On Error Bound Moduli for Locally Lipschitz and Regular Functions

Xiaoqi Yang

Department of Applied Mathematics,
Hong Kong Polytechnic University
Email: mayangxq@polyu.edu.hk

Joint work with Minghua Li and Kaiwen Meng

Outline

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1. Existing results

Let $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at $\bar{x} \in [\phi \leq 0]$[a].

Definition 1.1 *ϕ has a* **local error bound** *for $[\phi \leq 0]$ at some $\bar{x} \in [\phi \leq 0]$ if, there exist some $\tau > 0$ and $\varepsilon > 0$ such that for all $x \in R^n$ with $\|x - \bar{x}\| < \varepsilon$,*

$$\tau d(x, [\phi \leq 0]) \leq \phi(x). \tag{1.1}$$

ϕ is a **global error bound** *for $[\phi \leq 0]$ if (1.1) holds for all x.*

Closely related notions:

- calmness (variant of the Aubin property)
- subregularity (variant of metric regularity)
- (local) weak sharp minima

[Burke and Ferris (1993), Pang (1997), Dontchev and Rockafellar (2004), Rockafellar and Wets (1998), Henrion and Outrata (2005)]

....

[a] $[\phi \leq 0] := \{x \in \mathbb{R}^n | \phi(x) \leq 0\}$ denotes the level set.

The **local error bound modulus** of $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at $\bar{x} \in [\phi \leq 0]$ is defined by

$$\operatorname{ebm}(\phi, \bar{x}) := \liminf_{x \to \bar{x}, \phi(x) > 0} \frac{\phi(x)}{d(x, [\phi \leq 0])}.$$

Clearly, $0 \leq \operatorname{ebm}(\phi, \bar{x}) \leq +\infty$.

- **Case** $\mathrm{ebm}(\phi, \bar{x}) = 0$

  For any $0 < \tau$, there is no $V \in \mathcal{N}(\bar{x})$ such that

  $$\tau d(x, [\phi \le 0]) \le \phi(x)_+ \quad \forall x \in V.$$

  In this case, we say that ϕ has no *local error bound* at $\bar{x}$.

- **Case** $\mathrm{ebm}(\phi, \bar{x}) > 0$

  For $0 < \tau < \mathrm{ebm}(\phi, \bar{x})$, there is some $V_\tau \in \mathcal{N}(\bar{x})$ such that

  $$\tau d(x, [\phi \le 0]) \le \phi(x)_+ \quad \forall x \in V_\tau. \tag{1.2}$$

  In this case, we say that ϕ has a *local error bound* at $\bar{x}$.

Alternatively, we have

$$\mathrm{ebm}(\phi, \bar{x}) = \sup\{\tau > 0 \mid \exists V_\tau \in \mathcal{N}(\bar{x}) \text{ satisfying } (1.2)\}^a$$

[a] Here, $\sup \emptyset := 0$ is used by convention.

The set

$$\{\tau > 0 \mid \exists V_\tau \in \mathcal{N}(\bar{x}) \text{ satisfying } (1.2)\}$$

is an interval, either being $(0, \mathrm{ebm}(\phi, \bar{x})]$ or $(0, \mathrm{ebm}(\phi, \bar{x}))$.

For instance,

- in the case of $\phi(x) = x$ and $\bar{x} = 0$, it is $(0, 1]$ with $\mathrm{ebm}(\phi, \bar{x}) = 1$;
- in the case of $\phi(x) = \sin x$ and $\bar{x} = 0$, it is $(0, 1)$[a] with $\mathrm{ebm}(\phi, \bar{x}) = 1$.

[a] 1 is not included because it is impossible that $x \le \sin x$ when $x > 0$ is near 0.

