FULLY PIECEWISE LINEAR VECTOR OPTIMIZATION PROBLEM *

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Abstract. We distinguish two kinds of piecewise linear functions and provide an interesting representation for a piecewise linear function between infinite dimensional spaces. Based on such a representation, we study a fully piecewise linear vector optimization (PLP) with the objective and constraint functions being piecewise linear. We divide (PLP) into some linear subproblems and establish a finite dimensional reduction method to solve (PLP). Under some mild assumptions, we prove that the Pareto (resp. weak Pareto) solution set of (PLP) is the union of finitely many generalized polyhedra (resp. polyhedra), each of which is or is contained in a Pareto (resp. weak Pareto) face of some linear subproblem. Our main results are even new in the linear case and further generalize Arrow, Barankin and Blackwell's classical results on linear vector optimization problems in the framework of finite dimensional spaces.

Key words. *Polyhedron, piecewise linear function, Pareto solution, weak Pareto solution.*

AMS subject classifications. *52B60, 52B70, 90C29*

1. Introduction. Though vector optimization is often encountered in theory and practical application, the study of nonlinear vector optimization is far from deep and systemic (possibly because the vector ordering is much more complicated than the scalar one). On the other hand, linear vector optimization has been well studied (cf. [2, 4, 7, 8, 12, 13, 15, 21] and the references therein). In particular, in the finite-dimensional case, Arrow, Barankin and Blackwell [3] established the structure of the Pareto solution set and weak Pareto solution set of a linear vector optimization problem. However the linearity assumption is quite restrictive in both theory and application. To overcome the restriction of linearity, one sometimes adopts the piecewise linear functions (cf. [6, 20, 22]). The family of all piecewise linear functions is much larger than that of all linear functions and there exists a wide class of functions that can be approximated by piecewise linear functions. Therefore, from the viewpoint of theoretical interest as well as for applications, it is important to study piecewise linear problems. Given two normed spaces X and Y, the following piecewise linearity of a vector-valued function $f : X \to Y$ was adopted in the literature (cf. [20, 23]): *there exist finitely many polyhedra $\Lambda_1, \cdots, \Lambda_m$ in the product $X \times Y$ such that*

$$\mathrm{gph}(f) := \{(x, f(x)) : x \in X\} = \bigcup_{i=1}^{m} \Lambda_i. \tag{1.1}$$

*This research was supported by the National Natural Science Foundation of P. R. China (Grant No. 11771384) and the Research Grants Council of Hong Kong (PolyU 152128/17E).

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Throughout this paper, we will use $\mathcal{P}(Z)$ to denote the family of all polyhedra in a normed space Z. Another kind of piecewise linearity for a function f is as follows: *there exist* $T_i \in \mathcal{L}(X,Y)$, $P_i \in \mathcal{P}(X)$ *and* $b_i \in Y$ $(i = 1, \cdots, m)$ *such that*

$$X = \bigcup_{i=1}^{m} P_i \text{ and } f(x) = T_i(x) + b_i \quad \forall x \in P_i,\ i = 1, \cdots, m, \tag{1.2}$$

where $\mathcal{L}(X,Y)$ denotes the space of all continuous linear operators from X to Y. For convenience, let $\mathcal{PL}_1(X,Y)$ (resp. $\mathcal{PL}(X,Y)$) denote the family of all piecewise linear functions from X to Y in the sense of (1.1) (resp. (1.2)). It is clear that $\mathcal{L}(X,Y)$ is always contained in $\mathcal{PL}(X,Y)$; however if Y is infinite dimensional then every linear operator in $\mathcal{L}(X,Y)$ must not be in $\mathcal{PL}_1(X,Y)$. This motivates us to study the relationship between $\mathcal{PL}_1(X,Y)$ and $\mathcal{PL}(X,Y)$. To do this, we first consider polyhedra in infinite dimensional spaces. In Section 2, we provide several properties on polyhedra in infinite dimensional spaces. In particular, with the help of the notion of a prime generator group of a polyhedron (cf. [5, 18, 9]), we establish some results on the maximal faces of a polyhedron, which not only play a key role in the proof of the main theorem on piecewise linear functions but also should be valuable by themselves. In Section 3, using the results obtained in Section 2, we prove that

$$\dim(Y) < \infty \Leftrightarrow \mathcal{PL}_1(X,Y) = \mathcal{PL}(X,Y) \quad \text{and} \quad \dim(Y) = \infty \Leftrightarrow \mathcal{PL}_1(X,Y) = \emptyset.$$

As one of the mains results, we prove that for each $f \in \mathcal{PL}(X,Y)$ there exist two closed subspaces X_1 and X_2 of X, a closed subspace Y_2 of Y, $T \in \mathcal{L}(X_1,Y)$ and $g \in \mathcal{PL}_1(X_2,Y_2)$ such that $X = X_1 \oplus X_2$, $\dim(X_2) < \infty$, $\dim(Y_2) < \infty$ and

$$f(x_1 + x_2) = Tx_1 + g(x_2) \ \ \forall (x_1, x_2) \in X_1 \times X_2.$$

In Sections 4 and 5, we consider a fully piecewise linear vector optimization problem in the framework of infinite dimensional spaces. In the case when $f \in \mathcal{PL}(X,Y)$ and $\varphi_k \in \mathcal{PL}(X,\mathbb{R})$ $(k \in \overline{1m} := \{1, \cdots, m\})$, we study the structure of the (weak) Pareto solution set of the following fully piecewise linear vector optimization problem

$$C - \mathrm{Min} f(x) \ \text{ subject to } \varphi_k(x) \le 0,\ i = 1, \cdots, m, \tag{PLP}$$

where C is a closed convex cone in Y. In the case of finite dimensional spaces, the following well known result on the solution sets for linear vector optimization problems is based on the poineering work by Arrow et al. [3] (also see [12, Theorem 3.3] and [13, Theorems 4.1.20 and 4.3.8])

THEOREM 1.1. *Let* $X = \mathbb{R}^p$, $Y = \mathbb{R}^q$, $C = \mathbb{R}^q_+$, $f(x) = T(x) + b$ *and* $\varphi_k(x) = \langle x_k^*, x\rangle + r_k$ *for some* $T \in \mathcal{L}(X,Y)$, $x_k^* \in X^* = \mathcal{L}(X,\mathbb{R})$ *and* $(b, r_k) \in Y \times \mathbb{R}$ $(k \in \overline{1m})$. *Then the Pareto solution set and weak Pareto solution set of (PLP) are the union of finitely many faces of* A, *where* $A := \{x \in X :\ \varphi_k(x) \le 0,\ k = 1, \cdots, m\}$ *is the feasible set of (PLP).*

In the case when the objective f is further piecewise linear, several authors studied the structure of the Pareto solution set and weak Pareto soluiton set and proved that if the objective f is restricted in $\mathcal{PL}_1(X,Y)$ and each φ_k is linear then the weak Pareto solution set of the corresponding piecewise linear problem (PLP) is the union of finitely many polyhedra, while its Pareto solution set is the union of generalized polyhedra (cf. [23, 20, 21, 6] and the references therein). Noting that $\mathcal{PL}_1(X,Y)=\emptyset$ when $\dim(Y)=\infty$, in the case when $f\in\mathcal{PL}(X,Y)$ with $\dim(Y)=\infty$ and each $\varphi_i\in\mathcal{PL}(X,\mathbb{R})$, we will establish the structure of the Pareto solution set and weak Pareto solution set for fully piecewise linear vector oprimization problem (PLP). To the best of our knowledge, these results are new even in the case when each φ_k is linear.

2. Polyhedra in an infinite dimensional space. Let Z be a normed space with the dual space Z^*. Recall (cf.[1, 16]) that a subset P of Z is a (convex) polyhedron if there exist $u_1^*,\cdots,u_m^*\in Z^*$ and $s_1,\cdots,s_m\in\mathbb{R}$ such that

$$P=\{x\in Z:\ \langle u_i^*,x\rangle\le s_i,\ i=1,\cdots,m\}.$$

An exposed face of P is a set F such that

$$F=\{u\in P:\ \langle x^*,u\rangle=\sup_{x\in P}\langle x^*,x\rangle\}$$

for some $x^*\in Z^*$ (cf. [16, P.162]). It is known that each polyhedron has finitely many exposed faces. We say that a subset $\tilde{P}$ of Z is a generalized polyhedron if there exist a polyhedron P in Z, $v_1^*,\cdots,v_k^*\in Z^*$ and $t_1,\cdots,t_k\in\mathbb{R}$ such that

$$\tilde{P}=P\cap\{z\in Z:\ \langle v_i^*,z\rangle<t_i,\ 1\le i\le k\}.$$

Given $z^*\in Z^*\setminus\{0\}$, let $\mathcal{N}(z^*)$ denote the null space of z^*, that is,

$$\mathcal{N}(z^*):=\{z\in Z:\langle z^*,z\rangle=0\}.$$

Then $\mathcal{N}(z^*)$ is a closed subspace of Z with codimension $\mathrm{codim}(\mathcal{N}(z^*))=1$.

Recall that a normed space Z is a direct sum of its two closed subspaces Z_1 and Z_2, denoted by $Z=Z_1\oplus Z_2$, if $Z_1\cap Z_2=\{0\}$ and $Z=Z_1+Z_2$. It is easy to verify that if $Z=Z_1\oplus Z_2$ then for each $z\in Z$ there exists a unique $(z_1,z_2)\in Z_1\times Z_2$ such that $z=z_1+z_2$ and the projection mapping $\Pi_{Z_2}:\ Z=Z_1\oplus Z_2\to Z_2$ is linear, where

$$\Pi_{Z_2}(z_1+z_2):=z_2\quad\forall(z_1,z_2)\in Z_1\times Z_2.\tag{2.1}$$

It is known that if Q is a polyhedron in $Z_1\oplus Z_2$ then $\Pi_{Z_2}(Q)$ is a polyhedron in Z_2 (cf. [16, Theorem 19.3] and the following Proposition 2.1).

For a convex set C in Z, let $\mathrm{int}(C)$ (resp. $\mathrm{rint}(C)$) denote the interior (relative interior) of C. It is known that if $\dim(Z)<\infty$ and $C\neq\emptyset$ then $\mathrm{rint}(C)\neq\emptyset$. Throughout, let $\mathbb{N}$ denote the set of all natural numbers and

$$\overline{1m}:=\{1,\cdots,m\}\quad\forall m\in\mathbb{N}.$$

Now we provide some results on polyhedra which are useful for our analysis later.

PROPOSITION 2.1. *Let* $(z_1^*, s_1), \cdots, (z_m^*, s_m) \in Z^* \times \mathbb{R}$ *and* $P := \{z \in Z : \langle z_i^*, z\rangle \le s_i \ \forall i \in \overline{1m}\}$. *Let* Z_1 *and* Z_2 *be two closed subspaces of* Z *such that*

$$Z_1 \subset \bigcap_{i=1}^{m} \mathcal{N}(z_i^*),\ \dim(Z_2) = \mathrm{codim}(Z_1) < \infty \ \text{ and } \ Z = Z_1 \oplus Z_2. \tag{2.2}$$

Then

$$P = Z_1 + \hat{P} \ \text{ and } \ \mathrm{rint}(P) = Z_1 + \mathrm{rint}(\hat{P}), \tag{2.3}$$

where $\hat{P} := \{z \in Z_2 : \langle z_i^*, z\rangle \le s_i,\ i = 1, \cdots, m\}$.

