First- and Second-Order Necessary Conditions via Lower-order Exact Penalty Functions

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Outline

Consider the nonlinear programming problem (NLP):

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x) \leq 0, \quad i \in I := \{1, \ldots, m\}, \\ & h_j(x) = 0, \quad j \in J := \{m+1, \ldots, m+q\}, \end{array}$$

where $f, g_i, h_j : R^n \to R$ are assumed to be smooth functions.

KKT conditions, originated with [**?**] and [**?**], are the well-known first-order necessary conditions for local minima of (NLP).

KKT conditions are useful in the design of optimal algorithms as one can compute a KKT point at most.

We denote by C the feasible set and by $\bar{S}$ the set of optimal solutions of (NLP).

$$\left.\begin{array}{c}\text{(NLP) has a local minimum at } \bar{x} \\ \text{plus a constraint qualification}\end{array}\right\} \Longrightarrow \text{the following KKT condition a}$$

$$\nabla f(\bar{x}) + \sum_{i \in I(\bar{x})} \mu_i \nabla g_i(\bar{x}) + \sum_{j \in J} \eta_j \nabla h_j(\bar{x}) = 0 \quad (\mu_i \geq 0).$$

Constraint qualifications include:

- LICQ [**?**]
- MFCQ [**?**]
- ACQ [**?**]
- GCQ (weakest CQ) [**?**]

Literature review

Another approach to study optimality conditions is by virtue of exact penalty functions. [**?**] and [**?**] used l_1 exact penalty functions to derive KKT necessary optimality conditions.

On the other hand, [**?**] and [**?**] used $l_p(p \in [0, 1])$ exact penalty functions (see [**?**] to derive KKT necessary optimality conditions together with some nonpositivity condition on the second-order directional derivative of the constraints.

Let $0 \leq p \leq 1$, $0^0 := 0$ and $g_{i+}(x) = \max\{g_i(x), 0\}$. A particular penalty term associated with (NLP) is of the form

$$S^p(x) = \sum_{i \in I} g_{i+}^p(x) + \sum_{j \in J} |h_j(x)|^p \quad \forall x \in R^n,$$

while the l_p penalty function associated with (NLP) is of the form

$$\mathcal{F}_p(x) := f(x) + \mu S^p(x).$$

- $p = 1$, the classical l_1 penalty function, see [**?**] and [**?**].
- $p < 1$, referred to as the lower order l_p penalty function, first introduced in [**?**] for the study of MPEC and was rediscovered from a unified augmented Lagrangian scheme by [**?**] and [**?**].

A penalty function is said to be exact if any optimal solution of (NLP) is also one for the penalty problem.

By definition, $\mathcal{F}_0$ is exact at any local minimum of (NLP). It was shown in [**?**] that $\mathcal{F}_p$ with $0 < p \leq 1$ is exact if and only if the following generalized calmness-type condition holds:

$$\liminf_{u \to 0} \frac{\beta(u) - \beta(0)}{\|u\|^p} > -\infty,$$

where $\beta(u)$ is the optimal value of the optimization problem

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x) \leq u_i, \quad i \in I, \; h_j(x) = u_j, j \in J. \end{array}$$

When $p = 1$, this result was established in [**?**] and [**?**].

Let f be locally Lipschitz. If the following error bound condition holds

$$\tau d(x, \bar{S}) \leq S^p(x), \ x \in X$$

then $\mathcal{F}_p(x)$ is an exact penalty function.

The exact penalty function plays a key role in deriving KKT conditions.

$I(\bar{x}) := \{i \in I \mid g_i(\bar{x}) = 0\}.$
$I(\bar{x}, w) := \{i \in I \mid g_i(\bar{x}) = 0, \langle \nabla g_i(\bar{x}), w \rangle = 0\}.$

The first-order linearized tangent cone to C at $\bar{x}$ is

$$L_C(\bar{x}) := \left\{ w \in R^n \,\middle|\, \begin{array}{ll} \langle \nabla g_i(\bar{x}), w \rangle \leq 0 & \forall i \in I(\bar{x}) \\ \langle \nabla h_j(\bar{x}), w \rangle = 0 & \forall j \in J \end{array} \right\}.$$

The Dini upper directional derivative of a function $\phi : R^n \to R$ at $x \in R^n$ in the direction $u \in R^n$ is defined by

$$D_+\phi(x; u) = \limsup_{t \to 0+} \frac{\phi(x + tu) - \phi(x)}{t}.$$

The generalized Clarke second-order directional derivative of a $C^{1,1}$ function is

$$g^{\circ\circ}(x; w) = \limsup_{y \to x,\, t \to 0+} \frac{\nabla g(y + tu)^T w - \nabla g(y)^T w}{t}.$$

If $F_p(x) = f(x) + \mu \sum_{i \in I} g_{i+}^p(x)$ is exact at $\bar{x}$, then

$$D_+ F_p(\bar{x}; u) \geq 0, \quad \forall u \in R^n.$$

Thus

$$\nabla f(\bar{x})^\top u + \mu \sum_{i \in I} D_+ g_{i+}^p(\bar{x}; u) \geq 0, \quad \forall u \in R^n.$$

Then,

$$\sum_{i \in I} D_+ g_{i+}^p(\bar{x}; u) \leq 0 \Longrightarrow \nabla f(\bar{x})^\top u \geq 0, \quad \forall u \in R^n.$$

By Farkas lemma, which says that exactly one of the following two systems has a solution:

System 1 $\quad Au \leq 0,\ c^\top u > 0,\ $ for some u,

System 2 $\quad A^\top \mu = c,\ \mu \geq 0,\ $ for some μ,

we establish that the following KKT condition holds:

$$\nabla f(\bar{x}) + \sum_{i \in I(\bar{x})} \mu_i \nabla g_i(\bar{x}) = 0 \quad (\mu_i \geq 0).$$

The case $p \in (0,1)$.