- V_τ's in (1.2) often depend on τ;
- Let $V_\tau^{\max}$ be the maximal one among all possible V_τ's in (1.2), i.e.,

$$V_\tau^{\max} := \bigcup \{V_\tau \in \mathcal{N}(\bar{x}) \mid V_\tau \text{ satisfies } (1.2)\}.$$

  Then, $V_\tau^{\max} \searrow$ as $\tau \nearrow$.
- Outside the level set $[\phi \leq 0]$, there may exist no common point among all these neighborhoods $V_\tau^{\max}$, i.e., one may have

$$\left(\bigcap_{0<\tau<\mathrm{ebm}(\phi,\bar{x})} V_\tau^{\max} \right) \setminus [\phi \leq 0] = \emptyset.$$

Take $\phi(x) = \sqrt{x_+}$ and $\bar{x} = 0$ for example.

- $[\phi \le 0] = -\mathbb{R}_+$ and $d(0, [\phi \le 0]) = x_+$;
- $V_\tau^{\max} = (-\infty, \frac{1}{\tau^2}]$ for all $\tau > 0$;
- (1.2) can be written as $\tau x_+ \le \sqrt{x_+} \ \ \forall x \in (-\infty, \frac{1}{\tau^2}]$;
- $\mathrm{ebm}(\phi, \bar{x}) = +\infty$;
- $\left(\bigcap_{0<\tau<\mathrm{ebm}(\phi,\bar{x})} V_\tau^{\max}\right) \setminus [\phi \le 0] = \emptyset$.

As $\mathrm{ebm}(\phi, \bar{x}) = +\infty$ whenever $x \in \mathrm{int}([\phi \leq 0])$, we assume in what follows that $\bar{x} \in \mathrm{bdry}[\phi \leq 0]$.

A lower estimate via outer limiting subdifferential:

If $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ is lower semicontinuous, we have

$$d(0, \partial^> \phi(\bar{x})) \leq \mathrm{ebm}(\phi, \bar{x}),$$

where

$$\partial^> \phi(\bar{x}) := \left\{ \lim_{k \to +\infty} v_k \mid \exists x_k \to_\phi \bar{x},\ \phi(x_k) > \phi(\bar{x}),\ v_k \in \partial\phi(x_k) \right\}$$

is the outer limiting subdifferential of ϕ at $\bar{x}$. See [Kruger et al (2010), Fabian et al (2010), Iofe (2015)].

Smooth case

If ϕ is smooth, we have $\partial^{>}\phi(\bar{x}) = \nabla\phi(\bar{x})$ and

$$d(0, \partial^{>}\phi(\bar{x})) = \mathrm{ebm}(\phi, \bar{x}).$$

Pointwise max of $\mathcal{C}^1$

If $\phi(x) = \max_{1 \le i \le m} f_i(x)$ with f_i's being smooth, we have

$$d(0, \partial^{>}\phi(\bar{x})) = \mathrm{ebm}(\phi, \bar{x}).$$

Convex and lower semicontinuous case

If ϕ is lower semicontinuous and convex, and $\bar{x} \in \mathrm{bdry}[\phi \le 0]$, we have

$$d(0, \partial^{>}\phi(\bar{x})) = \mathrm{ebm}(\phi, \bar{x}).$$

Polyhedral case

- $\phi(x) = \max_{1 \le i \le m} (\langle a_i, x \rangle - b_i)$ and $\bar{x} \in \mathrm{bdry}[\phi \le 0]$.
- $I(x) := \{i \mid \phi(x) = \langle a_i, x \rangle - b_i\}$ is the active index set.

