta_0-\beta_1\xi_i}{\sigma}\right)},$$

$$\omega_i'(x) = \frac{\partial}{\partial z}\left[\frac{\phi'(z)}{\phi(z)}\right]\Bigg|_{z=\frac{\ln(x)-\beta_0-\beta_1\xi_i}{\sigma}}$$

and

$$\psi_i'(x) = \frac{\partial}{\partial z}\left[\frac{\phi(z)}{1-\Phi(z)}\right]\Bigg|_{z=\frac{\ln(x)-\beta_0-\beta_1\xi_i}{\sigma}}.$$

A.1. Step-Stress Condition

When $J \le m-1$, the log-likelihood function is

$$\begin{aligned} l(\boldsymbol{\theta} \mid \mathbf{X},\mathbf{Y}) = {} & D_0 - (R_1+R_2)\ln(\sigma) + \sum_{i=1}^{J} \ln\left[\phi\left(\frac{\ln(x_{i:n_L})-\beta_0-\beta_1\xi_L}{\sigma}\right)\right] - \sum_{i=1}^{R_1}\ln(x_{i:n_L}) \\ & + \sum_{i=J+1}^{R_1}\ln\left[\phi\left(\frac{\ln\left(x_{i:n_L}-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}\right)-\beta_0-\beta_1\xi_M}{\sigma}\right)\right] \\ & + (n_L-R_1)\ln\left[1-\Phi\left(\frac{\ln\left(\tau_2-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}\right)-\beta_0-\beta_1\xi_M}{\sigma}\right)\right] \\ & + \sum_{i=1}^{R_2}\ln\left[\phi\left(\frac{\ln(y_{i:n_H})-\beta_0-\beta_1\xi_H}{\sigma}\right)\right] \\ & - \sum_{i=1}^{R_2}\ln(y_{i:n_H}) + (n_H-R_2)\ln\left[1-\Phi\left(\frac{\ln(\tau_2)-\beta_0-\beta_1\xi_H}{\sigma}\right)\right]. \end{aligned}$$

The first-order partial derivatives of the log likelihood with respect to each parameters β_0, β_1, σ are expressed as

$$\begin{aligned} \frac{\partial l}{\partial \beta_0} = -\frac{1}{\sigma}\Bigg[& \sum_{i=1}^{J}\omega_L(x_{i:n_L}) + \sum_{i=J+1}^{R_1}\omega_M\left(x_{i:n_L}-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}\right) - (n_L-R_1)\psi_M\left(\tau_2-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}\right) \\ & + \sum_{i=1}^{R_2}\omega_H(y_{i:n_H}) - (n_H-R_2)\psi_H(\tau_2)\Bigg], \end{aligned}$$

$$\begin{aligned} \frac{\partial l}{\partial \beta_1} = \frac{1}{\sigma}\Bigg[& -\xi_L\sum_{i=1}^{J}\omega_L(x_{i:n_L}) + \sum_{i=J+1}^{R_1}\left(\frac{(\xi_M-\xi_L)\tau_1 e^{\beta_1(\xi_M-\xi_L)}}{x_{i:n_L}-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}}-\xi_M\right)\omega_M\left(x_{i:n_L}-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}\right) \\ & - (n_L-R_1)\left(\frac{(\xi_M-\xi_L)\tau_1 e^{\beta_1(\xi_M-\xi_L)}}{\tau_2-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}}-\xi_M\right)\psi_M\left(\tau_2-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}\right) \\ & -\xi_H\sum_{i=1}^{R_2}\omega_H(y_{i:n_H}) + (n_H-R_2)\xi_H\psi_H(\tau_2)\Bigg], \end{aligned}$$

$$
\begin{aligned}
\frac{\partial l}{\partial \sigma} = & -\frac{R_1+R_2}{\sigma} \\
& -\frac{1}{\sigma^2}\left[\sum_{i=1}^{J}\omega_L\left(x_{i:n_L}\right)\left(\log\left(x_{i:n_L}\right)-\beta_0-\beta_1\xi_L\right)\right. \\
& +\sum_{i=J+1}^{R_1}\omega_M\left(x_{i:n_L}-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}\right)\left(\log\left(x_{i:n_L}-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}\right)-\beta_0-\beta_1\xi_M\right) \\
& -\left(n_L-R_1\right)\psi_M\left(\tau_2-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}\right)\left(\log\left(\tau_2-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}\right)-\beta_0-\beta_1\xi_M\right) \\
& \left.+\sum_{i=1}^{R_2}\omega_H\left(y_{i:n_H}\right)\left(\log\left(y_{i:n_H}\right)-\beta_0-\beta_1\xi_H\right)-\left(n_H-R_2\right)\psi_H\left(\tau_2\right)\left(\log\left(\tau_2\right)-\beta_0-\beta_1\xi_H\right)\right].
\end{aligned}
$$