Lemma

[?]. Let $\bar{h}(x) = (\max\{h(x), 0\})^p$ *with* $p \in (0,1)$ *and* h *be continuously differentiable at* x.

(i) *If* $h(x) < 0$, *then* $D_+\bar{h}(x; d) = 0$;

(ii) *If* $h(x) = 0$ *and* $\langle \nabla h(x), d \rangle < 0$, *then* $D_+\bar{h}(x; d) = 0$;

(iii) *If* $p \in (0.5, 1)$, $h(x) = 0$, $\langle \nabla h(x), d \rangle = 0$ *and* $h^{\circ\circ}(x; d)$ *is finite, then* $D_+\bar{h}(x; d) = 0$;

(iv) *If* $p = 0.5$, $h(x) = 0$ *and* $\langle \nabla h(x), d \rangle = 0$, *then* $D_+\bar{h}(x; d) \leq \sqrt{\max\{\frac{1}{2} h^{\circ\circ}(x; d), 0\}}$;

(v) *If* $p \in (0, 0.5)$, $h(x) = 0$, $\langle \nabla h(x), d \rangle = 0$ *and* $h^{\circ\circ}(x; d) < 0$, *then* $D_+\bar{h}(x; d) = 0$.

By estimating the upper Dini-directional derivative of $\mathcal{F}_p(x)$, we have

Theorem

[?] If $F_p(x)$ is exact at $\bar{x}$ and in addition, one of the following conditions is satisfied:

(i) $p \in (\frac{1}{2}, 1]$, *$g_i (i \in I)$ and $h_j (j \in I)$ are $C^{1,1}$,*

(ii) $p = \frac{1}{2}$ *and, for every $w \in L_C(\bar{x})$, it follows that*

$$g_i^{\circ\circ}(\bar{x}; w) \leq 0, \qquad \forall i \in I(\bar{x}, w),$$
$$h_j^{\circ\circ}(\bar{x}; w) = 0, \qquad \forall j \in J,$$

(iii) $p \in [0, 1/2)$, $q = 0$ *(i.e., there is no equality constraint) and, for every $w \in L_C(\bar{x})$ with $w \neq 0$, it follows that*

$$g_i^{\circ\circ}(\bar{x}; w) < 0, \qquad \forall i \in I(\bar{x}, w),$$

then $\mathrm{KKT}(\bar{x}) \neq \emptyset$.

Let $M : R^n \rightrightarrows R^s$ be a set-valued map and $(x, y) \in gphM$. The contingent derivative of M at (x, y) is defined by the set-valued map $DM(x, y) : R^n \rightrightarrows R^s$ such that

$$gph(DM(x, y)) = T_{gphM}(x, y).$$

In particular, when M is single-valued at x, i.e., $M(x) = \{y\}$, we use $DM(x)$ to denote $DM(x, y)$ for simplicity, and define the kernel of $DM(x)$ by

$$KerDM(x) = \{u \in R^n \mid 0 \in DM(x)(u)\}.$$

Now, define an optimality indication set of (NLP) with respect to C and $\bar{x}$ as follows:

$$\Pi(C,\bar{x}) := \{p \in [0,1] \mid KerDS^p(\bar{x})^* \subset KerDS(\bar{x})^*\}.$$

By estimating the contingent derivative of $\mathcal{F}_p(x)$, we have

Theorem

[?] If there exists $p \in \Pi(C,\bar{x})$ such that the l_p penalty function $\mathcal{F}_p$ is exact at $\bar{x}$, then $\mathrm{KKT}(\bar{x}) \neq \emptyset$.

In what follows,

- we distinguish a point $\bar{x} \in C$ for consideration;
- let $\phi : R^n \to R_+ \cup \{+\infty\}$ be a lower semicontinuous function such that

$$C = \{x \in R^n \mid \phi(x) = 0\}.$$

- ϕ is called a **penalty term** associated with (NLP)
- The function of the form

$$f + \mu\phi$$

is called a **penalty function** associated with (NLP), where μ, a positive number, is often referred to as the **penalty parameter**.

Definition

We say that the penalty term ϕ is of **KKT-type at $\bar{x}$** if the KKT condition holds at $\bar{x}$ whenever the penalty function $f + \mu\phi$ is exact at $\bar{x}$.

Theorem

Consider the following conditions:

(i) $[\ker d\phi(\bar{x})]^* \subset L_C(\bar{x})^*$.

(ii) $\widehat{\partial}\phi(\bar{x}) \subset L_C(\bar{x})^*$.

(iii) *The penalty term* ϕ *is of* KKT-type *at* $\bar{x}$.

Then (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii).

Theorem

Let $0 \leq p < 1$. *Consider the following conditions:*

(i) $[\ker dS^p(\bar{x})]^* = L_C(\bar{x})^*$.

(ii) $\widehat{\partial} S^p(\bar{x}) = L_C(\bar{x})^*$.

(iii) S^p *is a* KKT-type *penalty term at* $\bar{x}$.

Then (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii).

In what follows,

- we distinguish a point $\bar{x} \in C$ for consideration;
- let $\phi : R^n \to R_+ \cup \{+\infty\}$ be a lower semicontinuous function such that

$$C = \{x \in R^n \mid \phi(x) = 0\}.$$

- ϕ is called a **penalty term** associated with (NLP)
- The function of the form

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is called a **penalty function** associated with (NLP), where μ, a positive number, is often referred to as the **penalty parameter**.

Definition

We say that the penalty function $f + \mu\phi$ is **exact at** $\bar{x}$ if, $f + \mu\phi$ admits a local minimum at $\bar{x}$ with some **finite penalty parameter**.

[exactness of penalty function at $\bar{x} \Longrightarrow \bar{x}$ being a local minimum of (NLP)]

It is well-known that [1]

$$\mathcal{F}_p \text{ with } p = 1 \text{ is exact at } \bar{x} \Longrightarrow \text{KKT condition at } \bar{x}.$$

But in general,

$$\mathcal{F}_p \text{ with } 0 < p < 1 \text{ is exact at } \bar{x} \not\Longrightarrow \text{KKT condition at } \bar{x}.$$

[1] See Theorem 4.8 of [**?**].

