Math. Program., Ser. A (2018) 171:463–487
https://doi.org/10.1007/s10107-017-1200-1

FULL LENGTH PAPER

# On error bound moduli for locally Lipschitz and regular functions

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Received: 11 August 2016 / Accepted: 2 October 2017 / Published online: 25 October 2017


**Abstract** In this paper we study local error bound moduli for a locally Lipschitz and regular function via outer limiting subdifferential sets. We show that the distance from 0 to the outer limiting subdifferential of the support function of the subdifferential set, which is essentially the distance from 0 to the end set of the subdifferential set, is an upper estimate of the local error bound modulus. This upper estimate becomes tight for a convex function under some regularity conditions. We show that the distance from 0 to the outer limiting subdifferential set of a lower $\mathcal{C}^1$ function is equal to the local error bound modulus.

**Keywords** Error bound modulus · Locally Lipschitz · Outer limiting subdifferential · Support function · End set · Lower $\mathcal{C}^1$ function

**Mathematics Subject Classification** 65K10 · 90C30 · 90C34

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## 1 Introduction

Error bounds play a key role in variational analysis. They are of great importance for subdifferential calculus, stability and sensitivity analysis, exact penalty functions, optimality conditions, and convergence of numerical methods, see the excellent survey papers [2,19,24] for more details. It should be noticed that the notion of error bounds is closely related to some other important concepts: weak sharp minima, calmness and metric subregularity, see [3,4,7,14,16,23,26].

In this paper, we study local error bound moduli in finite dimensional spaces. We say that a function $\phi : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ has a local error bound at $\bar{x} \in [\phi \leq 0]$ if there exist some $\tau > 0$ and some neighborhood $U$ of $\bar{x}$ such that

$$\tau d(x, [\phi \leq 0]) \leq \phi(x)_+ \;\forall x \in U, \tag{1}$$

where $[\phi \leq 0] := \{x \in \mathbb{R}^n | \phi(x) \leq 0\}$ and $t_+ := \max\{t, 0\}$ for all $t \in \mathbb{R}$. The supremum of all possible constants $\tau$ in (1) (for some associated $U$) is called the local error bound modulus of $\phi$ at $\bar{x}$, denoted by $\text{ebm}(\phi, \bar{x})$. We define $\text{ebm}(\phi, \bar{x})$ as 0 if $\phi$ does not have a local error bound at $\bar{x}$. Clearly, the local error bound modulus of $\phi$ at $\bar{x}$ can be alternatively defined as

$$\text{ebm}(\phi, \bar{x}) = \liminf_{x \to \bar{x}, \phi(x) > 0} \frac{\phi(x)}{d(x, [\phi \leq 0])}.$$

We know from the literature [9,15,17] that the distance from 0 to the outer limiting subdifferential of a lower semicontinuous function $\phi$ at $\bar{x}$, is a lower estimate of $\text{ebm}(\phi, \bar{x})$, which becomes tight when $\phi$ is convex.

In this paper we consider the local error bound modulus of a regular function $\phi$ and establish that the distance from 0 to the outer limiting subdifferential of the support function of the subdifferential $\partial\phi(\bar{x})$ at 0 is an upper estimate of $\text{ebm}(\phi, \bar{x})$. With additional Lipschitzian continuity assumption on $\phi$, we also investigate the geometric structure of this outer limiting subdifferential and show that it is equal to the closure of the end set of the subdifferential $\partial\phi(\bar{x})$, while the closure is surplus when the subdifferential set is a polyhedron. Thus the upper estimate is essentially the distance from 0 to the end set of $\partial\phi(\bar{x})$.

It is worth noting that the lower $\mathcal{C}^1$ functions and convex functions are two important examples of regular and locally Lipschitz functions. We prove that, for convex function $\phi$, under the Abadie's constraint qualification [20] and the assumption on exactness of tangent approximations [22] (see Definition 3.1 for their definitions), the upper estimate is tight, that is, $\text{ebm}(\phi, \bar{x})$ is equal to the distance from 0 to the end set of $\partial\phi(\bar{x})$. To the best of our knowledge, the first result of the kind is that Hu [13] proved for sublinear function $\phi$ that, $\text{ebm}(\phi, 0)$ is equal to the distance from 0 to the end set of $\partial\phi(0)$. For lower $\mathcal{C}^1$ function $\phi$, we show that the distance from 0 to the outer limiting subdifferential of $\phi$ at $\bar{x}$ is equal to $\text{ebm}(\phi, \bar{x})$. This generalizes the corresponding results in [9,15,17] for convex function $\phi$.

Throughout the paper we use the standard notations of variational analysis; see the seminal book [26] by Rockafellar and Wets. Let $A \subset \mathbb{R}^n$. We denote the interior, the closure, the boundary, the convex hull and the positive hull of $A$ respectively by int $A$, cl $A$, bdry $A$, conv $A$ and pos$A := \{0\} \cup \{\lambda x | x \in A \text{ and } \lambda > 0\}$.

The Euclidean norm of a vector $x$ is denoted by $\|x\|$, and the inner product of vectors $x$ and $y$ is denoted by $\langle x, y\rangle$. Let $B(x, \varepsilon)$ be a closed ball centered at $x$ with the radius $\varepsilon > 0$. We say that $A$ is locally closed at a point $x \in A$ if $A \cap U$ is closed for some closed neighborhood $U$ of $x$. The polar cone of $A$ is defined by

$$A^* := \left\{v \in \mathbb{R}^n | \langle v, x\rangle \leq 0 \ \forall x \in A\right\}.$$

The support function $\sigma_A : \mathbb{R}^n \to \overline{\mathbb{R}}$ of $A$ is defined by

$$\sigma_A(w) := \sup_{x\in A}\langle x, w\rangle.$$

For a closed and convex set $A$ with $0 \in A$, the gauge of $A$ is the function $\gamma_A : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$\gamma_A(x) := \inf\{\lambda \geq 0 | x \in \lambda A\}.$$

The distance from $x$ to $A$ is defined by

$$d(x, A) := \inf_{y\in A} \|y - x\|.$$

For $A = \emptyset$, we define $d(x, A) = +\infty$. The projection mapping $P_A$ is defined by

$$P_A(x) := \{y \in A | ||y - x|| = d(x, A)\}.$$

Let $x \in A$. We use $T_A(x)$ to denote the tangent cone to $A$ at $x$, i.e. $w \in T_A(x)$ if there exist sequences $t_k \downarrow 0$ and $\{w_k\} \subset \mathbb{R}^n$ with $w_k \to w$ and $x + t_k w_k \in A \ \forall k$. The regular normal cone $\widehat{N}_A(x)$ to $A$ at $x$ is the polar cone of $T_A(x)$. A vector $v \in \mathbb{R}^n$ belongs to the normal cone $N_A(x)$ to $A$ at $x$, if there exist sequences $x_k \to x$ and $v_k \to v$ with $x_k \in A$ and $v_k \in \widehat{N}_A(x_k)$ for all $k$. The set $A$ is said to be regular at $x$ in the sense of Clarke if it is locally closed at $x$ and $\widehat{N}_A(x) = N_A(x)$.

Let $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be an extended real-valued function and $\bar{x}$ a point with $g(\bar{x})$ finite. The epigraph of $g$ is the set

$$\text{epi}g := \left\{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} | g(x) \leq \alpha\right\}.$$

It is well known that $g$ is lower semicontinuous (lsc) if and only if epi$g$ is closed. The vector $v \in \mathbb{R}^n$ is a regular subgradient of $g$ at $\bar{x}$, written $v \in \widehat{\partial} g(\bar{x})$, if

$$g(x) \geq g(\bar{x}) + \langle v, x - \bar{x}\rangle + o\left(\|x - \bar{x}\|\right).$$

The vector $v \in \mathbb{R}^n$ is a (general) subgradient of $g$ at $\bar{x}$, written $v \in \partial g(\bar{x})$, if there exist sequences $x_k \to \bar{x}$ and $v_k \to v$ with $g(x_k) \to g(\bar{x})$ and $v_k \in \widehat{\partial} g(x_k)$. The outer limiting subdifferential of $g$ at $\bar{x}$ is defined in [9,15,17] by

$$\partial^{>} g(\bar{x}) = \left\{ \lim_{k\to+\infty} v_k \mid \exists x_k \xrightarrow[g]{} \bar{x},\ g(x_k) > g(\bar{x}),\ v_k \in \partial g(x_k) \right\}.$$

The subderivative function $dg(\bar{x}) : \mathbb{R}^n \to \overline{\mathbb{R}}$ is defined by

$$dg(\bar{x})(w) := \liminf_{t\downarrow 0, w'\to w} \frac{g(\bar{x}+tw') - g(\bar{x})}{t}\ \forall w \in \mathbb{R}^n.$$

Note that the subderivative $dg(\bar{x})$ is an lsc and positively homogeneous function and that the regular subdifferential set can be derived from the subderivative as follows:

$$\widehat{\partial} g(\bar{x}) = \left\{ v \in \mathbb{R}^n | \langle v, w\rangle \leq dg(\bar{x})(w)\ \forall w \in \mathbb{R}^n \right\}.$$

The function $g$ is said to be (subdifferentially) regular at $\bar{x}$ if epi$g$ is regular in the sense of Clarke at $(\bar{x}, g(\bar{x}))$ as a subset of $\mathbb{R}^n \times \mathbb{R}$.

For a sequence $\{A_k\}$ of subsets of $\mathbb{R}^n$, the outer limit $\limsup_{k\to\infty} A_k$ is the set consisting of all possible cluster points of sequences $x_k$ with $x_k \in A_k$ for all $k$, whereas the inner limit $\liminf_{k\to\infty} A_k$ is the set consisting of all possible limit points of such sequences. $\{A_k\}$ is said to converge to $A \subset \mathbb{R}^n$ in the sense of Painlevé–Kuratowski, written $A_k \to A$, if

$$\limsup_{k\to\infty} A_k = \liminf_{k\to\infty} A_k = A.$$

For a set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and a point $\bar{x} \in \mathbb{R}^n$, the outer limit of $S$ at $\bar{x}$ is defined by

$$\limsup_{x\to\bar{x}} S(x) := \left\{ u \in \mathbb{R}^m | \exists x_k \to \bar{x}, \exists u_k \to u \text{ with } u_k \in S(x_k) \right\}.$$

$S$ is outer semicontinuous (osc, for short) at $\bar{x}$ if and only if

$$\limsup_{x\to\bar{x}} S(x) \subset S(\bar{x}).$$

Let gph$S$ denote the graph of $S$. We recall that $S$ is said to be calm at $(\bar{x}, \bar{y}) \in$ gph$S$ if there exist a constant $\alpha > 0$ and neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{y}$ such that

$$d\left(y, S(\bar{x})\right) \leq \alpha \|x - \bar{x}\|\ \forall x \in U \text{ and } y \in S(x) \cap V.$$

The infimum of all possible constants $\alpha$ (for some associated $U$ and $V$ ) is called the calmness modulus of $S$ at $(\bar{x}, \bar{y})$, denoted as clm$S(\bar{x}, \bar{y})$, defined as $+\infty$ if $S$ is not calm at $(\bar{x}, \bar{y})$.