The first equality in (2.3) is a slight variant of [22, Lemma 2.1] and can be proved similar to the proof of [22, Lemma 2.1], while the second equality in (2.3) is immediate from the following observation: there exists $L \in (0,\ +\infty)$ such that $L(\|z_1\|+\|z_2\|) \le \|z_1+z_2\|$ for all $(z_1, z_2) \in Z_1 \times Z_2$ and the affine subspace $\mathrm{aff}(Z_1+\hat{P})$ is equal to $Z_1 + \mathrm{aff}(\hat{P})$ (thanks to (2.2) and the definition of $\hat{P}$).

From Proposition 2.1, one can see that many properties on polyhedra established in the finite dimension case also hold in the infinite dimension one. In particular, the following corollaries are consequences of Proposition 2.1 and [16, Corollary 6.5.1].

COROLLARY 2.1. *Let* $\{(u_1^*, s_1), \cdots, (u_n^*, s_n)\}$ *and* P *be as in Proposition 2.1. Then*

$$\mathrm{rint}(P) = \{z \in Z : \langle u_i^*, z\rangle < s_i,\ i \in \overline{1n} \setminus \bar{I}_P\} \cap \bigcap_{i \in \bar{I}_P} F_i, \tag{2.4}$$

where $\bar{I}_P := \{i \in \overline{1n} : \langle u_i^*, z\rangle = s_i \text{ for all } z \in P\}$ *and* $F_i := \{z \in Z : \langle u_i^*, z\rangle = s_i\}$.

COROLLARY 2.2. *Let* Z_1 *and* Z_2 *be two closed subspaces of* Z *such that*

$$Z = Z_1 + Z_2,\ Z_1 \cap Z_2 = \{0\} \ \text{ and } \ \dim(Z_2) < \infty. \tag{2.5}$$

Let $\hat{P}$ *be a polyhedron in* Z_2 *and* $\hat{F}$ *be a subset of* $\hat{P}$. *Then* $\hat{F}$ *is an exposed face of* $\hat{P}$ *if and only if* $Z_1 + \hat{F}$ *is an exposed face of the polyhedron* $Z_1 + \hat{P}$ *in* Z.

The following proposition is known and useful for us (cf. [22, Lemma 2.2]).

PROPOSITION 2.2. *Let* P_1 *and* P_2 *be two polyhedra (resp. generalized polyhedra) in* Z. *Then* $P_1 + P_2$ *and* $P_1 \cap P_2$ *are polyhedra (resp. generalized polyhedra).*

Note that a closed subspace of Z is not necessarily a polyhedron in Z. In fact, it is easy to verify that a closed subspace E of Z is a polyhedron in Z if and only if its codimension $\mathrm{codim}(E)$ is finite. Note that if E is a closed subspace of Z with $\mathrm{codim}(E) < +\infty$ and if H is a subspace of E then $E + H$ is a closed subspace of Z with $\mathrm{codim}(E + H) < +\infty$. The following proposition can be easily proved.

PROPOSITION 2.3. *Let* Z *be a normed space,* E *be a closed subspace of* Z *with* $\mathrm{codim}(E) < +\infty$, *and let* H *be a subspace of* Z. *Then the following statements hold:*
(i) $E + H + \hat{P}$ *is a polyhedron in* Z *for each polyhedron* $\hat{P}$ *in some finite dimensional*

subspace of Z.

(ii) $H + P$ *is a polyhedron for each polyhedron* P *in* Z.

The following lemma is useful in the proofs of some main results.

LEMMA 2.1. *Let* $C_1, \cdots, C_m$ *be closed sets in a normed space* Z *such that* $B(x_0, r_0) \subset \bigcup_{i=1}^{m} C_i$ *for some* $x_0 \in Z$ *and* $r_0 > 0$. *Then there exists* $i_0 \in \overline{1m}$ *such that* $B(x_0, r_0) \cap \mathrm{int}(C_{i_0}) \neq \emptyset$.

Proof. By the assumption, $B(x_0, r_0) \setminus \bigcup_{i=1}^{m-1} C_i$ is open, and $B(x_0, r_0) \setminus \bigcup_{i=1}^{m-1} C_i \subset B(x_0, r_0) \cap \mathrm{int}(C_m)$. Hence either $B(x_0, r_0) \cap \mathrm{int}(C_m) \neq \emptyset$ or $B(x_0, r_0) \subset \bigcup_{i=1}^{m-1} C_i$, which implies clearly that the conclusion holds. The proof is complete. □

With the help of Lemma 2.1, we can prove the following interesting proposition.

PROPOSITION 2.4. *Let* C *be a convex set in a normed space* Z *and let* $F_1, \cdots, F_\nu$ *be exposed faces of a polyhedron* P *in* Z *such that* $C \subset \bigcup_{j=1}^{\nu} F_j$. *Then there exists* $j_0 \in \overline{1\nu}$ *such that* $C \subset F_{j_0}$.

Proof. By Proposition 2.1, take two closed subspaces Z_1 and Z_2 of Z and a polyhedron $\hat{P}$ in Z_2 such that (2.2) and (2.3) hold. Thus, by Corollary 2.2, there exists an exposed face $\hat{F}_j$ of $\hat{P}$ such that $F_j = Z_1 + \hat{F}_j$ $(j \in \overline{1\nu})$. Hence $C \subset \bigcup_{j=1}^{\nu} F_j = \bigcup_{j=1}^{\nu} (Z_1 + \hat{F}_j)$. Noting that $\hat{C} := \Pi_{Z_2}(C)$ is a convex subset of $\hat{P}$ and $C \subset Z_1 + \hat{C}$, where Π_{Z_2} is the projection mapping from Z to Z_2 (see (2.1)), it follows from (2.2) that $\hat{C} \subset \bigcup_{j=1}^{\nu} \hat{F}_j$. Thus, it suffices to show that $\hat{C} \subset \hat{F}_{j_0}$ for some $j_0 \in \overline{1\nu}$. To prove this, take $(\hat{u}_j^*, \alpha_j) \in Z_2^* \times \mathbb{R}$ such that

$$(2.6) \qquad \alpha_j = \sup_{x_2 \in \hat{P}} \langle \hat{u}_j^*, x_2 \rangle \text{ and } \hat{F}_j = \{x_2 \in \hat{P} : \langle \hat{u}_j^*, x_2 \rangle = \alpha_j\} \quad \forall j \in \overline{1\nu}.$$

Since Z_2 is finite dimensional (cf.(2.2)), there exist $\hat{x} \in X_2$, a subspace Z_3 of Z_2 and $\delta > 0$ such that $\hat{C} \subset \hat{x} + Z_3$ and $\hat{x} + B_{Z_3}(0, \delta) \subset \hat{C} \subset \bigcup_{j=1}^{\nu} \hat{F}_j$. Thus, by Lemma 2.1, there exist $\hat{u} \in \hat{x} + B_{Z_3}(0, \delta)$, $\varepsilon \in (0, +\infty)$ and $j_0 \in \overline{1\nu}$ such that $\hat{u} + B_{Z_3}(0, \varepsilon) \subset \hat{F}_{j_0}$. This and (2.6) imply that $\langle \hat{u}_{j_0}^*, \hat{v} \rangle = 0$ for all $\hat{v} \in B_{Z_3}(0, \varepsilon)$ and so $\langle \hat{u}_{j_0}^*, \hat{v} \rangle = 0$ for all $\hat{v} \in Z_3$. Hence, $\hat{C} \subset \hat{x} + Z_3 = \hat{u} + Z_3 \subset \{x_2 \in Z_2 : \langle \hat{u}_{j_0}^*, x_2 \rangle = \alpha_{j_0}\}$. Since $\hat{C} \subset \hat{P}$, $\hat{C} \subset \hat{P} \cap \{x_2 \in Z_2 : \langle \hat{u}_{j_0}^*, x_2 \rangle = \alpha_{j_0}\} = \hat{F}_{j_0}$. The proof is complete. □

We also need the following proposition.

PROPOSITION 2.5. *Let* P_i *be polyhedra in a normed space* Z *such that* $\mathrm{int}(P_i) \neq \emptyset$ $(i = 1, \cdots, m)$. *Then there exist polyhedra* Q_j *in* Z *with* $\mathrm{int}(Q_j) \neq \emptyset$ $(j = 1, \cdots, \nu)$ *such that* $\bigcup_{i=1}^{m} P_i = \bigcup_{j=1}^{\nu} Q_j$ *and* $\mathrm{int}(Q_j) \cap Q_{j'} = \emptyset$ *for all* $j, j' \in \overline{1\nu}$ with $j \neq j'$.

Proof. The conclusion holds clearly when $m = 1$. Given a natural number n, suppose that the conclusion holds when $m = n$. Let $P_1, \cdots, P_n, P_{n+1}$ be arbitrary $n + 1$ polyhedra in Z such that each $\mathrm{int}(P_i)$ is nonempty. Then, by induction, it suffices to show that there exist polyhedra Q_j in Z with $\text{int}(Q_j) \neq \emptyset$ $(j = 1, \cdots, \nu)$ such that $\bigcup_{i=1}^{n+1} P_i = \bigcup_{j=1}^{\nu} Q_j$ and $\text{int}(Q_j) \cap Q_{j'} = \emptyset$ for all $j, j' \in \overline{1\nu}$ with $j \neq j'$. To do this, take polyhedra $H_1, \cdots, H_l$ in Z such that

$$(2.7)\quad \bigcup_{i=1}^{n} P_i = \bigcup_{i=1}^{l} H_i,\ \text{int}(H_i) \neq \emptyset \ \text{ and } \ H_i \cap \text{int}(H_{i'}) = \emptyset \quad \forall i, i' \in \overline{1l} \text{ with } i \neq i'.$$

If $\text{int}(P_{n+1}) \subset \bigcup_{i=1}^{l} H_i$, then $P_{n+1} \subset \bigcup_{i=1}^{l} H_i$ and so $\bigcup_{i=1}^{n+1} P_i = \bigcup_{i=1}^{l} H_i$; hence the conclusion is trivially true. Next suppose that $\text{int}(P_{n+1}) \not\subset \bigcup_{i=1}^{l} H_i$. Let $i \in \overline{1l}$, and take $(x_{ij}^*, t_{ij}) \in (Z^* \setminus \{0\}) \times \mathbb{R}$ $(j = 1, \cdots, \kappa_i)$ such that $H_i = \bigcap_{j=1}^{\kappa_i} H_{ij}$, where

$$H_{ij} := \{x \in Z : \langle x_{ij}^*, x\rangle \leq t_{ij}\} \quad \forall j \in \overline{1\kappa_i}.$$

Let $\Lambda_k^i := \bigcap_{j=1}^{k-1} H_{ij}$. Then $Z \setminus H_i = \bigcup_{k=1}^{\kappa_i} (Z \setminus H_{ik}) = \bigcup_{k=1}^{\kappa_i} \Lambda_k^i \cap (Z \setminus H_{ik})$, and so

$$\text{int}(P_{n+1}) \setminus H_i = \text{int}(P_{n+1}) \cap (Z \setminus H_i) = \bigcup_{k=1}^{\kappa_i} \text{int}(P_{n+1}) \cap \Lambda_k^i \cap (Z \setminus H_{ik}).$$