To derive the observed information matrix, we derive the second-order partial derivatives as

$$
\begin{aligned}
\frac{\partial^2 l}{\partial \beta_0^2} = & \frac{1}{\sigma^2}\left\{\sum_{i=1}^{J}\omega_L'\left(x_{i:n_L}\right)+\sum_{i=J+1}^{R_1}\omega_M'\left(x_{i:n_L}-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}\right)-\left(n_L-R_1\right)\psi_M'\left(\tau_2-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}\right)\right. \\
& \left.+\sum_{i=1}^{R_2}\omega_H'\left(y_{i:n_H}\right)-\left(n_H-R_2\right)\psi_H'\left(\tau_2\right)\right\}.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 l}{\partial \beta_1^2} = & \frac{1}{\sigma^2}\left[\xi_L^2\sum_{i=1}^{J}\omega_L'\left(x_{i:n_L}\right)+\sum_{i=J+1}^{R_1}\left(\frac{\left(\xi_M-\xi_L\right)\tau_1 e^{\beta_1(\xi_M-\xi_L)}}{x_{i:n_L}-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}}-\xi_M\right)^2\omega_M'\left(x_{i:n_L}-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}\right)\right. \\
& -\left(n_L-R_1\right)\left(\frac{\left(\xi_M-\xi_L\right)\tau_1 e^{\beta_1(\xi_M-\xi_L)}}{\tau_2-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}}-\xi_M\right)^2\psi_M'\left(\tau_2-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}\right) \\
& \left.+\xi_H^2\sum_{i=1}^{R_2}\omega_H'\left(y_{i:n_H}\right)-\left(n_H-R_2\right)\xi_H^2\psi_H'\left(\tau_2\right)\right] \\
& +\frac{1}{\sigma}\left[\sum_{i=J+1}^{R_1}\left(\frac{\left(\xi_M-\xi_L\right)^2\tau_1 e^{\beta_1(\xi_M-\xi_L)}\left(x_{i:n_L}-\tau_1\right)}{\left[x_{i:n_L}-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}\right]^2}\right)\omega_M\left(x_{i:n_L}-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}\right)\right. \\
& \left.-\left(n_L-R_1\right)\left(\frac{\left(\xi_M-\xi_L\right)^2\tau_1 e^{\beta_1(\xi_M-\xi_L)}\left(\tau_2-\tau_1\right)}{\left[\tau_2-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}\right]^2}\right)\psi_M\left(\tau_2-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 l}{\partial \sigma^2} = & \frac{R_1+R_2}{\sigma^2} \\
& +\frac{2}{\sigma^3}\left[\sum_{i=1}^{J}\omega_L\left(x_{i:n_L}\right)\left(\log\left(x_{i:n_L}\right)-\beta_0-\beta_1\xi_L\right)\right. \\
& +\sum_{i=J+1}^{R_1}\omega_M\left(x_{i:n_L}-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}\right)\left(\log\left(x_{i:n_L}-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}\right)-\beta_0-\beta_1\xi_M\right) \\
& -\left(n_L-R_1\right)\psi_M\left(\tau_2-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}\right)\left(\log\left(\tau_2-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}\right)-\beta_0-\beta_1\xi_M\right) \\
& \left.+\sum_{i=1}^{R_2}\omega_H\left(y_{i:n_H}\right)\left(\log\left(y_{i:n_H}\right)-\beta_0-\beta_1\xi_H\right)-\left(n_H-R_2\right)\psi_H\left(\tau_2\right)\left(\log\left(\tau_2\right)-\beta_0-\beta_1\xi_H\right)\right] \\
& +\frac{1}{\sigma^4}\left[\sum_{i=1}^{J}\omega_L'\left(x_{i:n_L}\right)\left(\log\left(x_{i:n_L}\right)-\beta_0-\beta_1\xi_L\right)^2\right. \\
& +\sum_{i=J+1}^{R_1}\omega_M'\left(x_{i:n_L}-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}\right)\left(\log\left(x_{i:n_L}-\tau_1+\tau_1 e^{\beta_1(\xi_M-\xi_L)}\right)-\beta_0-\beta_1\xi_M\right)^2
\end{aligned}
$$

$$- (n_L - R_1)\,\psi'_M\left(\tau_2 - \tau_1 + \tau_1 e^{\beta_1(\xi_M - \xi_L)}\right)\left(\log\left(\tau_2 - \tau_1 + \tau_1 e^{\beta_1(\xi_M - \xi_L)}\right) - \beta_0 - \beta_1\xi_M\right)^2$$

$$\left. + \sum_{i=1}^{R_2} \omega'_H(y_{i:n_H})(\log(y_{i:n_H}) - \beta_0 - \beta_1\xi_H)^2 - (n_H - R_2)\,\psi'_H(\tau_2)(\log(\tau_2) - \beta_0 - \beta_1\xi_H)^2\right],$$