A face of a convex set $A$ is a convex subset $A'$ of $A$ such that every closed line segment in $A$ with a relative interior point in $A'$ has both endpoints in $A'$. An exposed face of $A$ is the intersection of $A$ and a non-trivial supporting hyperplane to $A$. See [25]. For a nonempty and convex set $A \subset \mathbb{R}^n$, the end set of $A$ is defined in [11,12] by

$$\text{end}(A) := \{x \in \text{cl}A | tx \notin \text{cl}A \ \forall t > 1\}.$$

## 2 Support function and its outer limiting subdifferential

In this section, we study the outer limiting subdifferential of the support function of a general compact and convex set and show that it is the closure of the end set of this compact and convex set. In the next section, we shall apply these results to study the error bound modulus for a locally Lipschitz and regular function as its subdifferential set is compact and convex. It is worth noting that the results presented in this section have their own interest in the field of convex analysis and optimization.

Let $C$ be a compact and convex subset of $\mathbb{R}^n$. To begin with, we note from [26, Theorem 8.24, Proposition 8.29, Corollary 8.25] that

$$C = \left\{v \in \mathbb{R}^n \mid \langle v, w\rangle \le \sigma_C(w) \ \forall w\right\} \tag{2}$$

and

$$\partial\sigma_C(w) = \arg\max_{v\in C}\langle v, w\rangle = C \cap \left\{v \in \mathbb{R}^n \mid \langle v, w\rangle = \sigma_C(w)\right\}. \tag{3}$$

Since $C$ is compact and convex, we have $\sigma_C(w) < +\infty$ and $\partial\sigma_C(w) \neq \emptyset$ for all $w \in \mathbb{R}^n$.

We first consider the case that $0 \in C$ and then the general case that $C$ may not contain 0. Some basic properties of the end set of $C$ are listed in the following lemma.

**Lemma 2.1** *If $C \subset \mathbb{R}^n$ is compact and convex with $0 \in C$, then the following properties hold:*

(i) $\text{end}(C) \cap \text{ri}\, C = \emptyset$;
(ii) $C = \bigcup_{v\in\text{end}(C)}[0, v]$;
(iii) *For a subset $E \subset C$, $C = \bigcup_{v\in E}[0, v]$ if and only if* $\text{end}(C) \subset E$;
(iv) *$F$ is a nonempty exposed face of $C$ if and only if $F = \partial\sigma_C(w)$ for some $w \neq 0$.*

*Proof* (i) Suppose by contradiction that $v \in \text{end}(C) \cap \text{ri}\, C$. By the relative interior criterion [26, Exercise 2.41], there must exist some $v' \in C$ such that $v \in \text{ri}[0, v']$, which contradicts to the fact that $v \in \text{end}(C)$.

(ii) This equality holds because $C$ is convex and compact with $0 \in C$.

(iii) The 'if' part is trivial due to (ii). As for the 'only if' part, we only need to show $\text{end}(C) \subset E$. Let $v \in \text{end}(C)$. Since $C$ is compact, it is clear that $v \in C$. By $C = \bigcup_{v\in E}[0, v]$, there exists some $v' \in E$ such that $v \in [0, v']$. By the definition of the end set, we have $v = v'$. This entails that $\text{end}(C) \subset E$.

(iv) Clearly, any $\partial\sigma_C(w)$ with $w \neq 0$ is an exposed face of $C$. Conversely, if $F \neq \emptyset$ is exposed in $C$, then by definition there exist some $w \neq 0$ and $\alpha \in \mathbb{R}$ such that $F = C \cap \{v \in \mathbb{R}^n \mid \langle w, v\rangle = \alpha\}$ and $C \subset \{v \in \mathbb{R}^n \mid \langle w, v\rangle \le \alpha\}$. The latter inclusion holds if and only if $\sigma_C(w) \le \alpha$. In view of (2) and the fact that $F \subset C$, we have $\alpha = \langle w, v\rangle \le \sigma_C(w)$ for each $v \in F$. This entails that $\alpha = \sigma_C(w)$. In view of (3), we have $F = \partial\sigma_C(w)$. The proof is completed. □

Throughout this section, we use the following notation:

$$S := \bigcup_{\sigma_C(w)>0} \partial\sigma_C(w).$$

According to Lemma 2.1(iv), $S$ is the union of all the exposed faces $\partial\sigma_C(w)$ of $C$ with $\sigma_C(w) > 0$.

In next lemma, we prove that end$(C)$ is sandwiched between $S$ and its closure, the latter being exactly the same with the outer limiting subdifferential $\partial^>\sigma_C(0)$.

**Lemma 2.2** *If $C \subset \mathbb{R}^n$ is compact and convex with $0 \in C$, then*

$$S \subset \mathrm{end}(C) = \gamma_C^{-1}(1) \subset \partial^>\sigma_C(0) = \mathrm{cl}\, S,$$

*entailing that*

$$\mathrm{cl}(\mathrm{end}(C)) = \mathrm{cl}(\gamma_C^{-1}(1)) = \partial^>\sigma_C(0).$$

*Proof* First, we show $\mathrm{end}(C) = \gamma_C^{-1}(1)$ and $\mathrm{cl}\, S = \partial^>\sigma_C(0)$. The first equality follows from the definitions of the end set and the gauge function, while the second equality follows readily from the positive homogeneity of $\sigma_C$ and the definition of outer limiting subdifferential.

Next, we show $S \subset \mathrm{end}(C)$. Let $v \in S$, i.e., $v \in \partial\sigma_C(w)$ for some $w \in \mathbb{R}^n$ with $\sigma_C(w) > 0$. In view of (3), we have $v \in C$ and $\langle v, w\rangle = \sigma_C(w)$, implying that $\langle tv, w\rangle > \sigma_C(w)$ for all $t > 1$. By (2), we have $tv \notin C$ for all $t > 1$. That is, $v \in \mathrm{end}(C)$.

Finally we show $\mathrm{end}(C) \subset \mathrm{cl}\, S$. To begin with, we show that $\mathrm{end}(C) \subset S \cup S^0$, where

$$S^0 = \bigcup_{\sigma_C(w)=0,\ \partial\sigma_C(w)\neq C} \partial\sigma_C(w). \tag{4}$$

Let $v \in \mathrm{end}(C)$. By Lemma 2.1(i), $v \notin \mathrm{ri}\, C$. It then follows from [25, Theorem 11.6] that there exists a non-trivial supporting hyperplane $H$ to $C$ containing $v$. That is, we can find an exposed face $F := C \cap H$ of $C$ such that $v \in F$ and $F \neq C$. By Lemma 2.1(iv), we can find some $w \neq 0$ such that $F = \partial\sigma_C(w)$. This entails that $v \in S \cup S^0$. Therefore, we have $\mathrm{end}(C) \subset S \cup S^0$ as expected. By Lemma 2.1(iii), we have $C = \bigcup_{v\in S\cup S^0}[0, v]$. Observing that $0 \in \partial\sigma_C(w)$ for all $w \in \mathbb{R}^n$ with $\sigma_C(w) = 0$, we have $\bigcup_{v\in S^0}[0, v] = S^0$. This entails that $C = A \cup S^0$, where $A := \bigcup_{v\in S}[0, v]$. Since each $\partial\sigma_C(w)$ with $\sigma_C(w) = 0$ and $\partial\sigma_C(w) \neq C$ is a non-trivial exposed face of $C$, we confirm that $\mathrm{ri}\, C \cap S^0 = \emptyset$ (implying that $\mathrm{ri}\, C \subset A$ and hence $C \subset \mathrm{cl}\, A$). Clearly, we have $A \subset C$ and hence $\mathrm{cl}\, A \subset C$. That is, we have $C = \mathrm{cl}\, A$. On the basis of the fact that $S \subset C$ is bounded, it's easy to verify that $\mathrm{cl}\, A = \bigcup_{v\in \mathrm{cl}\, S}[0, v]$. Thus, we have $C = \bigcup_{v\in \mathrm{cl}\, S}[0, v]$. By Lemma 2.1(iii) again, we have $\mathrm{end}(C) \subset \mathrm{cl}\, S$.

To sum up, we have shown $S \subset \mathrm{end}(C) = \gamma_C^{-1}(1) \subset \partial^{>}\sigma_C(0) = \mathrm{cl}\, S$, which clearly implies that $\mathrm{cl}(\mathrm{end}(C)) = \mathrm{cl}(\gamma_C^{-1}(1)) = \partial^{>}\sigma_C(0)$. The proof is completed. □

*Remark 2.1* The closure operation in the equality $\partial^{>}\sigma_C(0) = \mathrm{cl}(\mathrm{end}(C))$ cannot in general be dropped, because $\partial^{>}\sigma_C(0)$ is always closed but $\mathrm{end}(C)$ may not be closed, taking for example the simple set $C = \{x \in \mathbb{R}^2 \mid 0 \le x_1 \le 1, x_1^2 \le x_2 \le x_1\}$.

Under some further conditions on the faces of $C$, we show that $S$ coincides with $\mathrm{end}(C)$.