Clearly, each $Q_k^i := P_{n+1} \cap \Lambda_k^i \cap \text{cl}(Z \setminus H_{ik})$ is a polyhedron in Z. Hence, by Corollary (2.1), $\text{int}(Q_k^i) = \text{int}(P_{n+1}) \cap \text{int}(\Lambda_k^i) \cap \{z \in Z : \langle x_{ik}^*, z\rangle > t_{ik}\}$,

$$(2.8)\qquad Q_k^i \cap \text{int}(Q_{k'}^i) = \emptyset \quad \forall k, k' \in I_i \text{ with } k \neq k'$$

and

$$(2.9)\qquad \bigcup_{k \in I_i} \text{int}(Q_k^i) \subset \text{int}(P_{n+1}) \setminus H_i \subset \bigcup_{k=1}^{\kappa_i} Q_k^i,$$

where $I_i := \{k \in \overline{1\kappa_i} : \text{int}(Q_k^i) \neq \emptyset\}$. Let

$$Q_{(k_1,\cdots,k_l)} := \bigcap_{i=1}^{l} Q_{k_i}^i \quad \forall (k_1, \cdots, k_l) \in I_1 \times \cdots \times I_l$$

and $\Gamma := \left\{(k_1, \cdots, k_l) \in I_1 \times \cdots \times I_l : \bigcap_{i=1}^{l} \text{int}(Q_{k_i}^i) \neq \emptyset\right\}$. Then, each $Q_{(k_1,\cdots,k_l)}$ is a polyhedron in Z with $\text{int}(Q_{(k_1,\cdots,k_l)}) = \bigcap_{i=1}^{l} \text{int}(Q_{k_i}^i)$ (thanks to Corollary 2.1). Hence, by (2.9) and (2.8), one has $P_{n+1} \setminus \bigcup_{i=1}^{l} H_i \subset \bigcup_{(k_1,\cdots,k_l)\in\Gamma} Q_{(k_1,\cdots,k_l)} \subset P_{n+1}$ and $Q_{(k_1,\cdots,k_l)} \cap \text{int}(Q_{(k_1',\cdots,k_l')}) = \emptyset$ whenever $(k_1, \cdots, k_l) \neq (k_1', \cdots, k_l')$. It follows from (2.7) that $\bigcup_{i=1}^{n+1} P_i = \bigcup_{i=1}^{l} H_i \cup \bigcup_{(k_1,\cdots,k_l)\in\Gamma} Q_{(k_1,\cdots,k_l)}$. This shows that the conclusion also holds when $m = n + 1$. The proof is complete. □

For $(u_1^*, s_1), \cdots, (u_n^*, s_n) \in Z^* \times \mathbb{R}$ and $P = \{z \in Z : \langle u_i^*, z\rangle \le s_i,\ i \in \overline{1n}\}$, we say that (u_i^*, s_i) is a redundant generator of P if $P = \left\{z \in Z : \langle u_j^*, z\rangle \le s_j,\ j \in \overline{1n} \setminus \{i\}\right\}$ (cf. [18, 9]). For convenience, we adopt the following notion.

Definition 2.1 *We say that* $\{(u_1^*, s_1), \cdots, (u_n^*, s_n)\} \subset Z^* \times \mathbb{R}$ *is a prime generator group of a polyhedron* P *in a normed space* Z *if*

$$P = \{z \in Z : \langle u_i^*, z\rangle \le s_i,\ i \in \overline{1n}\} \tag{2.10}$$

and (u_i^*, s_i) *is not a redundant generator of* P *for all* $i \in \overline{1n}$*.*

Every polyhedron has a prime generator group (cf. [5, 18]). It is clear that if $\{(u_1^*, s_1), \cdots, (u_n^*, s_n)\} \subset Z^* \times \mathbb{R}$ is a prime generator group of P then

$$P \ne \{z \in Z : \langle u_i^*, z\rangle \le s_i,\ i \in \overline{1n} \setminus \{j\}\} \quad \forall j \in \overline{1n}. \tag{2.11}$$

In the remainder of this paper, we assume that every polyhedron P of Z is not equal to Z. So, it is clear that $u_i^* \ne 0$ for all $i \in \overline{1n}$ whenever $\{(u_1^*, s_1), \cdots, (u_n^*, s_n)\}$ is a prime generator group of P.

The following lemma is immediate from Definition 2.1.

Lemma 2.2. *Let* $\{(u_1^*, s_1), \cdots, (u_n^*, s_n)\}$ *be a prime generator group of a polyhedron* P *in a normed space* Z*. Then, for each* $j \in \overline{1n}$*,*

$$F_j(P) := P \cap \{x \in Z : \langle u_j^*, x\rangle = s_j\} \ne \emptyset. \tag{2.12}$$

The following two lemmas will play an important role in the proof of our main result.

Lemma 2.3. *Let* $\{(u_1^*, s_1), \cdots, (u_n^*, s_n)\}$ *be a prime generator group of a polyhedron* P *in a normed space* Z*. Let* $F_j(P)$ *be as in (2.12) and*

$$F_j^\circ(P) := \{z \in Z : \langle u_j^*, z\rangle = s_j \text{ and } \langle u_i^*, z\rangle < s_i,\ i \in \overline{1n} \setminus \{j\}\} \tag{2.13}$$

for all $j \in \overline{1n}$*. Then the following statements are equivalent:*
(i) $\mathrm{int}(P) \ne \emptyset$.
(ii) $F_j(P) = \mathrm{cl}(F_j^\circ(P))$ *for all* $j \in \overline{1n}$*.*
(iii) $F_j^\circ(P) \ne \emptyset$ *for all* $j \in \overline{1n}$*.*
(iv) $F_{j_0}^\circ(P) \ne \emptyset$ *for some* $j_0 \in \overline{1n}$*.*

Proof. First suppose that (i) holds. Then, by Corollary 2.1, there exists $x_0 \in Z$ such that $\langle u_i^*, x_0\rangle < s_i$ for all $i \in \overline{1n}$. For each $j \in \overline{1n}$, by (2.11), there exists $v \in Z$ such that $\langle u_j^*, v\rangle > s_j$ and $\langle u_i^*, v\rangle \le s_i$ for all $i \in \overline{1n} \setminus \{j\}$. It follows that there exists $\lambda_0 \in (0,\ 1)$ such that

$$\langle u_j^*, \lambda_0 x_0 + (1 - \lambda_0)v\rangle = s_j \ \text{ and } \ \langle u_i^*, \lambda_0 x_0 + (1 - \lambda_0)v\rangle\rangle < s_i \quad \forall i \in \overline{1n} \setminus \{j\}.$$

Therefore, $\frac{kx}{1+k} + \frac{\lambda_0 x_0 + (1-\lambda_0)v}{k+1} \in F_j^\circ(P)$ for all $(x, k) \in F_j(P) \times \mathbb{N}$. Letting $k \to \infty$, it follows that $x \in \mathrm{cl}(F_j^\circ(P))$ for all $x \in F_j(P)$, that is, $F_j(P) \subset \mathrm{cl}(F_j^\circ(P))$. Since the converse inclusion holds trivially, this shows implication (i)$\Rightarrow$(ii). Since (ii)$\Rightarrow$(iii) is immediate from Lemma 2.2 and (iii)$\Rightarrow$(iv) is trivial, it suffices to show (iv)$\Rightarrow$(i). To prove this, let $\bar{x} \in F_{j_0}^{\circ}(P)$, that is, $\langle u_{j_0}^*, \bar{x}\rangle = s_{j_0}$ and $\langle u_i^*, \bar{x}\rangle < s_i$ for all $i \in \overline{1n} \setminus \{j_0\}$. Taking $h \in Z$ with $\langle u_{j_0}^*, h\rangle < 0$ (thanks to $u_{j_0}^* \neq 0$), it follows that there exists $t > 0$ sufficiently small such that $\langle u_k^*, \bar{x} + th\rangle < s_k$ for all $k \in \overline{1n}$. This shows that $\bar{x} + th \in \mathrm{int}(P)$, and hence (iv)$\Rightarrow$(i) holds. The proof is complete. □

LEMMA 2.4. *Let P_1 and P_2 be two polyhedra in a normed space Z such that $\mathrm{int}(P_1) \cap P_2 = \emptyset$, and let $\{(u_{i1}^*, s_{i1}), \cdots, (u_{in_i}^*, s_{in_i})\} \subset Z^* \times \mathbb{R}$ be a prime generator group of P_i ($i = 1, 2$). Then for any $(j_1, j_2) \in \overline{1n_1} \times \overline{1n_2}$ and $x_0 \in F_{j_1}^{\circ}(P_1) \cap F_{j_2}^{\circ}(P_2)$ there exists $r > 0$ such that $\mathcal{N}(u_{1j_1}^*) = \mathcal{N}(u_{2j_2}^*)$ and*

$$(2.14)\quad F_{j_1}^{\circ}(P_1) \cap B_Z(x_0, r) = F_{j_2}^{\circ}(P_2) \cap B_Z(x_0, r) = (x_0 + \mathcal{N}(u_{1j_1}^*)) \cap B_Z(x_0, r),$$

where $B_Z(x_0, r) := \{x \in Z : \|x - x_0\| < r\}$ and $F_{j_1}^{\circ}(P_1)$ is as in (2.13).

Proof. Let $(j_1, j_2) \in \overline{1n_1} \times \overline{1n_2}$ and $x_0 \in F_{j_1}^{\circ}(P_1) \cap F_{j_2}^{\circ}(P_2)$. Then $x_0 \in P_1 \cap P_2$. Since $\mathrm{int}(P_1) \cap P_2 = \emptyset$, the separation theorem implies that there exists $v^* \in Z^* \setminus \{0\}$ such that $\langle v^*, x_0\rangle = \inf\limits_{x \in P_1} \langle v^*, x\rangle = \sup\limits_{x \in P_2} \langle v^*, x\rangle$. Noting that

$$(2.15)\qquad F_{j_1}^{\circ}(P_1) \cap B_Z(x_0, r) = (x_0 + \mathcal{N}(u_{1j_1}^*)) \cap B_Z(x_0, r) \subset P_1$$

and

$$(2.16)\qquad F_{j_2}^{\circ}(P_2) \cap B_Z(x_0, r) = (x_0 + \mathcal{N}(u_{2j_2}^*)) \cap B_Z(x_0, r) \subset P_2$$

for some $r > 0$ (thanks to the definitions of $F_{j_1}^{\circ}(P_1)$ and $F_{j_2}^{\circ}(P_2)$), it follows that

$$\langle v^*, x_0\rangle = \inf_{x \in (x_0 + \mathcal{N}(u_{1j_1}^*)) \cap B_Z(x_0, r)} \langle v^*, x\rangle = \sup_{x \in (x_0 + \mathcal{N}(u_{2j_2}^*)) \cap B_Z(x_0, r)} \langle v^*, x\rangle.$$

Hence $\inf\limits_{x \in \mathcal{N}(u_{1j_1}^*) \cap B_Z(0, r)} \langle v^*, x\rangle = \sup\limits_{x \in \mathcal{N}(u_{2j_2}^*) \cap B_Z(0, r)} \langle v^*, x\rangle = 0$, and so

$$\mathcal{N}(v^*) = \mathcal{N}(u_{1j_1}^*) = \mathcal{N}(u_{2j_2}^*)$$

because v^* is linear and both $\mathcal{N}(u_{1j_1}^*)$ and $\mathcal{N}(u_{2j_2}^*)$ are maximal linear subspaces of Z. This, together with (2.15) and (2.16), implies that (2.14) holds. □

3. Piecewise linear vector-valued functions. In this section, we will distinguish $\mathcal{PL}_1(X, Y)$ and $\mathcal{PL}(X, Y)$ and consider the structure of a piecewise linear function.