Definition

We say that the penalty term ϕ is of **KKT-type at** $\bar{x}$ if the KKT condition holds at $\bar{x}$ whenever the penalty function $f + \mu\phi$ is exact at $\bar{x}$.

We will employ the tools from Variational Analysis, see [**?**].
For any $f : R^n \to \overline{R}$ and a point $\bar{x}$ with $f(\bar{x})$ finite,

- The vector $v \in R^n$ is a regular subgradient of f at $\bar{x}$, written $v \in \widehat{\partial} f(\bar{x})$, if

$$f(x) \geq f(\bar{x}) + \langle v, x - \bar{x} \rangle + o(\|x - \bar{x}\|).$$

- For any $w \in R^n$, the subderivative (or Hadamard directional derivative) of f at $\bar{x}$ for w is defined by

$$df(\bar{x})(w) := \liminf_{\tau \to 0+,\ w' \to w} \frac{f(\bar{x} + \tau w') - f(\bar{x})}{\tau}.$$

-

$$\widehat{\partial} f(\bar{x}) = \{v \in R^n \mid \langle v, w \rangle \leq df(\bar{x})(w)\ \forall w \in \mathrm{dom} df(\bar{x})\}.$$

Lemma

Suppose that the function $\psi : R^n \to \overline{R}$ has a local minimum at $\bar{x}$ with $\psi(\bar{x})$ finite. Then we have

$$[\mathrm{dom} d\psi(\bar{x})]^* \subset \widehat{\partial}\psi(\bar{x}) \subset [\ker d\psi(\bar{x})]^*. \tag{1}$$

Moreover,

- *The first inclusion in (1) is an equality if and only if the regular subdifferential $\widehat{\partial}\psi(\bar{x})$ is a cone;*
- *The second inclusion in (1) is an equality if and only if $[\mathrm{dom} d\psi(\bar{x})]^* = [\ker d\psi(\bar{x})]^*$;*
- *If the subderivative $d\psi(\bar{x})$ is a sublinear function as is true when ψ is regular at $\bar{x}$ (see Definition 7.25 of [?]), then*

$$\mathrm{clpos}(\widehat{\partial}\psi(\bar{x})) = [\ker d\psi(\bar{x})]^*. \tag{2}$$

We recall the **variational description of regular subgradients**:

Lemma

*([**?**], Proposition 8.5). A vector v belongs to $\widehat{\partial} f(\bar{x})$ if and only if, on some neighborhood of $\bar{x}$, there is a function $h \leq f$ with $h(\bar{x}) = f(\bar{x})$ such that h is differentiable at $\bar{x}$ with $\nabla h(\bar{x}) = v$. Moreover h can be taken to be continuously differentiable with $h(x) < f(x)$ for all $x \neq \bar{x}$ near $\bar{x}$.*

Remark

*This variational description is a contribution to the basics of variational analysis, as pointed out on p.347 of [**?**].*

We can obtain from Lemmas 9 and 10 the following.

Theorem

Consider the following conditions:

(i) $[\ker d\phi(\bar{x})]^* \subset L_C(\bar{x})^*$.

(ii) $\widehat{\partial}\phi(\bar{x}) \subset L_C(\bar{x})^*$.

(iii) *The penalty term* ϕ *is of* KKT-type *at* $\bar{x}$.

Then (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii).

Theorem

Let $0 \le p < 1$. Consider the following conditions:

(i) $[\mathrm{ker} dS^p(\bar{x})]^* = L_C(\bar{x})^*$.

(ii) $\widehat{\partial} S^p(\bar{x}) = L_C(\bar{x})^*$.

(iii) S^p *is a* KKT-type *penalty term at* $\bar{x}$.

Then (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii).

Remark

In the case of $p = 0$, (i) and (ii) are equivalent, and moreover Theorem 12 recovers a well-known result that the GCQ $[T_C(\bar{x})^* = L_C(\bar{x})^*]$ is the weakest one ensuring KKT conditions.

Remark

In the case of $0 < p < 1$, we are not aware of the equivalence of (i) and (ii), although they are the same in many situations.

By a direct calculation using the chain rule for second subderivatives of piecewise linear-quadratic functions [2], we have

$$dS^{\frac{1}{2}}(\bar{x})(w) = +\infty \quad \forall w \notin L_C(\bar{x}),$$

and if $w \in L_C(\bar{x})$, we have $dS^{\frac{1}{2}}(\bar{x})(w)$

$$= \frac{\sqrt{2}}{2}\sqrt{\max_{\rho \in \mathrm{KKT}_0(\bar{x}),\, \|\rho\|_\infty = 1} \left\langle \left[\sum_{i\in I} \rho_i \nabla^2 g_i(\bar{x}) + \sum_{j \in J} \rho_j \nabla^2 h_j(\bar{x})\right] w, w \right\rangle},$$

where

$$\mathrm{KKT}_0(\bar{x}) := \left\{ \rho \;\middle|\; \begin{array}{l} \sum_{i\in I} \rho_i \nabla g_i(\bar{x}) + \sum_{j\in J} \rho_j \nabla h_j(\bar{x}) = 0 \\ \rho_i \geq 0 \quad \forall i \in I(\bar{x}),\ \rho_i = 0\ \forall i \in I \backslash I(\bar{x}) \end{array} \right\}$$

denotes the degenerate KKT multiplier set at $\bar{x}$.
But we have no idea the explicit formula of $\widehat{\partial} S^p(\bar{x})$, though we are sure that

$$\widehat{\partial} S^{\frac{1}{2}}(\bar{x}) = \{v \mid \langle v, w\rangle \leq dS^{\frac{1}{2}}(\bar{x})(w) \quad \forall w\}.$$

[2] See Chapter 13 of [?]