**Lemma 2.3** *Assume that $C \subset \mathbb{R}^n$ is compact and convex with $0 \in C$. If, for any $w \in \mathbb{R}^n$ with $\sigma_C(w) = 0$ and $\partial\sigma_C(w) \neq C$, all the faces of $\partial\sigma_C(w)$ that do not contain 0 are exposed in $C$, then*

$$\mathrm{end}(C) = S.$$

*In particular, if $C$ is a polyhedral set, then*

$$S = \mathrm{end}(C) = \gamma_C^{-1}(1) = \partial^{>}\sigma_C(0) = \mathrm{cl}\, S,$$

*implying that the sets $S$, $\mathrm{end}(C)$ and $\gamma_C^{-1}(1)$ are all closed.*

*Proof* We first show the equality $\mathrm{end}(C) = S$ under the assumed conditions on the faces of $C$. Let $v \in \mathrm{end}(C) \cap S^0$, where $S^0$ is given by (4). By the definition of $S^0$, there exists some $w \in \mathbb{R}^n$ with $\sigma_C(w) = 0$ and $\partial\sigma_C(w) \neq C$ such that $v \in \partial\sigma_C(w)$. By [25, Theorem 18.2], there exists a unique face $F$ of $C$ such that $v \in \mathrm{ri}\, F$. By [25, Theorem 18.1], we have $F \subset \partial\sigma_C(w) \subset C$. Clearly, $F$ is also a face of $\partial\sigma_C(w)$. We claim that $0 \notin F$, for otherwise there must exist some $v' \in F$ such that $v \in \mathrm{ri}[0, v']$ (so that $t_0 v \in F \subset C$ for some $t_0 > 1$), contradicting to the assumption that $v \in \mathrm{end}(C)$. That is, $F$ is a face of $\partial\sigma_C(w)$ containing no 0, which is assumed to be exposed in $C$. It then follows from Lemma 2.1(iv) that $F = \partial\sigma_C(w')$ for some $w' \neq 0$. As $0 \notin F$, we have $\sigma_C(w') > 0$, implying that $F \subset S$. Then, we have $v \in S$. This entails that $\mathrm{end}(C) \cap S^0 \subset S$. As we have shown in the proof of Lemma 2.2 that $\mathrm{end}(C) \subset S \cup S^0$ and $S \subset \mathrm{end}(C)$, we get the equality $\mathrm{end}(C) = S$.

To complete the proof, it suffices to note that any polyhedral set has only finitely many faces and all non-trivial faces are exposed ones. The proof is completed. □

*Remark 2.2* Without the conditions imposed on faces of $C$ as in Lemma 2.3, the union set $S$ may not be closed as can be seen from Example 2.1 below, demonstrating that the closure operation in the equality $\partial^{>}\sigma_C(0) = \mathrm{cl}\, S$ cannot in general be dropped, and that the equality $\mathrm{end}(C) = S$ does not hold in general.

*Example 2.1* Let $C = \mathrm{conv}(0 \cup \{x \in \mathbb{R}^2 \mid (x_1 - 1)^2 + (x_2 - 1)^2 \le 1\})$. Clearly, $C$ is a compact and convex set with $0 \in C$. By some direct calculations, we have

$$\mathrm{end}(C) = \left\{x \in \mathbb{R}^2 \mid (x_1 - 1)^2 + (x_2 - 1)^2 = 1, x_1 + x_2 \ge 1\right\}$$

and

$$S = \left\{ x \in \mathbb{R}^2 \mid (x_1 - 1)^2 + (x_2 - 1)^2 = 1, x_1 + x_2 > 1 \right\}.$$

Let $v = (0, 1)^T$ and $w = (-1, 0)^T$. Clearly, we have $v \in \text{end}(C) \backslash S$, $\sigma_C(w) = 0$ and $\partial \sigma_C(w) = \{v \in \mathbb{R}^2 \mid v_1 = 0, 0 \le v_2 \le 1\}$. Moreover, it is easy to verify that the singleton set $\{v\}$ is a face of $\partial \sigma_C(w)$, but is not exposed in $C$.

In the following lemma, we present some equivalent conditions for the equality $\text{end}(C) = \partial^> \sigma_C(0)$ to hold. To this end, we recall from the literature two notions: relative continuity and a radiative set. We say that a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is continuous at a point $\bar{x}$ with $f(\bar{x})$ finite relative to a set $A \subset \mathbb{R}^n$ if, for each $x_k \to \bar{x}$ with $x_k \in A$ for all $k$, it holds that $f(x_k) \to f(\bar{x})$. Let $Q \subset \mathbb{R}^n$ be a nonempty cone. Consider $Q$ as a topological space equipped with the natural topology of a subspace: a fundamental system of neighborhoods of a point $x \in Q$ is the family of sets $N_\epsilon = \{y \in Q \mid \|y - x\| < \epsilon\} (\epsilon > 0)$. We denote by $\text{int}_Q X$ and $\text{bdry}_Q X$ the interior of a set $X \subset Q$ and its boundary in this topology, respectively. According to [27, Definition 4.1], we say that a closed nonempty set $A \subset Q$ is called a radiative subset of $Q$ if $0 \in \text{int}_Q A$ and for each $x \in Q$, the open ray $\{\lambda x \mid \lambda > 0\}$ does not intersect the boundary $\text{bdry}_Q A$ more than once.

**Lemma 2.4** *If $C \subset \mathbb{R}^n$ is compact and convex with $0 \in C$, then the following properties are equivalent:*

(i) $\text{end}(C) = \partial^> \sigma_C(0)$*;*
(ii) $\text{end}(C)$ *is closed;*
(iii) $\gamma_C$ *is continuous at every $x \in \text{pos}\, C$ relative to* $\text{pos}\, C$*;*
(iv) *$C$ is a radiative subset of* $\text{pos}\, C$*.*

*Proof* The equivalence of (i) and (ii) follows directly from Lemma 2.2, while the equivalence of (iii) and (iv) can be found in [27, Proposition 4.2]. It remains to show the equivalence of (ii) and (iii).

[(ii)$\Longrightarrow$(iii)]: Since $0 \notin \text{end}(C)$ and $\text{end}(C)$ is closed, we have $d(0, \text{end}(C)) > 0$. From [22, Theorem 4.1], it then follows that $\text{pos}\, C$ is closed, $\gamma_C$ is continuous at 0 relative to $\text{pos}\, C$, and there is no convergent sequence $\{x_k\} \subset \text{pos}\, C$ such that $\gamma_C(x_k) \to +\infty$. Let $x \in \text{pos}\, C$ with $x \neq 0$ and let $x_k \to x$ with $x_k \in \text{pos}\, C$ for all $k$. Without loss of generality, we may assume that $x_k \neq 0$ for all $k$ and that $\gamma_C(x_k) \to \beta$ (Note that the sequence $\gamma_C(x_k)$ is bounded). As $\gamma_C$ is lower semi-continuous, we have $\beta \geq \gamma_C(x) > 0$. Since $\gamma_C(x_k/\gamma_C(x_k)) = 1$, we have $x_k/\gamma_C(x_k) \in \text{end}(C)$. Since $\text{end}(C)$ is assumed to be closed, we have $x_k/\gamma_C(x_k) \to x/\beta \in \text{end}(C)$. Thus, we have $\gamma_C(x/\beta) = 1$ or $\beta = \gamma_C(x)$. This entails that $\gamma_C$ is continuous at $x$ relative to $\text{pos}\, C$. Therefore, we have (ii)$\Longrightarrow$(iii).

[(iii)$\Longrightarrow$(ii)]: From [22, Theorem 4.1], it follows that $\text{pos}\, C$ is closed and $d(0, \text{end}(C)) > 0$. Let $v_k \to v$ with $v_k \in \text{end}(C)$ (that is, $\gamma_C(v_k) = 1$) for all $k$. It's easy to verify that $v_k \in \text{pos}\, C$ for all $k$ and $v \in \text{pos}\, C$. Moreover, we have $v \neq 0$, for otherwise we have $d(0, \text{end}(C)) = 0$, a contradiction. By (iii), we have $\gamma_C(v_k) \to \gamma_C(v)$, implying that $\gamma_C(v) = 1$ or equivalently $v \in \text{end}(C)$. This entails the closedness of $\text{end}(C)$. The proof is completed. □

Now we present the results similar to the ones in Lemmas 2.2–2.4, for the case when $C$ may not contain $0 \in \mathbb{R}^n$.

**Theorem 2.1** *Let $C \subset \mathbb{R}^n$ be a compact and convex set not necessarily containing 0, and let $C' := \mathrm{conv}(C \cup \{0\})$. Then the following properties hold:*

(a) $S \subset \mathrm{end}(C) = \gamma_{C'}^{-1}(1) \subset \partial^> \sigma_C(0) = \mathrm{cl}\, S$, *entailing that* $\mathrm{cl}(\mathrm{end}(C)) = \mathrm{cl}(\gamma_{C'}^{-1}(1)) = \partial^> \sigma_C(0)$.

(b) *If, for any $w \in \mathbb{R}^n$ with $\sigma_C(w) = 0$ and $\partial\sigma_{C'}(w) \neq C'$, all the faces of $\partial\sigma_{C'}(w)$ that do not contain 0 are exposed in $C'$, then* $\mathrm{end}(C) = S$. *In particular, if $C$ is a polyhedral set, then* $S = \mathrm{end}(C) = \gamma_{C'}^{-1}(1) = \partial^> \sigma_C(0) = \mathrm{cl}\, S$, *implying that the sets $S$,* $\mathrm{end}(C)$ *and* $\gamma_{C'}^{-1}(1)$ *are all closed.*

(c) *The following properties are equivalent:*
  - (c1) $\mathrm{end}(C) = \partial^> \sigma_C(0)$*;*
  - (c2) $\mathrm{end}(C)$ *is closed;*
  - (c3) $\gamma_{C'}$ *is continuous at every $x \in \mathrm{pos}\, C'$ relative to* $\mathrm{pos}\, C'$*;*
  - (c4) $C'$ *is a radiative subset of* $\mathrm{pos}\, C'$.

*Proof* Clearly, $C' = \cup_{0 \le \lambda \le 1} \lambda C$ is a compact and convex set with $0 \in C'$, and $C'$ is polyhedral if $C$ is polyhedral. Moreover, it is easy to verify that $\sigma_{C'}(w) = \max\{\sigma_C(w), 0\}$ for all $w \in \mathbb{R}^n$, and that

$$\partial\sigma_{C'}(w) = \begin{cases} \partial\sigma_C(w) & \text{if } \sigma_C(w) > 0, \\ \cup_{0\le\lambda\le1} \lambda \partial\sigma_C(w) & \text{if } \sigma_C(w) = 0, \\ \{0\} & \text{if } \sigma_C(w) < 0. \end{cases}$$

This entails that $S = \cup_{\sigma_C(w)>0} \partial\sigma_C(w) = \cup_{\sigma_{C'}(w)>0} \partial\sigma_{C'}(w)$. By definition, we have $\mathrm{end}(C') = \mathrm{end}(C)$. All results then follow readily from Lemmas 2.2–2.4. □

By applying Theorem 2.1, we can give some formulas for calculating $\partial^> \sigma_C(0)$ when $C$ is the convex hull of a compact subset of $\mathbb{R}^n$.

