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Title	Reliability Estimation from Left-Truncated and Right-Censored Data Using Splines
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- Condition 1: The maximum spacing of the knots satisfies

$$\Delta = \max_{l+1 \le j \le m_n+l+1} |t_j - t_{j-1}| = O(n^{-\nu}).$$

  Moreover, there exists a constant $M > 0$ such that $\Delta/\delta \le M$ uniformly in n, where $\delta = \min_{l+1 \le j \le m_n+l+1} |t_j - t_{j-1}|$.

- Condition 2: The interval $[a, b]$ satisfies $P(\{Y \in [a, b]\}) = 1$.

- Condition 3: there exists a constant $C_0 > 0$ such that $\lambda_0(t) \ge C_0$ for $t \in [a, b]$. In add[illegible]ion, the true failure rate λ_0 is differentiable up to order r and all the [illegible]rivatives are uni[illegible] bounded by a constant M in $[a, b]$, where $r \ge 1$.

Remark 1. Condition 1 is a weak restriction on the [illegible] [illegible]ence, which is [illegible] [illegible]hen equally-spaced knots are used. This condition is als[illegible] [illegible]pted by Stone (19[illegible]) [illegible]n 2 requires that $[\underline{L}, \bar{C}] \subset [a, b]$. Condition 3 is needed i[illegible] the proof of the [illegible]rmality in Theorem 3. It usually holds in practice. Product l[illegible] [illegible]nd continuous random variable. Continuous parametric lifetime distributions, such [illegible]ull, lognormal and inverse Gaussian distributions have been widely used to model the [illegible]me data. See Balakrishnan and Mitra (2011, 2012, 2013), among others. All th[illegible]stributions have smooth hazard rate functions. As an extension from parametric to nonparametric estimation, the smoothness assumption in Condition 3 is nat[illegible] [illegible]nd reasonable. This assumption is also used in Wang (2005) and Zhao [illegible]d Zhang (2017).

Theorem 1 (Consistency) *Su[illegible]pose t[illegible] [illegible]nditi[illegible] [illegible] hold. Then the estimated failure rate $\hat{\lambda}_n$ converges to the [illegible]ue failure ra[illegible] [illegible], that is,*

$$\|[illegible] \lambda_0\| [illegible]\to_p 0, \quad as\ n \to \infty.$$

Theorem 2 (Rate of c[illegible]rg[illegible]ce) *[illegible]uppose that Conditions 1-3 hold. If ν is chosen to be* [illegible]$+1)$*, then*

$$[illegible] n^{\frac{r}{2r+1}} \|\hat{\lambda}_n - \lambda_0\|_2 = O_p(1).$$

[illegible] Th[illegible]m [illegible]ows that the spline likelihood estimators have a convergence rate slower than [illegible] [illegible]aster than $n^{-1/3}$.

To discuss [illegible]nptotic distributions of functions of $\hat{\lambda}_n$, define

$$\mathcal{H}_r = \left\{ h(\cdot) : |h^{(r-1)}(s) - h^{(r-1)}(t)| \le c_0 |s - t| \text{ for all } a \le s, t \le b \right\},$$

where $h^{(r-1)}$ is the $(r-1)$th derivative of h, and $c_0 > 0$ is a constant. Let $\mathcal{U}_\lambda$ denote a neighborhood of the failure rate λ_0. We also define a sequence of maps G_n, mapping $\mathcal{U}_\lambda$ in the parameter space for λ into $\mathcal{L}^\infty(\mathcal{H}_r)$ as

$$G_n(\lambda)[h] = n^{-1} \sum_{i=1}^{n} \left\{ \delta_i \frac{h(Y_i)}{\lambda(Y_i)} - \int_{L_i}^{Y_i} h(t)dt \right\} = \mathbb{P}_n \phi(\lambda; X)[h].$$