Proposition

$S^{\frac{1}{2}}$ is of KKT-type at $\bar{x}$ if one of the following conditions is satisfied:

(i) *For every $w \in L_C(\bar{x})$, it follows that*

$$\langle w, \nabla^2 g_i(\bar{x}) w \rangle \leq 0 \ \forall i \in I(\bar{x}, w), \quad \langle w, \nabla^2 h_j(\bar{x}) w \rangle = 0 \ \forall j \in J. \tag{3}$$

(ii) *For every $w \in L_C(\bar{x})$, there exists some $z \in R^n$ such that*

$$\begin{aligned} \langle \nabla g_i(\bar{x}), z \rangle + \langle w, \nabla^2 g_i(\bar{x}) w \rangle \leq 0 \quad & \forall i \in I(\bar{x}, w), \\ \langle \nabla h_j(\bar{x}), z \rangle + \langle w, \nabla^2 h_j(\bar{x}) w \rangle = 0 \quad & \forall j \in J. \end{aligned}$$

(iii) *For every $w \in L_C(\bar{x})$, it follows that*

$$\max_{\lambda \in \mathrm{KKT}_0(\bar{x})} \left\{ \sum_{i \in I} \lambda_i \langle w, \nabla^2 g_i(\bar{x}) w \rangle + \sum_{j \in J} \lambda_j \langle w, \nabla^2 h_j(\bar{x}) w \rangle \right\} = 0. \tag{4}$$

- Condition (3) was originally given in [?]. In general, we have
  - LICQ $\not\Longrightarrow$ (3). Consider $x_2^2 - x_1 \leq 0$ and $\bar{x} = (0,0)$.
  - (3) $\not\Longrightarrow$ LICQ. Consider $x^3 \leq 0$ and $\bar{x} = 0$.
- Condition (4) is newly obtained, and we have

$$\text{MFCQ} \Longrightarrow (4),$$

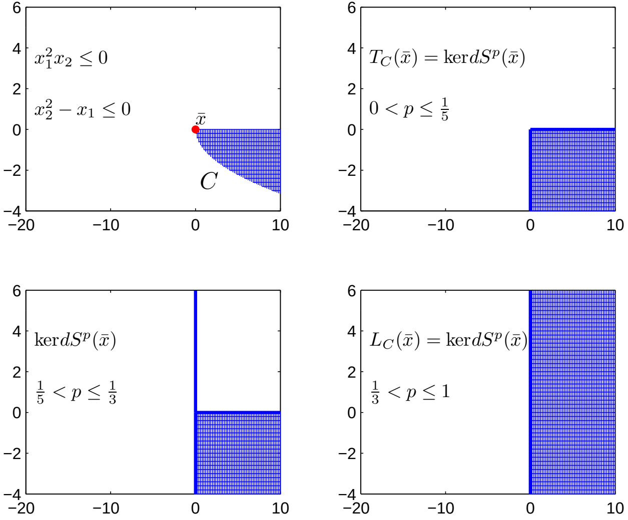
because the MFCQ at $\bar{x} \Longleftrightarrow \mathrm{KKT}_0(\bar{x}) = \{0\}$.

Example

Let $\bar{x} = (0,0)$ and let

$$C = \left\{ x \in R^n \;\middle|\; \begin{array}{c} x_1^2 x_2 \leq 0 \\ x_2^2 - x_1 \leq 0 \end{array} \right\}.$$

- Neither the GCQ nor (3) is satisfied at $\bar{x}$.
- (4) holds and $\mathrm{KKT}_0(\bar{x}) = R_+ \times \{0\}$.


$x_1^2 x_2 \leq 0$
$x_2^2 - x_1 \leq 0$
$\bar{x}$
C
$T_C(\bar{x}) = \ker dS^p(\bar{x})$
$0 < p \leq \frac{1}{5}$
$\ker dS^p(\bar{x})$
$\frac{1}{5} < p \leq \frac{1}{3}$
$L_C(\bar{x}) = \ker dS^p(\bar{x})$
$\frac{1}{3} < p \leq 1$

$T_C(\bar{x}) = R_+ \times (-R_+)$, $L_C(\bar{x}) = R_+ \times R$, and

$$\ker dS^p(\bar{x}) = \begin{cases} R_+ \times (-R_+) & \text{if } 0 < p \leq \frac{1}{5}, \\ R_+ \times (-R_+) \cup \{0\} \times R_+ & \text{if } \frac{1}{5} < p \leq \frac{1}{3}, \\ R_+ \times R & \text{if } \frac{1}{3} < p \leq 1. \end{cases}$$

Denote the set of all KKT multipliers at $\bar{x}$ by $\mathrm{KKT}(\bar{x})$ and the critical cone at $\bar{x}$ by

$$\mathcal{V}(\bar{x}) := \left\{ w \in R^n \;\middle|\; \begin{array}{ll} \langle \nabla f(\bar{x}), w \rangle \leq 0 & \\ \langle \nabla g_i(\bar{x}), w \rangle \leq 0 & \forall i \in I \text{ with } g_i(\bar{x}) = 0 \\ \langle \nabla h_j(\bar{x}), w \rangle = 0 & \forall j \in J \end{array} \right\}.$$

The second-order necessary condition (for short, SON), originated with [**?**], holds at a local minimum $\bar{x}$ of (NLP) if

$$\sup_{\lambda \in \mathrm{KKT}(\bar{x})} \langle w, \nabla^2_{xx} L(\bar{x}, \lambda) w \rangle \geq 0 \qquad \forall w \in \mathcal{V}(\bar{x}),$$

where the convention $\sup \emptyset := -\infty$ is used.

- l_1 exactness $\Longrightarrow$ (SON). See Corollary 4.5 of [**?**].