**Corollary 2.1** *Let $A$ be a nonempty compact subset of $\mathbb{R}^n$ such that $C = \mathrm{conv}\, A$. In terms of a collection of subsets of $A$ defined by $\mathcal{A} := \{A' \subset A \mid \exists w \in \mathbb{R}^n : A' = \arg\max_{a \in A} \langle a, w\rangle$ and $\max_{a\in A} \langle a, w\rangle > 0\}$, we have*

$$\bigcup_{A' \in \mathcal{A}} \mathrm{conv}\, A' \subset \mathrm{end}(C) = \gamma_C^{-1}(1) \subset \mathrm{cl}\left( \bigcup_{A' \in \mathcal{A}} \mathrm{conv}\, A' \right) = \partial^> \sigma_C(0). \tag{5}$$

*If $A$ is a finite set, all the inclusions in (5) become equalities.*

*Proof* It suffices to show that $A' \in \mathcal{A}$ if and only if there is some $w \in \mathbb{R}^n$ such that $\sigma_C(w) > 0$ and $\mathrm{conv}\, A' = \partial\sigma_C(w)$, and then apply Theorem 2.1 in a straightforward way. □

*Remark 2.3* It is easy to verify that $\mathcal{A}$ can be rewritten as

$$\left\{A' \subset A \mid \exists w \in \mathbb{R}^n \text{ such that } \langle a, w\rangle = 1\, \forall a \in A',\ \langle a, w\rangle < 1\, \forall a \in A \backslash A'\right\},$$

which is in the spirit of the index collection defined in Cánovas et al. [6] for the case that $A$ is a finite set. On the other hand, when $A$ is a finite set, the equalities in (5) provide a complete characterization of the set $\text{end}(C)$. From which, it is easy to see that

$$d(0, \text{end}(C)) > 0. \tag{6}$$

It is worth noting that (6) has been proved in [11,29].

## 3 Main results

In this section, for a given function $\phi : \mathbb{R}^n \to \overline{\mathbb{R}}$, which is regular at $\bar{x}$, a point on the boundary of the level set $[\phi \leq 0]$, we shall conduct some variational analysis on $\text{ebm}(\phi, \bar{x})$, the error bound modulus of $\phi$ at $\bar{x}$. We first show that the distance from 0 to $\partial^{>}\phi(\bar{x})$, the outer limiting subdifferential of $\phi$ at $\bar{x}$, is a lower estimate of $\text{ebm}(\phi, \bar{x})$, while the distance from 0 to $\partial^{>}\sigma_{\partial\phi(\bar{x})}(0)$, the outer limiting subdifferential of $\sigma_{\partial\phi(\bar{x})}$ (the support function of $\partial\phi(\bar{x})$) at 0, is an upper estimate of $\text{ebm}(\phi, \bar{x})$. We then show that the lower estimate is tight for a lower $\mathcal{C}^1$ function and the upper estimate is tight for a convex function under some regularity conditions.

To begin with, we recall that the inequality

$$\text{ebm}(f, x) \geq d\left(0, \partial^{>} f(x)\right) \tag{7}$$

holds for a lsc function $f$ on $\mathbb{R}^n$ and a point $x$ with $f(x)$ finite, and the equality

$$\text{ebm}(f, x) = d\left(0, \partial^{>} f(x)\right) \tag{8}$$

holds if, in addition, $f$ is convex. See [9,15,17].

**Theorem 3.1** *Consider a function $\phi : \mathbb{R}^n \to \overline{\mathbb{R}}$ and a point $\bar{x}$ on the boundary of the level set $[\phi \leq 0]$ with $\phi(\bar{x}) = 0$. If $\phi$ is regular at $\bar{x}$, then*

$$d\left(0, \partial^{>}\phi(\bar{x})\right) \leq \text{ebm}(\phi, \bar{x}) \leq d(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)). \tag{9}$$

*If in addition, $\partial\phi(\bar{x})$ is bounded as is true when $\phi$ is locally Lipschitz continuous at $\bar{x}$, we have*

$$\partial^{>}\sigma_{\partial\phi(\bar{x})}(0) = \text{cl}(\text{end}\,\partial\phi(\bar{x})) = \text{cl}\left(\bigcup_{\sigma_{\partial\phi(\bar{x})}(w)>0} \partial\sigma_{\partial\phi(\bar{x})}(w)\right). \tag{10}$$

*Proof* In view of (7) and the fact that both $d\left(0, \partial^{>}\phi(\bar{x})\right)$ and $\text{ebm}(\phi, \bar{x})$ reflect only local properties of $\phi$ near $\bar{x}$, we get the inequality $d\left(0, \partial^{>}\phi(\bar{x})\right) \leq \text{ebm}(\phi, \bar{x})$ immediately. If $\phi$ is regular at $\bar{x}$, then $\partial\phi(\bar{x})$ is closed and convex, and $d\phi(\bar{x}) = \sigma_{\partial\phi(\bar{x})}$ is lsc and convex. In view of (8), we have $\text{ebm}(d\phi(\bar{x}), 0) = d(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0))$. If in addition $\partial\phi(\bar{x})$ is bounded, (10) can be deduced directly from Theorem 2.1.

It remains to show the inequality $\mathrm{ebm}(\phi, \bar{x}) \leq \mathrm{ebm}(d\phi(\bar{x}), 0)$. This can be done by verifying that if there exist some $\tau > 0$ and some neighborhood $O$ of $\bar{x}$ such that

$$\tau d\,(x, [\phi \leq 0]) \leq \phi(x)_+ \ \forall x \in O, \tag{11}$$

then the following condition holds:

$$\tau d\,(w, [d\phi(\bar{x}) \leq 0]) \leq d\phi(\bar{x})(w)_+ \ \forall w \in \mathbb{R}^n.$$

In what follows, let $w \in \mathbb{R}^n$ be arbitrarily given and let $\kappa(x) := d(x, [\phi \leq 0])$. Assuming (11), we have

$$\begin{aligned}\tau d\kappa(\bar{x})(w) &= \tau \liminf_{t\downarrow 0, w'\to w} \frac{d(\bar{x}+tw', [\phi \leq 0])}{t}\\ &\leq \liminf_{t\downarrow 0, w'\to w} \frac{\phi(\bar{x}+tw')_+}{t} \leq d\phi(\bar{x})(w)_+,\end{aligned}$$

where the second inequality follows from the definition of lower limit. By definition, it is easy to verify that $T_{[\phi\leq 0]}(\bar{x}) \subset [d\phi(\bar{x}) \leq 0]$ and hence $d(w, [d\phi(\bar{x}) \leq 0]) \leq d(w, T_{[\phi\leq 0]}(\bar{x}))$. By [26, Example 8.53], we have $d(w, T_{[\phi\leq 0]}(\bar{x})) = d\kappa(\bar{x})(w)$. Therefore, we have

$$\tau d(w, [d\phi(\bar{x}) \leq 0]) \leq d\phi(\bar{x})(w)_+.$$

This completes the proof. □

In view of Theorem 3.1, when $\phi$ is regular and locally Lipschitz continuous at $\bar{x}$, the upper estimate $d\left(0, \partial^> \sigma_{\partial\phi(\bar{x})}(0)\right)$ in (9) is nothing else but the distance from 0 to the end set of $\partial\phi(\bar{x})$, or equivalently, the distance from 0 to the union of all the exposed faces of $\partial\phi(\bar{x})$ having normal vectors at which the support function $\sigma_{\partial\phi(\bar{x})}$ takes positive values.

However, the following examples show that, even if $\phi$ is regular and locally Lipschitz continuous, the lower estimate and upper estimate in (9) may not be tight, where the first example is taken from [28] (see also [21]) and the second one is taken from [6, Remark 3.6].

*Example 3.1* (underestimated lower estimate). Let $\bar{x} = 0$ and let $\phi : \mathbb{R} \to \mathbb{R}_+$ be defined by

$$\phi(x) = \begin{cases} 0 & \text{if } x \leq 0,\\ 2^{-n} & \text{if } 2^{-n-1} \leq x \leq 2^{-n} \text{ with } n \text{ being an odd integer},\\ 3x - 2^{-n} & \text{if } 2^{-n-1} \leq x \leq 2^{-n} \text{ with } n \text{ being an even integer},\\ x & \text{otherwise}.\end{cases}$$

It is clear to see that $\phi$ is Lipschitz continuous and regular at $\bar{x} = 0$. By some direct calculations, we have $\partial\phi(\bar{x}) = \partial^>\phi(\bar{x}) = [0, 1]$, $\partial^> \sigma_{\partial\phi(\bar{x})}(0) = \mathrm{end}(\partial\phi(\bar{x})) = \{1\}$, and $\mathrm{ebm}(\phi, \bar{x}) = 1$. It then follows that

$$0 = d\left(0, \partial^{>}\phi(\bar{x})\right) < \mathrm{ebm}(\phi, \bar{x}) = d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right) = 1.$$

That is, the lower estimate in (9) is underestimated.

*Example 3.2* (overestimated upper estimate). Let $\bar{x} = (0, 0)^T$, and let

$$\phi(x) = \max\{f_1(x), f_2(x)\},$$

where $f_1(x) = x_1^2 + x_2^2 + \frac{1}{2}(x_1 + x_2)$ and $f_2(x) = x_1 + x_2$. It is clear that $\phi$ is a convex function. Clearly, $\partial\phi(\bar{x}) = \mathrm{conv}\{(\frac{1}{2}, \frac{1}{2})^T, (1, 1)^T\}$. From Corollary 2.1, it follows that $\partial^{>}\sigma_{\partial\phi(\bar{x})}(0) = \mathrm{end}(\partial\phi(\bar{x})) = \{(1, 1)^T\}$. But from Remark 3.6 (i) of [6], we get $\partial^{>}\phi(\bar{x}) = \mathrm{conv}\{(\frac{1}{2}, \frac{1}{2})^T, (1, 1)^T\}$. Therefore,

$$\frac{\sqrt{2}}{2} = d\left(0, \partial^{>}\phi(\bar{x})\right) = \mathrm{ebm}(\phi, \bar{x}) < d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right) = \sqrt{2}.$$

That is, the upper estimate in (9) is overestimated.