$$\frac{\partial^2 l}{\partial\beta_0\partial\beta_1} = -\frac{1}{\sigma^2}\left[-\xi_L\sum_{i=1}^{J}\omega'_L(x_{i:n_L})\right.$$

$$+ \sum_{i=J+1}^{R_1}\left(\frac{(\xi_M - \xi_L)\,\tau_1 e^{\beta_1(\xi_M - \xi_L)}}{x_{i:n_L} - \tau_1 + \tau_1 e^{\beta_1(\xi_M - \xi_L)}} - \xi_M\right)\omega'_M\left(x_{i:n_L} - \tau_1 + \tau_1 e^{\beta_1(\xi_M - \xi_L)}\right)$$

$$- (n_L - R_1)\left(\frac{(\xi_M - \xi_L)\,\tau_1 e^{\beta_1(\xi_M - \xi_L)}}{\tau_2 - \tau_1 + \tau_1 e^{\beta_1(\xi_M - \xi_L)}} - \xi_M\right)\psi'_M\left(\tau_2 - \tau_1 + \tau_1 e^{\beta_1(\xi_M - \xi_L)}\right)$$

$$\left. -\xi_H\sum_{i=1}^{R_2}\omega'_H(y_{i:n_H}) + (n_H - R_2)\,\xi_H\psi'_H(\tau_2)\right],$$

$$\frac{\partial^2 l}{\partial\beta_0\partial\sigma} = \frac{1}{\sigma^2}\left[\sum_{i=1}^{J}\omega_L(x_{i:n_L})\right.$$

$$+ \sum_{i=J+1}^{R_1}\omega_M\left(x_{i:n_L} - \tau_1 + \tau_1 e^{\beta_1(\xi_M - \xi_L)}\right) - (n_L - R_1)\,\psi_M\left(\tau_2 - \tau_1 + \tau_1 e^{\beta_1(\xi_M - \xi_L)}\right)$$

$$\left. + \sum_{i=1}^{R_2}\omega_H(y_{i:n_H}) - (n_H - R_2)\,\psi_H(\tau_2)\right]$$

$$+ \frac{1}{\sigma^3}\left[\sum_{i=1}^{J}\omega'_L(x_{i:n_L})(\log(x_{i:n_L}) - \beta_0 - \beta_1\xi_L)\right.$$

$$+ \sum_{i=J+1}^{R_1}\omega'_M\left(x_{i:n_L} - \tau_1 + \tau_1 e^{\beta_1(\xi_M - \xi_L)}\right)\left(\log\left(x_{i:n_L} - \tau_1 + \tau_1 e^{\beta_1(\xi_M - \xi_L)}\right) - \beta_0 - \beta_1\xi_M\right)$$

$$- (n_L - R_1)\,\psi'_M\left(\tau_2 - \tau_1 + \tau_1 e^{\beta_1(\xi_M - \xi_L)}\right)\left(\log\left(\tau_2 - \tau_1 + \tau_1 e^{\beta_1(\xi_M - \xi_L)}\right) - \beta_0 - \beta_1\xi_M\right)$$

$$\left. + \sum_{i=1}^{R_2}\omega'_H(y_{i:n_H})(\log(y_{i:n_H}) - \beta_0 - \beta_1\xi_H) - (n_H - R_2)\,\psi'_H(\tau_2)(\log(\tau_2) - \beta_0 - \beta_1\xi_H)\right],$$

$$\frac{\partial l}{\partial\beta_1\partial\sigma} = -\frac{1}{\sigma^2}\left[-\xi_L\sum_{i=1}^{J}\omega_L(x_{i:n_L})\right.$$

$$+ \sum_{i=J+1}^{R_1}\left(\frac{(\xi_M - \xi_L)\,\tau_1 e^{\beta_1(\xi_M - \xi_L)}}{x_{i:n_L} - \tau_1 + \tau_1 e^{\beta_1(\xi_M - \xi_L)}} - \xi_M\right)\omega_M\left(x_{i:n_L} - \tau_1 + \tau_1 e^{\beta_1(\xi_M - \xi_L)}\right)$$

$$- (n_L - R_1)\left(\frac{(\xi_M - \xi_L)\,\tau_1 e^{\beta_1(\xi_M - \xi_L)}}{\tau_2 - \tau_1 + \tau_1 e^{\beta_1(\xi_M - \xi_L)}} - \xi_M\right)\psi_M\left(\tau_2 - \tau_1 + \tau_1 e^{\beta_1(\xi_M - \xi_L)}\right)$$

$$\left. - \xi_H\sum_{i=1}^{R_2}\omega_H(y_{i:n_H}) + (n_H - R_2)\,\xi_H\psi_H(\tau_2)\right]$$