For any w and z, let

$$
\begin{aligned}
I(\bar{x}, w) &:= \{i \in I(\bar{x}) \mid \langle w, \nabla g_i(\bar{x}) \rangle = 0\}, \\
I(\bar{x}, w, z) &:= \{i \in I(\bar{x}, w) \mid \langle z, \nabla g_i(\bar{x}) \rangle + \langle w, \nabla^2 g_i(\bar{x}) w \rangle = 0\},
\end{aligned}
$$

and let the second-order linearized tangent set to C at $\bar{x}$ in the direction $w \in L_C(\bar{x})$ be given by

$$
L_C^2(\bar{x} \mid w) := \left\{ z \,\middle|\, \begin{array}{ll} \langle \nabla g_i(\bar{x}), z \rangle + \langle w, \nabla^2 g_i(\bar{x}) w \rangle \le 0 & \forall i \in I(\bar{x}, w) \\ \langle \nabla h_j(\bar{x}), z \rangle + \langle w, \nabla^2 h_j(\bar{x}) w \rangle = 0 & \forall j \in J \end{array} \right\}
$$

The parabolic subderivative of f at $\bar{x}$ for w with respect to z is defined by, see [?]

$$d^2 f(\bar{x})(w \mid z) := \liminf_{\tau \to 0+,\, z' \to z} \frac{f(\bar{x} + \tau w + \frac{1}{2}\tau^2 z') - f(\bar{x}) - \tau df(\bar{x})(w)}{\frac{1}{2}\tau^2}.$$

Theorem

Let $\bar{x}$ be a local minimum of (NLP). Suppose that the penalty function $f + \mu\phi$ is exact at $\bar{x}$. If

$$L_C^2(\bar{x} \mid w) \subset \mathrm{clconv}[\mathrm{ker} d^2\phi(\bar{x})(w \mid \cdot)] \qquad \forall w \in \mathcal{V}(\bar{x}), \tag{5}$$

then the SON condition holds, and in particular when $L_C^2(\bar{x} \mid w) = \emptyset$, the supremum in the SON condition is $+\infty$.

Let $\bar{x} \in C$ and let $\phi = S^p$.
We shall give sufficient conditions in terms of the original data for the inclusion

$$L_C^2(\bar{x} \mid w) \subset \mathrm{ker} d^2 S^p(\bar{x})(w \mid \cdot) \qquad \forall w \in L_C(\bar{x}) \tag{6}$$

to hold, which is slightly stronger than (5) since in general $\mathrm{ker} d^2 S^p(\bar{x})(w \mid \cdot)$ is not a closed and convex set and $\mathcal{V}(\bar{x}) \subsetneq L_C(\bar{x})$.

Theorem

Let $\bar{x}$ be a local minimum of (NLP). *Suppose that the l_p penalty function is exact at $\bar{x}$. If, in addition, one of the following conditions is satisfied:*

(i) $p \in (\frac{2}{3}, 1]$,

(ii) $p = \frac{2}{3}$ *and, for every* $z \in L_C^2(\bar{x} \mid w)$, *it follows that*

$$\begin{cases} \langle w, \nabla^2 g_i(\bar{x})z \rangle + \dfrac{1}{3} g_i^{(3)}(\bar{x})(w, w, w) \leq 0 & \forall\, i \in I(\bar{x}, w, z), \\ \langle w, \nabla^2 h_j(\bar{x})z \rangle + \dfrac{1}{3} h_j^{(3)}(\bar{x})(w, w, w) = 0 & \forall\, j \in J, \end{cases} \tag{7}$$

(iii) $p \in [0, \frac{2}{3})$, $q = 0$ *(i.e., there is no equality constraint) and, for every* $z \in L_C^2(\bar{x} \mid w)$ *with* $(w, z) \neq 0$, *it follows that*

$$\langle w, \nabla^2 g_i(\bar{x})z \rangle + \frac{1}{3} g_i^{(3)}(\bar{x})(w, w, w) < 0 \qquad \forall\, i \in I(\bar{x}, w, z),$$

Remark

(a) *Let $p = 1$. By applying the second-order Taylor expansion we have*

$$L_C^2(\bar{x} \mid w) = \ker d^2 S(\bar{x})(w \mid \cdot) \quad \forall w \in L_C(\bar{x}), \tag{8}$$

*which implies that condition (6) holds. This recovers a well-known result that the (SON) condition holds at $\bar{x}$ when the l_1 penalty function is exact at $\bar{x}$, see [**?**].*

(b) *Let $0 \leq p < 1$. It can be shown that*

$$\mathrm{ker} d^2 S^p(\bar{x})(w \mid \cdot) \subset L_C^2(\bar{x} \mid w) \quad \forall w \in \mathrm{ker} dS^p(\bar{x}). \qquad (9)$$

Thus, condition (5) holds if and only if

$$L_C^2(\bar{x} \mid w) = \mathrm{clconv}[\mathrm{ker} d^2 S^p(\bar{x})(w \mid \cdot)] \quad \forall w \in \mathcal{V}(\bar{x}). \qquad (10)$$

Moreover, it is clear that

$$T_C^2(\bar{x} \mid w) = \mathrm{ker} d^2 S^0(\bar{x})(w \mid \cdot), \quad \forall w \in T_C(\bar{x}).$$

Condition (10) *with $p = 0$ reduces to the so-called* SGCQ, *originated with [?], which holds at $\bar{x}$ if by definition*

$$L_C^2(\bar{x} \mid w) = \mathrm{clconv}[T_C^2(\bar{x} \mid w)] \quad \forall w \in \mathcal{V}(\bar{x}).$$

(c) *It was shown by [**?**] that if the linear independent constraint qualification (for short, LICQ) holds at* $\bar{x}$*, then*

$$L_C^2(\bar{x} \mid w) = T_C^2(\bar{x} \mid w) \quad \forall w \in L_C(\bar{x}),$$

and hence (6) holds for any $p \in [0, 1]$.
Simple example can be given to demonstrate that condition (7) may not hold even if the LICQ holds at $\bar{x}$.

Consider the following semi-infinite program, denoted as (SIP):

$$\min f(x) \quad \text{s.t. } g(x,t) \leq 0, t \in \Omega,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \times \Omega \to \mathbb{R}$ are smooth functions, and Ω is a nonempty and compact set of parameters in $\mathbb{R}^m$.