It was shown in [Cánovas et al (2014), Theorem 4.1] that,

$$\partial^{>}\phi(\bar{x}) = \bigcup_{D \in \mathcal{D}(\bar{x})} \mathrm{conv}\{a_i, i \in D\},$$

where $D \in \mathcal{D}(\bar{x})$ if and only if $D \subset I(\bar{x})$ and there is some $d \in \mathbb{R}^n$ such that

$$\langle a_i, d \rangle = 1 \ \forall i \in D, \ \langle a_i, d \rangle < 1 \ \forall i \in I(\bar{x}) \backslash D.$$

For a nonempty and convex set $A \subset \mathbb{R}^n$, the end set of A is defined in [Hu (2005), Hu (2007)] by

$$\mathrm{end}(A) := \{x \in \mathrm{cl}A | tx \notin \mathrm{cl}A \ \forall t > 1\}.$$

Sublinear case

It was shown by [Hu and Wang (2011), Theorem 4.1] that, if ϕ is a sublinear and lower semicontinuous function, then

$$\mathrm{ebm}(\phi, 0) = d(0, \mathrm{end}(C)),$$

where C is the unique closed and convex set such that $\phi = \sigma_C$. In this case, we also have

$$\mathrm{ebm}(\phi, 0) = d(0, \partial^{>}\phi(0)) = d(0, \partial^{>}\sigma_C(0)).$$

2. Our contributions

2.1. Support function of a compact convex set and its outer limiting subdifferential

Let C be a compact and convex subset of $\mathbb{R}^n$. The support function $\sigma_C : \mathbb{R}^n \to \mathbb{R}$ of C is defined by

$$\sigma_C(w) := \sup_{x \in C} \langle x, w \rangle.$$

- $C = \{v \in \mathbb{R}^n \mid \langle v, w \rangle \leq \sigma_C(w)\ \forall w\}.$
- $$\partial \sigma_C(w) = \arg\max_{v \in C} \langle v, w \rangle = C \cap \{v \in \mathbb{R}^n \mid \langle v, w \rangle = \sigma_C(w)\}.$$
- F is a nonempty exposed face of C if and only if $F = \partial \sigma_C(w)$ for some $w \neq 0$.

Throughout this section, we use the following notation:

$$S := \bigcup_{\sigma_C(w)>0} \partial\sigma_C(w).$$

Theorem 2.1 *The following properties hold:*

(a)

$$S \subset \operatorname{end}(C) \subset \partial^{>}\sigma_C(0) = \operatorname{cl} S,$$

entailing that $\operatorname{cl}(\operatorname{end}(C)) = \partial^{>}\sigma_C(0)$.

(b) *If* C *is a polyhedral set, then*

$$S = \operatorname{end}(C) = \partial^{>}\sigma_C(0) = \operatorname{cl} S,$$

implying that the sets S *and* $\operatorname{end}(C)$ *are both closed.*

(c) $\operatorname{end}(C) = \partial^{>}\sigma_C(0)$ *if and only if* $\operatorname{end}(C)$ *is closed.*

Corollary 2.1 *Assume that $C = \operatorname{conv} A$ for a nonempty and compact set A. In terms of*

$$\mathcal{A} = \{\arg\max_{a\in A}\langle a, w\rangle \mid \max_{a\in A}\langle a, w\rangle > 0\},$$

we have

$$\bigcup_{A'\in\mathcal{A}} \operatorname{conv} A' \subset \operatorname{end}(C) \subset \partial^{>}\sigma_C(0) = \operatorname{cl}\left(\bigcup_{A'\in\mathcal{A}} \operatorname{conv} A'\right). \tag{2.3}$$

If A is a finite set, all the inclusions in (2.3) become equalities, entailing that $d(0, \operatorname{end}(C)) > 0$, see [Zheng and Ng (2004), Hu (2005)].

Remark: $A' \in \mathcal{A}$ if and only if there is some $w \in \mathbb{R}^n$ such that

$$\langle a, w\rangle = 1\,\forall a \in A',\ \langle a, w\rangle < 1\,\forall a \in A\backslash A'.$$

The latter construction was first introduced by [Cánovas et al (2016)] for the case when A is a finite set.