PROPOSITION 3.1. *Let X and Y be normed spaces. Then the following statements hold.*
(i) $\mathcal{L}(X, Y)$ is always contained in $\mathcal{PL}(X, Y)$.
(ii) $\mathcal{PL}_1(X, Y) \neq \emptyset$ if and only if $\dim(Y) < \infty$.
(iii) $\mathcal{PL}_1(X, Y) = \mathcal{PL}(X, Y)$ when $\dim(Y) < \infty$.

Proof. Since (i) is trivial and the sufficiency part of (ii) is a straightforward consequence of (i) and (iii), it suffices to show (iii) and the necessity part of (ii). First suppose that $\mathcal{PL}_1(X,Y) \neq \emptyset$, and let g be an element in $\mathcal{PL}_1(X,Y)$. Then there exist finitely many polyhedra $\Lambda_1, \cdots, \Lambda_k$ in the product $X \times Y$ such that

$$\text{gph}(g) = \bigcup_{i=1}^{k} \Lambda_i \ \text{ and } \ X = \bigcup_{i=1}^{k} \Lambda_i|_X, \tag{3.1}$$

where $\Lambda_i|_X := \{x \in X : \text{ there exists } y \in Y \text{ such that } (x,y) \in \Lambda_i\}$ is the projection of Λ_i to X. Given an $i \in \overline{1k}$, by Proposition 2.1, there exist two closed subspaces $X_i, \tilde{X}_i$ of X and two closed subspaces $Y_i, \tilde{Y}_i$ of Y such that

$$X \times Y = (X_i \times Y_i) \oplus (\tilde{X}_i \times \tilde{Y}_i), \ \ \text{codim}(X_i \times Y_i) = \dim(\tilde{X}_i \times \tilde{Y}_i) \leq \infty, \tag{3.2}$$

$$\Lambda_i = X_i \times Y_i + \tilde{\Lambda}_i, \tag{3.3}$$

where $\tilde{\Lambda}_i$ is a polyhedron in $\tilde{X}_i \times \tilde{Y}_i$. Thus, $\Lambda_i|_X = X_i + \tilde{\Lambda}_i|_{\tilde{X}_i}$ and so $\Lambda_i|_X$ is a polyhedron in X (thanks to Proposition 2.1). Since g is a single-valued function, it follows from (3.1) that $Y_i = \{0\}$ and $\tilde{Y}_i = Y$. Hence Y is finite-dimensional, and the necessity part of (ii) is proved. Next we prove $g \in \mathcal{PL}(X,Y)$. To prove this, we only need to show that there exist $T_i \in \mathcal{L}(X,Y)$ and $b_i \in Y$ such that

$$g(x) = T_i(x) + b_i \quad \forall x \in \Lambda_i|_X. \tag{3.4}$$

Since every convex set in a finite-dimensional space has a nonempty relative interior, $\text{rint}(\tilde{\Lambda}_i) \neq \emptyset$. Take a point $(\tilde{a}_i, \tilde{b}_i)$ in $\text{rint}(\tilde{\Lambda}_i)$. Thus, $\tilde{a}_i \in \text{rint}(\tilde{\Lambda}_i|_{\tilde{X}_i})$, and $E_i := \mathbb{R}_+(\tilde{\Lambda}_i|_{\tilde{X}_i} - \tilde{a}_i)$ and $Z_i := \mathbb{R}_+(\tilde{\Lambda}_i - (\tilde{a}_i, \tilde{b}_i))$ are linear subspaces of $\tilde{X}_i$ and $\tilde{X}_i \times \tilde{Y}_i$, respectively. Noting that $\tilde{\Lambda}_i \subset \text{gph}(g)$, define $\hat{T}_i : \tilde{\Lambda}_i|_{\tilde{X}_i} - \tilde{a}_i \to \tilde{Y}_i$ such that

$$\hat{T}_i(u_i) := g(u_i + \tilde{a}_i) - g(\tilde{a}_i) = g(u_i + \tilde{a}_i) - \tilde{b}_i \quad \forall u_i \in \tilde{\Lambda}_i|_{\tilde{X}_i} - \tilde{a}_i. \tag{3.5}$$

Then $\text{gph}(\hat{T}_i) = \tilde{\Lambda}_i - (\tilde{a}_i, \tilde{b}_i)$. Let $\tilde{T}_i : E_i \to \tilde{Y}_i$ be such that

$$\tilde{T}_i(tu_i) := t\hat{T}_i(u_i) \quad \forall (t, u_i) \in \mathbb{R}_+ \times (\tilde{\Lambda}_i|_{\tilde{X}_i} - \tilde{a}_i).$$

It is easy to verify that $\tilde{T}_i$ is well-defined and its graph is just the linear subspace $Z_i = \mathbb{R}_+(\tilde{\Lambda}_i - (\tilde{a}_i, \tilde{b}_i))$, and so $\tilde{T}_i$ is linear. Hence there exist $e_j \in Y$ and $e_{ij}^* \in E_i^*$ $(j = 1, \cdots, p)$ such that $e_1, \cdots, e_p$ are linearly independent and

$$\tilde{T}_i(x) = \sum_{j=1}^{p} \langle e_{ij}^*, x \rangle e_j \quad \forall x \in E_i.$$

For each $j \in \overline{1p}$, let $\tilde{e}_{ij}^* : X_i + E_i \to \mathbb{R}$ be such that

$$\langle \tilde{e}_{ij}^*, u + v \rangle = \langle e_{ij}^*, v \rangle \quad \forall (u,v) \in X_i \times E_i.$$

Then, by (3.2) and $E_i \subset \tilde{X}_i$, $\tilde{e}_{ij}^*$ is a linear functional on $X_i + E_i$, and its null space

$$\mathcal{N}(\tilde{e}_{ij}^*) := \{x \in X_i + E_i : \ \langle \tilde{e}_{ij}^*, x \rangle = 0\} = X_i + \{v \in E_i : \ \langle e_{ij}^*, v \rangle = 0\}.$$

Since X_i is a closed subspace of X and $\dim(E_i) < \infty$, it follows that $\mathcal{N}(\tilde{e}^*_{ij})$ is a closed subspace of X. Hence $\tilde{e}^*_{ij}$ is a continuous linear functional on $X_i + E_i$ (thanks to [17, Theorem 1.18]). By the Hahn-Banach theorem, there exists $x^*_{ij} \in X^*$ such that $x^*_{ij}|_{X_i+E_i} = \tilde{e}^*_{ij}$. Let $T_i : X \to Y$ be such that

$$T_i(x) = \sum_{j=1}^{p} \langle x^*_{ij}, x\rangle e_j \quad \forall x \in X.$$

Then $T_i \in \mathcal{L}(X, Y)$,

$$\mathcal{N}(T_i) \supset \bigcap_{j=1}^{p} \mathcal{N}(x^*_{ij}) \supset X_i \text{ and } T_i|_{\tilde{\Lambda}_i|_{\tilde{X}_i} - \tilde{a}_i} = \tilde{T}_i|_{\tilde{\Lambda}_i|_{\tilde{X}_i} - \tilde{a}_i} = \hat{T}_i. \tag{3.6}$$

Let x be an arbitrary element in $\Lambda_i|_X$ and take $y \in Y$ such that $(x, y) \in \Lambda_i$. Then, by (3.2) and (3.3), there exist $x_i \in X_i$ and $\tilde{x}_i \in \tilde{\Lambda}_i|_{\tilde{X}_i}$ such that $(\tilde{x}_i, y) \in \tilde{\Lambda}_i$ and $(x, y) = (x_i + \tilde{x}_i, y)$ (because $Y_i = \{0\}$). Hence, by (3.5) and (3.6), one has

$$g(x) = g(\tilde{x}_i) = y = \hat{T}_i(\tilde{x}_i - \tilde{a}_i) + \tilde{b}_i = T_i(\tilde{x}_i - \tilde{a}_i) + \tilde{b}_i = T_i(x) - T_i(\tilde{a}_i) + \tilde{b}_i.$$

This shows that (3.4) holds with $b_i = -T_i(\tilde{a}_i) + \tilde{b}_i$ and so $g \in \mathcal{PL}(X, Y)$. Therefore, $\mathcal{PL}_1(X, Y) \subset \mathcal{PL}(X, Y)$.

Now suppose that $\dim(Y) < \infty$. To prove the converse inclusion $\mathcal{PL}_1(X, Y) \supset \mathcal{PL}(X, Y)$, let $g \in \mathcal{PL}(X, Y)$. Then there exist $P_i \in \mathcal{P}(X)$, $T_i \in \mathcal{L}(X, Y)$ and $b_i \in Y$ $(i = 1, \cdots, n)$ such that

$$X = \bigcup_{i=1}^{n} P_i \text{ and } g(x) = T_i(x) + b_i \quad \forall x \in P_i \text{ and } \forall i \in \overline{1n}. \tag{3.7}$$

By $\dim(Y) < \infty$, there exist $y^*_1, \cdots, y^*_q \in Y^*$ such that $Y^* = \text{span}\{y^*_1, \cdots, y^*_q\}$. For any $x \in X$, since

$$T_i(x) = y \Leftrightarrow [\langle y^*, T_i(x)\rangle = \langle y^*, y\rangle \ \ \forall y^* \in Y^*] \Leftrightarrow [\langle y^*_j, T_i(x)\rangle = \langle y^*_j, y\rangle, \ j = 1, \cdots, q],$$

$$T_i(x) = y \Longleftrightarrow [\langle T^*_i(y^*_j), x\rangle = \langle y^*_j, y\rangle, \ j = 1, \cdots, q].$$

Hence $\text{gph}(T_i) = \{(x, y) \in X \times Y : \ \langle T^*_i(y^*_j), x\rangle - \langle y^*_j, y\rangle = 0, \ j = 1, \cdots, q\}$, and so $\text{gph}(T_i)$ is a polyhedron of $X \times Y$. Noting (by (3.7)) that

$$\text{gph}(g) = \bigcup_{i=1}^{n} (\text{gph}(T_i) + (0, b_i)) \cap (P_i \times Y),$$

it follows that $\text{gph}(g)$ is the union of finitely many polyhedra in $X \times Y$. Therefore, $g \in \mathcal{PL}_1(X, Y)$. The proof of (iii) is complete. □

Given $f \in \mathcal{PL}(X, Y)$, there exist $(P_1, T_1, b_1), \cdots, (P_m, T_m, b_m)$ in the product $\mathcal{P}(X) \times \mathcal{L}(X, Y) \times Y$ such that (1.2) holds. For $i \in \overline{1m}$, since each polyhedron is closed, the first equality of (1.2) implies that $\mathrm{int}(P_i) \supset X \setminus \bigcup_{j\in\overline{1m}\setminus\{i\}} P_j$ and so $X = \bigcup_{j\in\overline{1m}\setminus\{i\}} P_j$ whenever $\mathrm{int}(P_i) = \emptyset$. Hence, without loss of generality, we can assume that each P_i in (1.2) has a nonempty interior. Moreover, we assume without loss of generality that there exists $k \in \overline{1m}$ satisfying the following property:

$$(T_i, b_i) \neq (T_{i'}, b_{i'}) \quad \forall i, i' \in \overline{1k} \text{ with } i \neq i' \tag{3.8}$$

and for each $j \in \overline{1m}$ there exists $i \in \overline{1k}$ such that $(T_j, b_j) = (T_i, b_i)$. For each $i \in \overline{1k}$, let

$$I_i := \{j \in \overline{1m} : (T_j, b_j) = (T_i, b_i)\} \text{ and } Q_i := \bigcup_{j\in I_i} P_j. \tag{3.9}$$

Then $X = \bigcup_{i\in\overline{1k}} Q_i$, $X \neq \bigcup_{i\in\overline{1k}, i\neq j} Q_i$ and $f|_{Q_j} = T_j|_{Q_j} + b_j$ for all $j \in \overline{1k}$. We claim that

$$\mathrm{int}(Q_i) \cap \mathrm{int}(Q_{i'}) = \emptyset \quad \forall i, i' \in \overline{1k} \text{ with } i \neq i'. \tag{3.10}$$

Indeed, if this is not the case, there exist $i, i' \in \overline{1k}$ with $i \neq i'$, $x \in X$ and $r > 0$ such that $B(x, r) \subset Q_i \cap Q_{i'}$, and so

$$f(x) = T_i(u) + b_i = T_{i'}(u) + b_{i'} \quad \forall u \in B(x, r).$$

Since T_i and $T_{i'}$ are linear, it follows that $(T_i, b_i) = (T_{i'}, b_{i'})$, contradicting (3.8). Hence (3.10) holds. Since each Q_i is closed, (3.10) can be rewritten as

$$Q_i \cap \mathrm{int}(Q_{i'}) = \emptyset \quad \forall i, i' \in \overline{1k} \text{ with } i \neq i'.$$

Therefore, by Proposition 2.5, we have the following result.