Let x^* be a locally optimal solution of (SIP),

$$X := \{x \in R^n : g(x,t) \leq 0, t \in \Omega\}$$

be the feasible set and, for $x \in X$, let

$$\Omega_x := \{t \in \Omega : g(x,t) = 0\}.$$

Literature review

Three types of optimality conditions for (SIP):

$$0 \in \mathrm{conv}\{\nabla f(x^*), \nabla_x g(x^*, t)\ (t \in \Omega_{x^*})\},$$

(see Fritz John (1948), Pschenichnyi (1971), Hettich and Jongen (1978), and Borwein (1981),)

$$0 \in \nabla f(x^*) + \mathrm{cl\ cone}\{\nabla_x g(x^*, t)\ (t \in \Omega_{x^*})\},$$

(see [**?**], Li, et al (2000),)

$$0 \in \nabla f(x^*) + \mathrm{cone}\{\nabla_x g(x^*, t)\ (t \in \Omega_{x^*})\},$$

(see Pschenichnyi (1971), Lopez and Vercher (1983), Hettich and Kortanek (1993), Zheng and Yang (2007).)

Literature review

By a Farkas lemma, see [**?**],

$$0 \in \nabla f(x^*) + \mathrm{cl}\ \mathrm{cone}\{\nabla_x g(x^*, t)\ (t \in \Omega_{x^*})\}, \tag{11}$$

is equivalent to

$$\langle \nabla f(x^*), d \rangle \geq 0,\ \forall d \in D(x^*), \tag{12}$$

where $D(x) = \{0 \neq d \in R^n : \langle \nabla_x g(x, t), d \rangle \leq 0\ \forall t \in \Omega_x\}$.

In this talk, we will study (11) but using the form of (12).

Literature review

[?] introduced the following l_1 integral penalty function

$$f(x) + \rho \int_{\Omega(x)} g(x,t)\mathrm{d}\mu(t),$$

where $\Omega(x) := \{t \in \Omega : g(x,t) > 0\}$, but too weak penalty for infeasibility.

Let $p > 0$. For (SIP), [?] also introduced the following l_p integral penalty function

$$f(x) + \rho \int_{\Omega} g_+^p(x,t)\mathrm{d}\mu(t)$$

and established the convergence of the solution sequence of the penalty problems to an optimal solution of (SIP).

[?] established the exact l_1 integral penalty function

$$f(x) + \rho \int_{\Omega(x)} g(x,t)\mathrm{d}\mu(t) \Bigg/ \int_{\Omega(x)} \mathrm{d}\mu(t) \,.$$

Penalty Functions

A pth-order max-type penalty function for SIP is defined as,

$$F^p_{max}(x) = f(x) + \rho \max_{t\in\Omega} g^p_+(x,t).$$

Let μ be a non-negative regular Borel measure defined on Ω with the support of μ being equal to Ω, that is $\mathrm{supp}(\mu) = \Omega$, where the support of μ is defined as the set of the points $t \in \Omega$ such that any open neighbourhood V of t has a positive measure:

$$\mathrm{supp}\,(\mu) := \{t \in \Omega : \mu(V) > 0, \text{ for any open neighbourhood } V \text{ of } t\}.$$

Two pth-order integral-type penalty functions for SIP are defined by

$$\begin{aligned} F^p_{int}(x) &= f(x) + \rho \int_\Omega g^p_+(x,t)\,\mathrm{d}\mu(t), \\ \bar{F}^p_{int}(x) &= f(x) + \rho \left(\int_\Omega g_+(x,t)\,\mathrm{d}\mu(t)\right)^p. \end{aligned}$$

Exactness of $F^p_{int}(x) \Longrightarrow$ that of $\bar{F}^p_{int}(x) \Longrightarrow$ that of $F^p_{max}(x)$.

The case $p = 1$.

(SIP) can be rewritten as

$$\min f(x) \text{ s.t. } \max_{t \in \Omega} g(x, t) \leq 0. \tag{13}$$

The exactness of F^1_{max} is equivalent to saying that problem (13) has an l_1 exact penalty function in the usual sense, see Clarke (1983). Thus, if F^1_{max} is exact, then the KKT-type optimality condition (12).

The case $p \in (0,1)$.

Let $h : \mathbb{R}^n \to \mathbb{R}$. The upper Dini-directional derivative of h at a point x in the direction $d \in \mathbb{R}^n$ is defined by

$$D_+ h(x; d) = \limsup_{\lambda \downarrow 0} \frac{h(x + \lambda d) - h(x)}{\lambda}.$$

The generalized upper second-order directional derivatives of a $C^{1,1}$ function h at x in the direction $d \in \mathbb{R}^n$ is defined by

$$h^{\circ\circ}(x; d) = \limsup_{y \to x, \lambda \downarrow 0} \frac{\langle \nabla h(y + \lambda d), d \rangle - \langle \nabla h(y), d \rangle}{\lambda}.$$

Let $D(x) = \{0 \neq d \in R^n : \langle \nabla_x g(x,t), d \rangle \leq 0 \ \forall t \in \Omega_x\}$ and let

$$\begin{aligned} \Omega_x^{=}(d) &:= \{t \in \Omega_x : \langle \nabla_x g(x,t), d \rangle = 0\}, \\ \Omega_x^{<}(d) &:= \{t \in \Omega_x : \langle \nabla_x g(x,t).d \rangle < 0\}, \end{aligned}$$

The case $p \in (0,1)$.

Lemma

[?] Let $\bar{h}(x) = (\max\{h(x), 0\})^p$ with $p \in]0,1[$ and h be continuously differentiable at x.