### 3.1 Sharp lower estimation for lower $\mathcal{C}^1$ functions

Many functions expressed by pointwise max of infinite collections of smooth functions have the 'subsmoothness' property, which is between local Lipschitz continuity and strict differentiability. Our aim in this subsection is to show that the lower estimate in (9) is a tight one for lower $\mathcal{C}^1$ functions.

Throughout this subsection, let $\phi$ be lower $\mathcal{C}^1$ on an open subset $O$ of $\mathbb{R}^n$ (cf. [26, Definition 10.29]) and let $\bar{x} \in O$ be a fixed point on the boundary of the level set $[\phi \leq 0]$. Moreover, we assume that on some open neighborhood $V$ of $\bar{x}$ there is a representation

$$\phi(x) = \max_{y \in Y} f(x, y) \tag{12}$$

in which the functions $f(\cdot, y)$ are of class $\mathcal{C}^1$ on $V$ and the index set $Y \subset \mathbb{R}^m$ is a compact space such that $f(x, y)$ and $\nabla_x f(x, y)$ depend continuously not just on $x \in V$ but jointly on $(x, y) \in V \times Y$. In what follows, we shall show that the lower estimate $d(0, \partial^{>}\phi(\bar{x}))$ in (9) is equal to the error bound modulus $\mathrm{ebm}(\phi, \bar{x})$.

To begin with, we list some nice properties of $\phi$ as follows (cf. [26, Theorem 10.31]).

(a) $\phi$ is locally Lipschitz continuous and regular on $O$.
(b) $\partial\phi(x) = \mathrm{conv}\{\nabla_x f(x, y) | y \in Y(x)\}$ for all $x \in V$, where $Y : V \rightrightarrows \mathbb{R}^m$ is the active index set mapping defined by

$$Y(x) := \{y \in Y | f(x, y) = \phi(x)\}. \tag{13}$$

(c) $\sigma_{\partial\phi(x)}(w) = d\phi(x)(w) = \max_{y \in Y(x)} \langle \nabla_x f(x, y), w \rangle$ for all $x \in V$ and $w \in \mathbb{R}^n$.

(d) The set-valued mapping $Y$ defined by (13) is outer semicontinuous at $\bar{x}$, i.e.,

$$\limsup_{x \to \bar{x}} Y(x) \subset Y(\bar{x}).$$

Next we obtain some equivalent properties for $\phi$ defined by (12) having a local error bound.

**Proposition 3.1** *Let* $\tau > 0$ *and let*

$$\mathcal{Y}(\bar{x}) := \{Y' \subset Y(\bar{x}) \mid \exists \{x_k\} \subset [\phi > 0] \text{ with } x_k \to \bar{x} \text{ and } Y(x_k) \to Y'\}. \tag{14}$$

*The following properties are equivalent:*

(i) *There exists some* $\varepsilon > 0$ *such that for all* $x \in \mathbb{R}^n$ *with* $\|x - \bar{x}\| \le \varepsilon$,

$$\tau d(x, [\phi \le 0]) \le \phi(x)_+. \tag{15}$$

(ii) *For every* $Y' \in \mathcal{Y}(\bar{x})$, *there exists some* $u \in \mathbb{R}^n$ *with* $\|u\| = 1$ *such that*

$$\langle \nabla_x f(\bar{x}, y), u \rangle \ge \tau \ \forall y \in Y'.$$

(iii) *For every* $Y' \in \mathcal{Y}(\bar{x})$, $d(0, \operatorname{conv}\{\nabla_x f(\bar{x}, y) | y \in Y'\}) \ge \tau$.

(iv) *There exists some* $\delta > 0$ *such that the inequality* $d(0, \partial\phi(x)) \ge \tau$ *holds for all* $x \in \mathbb{R}^n$ *with* $\phi(x) > 0$ *and* $\|x - \bar{x}\| \le \delta$.

*Proof* For the sake of notation simplicity, we use $C$ to denote the level set $[\phi \le 0]$ in what follows. We shall prove step by step that (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii)$\Longrightarrow$(iv)$\Longrightarrow$(i).

[(i)$\Longrightarrow$(ii)]: Assume that there exists some $\varepsilon > 0$ such that (15) holds for all $x \in \mathbb{R}^n$ with $\|x - \bar{x}\| \le \varepsilon$. First, we show that for any $x \in \operatorname{bdry} C \cap B(\bar{x}, \frac{\varepsilon}{2})$ and any proximal normal vector $u$ to $C$ at $x$ with $\|u\| = 1$, there exists some $y \in Y(x)$ such that

$$\langle \nabla_x f(x, y), u \rangle \ge \tau. \tag{16}$$

By the definition of proximal normal vectors, there exist some $x' \in \mathbb{R}^n$ and $\beta > 0$ such that

$$u = \beta(x' - x) \text{ and } x \in P_C(x').$$

Take $\rho := \min\{\frac{\varepsilon}{2}, \|x' - x\|\}$. Then it is easy to verify that

$$x + tu \in B(\bar{x}, \varepsilon) \ \forall t \in (0, \rho] \text{ and } x \in P_C(x + tu) \ \forall t \in (0, \rho].$$

In view of (15), we have

$$\tau t = \tau \|x + tu - x\| = \tau d(x + tu, C) \le \phi(x + tu)_+ \ \forall t \in (0, \rho].$$

Thus, we have $\tau \leq \liminf_{t\to 0_+} \frac{\phi(x+tu)_+ - \phi(x)_+}{t}$. From [26, Theorems 9.16 and 10.31], it follows that $\phi(x)_+$ is locally Lipschitz continuous with

$$\liminf_{t\to 0_+} \frac{\phi(x+tu)_+ - \phi(x)_+}{t} = d\phi(x)(u)_+ = \max\{\max_{y\in Y(x)} \langle \nabla_x f(x,y), u\rangle, 0\}.$$

Therefore, we have $\tau \leq \max\{\max_{y\in Y(x)}\langle \nabla_x f(x,y), u\rangle, 0\}$. In view of $\tau > 0$, we have $\tau \leq \max_{y\in Y(x)}\langle \nabla_x f(x,y), u\rangle$. Since $Y(x)$ is compact, there exists some $y \in Y(x)$ such that (16) holds.

Next, we show (ii) by virtue of the previous result. Let $Y' \in \mathcal{Y}(\bar{x})$. By definition, there exists some sequence $\{x_k'\} \in \mathbb{R}^n \setminus C$ with $x_k' \to \bar{x}$ and $Y(x_k') \to Y'$, entailing that each $y \in Y'$ corresponds to a sequence $y_k' \to y$ such that $y_k' \in Y(x_k')$ for all $k$. Since $C$ is a closed set, there exists some $x_k \in \text{bdry}\, C$ such that $x_k \in P_C(x_k')$. Clearly, $x_k \to \bar{x}$ and $u_k := \frac{x_k' - x_k}{\|x_k' - x_k\|}$ is a proximal normal vector to $C$ at $x_k$. By taking a subsequence if necessary, we can assume that $u_k \to u$, implying that $\|u\| = 1$. In what follows, let $y \in Y'$ be given arbitrarily. To show (ii), it suffices to show

$$\langle \nabla_x f(\bar{x}, y), u\rangle \geq \tau. \tag{17}$$

According to the previous result, we can find some $y_k \in Y(x_k)$ such that for all sufficiently large $k$,

$$\langle \nabla_x f(x_k, y_k), u_k\rangle \geq \tau. \tag{18}$$

Since all $Y(x_k)$ are subsets of the compact set $Y$, by taking a subsequence if necessary, we can assume that $y_k \to \bar{y}$. By the mean value theorem, there is some $\theta_k \in [0, 1]$ such that

$$f\left(x_k', y_k'\right) - f\left(x_k, y_k'\right) = \left\langle \nabla_x f\left(x_k + \theta_k(x_k' - x_k), y_k'\right),\ x_k' - x_k\right\rangle,$$

which, by the continuity of $\nabla_x f$, implies that

$$\begin{aligned}
&\frac{\left|f\left(x_k', y_k'\right) - f\left(x_k, y_k'\right) - \left\langle \nabla_x f(x_k, y_k'), x_k' - x_k\right\rangle\right|}{\|x_k' - x_k\|} \\
&= \frac{\left\langle \nabla_x f\left(x_k + \theta_k(x_k' - x_k), y_k'\right) - \nabla_x f(x_k, y_k'),\ x_k' - x_k\right\rangle}{\|x_k' - x_k\|} \\
&\leq \left\|\nabla_x f\left(x_k + \theta_k(x_k' - x_k), y_k'\right) - \nabla_x f(x_k, y_k')\right\| \to 0.
\end{aligned}$$

Thus, we have

$$\lim_{k\to+\infty} \frac{f\left(x_k', y_k'\right) - f\left(x_k, y_k'\right)}{\|x_k' - x_k\|} = \lim_{k\to+\infty} \left\langle \nabla_x f(x_k, y_k'), u_k\right\rangle = \langle \nabla_x f(\bar{x}, y), u\rangle\,. \tag{19}$$

Similarly, we obtain

$$\lim_{k\to+\infty} \frac{f\left(x_k', y_k\right) - f\left(x_k, y_k\right)}{\|x_k' - x_k\|} = \lim_{k\to+\infty} \langle \nabla_x f(x_k, y_k), u_k\rangle \geq \tau, \tag{20}$$

where the inequality follows from (18). Observing that

$$f\left(x_k', y_k'\right) - f\left(x_k, y_k'\right) \geq \phi(x_k') - \phi(x_k) \geq f(x_k', y_k) - f(x_k, y_k),$$

we get from (19) and (20) that (17) holds. This completes the proof for (i)$\Longrightarrow$(ii).

[(ii)$\Longrightarrow$(iii)]: Let $Y' \in \mathcal{Y}(\bar{x})$. By (ii), there exists some $u \in \mathbb{R}^n$ with $\|u\| = 1$ that

$$\langle u, v\rangle \geq \tau \geq \langle u, w\rangle, \ \forall v \in \mathrm{conv}\{\nabla_x f(\bar{x}, y) | y \in Y'\}, \forall w \in B(0, \tau).$$

Then by a separation argument, we have

$$0 \notin \mathrm{int}\left(\mathrm{conv}\{\nabla_x f(\bar{x}, y) | y \in Y'\} - B(0, \tau)\right),$$

which clearly implies (iii).