Proposition 3.2. *For each $f \in \mathcal{PL}(X, Y)$ there exist $(P_i, T_i, b_i) \in \mathcal{P}(X) \times \mathcal{L}(X, Y) \times Y$ $(i = 1, \cdots, m)$ such that*

$$X = \bigcup_{i=1}^{m} P_i, \ \mathrm{int}(P_i) \neq \emptyset, \ P_i \cap \mathrm{int}(P_j) = \emptyset \ \forall i, j \in \overline{1m} \text{ with } i \neq j, \tag{3.11}$$

and

$$f|_{P_i} = T_i|_{P_i} + b_i \quad \forall i \in \overline{1m}, \tag{3.12}$$

that is, $f(x) = T_i x + b_i$ for all $x \in P_i$ and $i \in \overline{1m}$.

Now we are ready to establish the main result in this section, which shows that any piecewise linear function defined on an infinite dimensional space X can be decomposed into the sum of a linear function on an infinite dimensional closed subspace of X and a piecewise linear function on a finite dimensional subspace of X.

THEOREM 3.1. *Let $f \in \mathcal{PL}(X,Y)$. Then there exist two closed subspaces X_1 and X_2 of X, $(\hat{P}_i, T_i, b_i) \in \mathcal{P}(X_2) \times \mathcal{L}(X,Y) \times Y$ $(i = 1, \cdots, m)$ and $\hat{T} \in \mathcal{L}(X_1, Y)$ such that*

$$X = X_1 \oplus X_2,\ \operatorname{codim}(X_1) = \dim(X_2) < \infty,\ X_2 = \bigcup_{i=1}^{m} \hat{P}_i, \tag{3.13}$$

$$\operatorname{int}_{X_2}(\hat{P}_i) \neq \emptyset,\quad \hat{P}_i \cap \operatorname{int}_{X_2}(\hat{P}_j) = \emptyset\ \ \forall i,j \in \overline{1m} \text{ with } i \neq j, \tag{3.14}$$

$$T_i|_{X_1} = \hat{T}\ \text{ and }\ f|_{X_1+\hat{P}_i} = T_i|_{X_1+\hat{P}_i} + b_i \quad \forall i \in \overline{1m}. \tag{3.15}$$

Consequently, there exist a finite dimensional subspace Y_2 of Y and a piecewise linear function g between the finite dimensional spaces X_2 and Y_2 such that

$$f(x_1 + x_2) = \hat{T}(x_1) + g(x_2) \quad \forall (x_1, x_2) \in X_1 \times X_2.$$

Proof. Since f is in $\mathcal{PL}(X,Y)$, Proposition 3.2 implies that there exist $(P_i, T_i, b_i) \in \mathcal{P}(X) \times \mathcal{L}(X,Y) \times Y$ $(i = 1, \cdots, m)$ such that (3.11) and (3.12) hold. For each $i \in \overline{1m}$, take a prime generator group $\{(x_{i1}^*, t_{i1}), \cdots, (x_{i\nu_i}^*, t_{i\nu_i})\}$ of P_i, that is,

$$P_i = \{x \in X :\ \langle x_{ij}^*, x\rangle \le t_{ij},\ j \in \overline{1\nu_i}\} \tag{3.16}$$

and

$$P_i \neq \{x \in X :\ \langle x_{ij}^*, x\rangle \le t_{ij},\ j \in \overline{1\nu_i} \setminus \{j'\}\} \quad \forall j' \in \overline{1\nu_i}. \tag{3.17}$$

Let $X_1 := \bigcap\limits_{i \in \overline{1m}} \bigcap\limits_{j \in \overline{1\nu_i}} \mathcal{N}(x_{ij}^*)$. Then X_1 is a closed subspace of X with $\operatorname{codim}(X_1) \le \sum\limits_{i=1}^{m} \nu_i$ and so there exists a closed subspace X_2 of X such that

$$X = X_1 \oplus X_2\ \text{ and }\ \operatorname{codim}(X_1) = \dim(X_2) < \infty. \tag{3.18}$$

Let

$$\hat{P}_i := \{x \in X_2 :\ \langle x_{ij}^*, x\rangle \le t_{ij},\ \ j \in \overline{1\nu_i}\}. \tag{3.19}$$

By (3.16) and the definition of X_1, one has $P_i = X_1 + \hat{P}_i$. It follows from (3.11), (3.18) and Proposition 2.1 that (3.13) holds, $\operatorname{int}(P_i) = X_1 + \operatorname{int}_{X_2}(\hat{P}_i)$ for all $i \in \overline{1m}$, and so (3.14) also holds. Thus, by (3.12), it remains to show the first equality of (3.15). For any $i \in \overline{1m}$ and $j \in \overline{1\nu_i}$, let

$$F_j^\circ(P_i) := \big\{x \in X :\ \langle x_{ij}^*, x\rangle = t_{ij} \text{ and } \langle x_{il}^*, x\rangle < t_{il} \text{ for all } l \in \overline{1\nu_i} \setminus \{j\}\big\}$$

and

$$F_j^\circ(\hat{P}_i) := \{x \in X_2 :\ \langle x_{ij}^*, x\rangle = t_{ij} \text{ and } \langle x_{il}^*, x\rangle < t_{il} \text{ for all } l \in \overline{1\nu_i} \setminus \{j\}\}. \tag{3.20}$$

Then, $F_j^\circ(P_i) = X_1 + F_j^\circ(\hat{P}_i)$ and $F_j^\circ(\hat{P}_i) \neq \emptyset$ (thanks to Lemma 2.3). Let i and i' be two arbitrary indices in $\overline{1m}$ such that $i \neq i'$. Then, to prove the first equality of (3.15), we only need to show $T_i|_{X_1} = T_{i'}|_{X_1}$. To do this, take $(\bar{u}, \bar{u}') \in \mathrm{int}(\hat{P}_i) \times \mathrm{int}(\hat{P}_{i'})$ and $u^* \in X_2^* \setminus \{0\}$ such that $\langle u^*, \bar{u}' - \bar{u}\rangle \neq 0$. Then there exists $\delta > 0$ such that

$$\bar{u} + B_{X_3}(0,\delta) \subset \mathrm{int}_{X_2}(\hat{P}_i) \ \text{ and } \ \bar{u}' + B_{X_3}(0,\delta) \subset \mathrm{int}_{X_2}(\hat{P}_{i'}), \tag{3.21}$$

where $X_3 := \mathcal{N}(u^*) = \{x \in X_2 :\ \langle u^*, x\rangle = 0\}$. Hence

$$\dim(X_3) = \dim(X_2) - 1, \ \ X_2 = X_3 \oplus \mathbb{R}(\bar{u}' - \bar{u}) \tag{3.22}$$

and

$$\mathrm{int}_{X_2}\big([\bar{u},\ \bar{u}'] + B_{X_3}(0,\delta)\big) = (\bar{u},\ \bar{u}) + B_{X_3}(0,\delta) \neq \emptyset, \tag{3.23}$$

where $[\bar{u},\ \bar{u}'] := \{\bar{u} + t(\bar{u}' - \bar{u}) :\ 0 \le t \le 1\}$ and $(\bar{u},\ \bar{u}') := \{\bar{u} + t(\bar{u}' - \bar{u}) :\ 0 < t < 1\}$. For each $z \in B_{X_3}(0,\delta)$, let

$$I_z := \{i \in \overline{1m} :\ \{x\} \neq \hat{P}_i \cap (z + [\bar{u},\ \bar{u}']) \neq \emptyset \text{ for all } x \in X_2\}$$

and

$$I_z^\circ := \{i \in \overline{1m} :\ \mathrm{int}_{X_2}(\hat{P}_i) \cap (z + [\bar{u},\ \bar{u}']) \neq \emptyset\}.$$

Then $I_z^\circ \subset I_z$, and $\hat{P}_i \cap (z + [\bar{u},\ \bar{u}'])$ contains at most an element for all $i \in \overline{1m} \setminus I_z$. Noting that $X_2 = \bigcup\limits_{i \in \overline{1m}} \hat{P}_i$ (thanks to (3.13)), it follows that

$$z + [\bar{u},\ \bar{u}'] = \bigcup_{i \in I_z} \hat{P}_i \cap (z + [\bar{u},\ \bar{u}']) \quad \forall z \in B_{X_3}(0,\delta). \tag{3.24}$$

Regarding X_2 as the Euclidean space $\mathbb{R}^{\dim(X_2)}$ (without loss of generality), let μ_{X_2} and μ_{X_3} denote the Lebesgue measures on X_2 and X_3, respectively. Setting $E_0 := \{z \in B_{X_3}(0,\delta) :\ I_z^\circ \neq I_z\}$, we claim that $\mu_{X_3}(E_0) = 0$. To prove this, let z be an arbitrary element in E_0. Then there exists $i_z \in I_z$ such that $i_z \notin I_z^\circ$. This implies that $\hat{P}_{i_z} \cap (z + [\bar{u},\ \bar{u}']) \subset \hat{P}_{i_z} \setminus \mathrm{int}_{X_2}(\hat{P}_{i_z})$. Noting that $\hat{P}_{i_z} \setminus \mathrm{int}_{X_2}(\hat{P}_{i_z})$ is the union of finitely many faces of $\hat{P}_{i_z}$, it follows from Proposition 2.4 that there exists a face of $\hat{P}_{i_z}$ containing the convex set $\hat{P}_{i_z} \cap (z + [\bar{u},\ \bar{u}'])$. Since $\hat{P}_{i_z} \cap (z + [\bar{u},\ \bar{u}'])$ is a segment containing at least two points (thanks to the definition of I_z) and each $\hat{P}_i$ (as a polyhedron in X_2) has finitely many faces, there exist $v_1^*, \cdots, v_q^* \in X_2^* \setminus \{0\}$ and $v_1, \cdots, v_q \in X_2$ such that $z + [\bar{u},\ \bar{u}'] \subset \bigcup\limits_{k=1}^{q} (v_k + \mathcal{N}(v_k^*))$ for all $z \in E_0$, that is, $E_0 + [\bar{u},\ \bar{u}'] \subset \bigcup\limits_{k=1}^{q} (v_k + \mathcal{N}(v_k^*))$. Since each $\mathcal{N}(v_k^*)$ is of dimension $\dim(X_2) - 1$, $\mu_{X_2}(E_0 + [\bar{u},\ \bar{u}']) \le \mu_{X_2}\left(\bigcup\limits_{k=1}^{q} (v_k + \mathcal{N}(v_k^*))\right) \le \sum\limits_{k=1}^{q} \mu_{X_2}(v_k + \mathcal{N}(v_k^*)) = 0$. This and (3.22) show that $\mu_{X_3}(E_0) = 0$. Next, let

$$z \in B_{X_3}(0,\delta) \setminus E_0.$$

Then $I_z = I_z^\circ$. Thus, by (3.24) and the definition of I_z°,

$$z + [\bar{u},\ \bar{u}'] = \bigcup_{\kappa \in I_z^\circ} \hat{P}_\kappa \cap (z + [\bar{u},\ \bar{u}']) \ \text{ and } \ \mathrm{int}_{X_2}(\hat{P}_\kappa) \cap (z + [\bar{u},\ \bar{u}']) \neq \emptyset \quad \forall \kappa \in I_z^\circ.$$