(i) *If $h(x) < 0$, then $D_+\bar{h}(x; d) = 0$;*

(ii) *If $h(x) = 0$ and $\langle \nabla h(x), d\rangle < 0$, then $D_+\bar{h}(x; d) = 0$;*

(iii) *If $p \in (0.5, 1)$, $h(x) = 0$, $\langle \nabla h(x), d\rangle = 0$ and $h^{\circ\circ}(x; d)$ is finite, then $D_+\bar{h}(x; d) = 0$;*

(iv) *If $p = 0.5$, $h(x) = 0$ and $\langle \nabla h(x), d\rangle = 0$, then $D_+\bar{h}(x; d) \leq \sqrt{\max\{\frac{1}{2}h^{\circ\circ}(x; d), 0\}}$;*

(v) *If $p \in (0, 0.5)$, $h(x) = 0$, $\langle \nabla h(x), d\rangle = 0$ and $h^{\circ\circ}(x; d) < 0$, then $D_+\bar{h}(x; d) = 0$.*

The case $p \in (0,1)$.

Now we establish a necessary optimality condition for SIP by virtue of the exact penalty function F^p_{int}.

Theorem

Let $p \in (0,1)$ and F^p_{int} be exact at x^*. Under any one of the three assumptions below,
(i) $p \in (0.5,1)$ *and* $g(\cdot,t)$ *is* $C^{1,1}$, *for all* $t \in \Omega^{=}_{x^*}(d)$,
(ii) $p = 0.5$ *and* $g^{\circ\circ}(x^*,t;d) \leq 0$ *for all* $t \in \Omega^{=}_{x^*}(d)$ *and* $d \in D(x^*)$,*and*
(iii) $p \in (0,0.5)$ *and* $g^{\circ\circ}(x^*,t;d) < 0$, *for all* $t \in \Omega^{=}_{x^*}(d)$ *and* $d \in D(x^*)$,
we have

$$\langle \nabla f(x^*), d\rangle \geq 0, \forall d \in D(x^*).$$

The case $p \in (0,1)$.

Next we employ the exactness of $\bar{F}_{int}(p \in (0,1))$ to develop the optimality condition (12) of (SIP).

Theorem

Let $p \in (0,1)$ and $\bar{F}^p_{int}$ be exact at x^*. Under any one of the three assumptions below,
(i) $p \in (0.5,1)$ *and* $g(\cdot,t)$ *is* $C^{1,1}$, *for all* $t \in \Omega^{=}_{x^*}(d)$,
(ii) $p = 0.5$ *and* $g^{\circ\circ}(x^*,t;d) \leq 0$ *for all* $t \in \Omega^{=}_{x^*}(d)$ *and* $d \in D(x^*)$, *and*
(iii) $p \in (0,0.5)$ *and* $g^{\circ\circ}(x^*,t;d) < 0$, *for all* $t \in \Omega^{=}_{x^*}(d)$ *and* $d \in D(x^*)$,
we have

$$\langle \nabla f(x^*), d\rangle \geq 0, \forall d \in D(x^*).$$

The case $p \in (0,1)$.

We need the following lemma in the proof of the above theorem.

Proposition

Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ *be a non-negative function,* $f : \mathbb{R}_+ \to \mathbb{R}$ *be a continuous and strictly increasing function and* $\lambda_0 \in \mathbb{R}_+$*. Then*

$$\limsup_{\lambda \to \lambda_0} f(g(\lambda)) \leq f(\limsup_{\lambda \to \lambda_0} g(\lambda)).$$

We will also consider the following generalized semi-infinite program, denoted as (GSIP),

$$\min f(x) \quad \text{s.t. } g(x,t) \le 0, t \in \Omega \cap \Omega(x),$$

where Ω is a compact subset of $\mathbb{R}^m$,

$$\Omega(x) := \{t \in \mathbb{R}^m : v_i(x,t) \le 0, i = 1, \cdots, l\}$$

and the functions $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, and $v_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}(i = 1, \cdots, l)$ are smooth.

[?] associated (GSIP) with an (SIP) problem via augmented Lagrangians of the lower level problem.

The lower level problem associated with (GSIP) is

$$Q(x) \qquad \max_{t\in\Omega} g(x,t) \quad \text{s.t.} \quad v_i(x,t) \leq 0, i = 1, \cdots, l.$$

Let $\mathrm{val}Q(x)$ be the optimal value of the problem $Q(x)$. It is clear that

$$x \in X_{(\mathrm{GSIP})} \quad \text{iff} \quad \mathrm{val}Q(x) \leq 0.$$

Let $\bar{f}(x,\mu,c) = f(x)$ and, for $(x,\mu,c) \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}_{++}$,

$$\bar{g}(x,t,\mu,c) = g(x,t) - \frac{1}{2c}\sum_{i=1}^{l}\{([cv_i(x,t)+\mu_i]_+)^2 - \mu_i^2\}.$$

Then $\bar{g}$ is of $C^{1,1}$, see Hiriart-Urruty et al (1984).

Next we recall some concepts from [**?**].

Problem $Q(x)$ is said to satisfy the *quadratic growth condition* iff there is a $c \geq 0$ such that $\bar{g}(x, t, 0, c)$ is bounded above as a function of $t \in \Omega$.

Problem $Q(x)$ is said to be *stable of degree* 2 iff there is a neighbourhood U of the origin in $\mathbb{R}^l$ and a C^2 function $\pi_x : U \to \mathbb{R}$ such that

$$\nu(x, u) \leq \pi_x(u), \forall u \in U, \text{ and } \nu(x, 0) = \pi_x(0).$$

Let $\bar{H}(x, \mu, c) := \max_{t \in \Omega} \bar{g}(x, t, \mu, c)$.

Lemma

[?] Under the quadratic growth condition of $Q(x)$, we have

$$\mathrm{val}\, Q(x) = \min_{(\mu,c) \in \mathbb{R}^l \times \mathbb{R}_{++}} \bar{H}(x, \mu, c)$$

iff the problem $Q(x)$ is stable of degree 2.