$$- \frac{1}{\sigma^3}\left[-\xi_L\sum_{i=1}^{J}\omega'_L(x_{i:n_L})(\log(x_{i:n_L}) - \beta_0 - \beta_1\xi_L)\right.$$

$$+ \sum_{i=J+1}^{R_1}\left(\frac{(\xi_M - \xi_L)\,\tau_1 e^{\beta_1(\xi_M - \xi_L)}}{x_{i:n_L} - \tau_1 + \tau_1 e^{\beta_1(\xi_M - \xi_L)}} - \xi_M\right)\omega'_M\left(x_{i:n_L} - \tau_1 + \tau_1 e^{\beta_1(\xi_M - \xi_L)}\right)$$

$$\times \left(\log\left(x_{i:n_L} - \tau_1 + \tau_1 e^{\beta_1(\xi_M - \xi_L)}\right) - \beta_0 - \beta_1\xi_M\right)$$
$$- (n_L - R_1)\left(\frac{(\xi_M - \xi_L)\tau_1 e^{\beta_1(\xi_M-\xi_L)}}{\tau_2 - \tau_1 + \tau_1 e^{\beta_1(\xi_M-\xi_L)}} - \xi_M\right)\psi'_M\left(\tau_2 - \tau_1 + \tau_1 e^{\beta_1(\xi_M-\xi_L)}\right)$$
$$\times \left(\log\left(\tau_2 - \tau_1 + \tau_1 e^{\beta_1(\xi_M-\xi_L)}\right) - \beta_0 - \beta_1\xi_M\right)$$
$$\left. - \xi_H \sum_{i=1}^{R_2} \omega'_H(y_{i:n_H})(\log(y_{i:n_H}) - \beta_0 - \beta_1\xi_H) + (n_H - R_2)\xi_H\psi'_H(\tau_2)(\log(\tau_2) - \beta_0 - \beta_1\xi_H)\right].$$

A.2. Constant-Stress Condition

When $J \geq m$, we have the log-likelihood function as

$$l(\boldsymbol{\theta} \mid \mathbf{X}, \mathbf{Y}) = D_0 - (R_1 + R_2)\ln(\sigma) + \sum_{i=1}^{R_1} \ln\left[\phi\left(\frac{\ln(x_{i:n_L}) - \beta_0 - \beta_1\xi_L}{\sigma}\right)\right]$$
$$- \sum_{i=1}^{R_1} \ln(x_{i:n_L}) + (n_L - R_1)\ln\left[1 - \Phi\left(\frac{\ln(\tau_2) - \beta_0 - \beta_1\xi_L}{\sigma}\right)\right]$$
$$+ \sum_{i=1}^{R_2} \ln\left[\phi\left(\frac{\ln(y_{i:n_H}) - \beta_0 - \beta_1\xi_H}{\sigma}\right)\right]$$
$$- \sum_{i=1}^{R_2} \ln(y_{i:n_H}) + (n_H - R_2)\ln\left[1 - \Phi\left(\frac{\ln(\tau_2) - \beta_0 - \beta_1\xi_H}{\sigma}\right)\right].$$

The first-order partial derivatives are

$$\frac{\partial l}{\partial \beta_0} = -\frac{1}{\sigma}\left[\sum_{i=1}^{R_1} \omega_L(x_{i:n_L}) - (n_L - R_1)\psi_L(\tau_2) + \sum_{i=1}^{R_2} \omega_H(y_{i:n_H}) - (n_H - R_2)\psi_H(\tau_2)\right],$$
$$\frac{\partial l}{\partial \beta_1} = \frac{1}{\sigma}\left[-\xi_L \sum_{i=1}^{R_1} \omega_L(x_{i:n_L}) - (n_L - R_1)(-\xi_L)\psi_L(\tau_2) - \xi_H \sum_{i=1}^{R_2} \omega_H(y_{i:n_H}) + (n_H - R_2)\xi_H\psi_H(\tau_2)\right],$$
$$\frac{\partial l}{\partial \sigma} = -\frac{R_1 + R_2}{\sigma}$$
$$- \frac{1}{\sigma^2}\left[\sum_{i=1}^{R_1} \omega_L(x_{i:n_L})(\log(x_{i:n_L}) - \beta_0 - \beta_1\xi_L) - (n_L - R_1)\psi_L(\tau_2)(\log(\tau_2) - \beta_0 - \beta_1\xi_L)\right.$$
$$\left. + \sum_{i=1}^{R_2} \omega_H(y_{i:n_H})(\log(y_{i:n_H}) - \beta_0 - \beta_1\xi_H) - (n_H - R_2)\psi_H(\tau_2)(\log(\tau_2) - \beta_0 - \beta_1\xi_H)\right].$$