Noting that $\hat{P}_\kappa \cap \mathrm{int}_{X_2}(\hat{P}_{\kappa'}) = \emptyset$ for any $\kappa, \kappa' \in I_z^\circ$ with $\kappa \neq \kappa'$, it follows from (3.21) that there exist $\iota_0^z, \iota_1^z, \cdots, \iota_{\gamma_z}^z \in \overline{1m}$ and $\lambda_0^z, \lambda_1^z, \cdots, \lambda_{\gamma_z}^z \in [0,\ 1)$ such that

$$I_z = I_z^\circ = \{\iota_0^z, \iota_1^z, \cdots, \iota_{\gamma_z}^z\},\ \iota_0^z = i,\ \iota_{\gamma_z}^z = i',\ \lambda_0^z = 0,\ \lambda_{k-1}^z < \lambda_k^z, \tag{3.25}$$

$$z + \bar{u} + [0,\ \lambda_1^z)(\bar{u}' - \bar{u}) = (z + [\bar{u},\ \bar{u}']) \cap \mathrm{int}_{X_2}(\hat{P}_i),$$

$$z + \bar{u} + (\lambda_{\gamma_z}^z,\ 1](\bar{u}' - \bar{u}) = (z + [\bar{u},\ \bar{u}']) \cap \mathrm{int}_{X_2}(\hat{P}_{i'}),$$

$$z + \bar{u} + [\lambda_{k-1}^z,\ \lambda_k^z](\bar{u}' - \bar{u}) = (z + [\bar{u},\ \bar{u}']) \cap \hat{P}_{\iota_{k-1}^z}$$

and

$$z + \bar{u} + (\lambda_{k-1}^z,\ \lambda_k^z)(\bar{u}' - \bar{u}) = (z + [\bar{u},\ \bar{u}']) \cap \mathrm{int}_{X_2}(\hat{P}_{\iota_{k-1}^z})$$

for all $k \in \overline{1\gamma_z}$. Therefore

$$z + \bar{u} + \lambda_k^z(\bar{u}' - \bar{u}) \in \hat{P}_{\iota_{k-1}^z} \cap \hat{P}_{\iota_k^z} \quad \forall k \in \overline{1\gamma_z}. \tag{3.26}$$

This and (3.14) imply that $z + \bar{u} + \lambda_k^z(\bar{u}' - \bar{u}) \notin \mathrm{int}_{X_2}(\hat{P}_{\iota_{k-1}^z}) \cup \mathrm{int}_{X_2}(\hat{P}_{\iota_k^z})$ for all $k \in \overline{1\gamma_z}$. Letting

$$J_{(z,k)}^- := \{j \in \overline{1\nu_{\iota_{k-1}^z}} :\ \langle x_{\iota_{k-1}^z j}^*, z + \bar{u} + \lambda_k^z(\bar{u}' - \bar{u})\rangle = t_{\iota_{k-1}^z j}\} \tag{3.27}$$

and

$$J_{(z,k)} := \{j \in \overline{1\nu_{\iota_k^z}} :\ \langle x_{\iota_k^z j}^*, z + \bar{u} + \lambda_k^z(\bar{u}' - \bar{u})\rangle = t_{\iota_k^z j}\}, \tag{3.28}$$

it follows from (3.19) and Corollary 2.1 that $J_{(z,k)}^- \neq \emptyset$ and $J_{(z,k)} \neq \emptyset$ for all $k \in \overline{1\gamma_z}$. We claim that there exist $\bar{z} \in B_{X_3}(0, \delta) \setminus E_0$ and $(j_k^-, j_k) \in \overline{1\nu_{\iota_{k-1}^{\bar{z}}}} \times \overline{1\nu_{\iota_k^{\bar{z}}}}$ such that

$$J_{(\bar{z},k)}^- = \{j_k^-\} \ \text{ and } \ J_{(\bar{z},k)} = \{j_k\} \quad \forall k \in \overline{1\gamma_{\bar{z}}}. \tag{3.29}$$

Indeed, if this is not the case, for each $z \in B_{X_3}(0, \delta) \setminus E_0$ there exists $k \in \overline{1\gamma_z}$ such that either $J_{(z,k)}^-$ or $J_{(z,k)}$ contains at least two elements; we assume without loss of generality that there exist $k \in \overline{1\gamma_z}$ and $j_1, j_2 \in J_{(z,k)}$ such that $j_1 \neq j_2$. Then, by (3.26) and (3.28),

$$z + \bar{u} + \lambda_k^z(\bar{u}' - \bar{u}) \in \big\{x \in \hat{P}_{\iota_k^z} :\ \langle x_{\iota_k^z j_1}^*, x\rangle = t_{\iota_k^z j_1} \ \text{ and } \ \langle x_{\iota_k^z j_2}^*, x_2\rangle = t_{\iota_k^z j_2}\big\}. \tag{3.30}$$

Since $\big\{(x_{\iota_k^z 1}^*, t_{\iota_k^z 1}), (x_{\iota_k^z 2}^*, t_{\iota_k^z 2}), \cdots, (x_{\iota_k^z \nu_{\iota_k^z}}^*, t_{\iota_k^z \nu_{\iota_k^z}})\big\}$ is a prime generator group of $P_{\iota_k^z}$ and $\mathrm{int}(P_{\iota_k^z})$ is nonempty, it is easy to verify that $x_{\iota_k^z j_1}^*$ and $x_{\iota_k^z j_2}^*$ are linearly independent. Hence $\mathrm{codim}(\mathcal{N}(x_{\iota_k^z j_1}^*) \cap \mathcal{N}(x_{\iota_k^z j_2}^*)) = 2$. Noting that X_1 is a subspace of

$\mathcal{N}(x^*_{\iota^z_k j_1})\cap\mathcal{N}(x^*_{\iota^z_k j_2})$, it follows from (3.13) that $X_2\cap\mathcal{N}(x^*_{\iota^z_k j_1})\cap\mathcal{N}(x^*_{\iota^z_k j_2})$, as a linear subspace of X_2, is of codimension 2. This and (3.30) imply that there exists a face $\hat{F}$ of $\hat{P}_{\iota^z_k}$ such that $\dim(\hat{F})\le\dim(X_2)-2$, $z+\bar{u}+\lambda^z_k(\bar{u}'-\bar{u})\in\hat{F}$, and so

$$z+[\bar{u},\ \bar{u}']\subset\hat{F}-(\bar{u}+\lambda^z_{k'}(\bar{u}'-\bar{u}))+[\bar{u},\ \bar{u}']\subset\hat{F}+[\bar{u}'-\bar{u},\ \bar{u}-\bar{u}'].$$

Since each polyhedron has finitely many faces (cf. [11, 16]) , there exist finitely many linear subspaces $S_1,\cdots,S_l$ of X_2 and $\omega_1,\cdots,\omega_l\in X_2$ such that $\dim(S_j)\le\mathrm{dom}(X_2)-2$ $(j=1,\cdots,l)$ and $z+[\bar{u},\ \bar{u}']\subset\bigcup_{j=1}^{l}(S_j+\omega_j+[\bar{u}'-\bar{u},\ \bar{u}-\bar{u}'])$ for all $z\in B_{X_3}(0,\delta)\setminus E_0$. This means that $(B_{X_3}(0,\delta)\setminus E_0)+[\bar{u},\ \bar{u}']\subset\bigcup_{j=1}^{l}(S_j+\omega_j+[\bar{u}'-\bar{u},\ \bar{u}-\bar{u}'])$ and so

$$\mu_{X_2}((B_{X_3}(0,\delta)\setminus E_0)+[\bar{u},\ \bar{u}'])\le\sum_{j=1}^{l}\mu_{X_2}(S_j+\omega_j+[\bar{u}'-\bar{u},\ \bar{u}-\bar{u}'])=0.$$

Thus, by (3.22), $\mu_{X_3}(B_{X_3}(0,\delta)\setminus E_0)=0$. Hence $\mu_{X_3}(E_0)\ge\mu_{X_3}(B_{X_3}(0,\delta))>0$, contradicting $\mu_{X_3}(E_0)=0$. This shows that (3.29) holds, that is, there exist $\bar{z}\in B_{X_3}(0,\delta)\setminus E_0$ and $(j^-_k,j_k)\in\overline{1\nu_{\iota^{\bar{z}}_{k-1}}}\times\overline{1\nu_{\iota^{\bar{z}}_k}}$ such that

$$\bar{x}_k:=\bar{z}+\bar{u}+\lambda^{\bar{z}}_k(\bar{u}'-\bar{u})\in F^\circ_{j^-_k}(\hat{P}_{\iota^{\bar{z}}_{k-1}})\cap F^\circ_{j_k}(\hat{P}_{\iota^{\bar{z}}_k})\quad\forall k\in\overline{1\gamma_{\bar{z}}}.$$

Noting that $F^\circ_{j^-_k}(P_{\iota^{\bar{z}}_{k-1}})=X_1+F^\circ_{j^-_k}(\hat{P}_{\iota^{\bar{z}}_{k-1}})$ and $F^\circ_{j_k}(P_{\iota^{\bar{z}}_k})=X_1+F^\circ_{j_k}(\hat{P}_{\iota^{\bar{z}}_k})$, one has

$$\bar{x}_k\in F^\circ_{j^-_k}(P_{\iota^{\bar{z}}_{k-1}})\cap F^\circ_{j_k}(P_{\iota^{\bar{z}}_k})\quad\forall k\in\overline{1\gamma_{\bar{z}}}.$$

It follows from Lemma 2.4 that for each $k\in\overline{1\gamma_{\bar{z}}}$,

$$\mathcal{N}_k:=\mathcal{N}(x^*_{\iota^{\bar{z}}_{k-1}j^-_k})=\mathcal{N}(x^*_{\iota^{\bar{z}}_k j_k})$$

and

$$F^\circ_{j^-_k}(P_{\iota^{\bar{z}}_{k-1}})\cap B_X(\bar{x}_k,r_k)=F^\circ_{j_k}(P_{\iota^{\bar{z}}_k})\cap B_X(\bar{x}_k,r_k)=(\bar{x}_k+\mathcal{N}_k)\cap B_X(\bar{x}_k,r_k)$$

for some $r_k>0$. Thus, by (3.12), one has

$$T_{\iota^{\bar{z}}_{k-1}}|_{(\bar{x}_k+\mathcal{N}_k)\cap B_X(\bar{x}_k,r_k)}+b_{\iota^{\bar{z}}_{k-1}}=T_{\iota^{\bar{z}}_k}|_{(\bar{x}_k+\mathcal{N}_k)\cap B_X(\bar{x}_k,r_k)}+b_{\iota^{\bar{z}}_k}\quad\forall k\in\overline{1\gamma_{\bar{z}}}.$$

Since $\mathcal{N}_k$ is a maximal subspace of X and both $T_{\iota^{\bar{z}}_{k-1}}$ and $T_{\iota^{\bar{z}}_k}$ are linear,

$$T_{\iota^{\bar{z}}_{k-1}}|_{\mathcal{N}_k}=T_{\iota^{\bar{z}}_k}|_{\mathcal{N}_k}\quad\forall k\in\overline{1\gamma_{\bar{z}}}.$$

Noting that $X_1\subset\mathcal{N}_k$ (thanks to the definitions of $\mathcal{N}_k$ and X_1), it follows that $T_{\iota^{\bar{z}}_{k-1}}|_{X_1}=T_{\iota^{\bar{z}}_k}|_{X_1}$ for all $k\in\overline{1\gamma_{\bar{z}}}$, and so $T_i|_{X_1}=T_{\iota^{\bar{z}}_0}|_{X_1}=T_{\iota^{\bar{z}}_{\gamma_{\bar{z}}}}|_{X_1}=T_{i'}|_{X_1}$ (thanks to (3.25)). This shows that the first equality of (3.15) holds. The proof is complete. □

The following corollary is a consequence of Theorem 3.1 and Propositions 2.1 and 2.5.