Consider the following (SIP) problem, denoted as (SIPg),

$$\min_{(x,\mu,c)\in\mathbb{R}^n\times\mathbb{R}^l\times\mathbb{R}_{++}} \bar{f}(x,\mu,c) \quad \text{s.t.} \quad \bar{g}(x,t,\mu,c)\leq 0, t\in\Omega.$$

Therefore we have

Proposition

Assume that, for all $x\in\mathbb{R}^n$, $Q(x)$ satisfies the quadratic growth condition and is stable of degree 2. Then problems (GSIP) and (SIPg) have the same optimal value, i.e., val(GSIP) = val(SIPg), and furthermore,
(i) *if $\hat{x}$ is a locally optimal solution of (GSIP), then there exists $(\hat{\mu},\hat{c})\in\mathbb{R}^l\times\mathbb{R}_{++}$ such that $(\hat{x},\hat{\mu},\hat{c})$ is a locally optimal solution of (SIPg);*
(ii) *if $(\hat{x},\hat{\mu},\hat{c})$ is a locally optimal solution of (SIPg), then $\hat{x}$ is a locally optimal solution of (GSIP).*

For $p \in (0,1)$, let

$$G^p_{int}(x,\mu,c) := \bar{f}(x,\mu,c) + \rho \int_\Omega \bar{g}^p_+(x,t,\mu,c)\,\mathrm{d}\mu(t).$$

By applying previous Theorem for SIP, we have.

Theorem

Let the assumptions of the previous Proposition hold. Let $\hat{x}$ be a locally optimal solution of (GSIP) and G^p_{int} be exact at the point $(\hat{x}, \hat{\mu}, \hat{c})$. Then, under one of the following assumptions,
(i) $p \in (0.5, 1)$,
(ii) $p = 0.5$ *and* $\bar{g}^{\circ\circ}_{(x,\mu,c)}(\hat{x}, t, \hat{\mu}, \hat{c}; d) \leq 0$ *for all* $d \in D(\hat{x}, \hat{\mu}, \hat{c})$ *and* $t \in \Omega_{(\hat{x},\hat{\mu},\hat{c})}$ *with* $\langle \nabla_{(x,\mu,c)}\bar{g}(\hat{x}, t, \hat{\mu}, \hat{c}), d\rangle = 0$, *and*
(iii) $p \in (0, 0.5)$ *and* $\bar{g}^{\circ\circ}_{(x,\mu,c)}(\hat{x}, t, \hat{\mu}, \hat{c}; d) < 0$ *for all* $d \in D(\hat{x}, \hat{\mu}, \hat{c})$ *and* $t \in \Omega_{(\hat{x},\hat{\mu},\hat{c})}$ *with* $\langle \nabla_{(x,\mu,c)}\bar{g}(\hat{x}, t, \hat{\mu}, \hat{c}), d\rangle = 0$,
we have

$$\langle \nabla f(\hat{x}), d_1\rangle \geq 0,$$

for all $d_1 \in \mathbb{R}^n$ *satisfying* $\langle \nabla_x g(\hat{x}, t) - \nabla_x^T v(\hat{x}, t)\hat{\mu}, d_1\rangle \leq 0, t \in \Omega_{(\hat{x},\hat{\mu},\hat{c})}$.

We now compute the generalized second-order directional derivative $\bar{g}^{\circ\circ}_{(x,\mu,c)}(\hat{x}, t, \hat{\mu}, \hat{c}; d)$ for $d \in D(\hat{x}, \hat{\mu}, \hat{c})$ and $t \in \Omega(\hat{x}, \hat{\mu}, \hat{c})$.

Lemma

Let $d \in D(\hat{x}, \hat{\mu}, \hat{c})$ and $t \in \Omega(\hat{x}, \hat{\mu}, \hat{c})$. Then the following formula holds:

$$\bar{g}^{\circ\circ}_{(x,\mu,c)}(\hat{x}, t, \hat{\mu}, \hat{c}; d) = d_1^T[\nabla^2_{xx} g(\hat{x}, t) - \sum_{i=1}^{l} \hat{\mu}_i \nabla^2_{xx} v_i(\hat{x}, t)] d_1$$

$$- \sum_{i \in \hat{I}^+_{(\hat{x},\hat{\mu},\hat{c})}(t)} (\sqrt{\hat{c}} d_1^T \nabla_x v_i(\hat{x}, t) + \frac{d_{2i}}{\sqrt{\hat{c}}})^2 + \sum_{i=1}^{l} \frac{d_{2i}^2}{\hat{c}},$$

where $\hat{I}^+_{(\hat{x},\hat{\mu},\hat{c})}(t) = \{i \in \{i, \cdots, l\} : \hat{c} v_i(\hat{x}, t) + \hat{\mu}_i > 0\}$.

We have the following corollary.

Corollary

Assume that the following conditions hold:
(i) $G_{int}^{\frac{1}{2}}(x,\mu,c)$ *is exact at* $(\hat{x},\hat{\mu},\hat{c})$;
(ii) $g(\cdot,t)$ *and* $-v_i(\cdot,t)$ $(i=1,\cdots,l)$ *are concave for each* $t\in\Omega$; *and*
(iii) $\hat{I}^{+}_{(\hat{x},\hat{\mu},\hat{c})}(t)=\{1,\cdots,l\}$ *and* $\langle\nabla_x v_i(\hat{x},t),d_1\rangle=0$ *for* $d\in D(\hat{x},\hat{\mu},\hat{c}), t\in\Omega(\hat{x},\hat{\mu},\hat{c})$ *and* $i\in\hat{I}^{+}(t)$. *Then we have*

$$\langle\nabla f(\hat{x}),d_1\rangle\geq 0,$$

for all $d_1\in\mathbb{R}^n$ *satisfying* $\langle\nabla_x g(\hat{x},t)-\nabla_x^T v(\hat{x},t)\hat{\mu},d_1\rangle\leq 0, t\in\Omega_{(\hat{x},\hat{\mu},\hat{c})}$.

• For (NLP), we discussed the first-order optimality conditions by Dini-directional derivative, contingent directional derivative and subderivative respectively.

• For (SIP) and (GSIP), we investigated the first-order optimality conditions by Dini-directional derivative.