The second-order partial derivatives are

$$\frac{\partial^2 l}{\partial \beta_0^2} = \frac{1}{\sigma^2}\left\{\sum_{i=1}^{R_1} \omega'_L(x_{i:n_L}) - (n_L - R_1)\psi'_L(\tau_2) + \sum_{i=1}^{R_2} \omega'_H(y_{i:n_H}) - (n_H - R_2)\psi'_H(\tau_2)\right\},$$
$$\frac{\partial^2 l}{\partial \beta_1^2} = \frac{1}{\sigma^2}\left[\xi_L^2 \sum_{i=1}^{R_1} \omega'_L(x_{i:n_L}) - (n_L - R_1)\xi_L^2\psi'_L(\tau_2) + \xi_H^2 \sum_{i=1}^{R_2} \omega'_H(y_{i:n_H}) - (n_H - R_2)\xi_H^2\psi'_H(\tau_2)\right],$$
$$\frac{\partial^2 l}{\partial \sigma^2} = \frac{R_1 + R_2}{\sigma^2}$$
$$+ \frac{2}{\sigma^3}\left[\sum_{i=1}^{R_1} \omega_L(x_{i:n_L})(\log(x_{i:n_L}) - \beta_0 - \beta_1\xi_L) - (n_L - R_1)\psi_L(\tau_2)(\log(\tau_2) - \beta_0 - \beta_1\xi_L)\right.$$

$$+ \sum_{i=1}^{R_2} \omega_H\left(y_{i:n_H}\right)\left(\log\left(y_{i:n_H}\right)-\beta_0-\beta_1\xi_H\right)-\left(n_H-R_2\right)\psi_H\left(\tau_2\right)\left(\log\left(\tau_2\right)-\beta_0-\beta_1\xi_H\right)\Bigg]$$

$$+\frac{1}{\sigma^4}\left[\sum_{i=1}^{R_1}\left[\omega_L\left(x_{i:n_L}\right)\right]'\left(\log\left(x_{i:n_L}\right)-\beta_0-\beta_1\xi_L\right)^2-\left(n_L-R_1\right)\left[\psi_L\left(\tau_2\right)\right]'\left(\log\left(\tau_2\right)-\beta_0-\beta_1\xi_L\right)^2\right.$$

$$\left.+\sum_{i=1}^{R_2}\left[\omega_H\left(y_{i:n_H}\right)\right]'\left(\log\left(y_{i:n_H}\right)-\beta_0-\beta_1\xi_H\right)^2-\left(n_H-R_2\right)\left[\psi_H\left(\tau_2\right)\right]'\left(\log\left(\tau_2\right)-\beta_0-\beta_1\xi_H\right)^2\right],$$

$$\frac{\partial^2 l}{\partial\beta_0\partial\beta_1}=-\frac{1}{\sigma^2}\left[-\xi_L\sum_{i=1}^{R_1}\omega_L'\left(x_{i:n_L}\right)+\left(n_L-R_1\right)\xi_L\psi_L'\left(\tau_2\right)-\xi_H\sum_{i=1}^{R_2}\omega_H'\left(y_{i:n_H}\right)+\left(n_H-R_2\right)\xi_H\psi_H'\left(\tau_2\right)\right],$$

$$\frac{\partial^2 l}{\partial\beta_0\partial\sigma}=\frac{1}{\sigma^2}\left[\sum_{i=1}^{R_1}\omega_L\left(x_{i:n_L}\right)-\left(n_L-R_1\right)\psi_L\left(\tau_2\right)+\sum_{i=1}^{R_2}\omega_H\left(y_{i:n_H}\right)-\left(n_H-R_2\right)\psi_H\left(\tau_2\right)\right]$$

$$+\frac{1}{\sigma^3}\left[\sum_{i=1}^{R_1}\omega_L'\left(x_{i:n_L}\right)\left(\log\left(x_{i:n_L}\right)-\beta_0-\beta_1\xi_L\right)-\left(n_L-R_1\right)\psi_L'\left(\tau_2\right)\left(\log\left(\tau_2\right)-\beta_0-\beta_1\xi_L\right)\right.$$

$$\left.+\sum_{i=1}^{R_2}\omega_H'\left(y_{i:n_H}\right)\left(\log\left(y_{i:n_H}\right)-\beta_0-\beta_1\xi_H\right)-\left(n_H-R_2\right)\psi_H'\left(\tau_2\right)\left(\log\left(\tau_2\right)-\beta_0-\beta_1\xi_H\right)\right],$$

$$\frac{\partial l}{\partial\beta_1\partial\sigma}=-\frac{1}{\sigma^2}\left[-\xi_L\sum_{i=1}^{R_1}\omega_L\left(x_{i:n_L}\right)+\left(n_L-R_1\right)\xi_L\psi_L\left(\tau_2\right)-\xi_H\sum_{i=1}^{R_2}\omega_H\left(y_{i:n_H}\right)+\left(n_H-R_2\right)\xi_H\psi_H\left(\tau_2\right)\right]$$