[(iii)$\Longrightarrow$(iv)]: Let $\tau' \in (0, \tau)$ be given arbitrarily. First, we shall prove that, there exists some $\delta > 0$ such that for all $x \notin C$ with $\|x - \bar{x}\| \leq \delta$,

$$d\left(0, \mathrm{conv}\left\{\nabla_x f(x, y) | y \in Y(x)\right\}\right) \geq \tau'. \tag{21}$$

Suppose by contradiction that (21) does not hold, i.e., there exists a sequence $\{x_k\} \subset \mathbb{R}^n \setminus C$ with $x_k \to \bar{x}$ and

$$d\left(0, \mathrm{conv}\left\{\nabla_x f(x_k, y) | y \in Y(x_k)\right\}\right) < \tau'.$$

It follows from the Carathéodory theorem that, there exist some $t_k^j \geq 0$ and $y_k^j \in Y(x_k)$ with $j = 1, 2, \cdots, n+1$ such that

$$\Sigma_{j=1}^{n+1} t_k^j = 1 \text{ and } \left\|\Sigma_{j=1}^{n+1} t_k^j \nabla_x f\left(x_k, y_k^j\right)\right\| \leq \tau'. \tag{22}$$

Since $Y(x_k) \subset Y$ for all $k$ and $Y$ is compact, it follows from [26, Theorem 4.18] that $Y(x_k)$ has a subsequence converging to $Y^*$, a subset of $Y$. By taking a subsequence if necessary, we assume that

$$Y(x_k) \to Y^*, \quad t_k^j \to t^j \geq 0, \quad \text{and} \quad y_k^j \to y^j \in Y^*.$$

Since $Y : V \rightrightarrows \mathbb{R}^m$ defined by (13) is osc at $\bar{x}$, it follows from [26, Exercise 5.3] that $Y^* \subset Y(\bar{x})$, entailing that $Y^* \in \mathcal{Y}(\bar{x})$. By (22) and the continuity of $\nabla_x f$, we have

$$\sum_{j=1}^{n+1} t^j = 1 \text{ and } \left\|\sum_{j=1}^{n+1} t^j \nabla_x f(\bar{x}, y^j)\right\| \leq \tau'.$$

Thus, we have $d(0, \mathrm{conv}\{\nabla_x f(\bar{x}, y) | y \in Y^*\}) \leq \tau'$, contradicting to (ii). This contradiction implies that (21) holds. Since $\tau' \in (0, \tau)$ is given arbitrarily, we confirm that there exists some $\delta > 0$ such that the following inequality holds for all $x \notin C$ with $\|x - \bar{x}\| \le \delta$:

$$d\left(0, \operatorname{conv}\{\nabla_x f(x, y) | y \in Y(x)\}\right) \ge \tau. \tag{23}$$

In view of (b), we can reformulate (23) as $d(0, \partial\phi(x)) \ge \tau$.

[(iv)$\Longrightarrow$(i)]: This implication follows readily from [21, Proposition 2.1]. □

*Remark 3.1* When $Y$ is a finite set, the results in Proposition 3.1 can be found in [21, Theorem 2.1]. See also Kummer [18]. In the semi-infinite setting, Proposition 3.1 improves the corresponding results in Henrion and Outrata [10] and Zheng and Yang [31].

Next theorem shows that the lower estimate in (9) is a tight one.

**Theorem 3.2** *The following equalities hold:*

$$\partial^{>}\phi(\bar{x}) = \bigcup_{Y' \in \mathcal{Y}(\bar{x})} \operatorname{conv}\left\{\nabla_x f(\bar{x}, y) | y \in Y'\right\} \tag{24}$$

*and*

$$\operatorname{ebm}(\phi, \bar{x}) = d(0, \partial^{>}\phi(\bar{x})).$$

*Proof* The equality (24) follows readily from the definition of outer limiting subdifferential and the fact that all $Y(x)$ are compact and convex subsets of $Y(\bar{x})$ when $x$ is close enough to $\bar{x}$. The equality $\operatorname{ebm}(\phi, \bar{x}) = d(0, \partial^{>}\phi(\bar{x}))$ follows from (24) and the equivalence of (i) and (iii) in Proposition 3.1.

The upper estimate $d(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0))$ in (9) has an alternative expression in terms of a collection of subsets of the index set $Y(\bar{x})$ defined by

$$\begin{aligned}\mathcal{Y}^{>}(\bar{x}) := \Big\{ Y' \subset Y(\bar{x}) \,\Big|\, \exists w \in \mathbb{R}^n : Y' = \arg\max_{y \in Y(\bar{x})} \langle \nabla_x f(\bar{x}, y), w \rangle, \\ \max_{y \in Y(\bar{x})} \langle \nabla_x f(\bar{x}, y), w \rangle > 0 \Big\}.\end{aligned} \tag{25}$$

By applying Corollary 2.1, we have

$$\begin{aligned}\bigcup_{Y' \in \mathcal{Y}^{>}(\bar{x})} \operatorname{conv}\left\{\nabla_x f(\bar{x}, y) | y \in Y'\right\} \subset \operatorname{end}(\partial\phi(\bar{x})) = \gamma^{-1}_{\partial\phi(\bar{x})}(1) \\ \subset \operatorname{cl}\left(\bigcup_{Y' \in \mathcal{Y}^{>}(\bar{x})} \operatorname{conv}\left\{\nabla_x f(\bar{x}, y) | y \in Y'\right\}\right) = \partial^{>}\sigma_{\partial\phi(\bar{x})}(0),\end{aligned} \tag{26}$$

where each $\operatorname{conv}\{\nabla_x f(\bar{x}, y) | y \in Y'\}$ is an exposed face of $\partial\phi(\bar{x})$, and all the inclusions in (26) become equalities when the index set $Y(\bar{x})$ is finite. Combining Theorems 3.1

and 3.2, we have

$$\begin{aligned}\mathrm{ebm}(\phi,\bar{x}) &= d\left(0, \bigcup_{Y'\in\mathcal{Y}(\bar{x})} \mathrm{conv}\left\{\nabla_x f(\bar{x},y)|y\in Y'\right\}\right)\\ &\le d\left(0, \bigcup_{Y'\in\mathcal{Y}^{>}(\bar{x})} \mathrm{conv}\left\{\nabla_x f(\bar{x},y)|y\in Y'\right\}\right). \qquad (27)\end{aligned}$$

It should be noticed that the index collection $\mathcal{Y}^{>}(\bar{x})$ has been used in [6] for the study of the calmness modulus of a finite $\mathcal{C}^1$ system in the context of right-hand-side perturbations. Taking our results presented in this subsection and [8, Corollary 2] into account, the corresponding results in [6] for the calmness modulus can be slightly improved, as can been seen from Corollary 3.1 below.

**Corollary 3.1** *Consider the parametrized $\mathcal{C}^1$ system*

$$\sigma(b) := \{f_i(x) \le b_i \ \forall i \in I := \{1,\ldots,m\}\},$$

*and its associated feasible set mapping $\mathcal{F}:\mathbb{R}^m \rightrightarrows \mathbb{R}^n$ defined by*

$$\mathcal{F}(b) := \{x\in\mathbb{R}^n \mid f_i(x)\le b_i \ \forall i\in I\},$$

*where $f_i \in \mathcal{C}^1$ and $b_i \in \mathbb{R}$ for all $i \in I$. Let $(\bar{b},\bar{x}) \in \mathrm{gph}\mathcal{F}$ be given such that $\bar{x} \in \mathrm{bdry}\,\mathcal{F}(\bar{b})$. In terms of $\phi(x) := \max_{i\in I}\{f_i(x)-\bar{b}_i\}$, $I(x) := \{i \in I \mid f_i(x)-\bar{b}_i = \phi(x)\}$,*

$$\begin{aligned}\mathcal{I}^{>}(\bar{x}) := \Big\{I' \subset I(\bar{x}) \mid \exists w\in\mathbb{R}^n : \langle\nabla f_i(\bar{x}),w\rangle = 1\,\forall i\in I',\\ \langle\nabla f_i(\bar{x}),w\rangle < 1\,\forall i\in I(\bar{x})\backslash I'\Big\}, \qquad (28)\\ \mathcal{I}^{=}(\bar{x}) := \Big\{I' \subset I(\bar{x}) \mid \exists w\in\mathbb{R}^n\backslash\{0\} : \langle\nabla f_i(\bar{x}),w\rangle = 0\,\forall i\in I',\\ \langle\nabla f_i(\bar{x}),w\rangle < 0\,\forall i\in I(\bar{x})\backslash I'\Big\} \qquad (29)\end{aligned}$$

*and*

$$\mathcal{I}(\bar{x}) := \{I' \subset I(\bar{x}) \mid \exists\,\{x_k\} \subset [\phi > 0] \text{ with } x_k \to \bar{x} \text{ and } I(x_k) \equiv I'\}, \qquad (30)$$

*we have*

$$\begin{aligned}\bigcup_{I'\in\mathcal{I}^{>}(\bar{x})} \mathrm{conv}\left\{\nabla f_i(\bar{x})|i\in I'\right\} &\subseteq \bigcup_{I'\in\mathcal{I}(\bar{x})} \mathrm{conv}\left\{\nabla f_i(\bar{x})|i\in I'\right\}\\ &\subseteq \bigcup_{I'\in\mathcal{I}^{>}(\bar{x})\cup\mathcal{I}^{=}(\bar{x})} \mathrm{conv}\left\{\nabla f_i(\bar{x})|i\in I'\right\}, \qquad (31)\end{aligned}$$

*and*

$$d\left(0, \bigcup_{I'\in\mathcal{I}^{>}(\bar{x})\cup\mathcal{I}^{=}(\bar{x})} \operatorname{conv}\left\{\nabla f_i(\bar{x})|i \in I'\right\}\right) \leq \operatorname{ebm}(\phi, \bar{x})$$
$$= (\operatorname{clm}\mathcal{F}(\bar{b}, \bar{x}))^{-1}$$
$$= d\left(0, \bigcup_{I'\in\mathcal{I}(\bar{x})} \operatorname{conv}\left\{\nabla f_i(\bar{x})|i \in I'\right\}\right)$$
$$\leq d\left(0, \bigcup_{I'\in\mathcal{I}^{>}(\bar{x})} \operatorname{conv}\left\{\nabla f_i(\bar{x})|i \in I'\right\}\right).$$