Corollary 3.1. *For any two $f, f' \in \mathcal{PL}(X,Y)$ there exist two closed subspaces X_1 and X_2 of X and $(\hat{P}_i, T_i, T_i', b_i, b_i') \in \mathcal{P}(X_2) \times \mathcal{L}(X,Y)^2 \times Y^2$ $(i = 1, \cdots, m)$ such that* $\operatorname{codim}(X_1) = \dim(X_2) < \infty$*,*

$$X = X_1 \oplus X_2,\ \ X_2 = \bigcup_{i=1}^{m} \hat{P}_i,\ \ \operatorname{int}_{X_2}(\hat{P}_i) \neq \emptyset,\ \ \operatorname{int}_{X_2}(\hat{P}_i) \cap \hat{P}_j = \emptyset,$$

$T_i|_{X_1} = T_j|_{X_1},\ T_i'|_{X_1} = T_j'|_{X_1},\ f|_{X_1+\hat{P}_i} = T_i|_{X_1+\hat{P}_i} + b_i$ and $f'|_{X_1+\hat{P}_i} = T_i'|_{X_1+\hat{P}_i} + b_i'$ *for all $i, j \in \overline{1m}$ with $i \neq j$.*

4. Fully piecewise linear vector optimization problem (PLP). Let Y be a normed linear space and C be a nontrivial convex cone in Y. Let $\leq_C$ denote the preorder induced by C in Y, that is, for $y_1, y_2 \in Y$, $y_1 \leq_C y_2 \Leftrightarrow y_2 - y_1 \in C$. When the interior $\operatorname{int}(C)$ of C is nonempty, $y_1 <_C y_2$ is defined as $y_2 - y_1 \in \operatorname{int}(C)$.

For a subset Ω of Y and a point ω in Ω, we say that ω is a Pareto efficient point of Ω (with respect to C), denoted by $\omega \in E(\Omega, C)$, if there is no element $v \in \Omega \setminus \{\omega\}$ such that $v \leq_C \omega$. In the case when $\operatorname{int}(C) \neq \emptyset$, we say that ω is a weak Pareto efficient point of Ω, denoted by $\omega \in \mathrm{WE}(\Omega, C)$, if there is no element $v \in \Omega$ such that $v <_C \omega$. Clearly,

$$a \in \mathrm{E}(\Omega, C) \Leftrightarrow (\omega - C) \cap \Omega = \{\omega\} \quad \text{and} \quad a \in \mathrm{WE}(\Omega, C) \Leftrightarrow (\omega - \operatorname{int}(C)) \cap \Omega = \emptyset.$$

In the remainder, let X and Y be normed spaces, $C \subset Y$ be a nontrivial convex cone such that $\operatorname{int}(C) \neq \emptyset$, and let $(f, \varphi_i) \in \mathcal{PL}(X,Y) \times \mathcal{PL}(X, \mathbb{R})$ $(i = 1, \cdots, l)$. We consider the following fully piecewise linear vector optimization problem:

$$\text{(PLP)} \qquad C - \min f(x) \quad \text{subject to } \varphi_1(x) \leq 0, \cdots, \varphi_l(x) \leq 0.$$

Let A denote the feasible set of (PLP), that is,

$$A := \{x \in X :\ \varphi_1(x) \leq 0, \cdots, \varphi_l(x) \leq 0\}.$$

We say that $\bar{x} \in A$ is a Pareto (resp. weak Pareto) solution of (PLP) if $f(\bar{x}) \in \mathrm{E}(f(A), C)$ (resp. $f(\bar{x}) \in \mathrm{WE}(f(A), C)$). Let S (resp. S^w) denote the set of all Pareto (resp. weak Pareto) solutions of (PLP).

Since the objective f and each φ_i in problem (PLP) are piecewise linear, Corollary 3.1 implies that there exist $(P_i, T_i, b_i, x_{ij}^*, c_{ij}) \in \mathcal{P}(X) \times \mathcal{L}(X,Y) \times Y \times X^* \times \mathbb{R}$ ($i = 1, \cdots, m$ and $j = 1, \cdots, l$) such that

$$X = \bigcup_{i=1}^{m} P_i,\ \operatorname{int}(P_i) \neq \emptyset,\ P_i \cap \operatorname{int}(P_{i'}) = \emptyset \quad \forall i, i' \in \overline{1m} \text{ with } i \neq i', \tag{4.1}$$

$$f|_{P_i} = T_i|_{P_i} + b_i \ \text{ and } \ \varphi_j|_{P_i} = x_{ij}^*|_{P_i} - c_{ij} \quad \forall (i,j) \in \overline{1m} \times \overline{1l}. \tag{4.2}$$

For each $i \in \overline{1m}$, let

$$A_i := \{x \in P_i : \langle x_{ij}^*, x\rangle \le c_{ij} \ \ \forall j \in \overline{1l}\}. \tag{4.3}$$

Then each A_i is a polyhedron in X and

$$A = \bigcup_{i\in\overline{1m}} A_i. \tag{4.4}$$

Take a prime generator group $\{(u_{ik}^*, t_{ik}) \in X^* \times \mathbb{R} : \ k = 1, \cdots, q_i\}$ of P_i (where P_i is as in (4.1) and (4.2)). Then

$$P_i = \{x \in X : \ \langle u_{ik}^*, x\rangle \le t_{ik} \ \ \forall k \in \overline{1q_i}\} \ne \{x \in X : \ \langle u_{ik}^*, x\rangle \le t_{ik}, \ k \in \overline{1q_i} \setminus \{k'\}\}$$

for all $k' \in \overline{1q_i}$. It follows from (4.3) that

$$A_i = \bigcap_{(j,k)\in\overline{1l}\times\overline{1q_i}} \{x \in X : \ \langle x_{ij}^*, x\rangle \le c_{ij}\} \cap \{x \in X : \ \langle u_{ik}^*, x\rangle \le t_{ik}\} \ \ \forall i \in \overline{1m}. \tag{4.5}$$

Let

$$X_1 := \bigcap_{i=1}^{m} \bigcap_{(j,k)\in\overline{1l}\times\overline{1q_i}} \mathcal{N}(x_{ij}^*) \cap \mathcal{N}(u_{ik}^*). \tag{4.6}$$

Then X_1 is a closed subspace of X such that $\operatorname{codim}(X_1) < \infty$. Thus, one can take another closed subspace X_2 of X such that

$$X = X_1 \oplus X_2 \ \text{ and } \ \dim(X_2) = \operatorname{codim}(X_1) < \infty. \tag{4.7}$$

By Theorem 3.1 and its proof, there exists $\hat{T} \in \mathcal{L}(X_1, Y)$ such that

$$T_i|_{X_1} = \hat{T} \quad \forall i \in \overline{1m}. \tag{4.8}$$

For each $i \in \overline{1m}$, let

$$\hat{A}_i := \bigcap_{(j,k)\in\overline{1l}\times\overline{1q_i}} \{x \in X_2 : \ \langle x_{ij}^*, x\rangle \le c_{ij}\} \cap \{x \in X_2 : \ \langle u_{ik}^*, x\rangle \le t_{ik}\} \tag{4.9}$$

Then each $\hat{A}_i$ is a polyhedron in the finite dimensional space X_2 and

$$A_i = X_1 + \hat{A}_i \quad \forall i \in \overline{1m}. \tag{4.10}$$

Hence, by (4.4), the feasible set A of piecewise linear problem (PLP) can be rewritten as

$$A = X_1 + \bigcup_{i\in\overline{1m}} \hat{A}_i. \tag{4.11}$$

To study piecewise linear problem (PLP), we consider the following linear subproblems

$$(\text{LP})_i \qquad C - \min T_i x + b_i \ \text{ subject to } x \in A_i,$$

where $i \in \overline{1m}$. Recall that a weak Pareto face (resp. Pareto face) F of linear problem $(\text{LP})_i$ is a face of A_i such that each point in F is a weak Pareto solution (resp. Pareto solution) of $(\text{LP})_i$.

THEOREM 4.1. *Let C be a convex cone in Y such that $f(A)$ is C-convex, that is, $f(A)+C$ is a convex subset of Y. Then there exist finitely many polyhedra $F_1, \cdots, F_p$ in X satisfying the following properties:*
(i) $S^w = \bigcup_{k=1}^{p} F_k$.
(ii) For each k there exists $i \in \bar{I}$ such that F_k is a face of A_i and $F_k \subset S_i^w$, where $\bar{I} := \{i \in \overline{1m} : A_i \neq \emptyset\}$ and S_i^w is the weak Pareto solution set of linear subproblem (LP)$_i$.
Consequently each F_k is just a weak Pareto face of linear subproblem (LP)$_i$ for some $i \in \bar{I}$.