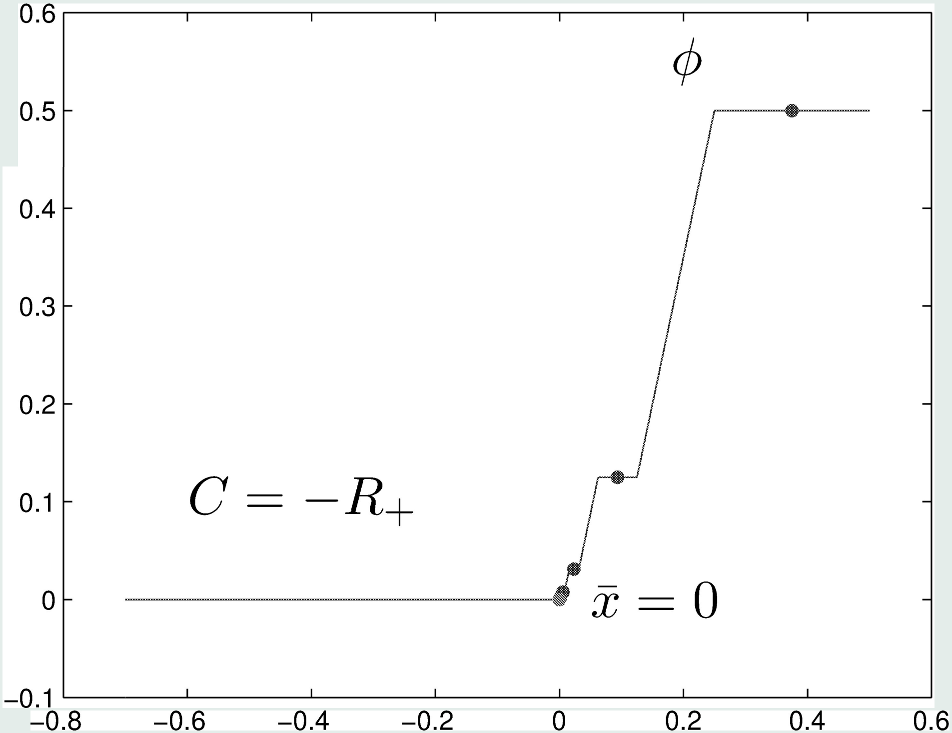
2.2. Lower and upper bounds of error bound moduli for locally Lipschitz and regular functions

Theorem 2.2 *Let* $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *and* $\bar{x} \in \mathrm{bdry}[\phi \leq 0]$. *If* ϕ *is* **regular** *and* **locally Lipschitz** *at* $\bar{x}$, *then*

$$d\left(0, \partial^{>}\phi(\bar{x})\right) \leq \mathrm{ebm}(\phi, \bar{x}) \leq d(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)) \equiv d(0, \mathrm{end}(\partial\phi(\bar{x}))). \tag{2.4}$$

Example 2.1 *[Studnariski and Ward (1999)] (underestimated lower estimate).*

$$\phi(x) = \begin{cases} 0 & \text{if } x \le 0, \\ 2^{-n} & \text{if } 2^{-n-1} \le x \le 2^{-n} \text{ with } n \text{ being an odd integer}, \\ 3x - 2^{-n} & \text{if } 2^{-n-1} \le x \le 2^{-n} \text{ with } n \text{ being an even integer}, \\ x & \text{otherwise}. \end{cases}$$



Let $\bar{x} = 0$. It is clear to see that ϕ is Lipschitz continuous and regular at $\bar{x} = 0$. By some direct calculations, we have

$$\partial\phi(\bar{x}) = \partial^{>}\phi(\bar{x}) = [0, 1]$$

$$\partial^{>}\sigma_{\partial\phi(\bar{x})}(0) = \mathrm{end}(\partial\phi(\bar{x})) = \{1\}$$

$$\mathrm{ebm}(\phi, \bar{x}) = 1.$$

It then follows that

$$0 = d\left(0, \partial^{>}\phi(\bar{x})\right) < \mathrm{ebm}(\phi, \bar{x}) = d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right) = 1.$$

That is, the lower estimate in (2.4) is underestimated.