$$-\frac{1}{\sigma^3}\left[-\xi_L\sum_{i=1}^{R_1}\omega_L'\left(x_{i:n_L}\right)\left(\log\left(x_{i:n_L}\right)-\beta_0-\beta_1\xi_L\right)+\left(n_L-R_1\right)\xi_L\psi_L'\left(\tau_2\right)\left(\log\left(\tau_2\right)-\beta_0-\beta_1\xi_L\right)\right.$$

$$\left.-\xi_H\sum_{i=1}^{R_2}\omega_H'\left(y_{i:n_H}\right)\left(\log\left(y_{i:n_H}\right)-\beta_0-\beta_1\xi_H\right)+\left(n_H-R_2\right)\xi_H\psi_H'\left(\tau_2\right)\left(\log\left(\tau_2\right)-\beta_0-\beta_1\xi_H\right)\right].$$

Table 13 gives the detailed formulas for commonly used log-location-scale distributions.

B. Derivation of Fisher Information Matrix Under Exponential Distribution

Under the exponential model with $\boldsymbol{\theta}=(\beta_0,\beta_1)$, the mean failure time under each stress level is e^{μ_i}, where $\mu_i=\beta_0+\beta_1\xi_i$ and the parameter $\sigma=1$.

If $0\le J\le m-1$, the derivatives of Equation (6) with respect to β_0 and β_1 are

$$\frac{\partial l_1}{\partial\beta_0}=-(R_1+R_2)+D_1+D_2+D_3,$$

$$\frac{\partial l_1}{\partial\beta_1}=-J\xi_L-(R_1-J)\xi_M-R_2\xi_H+\xi_L D_1+\xi_M D_2+\xi_H D_3,$$

respectively. Then, the second-order derivatives are

$$\frac{\partial^2 l_1}{\partial\beta_0^{2-i}\partial\beta_1^{i}}=-\xi_L^i D_1-\xi_M^i D_2-\xi_H^i D_3,$$

where $i=0,1,2$.

When $m\le J\le n_L$, the first-order partial derivatives are

$$\frac{\partial l}{\partial\beta_0}=-(R_1+R_2)+D_1+D_4+D_3,$$

$$\frac{\partial l}{\partial\beta_1}=-(R_1\xi_L+R_2\xi_H)+\xi_L\left(D_1+D_4\right)+\xi_H D_3.$$

TABLE 13. Detailed Formulas Under Some Commonly Used Log-Location-Scale Distributions

	Log-normal	Weibull
Φ	$\int_{-\infty}^{x}\phi(t)dt$	$1-e^{-e^x}$
ϕ	$(1/\sqrt{2\pi})e^{-x^2/2}$	e^{x-e^x}
$\phi'(x)/\phi(x)$	$-x$	$1-e^x$
$[\phi'(x)/\phi(x)]'$	-1	$-e^x$
$\phi(x)/[1-\Phi(x)]$	$\phi(x)/[1-\Phi(x)]$	e^x
$[\phi(x)/[1-\Phi(x)]]'$	$\dfrac{-x\phi\,(1-\Phi)+\phi^2}{(1-\Phi)^2}$	e^x

The second-order partial derivatives are

$$\frac{\partial^2 l_1}{\partial \beta_0^{2-i} \partial \beta_1^i} = -\xi_L^i (D_1 + D_4) - \xi_H^i D_3,$$

where $i = 0, 1, 2$.

With the fact that

$$\mathbb{E}[\partial l/\partial \boldsymbol{\theta}] = \sum_{i=0}^{m-1} \mathbb{E}[\partial l_1/\partial \boldsymbol{\theta} \mid j] \Pr(J = j) + \sum_{i=m}^{n_L} \mathbb{E}[\partial l_2/\partial \boldsymbol{\theta} \mid J] \Pr(J = j) = 0.$$

we have

$$\begin{aligned}\mathbb{E}[D_1] &+ \sum_{j=m}^{n_L} \mathbb{E}[D_4 \mid J = j] \Pr(J = j) + \sum_{j=0}^{m-1} \mathbb{E}[D_2 \mid J = j] \Pr(J = j) + \mathbb{E}[D_3] \\ &= \mathbb{E}(J) + \sum_{j=m}^{n_L} \mathbb{E}[R_1 - J \mid J = j] \Pr(J = j) + \sum_{j=0}^{m-1} \mathbb{E}[R_1 - J \mid J = j] \Pr(J = j) + \mathbb{E}[R_2]\end{aligned}$$

and

$$\begin{aligned}\xi_L &\left(\mathbb{E}[D_1] + \sum_{j=m}^{n_L} \mathbb{E}[D_4 \mid J = j] \Pr(J = j)\right) + \xi_M \sum_{j=0}^{m-1} \mathbb{E}[D_2 \mid J = j] \Pr(J = j) + \xi_H \mathbb{E}[D_3] \\ &= \xi_L \left(\mathbb{E}[J] + \sum_{j=m}^{n_L} \mathbb{E}[R_1 - J \mid J = j] \Pr(J = j)\right) + \xi_M \sum_{j=0}^{m-1} \mathbb{E}[R_1 - J \mid J = j] \Pr(J = j) + \xi_H \mathbb{E}[R_2].\end{aligned}$$