*Proof* Clearly, $\phi$ is a lower $\mathcal{C}^1$ function. (This is the case where $Y$ in (12) is the index set $\{1,\ldots,m\}$ in the discrete topology, $\mathcal{Y}(\bar{x})$ defined by (14) is the index collection $\mathcal{I}(\bar{x})$, and $\mathcal{Y}^{>}(\bar{x})$ defined by (25) is the index collection $\mathcal{I}^{>}(\bar{x})$.) Moreover, we have $\bar{x} \in \operatorname{bdry}[\phi \leq 0]$ as $\bar{x} \in \operatorname{bdry} \mathcal{F}(\bar{b})$. Therefore, Theorem 3.2 and (27) are applicable, implying in particular that

$$\operatorname{ebm}(\phi, \bar{x}) = d\left(0, \bigcup_{I'\in\mathcal{I}(\bar{x})} \operatorname{conv}\left\{\nabla f_i(\bar{x})|i \in I'\right\}\right)$$
$$\leq d\left(0, \bigcup_{I'\in\mathcal{I}^{>}(\bar{x})} \operatorname{conv}\left\{\nabla f_i(\bar{x})|i \in I'\right\}\right)$$

and

$$\partial^{>}\phi(\bar{x}) = \bigcup_{I'\in\mathcal{I}(\bar{x})} \operatorname{conv}\left\{\nabla f_i(\bar{x})|i \in I'\right\}.$$

In view of [6, Theorem 3.2(i), Corollary 3.1, (6)], we have

$$d\left(0, \bigcup_{I'\in\mathcal{I}^{>}(\bar{x})\cup\mathcal{I}^{=}(\bar{x})} \operatorname{conv}\left\{\nabla f_i(\bar{x})|i \in I'\right\}\right) \leq (\operatorname{clm}\mathcal{F}(\bar{b}, \bar{x}))^{-1} = \operatorname{ebm}(\phi, \bar{x})$$

and

$$\partial^{>}\phi(\bar{x}) \subseteq \bigcup_{I'\in\mathcal{I}^{>}(\bar{x})\cup\mathcal{I}^{=}(\bar{x})} \operatorname{conv}\{\nabla f_i(\bar{x})|i \in I'\}.$$

In view of [8, Corollary 2 ], we have

$$\bigcup_{I' \in \mathcal{I}^{>}(\bar{x})} \text{conv} \left\{ \nabla f_i(\bar{x}) | i \in I' \right\} \subseteq \partial^{>} \phi(\bar{x}).$$

This completes the proof. □

*Remark 3.2* Consider the functions $f_i$'s defined in Example 3.2, and set $I = \{1, 2\}$, $\bar{b} = (0, 0)^T$ and $\bar{x} = (0, 0)^T$. Then, we have $\mathcal{I}^{>}(\bar{x}) = \{\{2\}\}$, $\mathcal{I}^{=}(\bar{x}) = \{\{1, 2\}\}$ and $\mathcal{I}(\bar{x}) = \{\{1\}, \{2\}, \{1, 2\}\}$. This example demonstrates that the first inclusion in (31) cannot be an equality in the general case, and that the index collection $\mathcal{I}(\bar{x})$ may not be included in the index collection $\mathcal{I}^{>}(\bar{x}) \cup \mathcal{I}^{=}(\bar{x})$ even in the case that the second inclusion in (31) becomes an equality.

### 3.2 Sharp upper estimation for convex functions

In the case of $\phi$ being finite and convex on some convex neighborhood of $\bar{x}$, entailing that $\phi$ is regular and locally Lipschitz continuous on some open neighborhood of $\bar{x}$ (cf. [26, Examples 7.27 and 9.14]), the lower estimate in (9) is tight, but the upper estimate in (9) could be overestimated, as has been seen in Example 3.2.

In general, we cannot expect that the upper estimate in (9) is a tight one, unless some regularity conditions are imposed as we have done in the theorem below.

**Definition 3.1** [20,22] Let $\phi$ be finite and convex on some convex neighborhood of $\bar{x} \in [\phi \leq 0]$. We say that (i) the Abadie's constraint qualification (ACQ, for short) holds for the level set $[\phi \leq 0]$ at $\bar{x}$ if

$$[d\phi(\bar{x}) \leq 0] = T_{[\phi \leq 0]}(\bar{x}); \tag{32}$$

(ii) the level set $[\phi \leq 0]$ admits exactness of tangent approximation (ETA, for short) at $\bar{x}$ if there exists some neighborhood $V$ of $\bar{x}$ such that

$$[\phi \leq 0] \cap V = \left(\bar{x} + T_{[\phi \leq 0]}(\bar{x})\right) \cap V. \tag{33}$$

Let $\bar{x} \in [\phi \leq 0]$. By the definitions of tangent cone and subderivative, we can easily verify that $[\phi \leq 0] \subset \bar{x} + T_{[\phi \leq 0]}(\bar{x})$ and $T_{[\phi \leq 0]}(\bar{x}) \subset [d\phi(\bar{x}) \leq 0]$. Thus, the ACQ and ETA properties amount to the following regularity condition:

$$[\phi \leq 0] \cap V = (\bar{x} + [d\phi(\bar{x}) \leq 0]) \cap V. \tag{34}$$

**Theorem 3.3** *Assume that $\phi$ is finite and convex on some convex neighborhood of $\bar{x}$. If ACQ holds at $\bar{x}$ and the level set $[\phi \leq 0]$ admits ETA at $\bar{x}$, then the following equalities hold:*

$$d\left(0, \partial^{>} \phi(\bar{x})\right) = \text{ebm}(\phi, \bar{x}) = d\left(0, \partial^{>} \sigma_{\partial \phi(\bar{x})}(0)\right). \tag{35}$$

*Proof* In view of (8) and the assumption that $\phi$ is finite and convex on some convex neighborhood of $\bar{x}$, we get the first equality in (35) immediately. It remains to show that the second equality holds. Without loss of generality, we assume that there exists an open ball $O := \{x \in \mathbb{R}^n \mid \|x\| < \delta\}$ of radius $\delta > 0$ such that $\phi$ is finite and convex on $\bar{x} + O$ and that $V$ in (33) can be replaced by $\bar{x} + O$. As $\phi$ is assumed to be finite and convex on some convex neighborhood of $\bar{x}$, it follows from [26, Examples 7.27 and 9.14, Theorem 9.16] that $\phi$ is regular and locally Lipschitz continuous on some open neighborhood of $\bar{x}$, and hence that $\sigma_{\partial\phi(\bar{x})} = d\phi(\bar{x})$ and

$$\text{ebm}(d\phi(\bar{x}), 0) = \text{ebm}(\sigma_{\partial\phi(\bar{x})}, 0) = d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right). \tag{36}$$

Moreover, we get from Theorem 3.1 that $\text{ebm}(\phi, \bar{x}) \leq d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right)$. In the case of $d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right) = 0$, the second equality in (35) holds trivially. So in what follows we assume that $d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right) > 0$.

Let $0 < \tau < d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right)$. In view of (36) and the positive homogeneity of $d\phi(\bar{x})$, the following condition holds:

$$\tau d(w, [d\phi(\bar{x}) \leq 0]) \leq d\phi(\bar{x})(w)_+ \ \forall w \in \mathbb{R}^n. \tag{37}$$

Let $x \in \bar{x} + \frac{1}{2}O$ be arbitrarily chosen. It is straightforward to verify that

$$d\left(x - \bar{x}, T_{[\phi \leq 0]}(\bar{x})\right) = d\left(x - \bar{x}, T_{[\phi \leq 0]}(\bar{x}) \cap O\right) = d\left(x, \left(\bar{x} + T_{[\phi \leq 0]}(\bar{x})\right) \cap (\bar{x} + O)\right),$$

and

$$d\,(x, [\phi \leq 0]) = d\,(x, [\phi \leq 0] \cap (\bar{x} + O)).$$

In view of (33), we have

$$d(x, [\phi \leq 0]) = d\left(x - \bar{x}, T_{[\phi \leq 0]}(\bar{x})\right),$$

which implies by (32) that

$$d(x, [\phi \leq 0]) \leq d\,(x - \bar{x}, [d\phi(\bar{x}) \leq 0]).$$

By (37), we have

$$\tau d(x, [\phi \leq 0]) \leq d\phi(\bar{x})(x - \bar{x})_+. \tag{38}$$

Since $\phi$ is finite and convex on $\bar{x} + O$, we get from [26, Proposition 8.21] that

$$d\phi(\bar{x})(x - \bar{x}) \leq \phi(x) - \phi(\bar{x}) = \phi(x). \tag{39}$$

In view of (38) and (39), we have $\tau d(x, [\phi \leq 0]) \leq \phi(x)_+$. Since $x \in \bar{x} + \frac{1}{2}O$ is chosen arbitrarily, we thus have $\tau \leq \text{ebm}(\phi, \bar{x})$, entailing that $d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right) \leq \text{ebm}(\phi, \bar{x})$. This completes the proof. ◻

*Remark 3.3* It turns out in the last section that, the outer limiting subdifferential set $\partial^{>}\sigma_{\partial\phi(\bar{x})}(0)$, unlike the outer limiting subdifferential set $\partial^{>}\phi(\bar{x})$, depends on the nominal point $\bar{x}$ only and does not get the nearby points involved. As can be seen from Theorem 3.3, it is the ETA property that makes it possible for $d(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0))$ to serve as the error bound modulus $\mathrm{ebm}(\phi, \bar{x})$ which normally depends on not only $\bar{x}$ but its nearby points. Note that the idea of using the ETA property has already appeared in Zheng and Ng [30] and that various characterizations of the ETA property have been presented in [22]. If the ETA property (33) does not hold, the upper estimate $d(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0))$ may be overestimated as can be seen from Example 3.2, in which $[\phi \leq 0] = \{x \in \mathbb{R}^2 | x_1^2 + x_2^2 + \frac{1}{2}(x_1 + x_2) \leq 0\}$ and the ETA property does not hold at any $x \in [\phi \leq 0]$.