Example 2.2 *(overestimated upper estimate). Let* $\bar{x} = (0,0)^T$*, and let*

$$\phi(x) = \max\{f_1(x), f_2(x)\},$$

where $f_1(x) = x_1^2 + x_2^2 + \frac{1}{2}(x_1 + x_2)$ *and* $f_2(x) = x_1 + x_2$*. It is clear that* ϕ *is a convex function. Clearly,* $\partial\phi(\bar{x}) = \mathrm{conv}\{(\frac{1}{2}, \frac{1}{2})^T, (1,1)^T\}$*. From Corollary 2.1, it follows that*

$$\partial^{>}\sigma_{\partial\phi(\bar{x})}(0) = \mathrm{end}(\partial\phi(\bar{x})) = \{(1,1)^T\}.$$

But from Remark 3.6 (i) of [Cánovas et al (2016)], we get $\partial^{>}\phi(\bar{x}) = \mathrm{conv}\{(\frac{1}{2}, \frac{1}{2})^T, (1,1)^T\}$*. Therefore,*

$$\frac{\sqrt{2}}{2} = d(0, \partial^{>}\phi(\bar{x})) = \mathrm{ebm}(\phi, \bar{x}) < d(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)) = \sqrt{2}.$$

That is, the upper estimate in (2.4) is overestimated.

2.3. Sharp lower bound for lower-$\mathcal{C}^1$ functions

let ϕ be lower-$\mathcal{C}^1$ on an open subset O of $\mathbb{R}^n$ (cf. [Rockafellar and Wets (1998), Definition 10.29]) and let $\bar{x} \in O$ be a fixed point on the boundary of the level set $[\phi \leq 0]$. Moreover, we assume that on some open neighborhood V of $\bar{x}$ there is a representation

$$\phi(x) = \max_{y \in Y} f(x, y) \tag{2.5}$$

in which the functions $f(\cdot, y)$ are of class $\mathcal{C}^1$ on V and the index set $Y \subset \mathbb{R}^m$ is a compact space such that $f(x, y)$ and $\nabla_x f(x, y)$ depend continuously not just on $x \in V$ but jointly on $(x, y) \in V \times Y$.

Let the active index set mapping $Y : V \rightrightarrows \mathbb{R}^m$ be defined by

$$Y(x) := \{y \in Y | f(x, y) = \phi(x)\}.$$

We introduce two collections of index sets as follows:

$$\mathcal{Y}(\bar{x}) := \left\{ \lim_{k \to +\infty} Y(x_k) \mid x_k \to \bar{x} \text{ and } \phi(x_k) > 0 \ \forall k \right\},$$

$$\mathcal{Y}^{>}(\bar{x}) := \left\{ \arg\max_{y \in Y(\bar{x})} \langle \nabla_x f(\bar{x}, y), w \rangle \,\middle|\, \max_{y \in Y(\bar{x})} \langle \nabla_x f(\bar{x}, y), w \rangle > 0 \right\}.$$

Theorem 2.3 *If ϕ is lower-$\mathcal{C}^1$ on an open subset O of $\mathbb{R}^n$ and $\bar{x} \in O$ is on the boundary of the level set $[\phi \leq 0]$, then*

$$d(0, \partial^{>}\phi(\bar{x})) = \mathrm{ebm}(\phi, \bar{x}) \leq d(0, \mathrm{end}(\partial\phi(\bar{x}))).$$

If, in addition, f has a representation by (2.5), we have

$$\partial^{>}\phi(\bar{x}) = \bigcup_{Y' \in \mathcal{Y}(\bar{x})} \mathrm{conv}\{\nabla_x f(\bar{x}, y) | y \in Y'\},$$

and

$$\mathrm{end}(\partial\phi(\bar{x})) \cong \bigcup_{Y' \in \mathcal{Y}^{>}(\bar{x})} \mathrm{conv}\{\nabla_x f(\bar{x}, y) | y \in Y'\}.$$