As $\mathbf{X}$ and $\mathbf{Y}$ are independent, expectations of the first derivatives are still zero without terms involving $\mathbf{X}$. Therefore, $\mathbb{E}[D_3] = \mathbb{E}[R_2]$ and then we have

$$\mathbb{E}[D_1] + \sum_{j=m}^{n_L} \mathbb{E}[D_4 \mid J = j] \Pr(J = j) = \mathbb{E}[J] + \sum_{j=m}^{n_L} \mathbb{E}[R_1 - J \mid J = j] \Pr(J = j),$$

$$\sum_{j=0}^{m-1} \mathbb{E}[D_2 \mid J = j] \Pr(J = j) = \sum_{j=0}^{m-1} \mathbb{E}[R_1 - J \mid J = j] \Pr(J = j).$$

Given those equations, we can obtain Equation (7). To calculate Equation (7), the joint distribution of J and R_1 and the distribution of R_2 are needed. It is obvious that the joint probability mass function of J and R_1 is

$$\Pr_{\boldsymbol{\theta}}(J = j, R_1 = r_1) = \begin{cases} \frac{n_L!}{j!(r_1-j)!(n_L-r_1)!} [F_L(\tau_1; \boldsymbol{\theta})]^j [F_L(\tau_2; \boldsymbol{\theta}) - F_L(\tau_1; \boldsymbol{\theta})]^{r_1-j} [1 - F_L(\tau_2; \boldsymbol{\theta})]^{n_L - r_1}, & m \le j \le r_1 \le n_L; \\ \frac{n_L!}{j!(r_1-j)!(n_L-r_1)!} [F_L(\tau_1; \boldsymbol{\theta})]^j [F_M(\tau_2 - \tau_1 + \tau'; \boldsymbol{\theta}) - F_L(\tau_1; \boldsymbol{\theta})]^{r_1-j} \\ \times [1 - F_M(\tau_2 - \tau_1 + \tau'; \boldsymbol{\theta})]^{n_L - r_1}, & 0 \le j < m, j \le r_1 \le n_L; \\ 0, & \text{elsewhere.} \end{cases}$$

Besides, $J \sim B(n_L, 1 - \exp(-\tau_1/\exp(\beta_0 + \xi_L \beta_1)))$ and $R_2 \sim B(n_H, 1 - \exp(-\tau_2/\exp(\beta_0 + \xi_H \beta_1)))$. Here, $B(n, p)$ stands for a binomial distribution with the number of trials n and the success probability p. Hence, those expectations are given by

$$\mathbb{E}[J] = n_L (1 - \exp(-\tau_1/\exp(\beta_0 + \xi_L \beta_1)))$$

$$\mathbb{E}[R_2] = n_H (1 - \exp(-\tau_2/(\exp(\beta_0 + \xi_H \beta_1)))$$

$$\mathbb{E}[R_1 - J \mid J = j] = \begin{cases} (n_L - j) \frac{F_M(\tau_2 - \tau_1 + \tau'; \boldsymbol{\theta}) - F_L(\tau_1; \boldsymbol{\theta})}{1 - F_L(\tau_1; \boldsymbol{\theta})}, & \text{if } J < m; \\ (n_L - j) \frac{F_L(\tau_2; \boldsymbol{\theta}) - F_L(\tau_1; \boldsymbol{\theta})}{1 - F_L(\tau_1; \boldsymbol{\theta})}, & \text{if } J \ge m. \end{cases}$$

Acknowledgment

We are grateful to the editor and two referees for their insightful comments that have led to a substantial improvement in an earlier version of the paper. Sun and Zhao were supported by the National Natural Science Foundation of China (11371299). Ye is supported by Natural Science Foundation of China (71601138), and Singapore AcRF Tier 1 funding (R-266-000-095-112).

References

ARNOLD, B. C.; BALAKRISHNAN, N.; and NAGARAJA, H. N. (1992). *A First Course in Order Statistics.* Philadelphia, PA: SIAM.

BAGDONAVICIUS, V. and NIKULIN, M. (2010).*Accelerated Life Mode: Modeling and Statistical Analysis.* Boca Raton, FL: CRC Press.