*Remark 3.4* From [9,15,17], it is clear that the lower estimate in (9) is tight for convex function $\phi$, which can also be obtained from the results for lower $\mathcal{C}^1$ functions presented in the last subsection as convex functions are lower $\mathcal{C}^1$ functions. In this paper, we provide an upper estimate for $\mathrm{ebm}(\phi, \bar{x})$ when $\phi$ is regular and show that this upper estimate is tight for a convex function under some regularity conditions, which recovers the corresponding results in [5] and [13], where $\phi$ is considered as, respectively, the pointwise max of a finite collection of affine functions, and a sublinear function.

In the remainder of this subsection, we apply Theorem 3.3 to the linear system

$$\langle a_t, x\rangle \leq b_t \ \forall t \in T, \tag{40}$$

where $a_t \in \mathbb{R}^n$, $b_t \in \mathbb{R}$, and $T$ is a compact space such that $a_t$ and $b_t$ depend continuously on $t \in T$. In what follows, let $\phi(x) := \max_{t\in T}\{\langle a_t, x\rangle - b_t\}$ and let $T(x) := \{t \in T \mid \langle a_t, x\rangle - b_t = \phi(x)\}$. Clearly, the level set $[\phi \leq 0]$ is the solution set of the linear system (40), and the regularity condition (34) specified for $x \in [\phi \leq 0]$ can be reformulated as

$$\{y \mid \langle a_t, y\rangle \leq b_t \ \forall t \in T\} \cap V = (x + \{w \mid \langle a_t, w\rangle \leq 0 \ \forall t \in T(x)\}) \cap V, \tag{41}$$

where $V$ is a neighborhood of $x$.

Our first result for the linear system (40) assumes the regularity condition (41) on one nominal point in the solution set only.

**Corollary 3.2** *Consider a solution $x$ to the linear system (40). If the regularity condition (41) holds, then*

$$\begin{aligned} d\left(0, \partial^{>}\phi(x)\right) = \mathrm{ebm}(\phi, x) &= d\left(0, \partial^{>}\sigma_{\partial\phi(x)}(0)\right) \\ &= d\left(0, \bigcup_{T'\in\mathcal{T}(x)} \mathrm{conv}\left\{a_t \mid t \in T'\right\}\right), \end{aligned} \tag{42}$$

*where*

$$\mathcal{T}(x) := \{T' \subset T(x) \mid \exists w \in \mathbb{R}^n : \langle a_t, w\rangle = 1 \ \forall t \in T',\ \langle a_t, w\rangle < 1 \ \forall t \in T(x)\backslash T'\}.$$

*Proof* Applying Theorem 3.3, we get the first two equalities in (42). Applying Corollary 2.1, we get the third equality in (42) by taking Remark 2.3 into account. This completes the proof. □

Our second result for the linear system (40) assumes the regularity condition (41) on the whole solution set, leading to a locally polyhedral linear system as defined in [1], which requires that

$$(\text{pos conv}\{a_t \mid t \in T(x)\})^* = \text{pos}([\phi \le 0] - x)\ \forall x \in [\phi \le 0]. \tag{43}$$

As a finite linear system is naturally locally polyhedral, our result below recovers [5, Theorem 4.1] for the case of a finite linear system.

**Corollary 3.3** *Consider the linear system (40). The equalities in (42) hold for all* $x \in [\phi \le 0]$ *if one of the following equivalent properties is satisfied:*

(a) *The regularity condition* (41) *holds for all* $x$ *in the solution set* $[\phi \le 0]$*;*
(b) *The linear system* (40) *is locally polyhedral, i.e.,* (43) *holds.*

*Proof* It suffices to show the equivalence of (a) and (b). To begin with, we point out that $d\phi(x)(w) = \max_{t\in T(x)}\langle a_t, w\rangle$ as can be seen from [26, Theorem 10.31], and that $[\phi \le 0]$ is convex (implying that $T_{[\phi\le 0]}(x) = \text{cl pos}([\phi \le 0] - x)$). Moreover, we have

$$[\phi \le 0] - x \subset \text{pos}([\phi \le 0] - x) \subset T_{[\phi\le 0]}(x) \subset [d\phi(x) \le 0], \tag{44}$$

and

$$\begin{aligned}(\text{pos conv}\{a_t \mid t \in T(x)\})^* &= \{a_t \mid t \in T(x)\}^* \\ &= \{w \in \mathbb{R}^n \mid \langle a_t, w\rangle \le 0\ \forall t \in T(x)\} \\ &= [d\phi(x) \le 0].\end{aligned} \tag{45}$$

First, we show $(b) \Longrightarrow (a)$. Condition (43) implies that $\text{pos}([\phi \le 0]-x)$ is closed for all $x \in [\phi \le 0]$. In view of [22, Proposition 4.1], the level set $[\phi \le 0]$ admits the ETA property (33) at every $x \in [\phi \le 0]$. By (43) and (45), the ACQ (32) holds for all $x \in [\phi \le 0]$. Thus, the regularity condition (41) holds for all $x \in [\phi \le 0]$.

Now we show $(a) \Longrightarrow (b)$. Let $x \in [\phi \le 0]$. Assume that the regularity condition (41) holds at $x$. It then follows from (44) that

$$\text{pos}([\phi \le 0] - x) = [d\phi(x) \le 0],$$

which together with (45) implies (43). This completes the proof. □

To end this subsection, we illustrate two examples selected from [5]. By Example 3.3, we demonstrate that (42) may not hold if the linear system (40) is not locally polyhedral, and by Example 3.4, we demonstrate that (42) may still hold even if the linear system (40) is not locally polyhedral.

*Example 3.3* Let $\bar{x} = (1, 0)^T$ and $\phi(x) = \max_{t\in T}\{\langle a_t, x\rangle - b_t\}$, where $T = [0, 2\pi]$, $a_t = (t\cos t, t\sin t)^T$ and $b_t = t$. Clearly, $[\phi \le 0] = \{x \in \mathbb{R}^n \mid \|x\| \le 1\}$. Thus,

$\text{pos}([\phi \leq 0] - \bar{x})$ is not closed, implying that (43) does not hold at $\bar{x}$ and that the linear system (40) cannot be locally polyhedral. From Example 1 of [5], it follows that $d\,(0, \partial^{>}\phi(\bar{x})) = \text{ebm}(\phi, \bar{x}) = 0$. Observing that $T(\bar{x}) = \{0, 2\pi\}$ and $\mathcal{T}(\bar{x}) = \{\{2\pi\}\}$, we get

$$d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right) = d\left(0, \bigcup_{T' \in \mathcal{T}(\bar{x})} \text{conv}\left\{a_t \mid t \in T'\right\}\right) = 2\pi.$$

That is, the upper estimate $d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right)$ is overestimated.

*Example 3.4* Let $\bar{x} = (1, 0)^T$ and $\phi(x) = \max_{t \in T}\{\langle a_t, x\rangle - b_t\}$, where $T = [0, 2\pi]$, $a_t = (\cos t, \sin t)^T$ and $b_t = 1$. Clearly, $[\phi \leq 0] = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$. Thus, $\text{pos}([\phi \leq 0] - \bar{x})$ is not closed, implying that (43) does not hold at $\bar{x}$ and that the linear system (40) cannot be locally polyhedral. By some direct calculations, we have $T(\bar{x}) = \{0, 2\pi\}$, $\mathcal{T}(\bar{x}) = \{\{0, 2\pi\}\}$, and

$$d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right) = d\left(0, \bigcup_{T' \in \mathcal{T}(\bar{x})} \text{conv}\left\{a_t \mid t \in T'\right\}\right) = 1.$$

Moreover, we have $\partial\phi(\bar{x}) = \text{conv}\{a_t \mid t \in T(\bar{x})\} = (1, 0)^T$ and hence $\partial^{>}\phi(\bar{x}) = (1, 0)^T$, entailing that

$$d\left(0, \partial^{>}\phi(\bar{x})\right) = \text{ebm}(\phi, \bar{x}) = 1.$$

That is, (42) still holds even when the linear system (40) is not locally polyhedral.

## 4 Conclusions and perspectives

When $\phi$ is regular at some $\bar{x}$ with $\phi(\bar{x}) = 0$, we obtained in Theorem 3.1 a lower estimate and an upper estimate of the local error bound modulus $\text{ebm}(\phi, \bar{x})$ as follows:

$$d\left(0, \partial^{>}\phi(\bar{x})\right) \leq \text{ebm}(\phi, \bar{x}) \leq d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right).$$

In particular, when $\phi$ is finite and convex on some convex neighborhood of $\bar{x}$, we obtained in Theorem 3.3 under the ACQ and ETA properties the following:

$$d\left(0, \partial^{>}\phi(\bar{x})\right) = \text{ebm}(\phi, \bar{x}) = d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right),$$

and when $\phi$ is a lower $\mathcal{C}^1$ function, we obtained in Theorem 3.2 the following:

$$d\left(0, \partial^{>}\phi(\bar{x})\right) = \text{ebm}(\phi, \bar{x}) \leq d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right).$$

One open question is whether the inclusion

$$\partial^{>} \sigma_{\partial\phi(\bar{x})}(0) \subset \partial^{>} \phi(\bar{x}) \tag{46}$$

holds or not when $\phi$ is regular at $\bar{x}$. By trying to find answers to this open question, one may need to look into the differential structure of the functions in question and need to apply some delicate modern variational tools. It is worth noting that [6, Theorem 3.1] shows that (46) holds as an equality when $\phi$ is the pointwise max of a finite collection of affine functions. Moreover, when $\phi$ is regular and locally Lipschitz on some neighborhood of $\bar{x}$, this open question has affirmatively been answered in [8, Remark 2].

**Acknowledgements** The authors would like to thank the two anonymous reviewers for their valuable comments and suggestions, which have helped to improve the paper. Li's work was supported in part by the National Natural Science Foundation of China (Grant number: 11301418), the Natural Science Foundation of Chongqing Municipal Science and Technology Commission (Grant number: cstc2016jcyjA0141), the science and technology research program of Chongqing Education Commission of China (Grant number: KJ1601102), and the Foundation for High-level Talents of Chongqing University of Art and Sciences (Grant number: R2016SC13). Meng's work was supported in part by the National Natural Science Foundation of China (11671329, 31601066, 71601162). Yang's work was supported in part by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. PolyU 152167/15E).

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