2.4. Sharp upper bound for convex functions

Theorem 2.4 *Assume that ϕ is finite and convex on some convex neighborhood of $\bar{x}$. If there is a neighborhood V of $\bar{x}$ such that*

$$[\phi \leq 0] \cap V = (\bar{x} + [d\phi(\bar{x}) \leq 0]) \cap V, \tag{2.6}$$

then the following equalities hold:

$$d\left(0, \partial^{>}\phi(\bar{x})\right) = \mathrm{ebm}(\phi, \bar{x}) = d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right).$$

Consider the linear system

$$\langle a_t, x\rangle \le b_t \quad \forall t \in T, \tag{2.7}$$

where $a_t \in \mathbb{R}^n$, $b_t \in \mathbb{R}$, and T is a compact space such that a_t and b_t depend continuously on $t \in T$.

- $\phi(x) := \max_{t\in T}\{\langle a_t, x\rangle - b_t\}$;
- $T(x) := \{t \in T \mid \langle a_t, x\rangle - b_t = \phi(x)\}$.

According to [Anderson et al (1998)], (2.7) is a locally polyhedral linear system if

$$(\operatorname{pos}\operatorname{conv}\{a_t \mid t \in T(x)\})^* = \operatorname{pos}([\phi \le 0]-x) \quad \forall x \in [\phi \le 0]. \tag{2.8}$$

Corollary 2.2 *Consider the linear system (2.7). If one of the following equivalent properties is satisfied:*

(a) *The regularity condition (2.6) holds for all x in the solution set $[\phi \leq 0]$,*

(b) *The linear system (2.7) is locally polyhedral, i.e., (2.8) holds,*

then,

$$\operatorname{ebm}(\phi, x) = d\left(0, \bigcup_{T' \in \mathcal{T}(x)} \operatorname{conv}\{a_t \mid t \in T'\}\right) \quad \forall x \in [\phi \leq 0], \tag{2.9}$$

where

$$\mathcal{T}(x) := \{T' | \exists w : \langle a_t, w \rangle = 1\, \forall t \in T',\, \langle a_t, w \rangle < 1\, \forall t \in T(x) \backslash T'\}.$$

Remark: As a finite linear system is naturally locally polyhedral, our result below recovers [Cánovas et al (2014), Theorem 4.1] for the case of a finite linear system.

3. Perspectives and open questions

When ϕ is regular and locally Lipschitz continuous on some neighborhood of $\bar{x} \in \mathrm{bdry}([\phi \leq 0])$, we obtained in Theorem 2.2 a lower estimate and an upper estimate of the local error bound modulus $\mathrm{ebm}(\phi, \bar{x})$ as follows:

$$d\left(0, \partial^{>} \phi(\bar{x})\right) \leq \mathrm{ebm}(\phi, \bar{x}) \leq d\left(0, \partial^{>} \sigma_{\partial \phi(\bar{x})}(0)\right).$$

In particular, when ϕ is finite and convex on some convex neighborhood of $\bar{x} \in \mathrm{bdry}([\phi \leq 0])$, we obtained in Theorem 2.4 under the ACQ and ETA properties the following:

$$d\left(0, \partial^{>} \phi(\bar{x})\right) = \mathrm{ebm}(\phi, \bar{x}) = d\left(0, \partial^{>} \sigma_{\partial \phi(\bar{x})}(0)\right),$$

and when ϕ is a lower $\mathcal{C}^1$ functions, we obtained in Theorem 2.3 the following:

$$d\left(0, \partial^{>} \phi(\bar{x})\right) = \mathrm{ebm}(\phi, \bar{x}) \leq d\left(0, \partial^{>} \sigma_{\partial \phi(\bar{x})}(0)\right).$$

One open question is whether the inclusion

$$\partial^{>} \sigma_{\partial \phi(\bar{x})}(0) \subset \partial^{>} \phi(\bar{x}) \tag{3.10}$$

holds or not in the general case or in some particular settings.

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