BALAKRISHNAN, N.; KAMPS, U.; and KATERI, M. (2012). "A Sequential Order Statistics Approach to Step-Stress Testing". *Annals of the Institute of Statistical Mathematics* 64(2), pp. 303-318.

CHEN, N.; TANG, Y.; and YE, Z.-S. (2016a). "Robust Quantile Analysis for Accelerated Life Test Data". *IEEE Transactions on Reliability* 65(2), pp. 901–913.

CHEN, P.; XU, A.; and YE, Z.-S. (2016b). "Generalized Fiducial Inference for Accelerated Life Tests with Weibull Distribution and Progressively Type-II Censoring". *IEEE Transactions on Reliability* 65(4), pp. 1737–1744.

EFRON, B. and TIBSHIRANI, R. J. (1994). *An Introduction to the Bootstrap.* New York, NY: CRC Press.

GINEBRA, J.; BARCELONA, D.; and SEN, A. (1998). "Minimax Approach to Accelerated Life Tests". *IEEE Transactions on Reliability* 47(3), pp. 261–267.

GUAN, Q.; TANG, Y.; FU, J.; and XU, A. (2014). "Optimal Multiple Constant-Stress Accelerated Life Tests for Generalized Exponential Distribution". *Communications in Statistics—Simulation and Computation* 43(8), pp. 1852–1865.

HAN, D. and NG, H. (2013). "Comparison Between Constant-Stress and Step-Stress Accelerated Life Tests Under Time Constraint". *Naval Research Logistics* 60(7), pp. 541–556.

LIAO, H. and ELSAYED, E. A. (2010). "Equivalent Accelerated Life Testing Plans for Log-Location-Scale Distributions". *Naval Research Logistics* 57(5), pp. 472–488.

LIN, C.-T.; CHOU, C.-C.; and BALAKRISHNAN, N. (2014). "Planning Step-Stress Test Under Type-I Censoring for the Exponential Case". *Journal of Statistical Computation and Simulation* 84(4), pp. 819–832.

LIU, X. (2012). "Planning of Accelerated Life Tests with Dependent Failure Modes Based on a Gamma Frailty Model". *Technometrics* 54(4), pp. 398–409. LIU, X. and Qiu, W. S. (2011). "MODELING AND PLANNING OF STEP-STRESS ACCELERATED LIFE TESTS WITH INDEPENDENT COMPETING RISKS". *IEEE Transactions on Reliability* 60(4), PP. 712–720.

MEEKER, W. Q. and HAHN, G. J. (1985). *How to Plan an Accelerated Life Test: Some Practical Guidelines*, vol. 10. Milwaukee, WI: American Society for Quality Control.

NELSON, W. (1980). "Accelerated Life Testing Step-Stress Models and Data Analyses". *IEEE Transactions on Reliability* R-29(2), pp. 103–108.

NELSON, W. and KIELPINSKI, T. J. (1976). "Theory for Optimum Censored Accelerated Life Tests for Normal and Lognormal Life Distributions". *Technometrics* 18(1), pp. 105–114.

NELSON, W. B. (2009). *Accelerated Testing: Statistical Models, Test Plans, and Data Analyses.* Hoboken, NJ: John Wiley & Sons.

PAN, R. and YANG, T. (2014). "Design and Evaluation of Accelerated Life Testing Plans with Dual Objectives". *Journal of Quality Technology* 46(2), pp. 114–126.

Q/GDW 374.2 (2009). *Power User Electric Energy Data Acquire System Technical Specification.* State Grid Corporation of China.

TANG, L. and LIU, X. (2010). "Planning and Inference for a Sequential Accelerated Life Test". *Journal of Quality Technology* 42(1), pp. 103–118.

WANG, B. X.; YU, K.; and SHENG, Z. (2014). "New Inference for Constant-Stress Accelerated Life Tests with Weibull Distribution and Progressively Type-II Censoring". *IEEE Transactions on Reliability* 63(3), pp. 807–815.

XU, A.; FU, J.; TANG, Y.; and GUAN, Q. (2015). "Bayesian Analysis of Constant-Stress Accelerated Life Test for the Weibull Distribution Using Noninformative Priors". *Applied Mathematical Modelling* 39(20), pp. 6183–6195.

YE, Z.-S.; HONG, Y.; and XIE, Y. (2013). "How Do Heterogeneities in Operating Environments Affect Field Failure Predictions and Test Planning?" *The Annals of Applied Statistics* 7(4), pp. 2249–2271.

YUAN, T.; LIU, X.; and KUO, W. (2012). "Planning Simple Step-Stress Accelerated Life Tests Using Bayesian Methods". *IEEE Transactions on Reliability* 61(1), pp. 254–263.

ZHANG, D. and LIAO, H. (2014). "Design of Statistically and Energy Efficient Accelerated Life Testing Experiments". *IIE Transactions* 46(10), pp. 1031–1049.