Proof. Let $x \in A$. Then $x \in S^w$ if and only if $f(A) \cap (f(x) - \text{int}(C)) = \emptyset$, which is equivalent to $(f(A) + C) \cap (f(x) - \text{int}(C)) = \emptyset$. Thus, by the separation theorem and the convexity of $f(A) + C$, $x \in S^w$ if and only if there exists $c^* \in C^+ \setminus \{0\}$ such that $\langle c^*, f(x)\rangle = \inf_{u \in A} \langle c^*, f(u)\rangle$. Let $S^w(c^*) := \{x \in A : \langle c^*, f(x)\rangle = \inf_{u \in A} \langle c^*, f(u)\rangle\}$ for each $c^* \in C^+ \setminus \{0\}$, and $C^+(f, A) := \{c^* \in C^* \setminus \{0\} : S^w(c^*) \neq \emptyset\}$. Then, by (4.4), one has $S^w = \bigcup_{c^* \in C^+(f,A)} S^w(c^*) = \bigcup_{c^* \in C^+(f,A)} \bigcup_{i \in \Lambda(c^*)} S^w(c^*) \cap A_i$, where $\Lambda(c^*) := \{i \in \bar{I} : S^w(c^*) \cap A_i \neq \emptyset\}$. On the other hand, for $c^* \in C^+(f, A)$ and $i \in \Lambda(c^*)$,

$$\begin{aligned} S^w(c^*) \cap A_i &= \{x \in A_i : \langle c^*, f(x) = \min_{u \in A_i} \langle c^*, f(u)\rangle\} \\ &= \{x \in A_i : \langle c^*, T_i x + b_i\rangle = \min_{u \in A_i} \langle c^*, T_i u + b_i\rangle\} \\ &= \{x \in A_i : \langle c^*, T_i x\rangle = \min_{u \in A_i} \langle c^*, T_i u\rangle\} \\ &= \{x \in A_i : \langle T_i^*(c^*), x\rangle = \min_{u \in A_i} \langle T_i^*(c^*), u\rangle\} \end{aligned}$$

(thanks to (4.2) and (4.3)) is a face of A_i and a subset of the weak Pareto solution set of linear subproblem $(\text{LP})_i$. Therefore, since every polyhedron only has finitely many faces, there exist $c_1^*, \cdots, c_p^* \in C^+(f, A)$ such that

$$S^w = \bigcup_{c^* \in C^+(f,A)} \bigcup_{i \in \Lambda(c^*)} S^w(c^*) \cap A_i = \bigcup_{k=1}^{p} \bigcup_{i \in \Lambda(c_k^*)} S^w(c_k^*) \cap A_i.$$

The proof is complete. □

Remark. If $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, then each set in Y is trivially C-convex. Moreover, if f is C-convex (i.e. $\text{epi}_C(f) = \{(x, y) : y \in f(x) + C\}$ is convex) then $f(A)$ is C-convex.

Dropping the C-convexity assumption on $f(A)$ but imposing the polyhedral assumption on the ordering cone C, the following theorems show that the weak Pareto solution set (resp. Pareto solution set) of (PLP) is the union of finitely many polyhedra (resp. generalized polyhedra), each of which is contained in a face of some A_i.

Theorem 4.2. *Let S^w be the set of all weak Pareto solutions of piecewise linear problem (PLP). Suppose that the ordering cone C is polyhedral. Then there exist finitely many polyhedra $F_1, \cdots, F_p$ in X such that $S^w = \bigcup_{k=1}^{p} F_k$ and each F_k is contained in a weak Pareto face of some linear subproblem (LP)$_i$.*

Theorem 4.3. *Let S be the set of all Pareto solutions of piecewise linear problem (PLP). Suppose that the ordering cone C is polyhedral. Then there exist finitely many generalized polyhedra $F_1, \cdots, F_p$ in X such that $S = \bigcup_{k=1}^{p} F_k$ and F_k is contained in a Pareto face of some linear subproblem (LP)$_i$.*

Remark. In the special case when the feasible set A of (PLP) is a polyhedron in X (i.e., each function φ_k is linear in the constraint system of (PLP)), Luan [10] proved that the weak Pareto solution set (resp. Pareto solution set) of (PLP) is the union of finitely many polyhedra (resp. generalized polyhedra) in X; in contrast, Theorem 4.2 (resp. Theorem 4.3) implies that the weak Pareto solution set (resp. Pareto solution set) of (PLP) is the union of finitely many polyhedra (resp. generalized polyhedra) in X with each of these polyhedra (resp. generalized polyhedra) contained in some face of A.

We postpone the proofs of Theorems 4.2 and 4.3 to the next section which establish a kind of finite dimensional reduction method to solve (PLP).

5. Finite dimension reduction method to solve (PLP). In this section, with the help of Theorem 3.1, we reduce fully piecewise linear problem (PLP) and linear subproblem (LP)$_i$ in the general normed space framework to the corresponding ones in the finite-dimensional space framework.

Throughout this section, we assume that the objective function f and all constraint functions φ_j in (PLP) are completely known, that is, $T_i \in \mathcal{L}(X, Y)$, $b_i \in Y$, $u_{ik}^*, x_{ij}^* \in X^*$, $b_i \in Y$ and $t_{ik}, c_{ij} \in \mathbb{R}$ are known data such that

$$X = \bigcup_{i=1}^{m} P_i,\ \mathrm{int}(P_i) \neq \emptyset,\ P_i \cap \mathrm{int}(P_{i'}) = \emptyset \quad \forall i, i' \in \overline{1m} \text{ with } i \neq i', \tag{5.1}$$

$$f|_{P_i} = T_i|_{P_i} + b_i \ \text{ and } \ \varphi_j|_{P_i} = x_{ij}^*|_{P_i} - c_{ij} \quad \forall (i, j) \in \overline{1m} \times \overline{1l} \tag{5.2}$$

where

$$P_i = \{x \in X : \langle u_{ik}^*, x\rangle \leq t_{ik},\ k = 1, \cdots, q_i\},\ \ i \in \overline{1m}. \tag{5.3}$$

We first provide a procedure to obtain exact formulas for optimal value sets and solution sets of (PLP):

Step 1 (Decomposing the space X): Let

$$X_1 := \bigcap_{i=1}^{m} \bigcap_{(j,k)\in\overline{1l}\times\overline{1q_i}} \mathcal{N}(x_{ij}^*) \cap \mathcal{N}(u_{ik}^*),$$

namely, X_1 is the solution space of the following system of homogeneous linear equations

$$\langle u_{ik}^*, x\rangle = \langle x_{ij}^*, x\rangle = 0,\ i = 1, \cdots, m,\ j = 1, \cdots, l,\ k = 1, \cdots, q_i.$$

Take a maximal linearly independent subset $\{e_1^*, \cdots, e_\nu^*\}$ of the finite set $\{u_{ik}^*, x_{ij}^* : i \in \overline{1m},\ j \in \overline{1l},\ k \in \overline{1q_i}\}$. For each $\iota \in \overline{1\nu}$, let h_ι be a solution of the following system of linear equations

$$\langle e_\iota^*, x\rangle = 1 \quad \text{and} \quad \langle e_{\iota'}^*, x\rangle = 0 \ \ \forall \iota' \in \overline{1\nu} \setminus \{\iota\}.$$

In particular, in the case that X is a Hilbert space, $h_\iota = x_\iota^*$. Let

$$X_2 := \operatorname{span}\{h_1, \cdots, h_\nu\} = \left\{ \sum_{\iota=1}^{\nu} t_\iota h_\iota : \ t_1, \cdots, t_\nu \in \mathbb{R} \right\}.$$

Then

$$X = X_1 + X_2 \quad \text{and} \quad X_1 \cap X_2 = \{0\}. \tag{5.4}$$

Step 2 (Constructing finite dimensional subspace Z of Y): Thanks to Corollary 3.1 and (5.4),

$$\hat{T} := T_1|_{X_1} = T_2|_{X_1} = \cdots = T_m|_{X_1}. \tag{5.5}$$

Let D denote the finite set $\bigcup\limits_{i=1}^{m} \{T_i(h_1), \cdots, T_i(h_\nu),\, b_i\}$ and take $u_1, \cdots, u_\varsigma$ in D with ς being the maximal integer such that $u_1 \in D \setminus \hat{T}(X_1)$,

$$u_2 \in D \setminus (\hat{T}(X_1) + \operatorname{span}\{u_1\}), \cdots, u_\varsigma \in D \setminus (\hat{T}(X_1) + \operatorname{span}\{u_1, \cdots, u_{\varsigma-1}\}),$$

where X_1 and $h_1, \cdots, h_\nu$ are as in Step 1. Let $Z := \operatorname{span}\{u_1, \cdots, u_\varsigma\}$. Clearly, Z is a subspace of Y such that $\dim(Z) = \varsigma$,

$$\hat{T}(X_1) \cap Z = \{0\} \ \text{ and } \ f(X) = \bigcup_{i=1}^{m} (\hat{T}(X_1) + T_i(\hat{P}_i) + b_i)) \subset \hat{T}(X_1) \oplus Z, \tag{5.6}$$

where $\hat{P}_i := \{x_2 \in X_2 : \langle u_{ik}^*, x_2\rangle \le t_{ik} \ \ \forall k \in \overline{1q_i}\}$. Let Π_Z denote the projection from $\hat{T}(X_1) \oplus Z$ onto Z, that is,

$$\Pi_Z(y + z) := z \quad \forall (y, z) \in \hat{T}(X_1) \times Z, \tag{5.7}$$

and let C_Z be a convex cone in the finite dimensional space Z defined by

$$C_Z := \Pi_Z((\hat{T}(X_1) \oplus Z) \cap C). \tag{5.8}$$

Step 3 (Exact formulas for weak Pareto optimal value set and weak Pareto set of (PLP)): For each $i \in \overline{1m}$, let

$$\hat{A}_i := \{x_2 \in \hat{P}_i : \langle x_{ij}^*, x_2\rangle \le c_{ij} \ \ \forall j \in \overline{1l}\}$$

and let $\hat{A} := \bigcup_{i=1}^{m} \hat{A}_i$. The weak Pareto optimal value set $\mathrm{WE}(f(A), C)$ and weak Pareto solution set S^w of (PLP) can be formulized as follows:
(i) If $(\hat{T}(X_1) \oplus Z) \cap \mathrm{int}(C) = \emptyset$ then $\mathrm{WE}(f(A), C) = f(A)$ and $S^w = A$.
(ii) If $(\hat{T}(X_1) \oplus Z) \cap \mathrm{int}(C) \neq \emptyset$ then

$$\mathrm{WE}(f(A), C) = \hat{T}(X_1) + \bigcup_{i=1}^{m} \hat{V}_i^w \text{ and } S^w = X_1 + \bigcup_{i=1}^{m} \hat{A}_i \cap (\Pi_Z \circ T_i)^{-1}(\hat{V}_i^w - \Pi_Z(b_i)),$$

where $\hat{V}_i^w := \Pi_Z\big(T_i(\hat{A}_i) + b_i\big) \setminus (\hat{f}(\hat{A}) + \mathrm{int}_Z(C_Z))$.
Formulas (i) and (ii) are immediate from Theorems 5.1 and 5.3. Similarly, with Theorems 5.1 and 5.3 being replaced by Corollary 5.1, Propositions 5.2 and their proofs, we can also obtain the formulas for the Pareto optimal value set and Pareto solution set of (PLP).

To establish the main results in this section, we need the following lemma.

LEMMA 5.1. *Suppose that* $(\hat{T}(X_1) \oplus Z) \cap \mathrm{int}(C)$ *is nonempty. Then*

$$\mathrm{int}_Z(C_Z) = \Pi_Z((\hat{T}(X_1) \oplus Z) \cap \mathrm{int}(C)). \tag{5.9}$$

Proof. By the assumption, take $(\bar{x}_1, \bar{z}) \in X_1 \times Z$ and $r > 0$ such that

$$\hat{T}(\bar{x}_1) + \bar{z} + rB_{\hat{T}(X_1)\oplus Z} \subset C. \tag{5.10}$$

Noting that the projection Π_Z is an open mapping from $\hat{T}(X_1) \oplus Z$ to Z, (5.8) implies that $\mathrm{int}_Z(C_Z) \supset \Pi_Z((\hat{T}(X_1) \oplus Z) \cap \mathrm{int}(C))$. Hence it suffices to show the converse inclusion. To do this, let $z \in \mathrm{int}_Z(C_Z)$. Then there exists $\sigma > 0$ such that $z + \sigma(z - \bar{z}) \in C_Z$, that is, $\hat{T}(x_1) + z + \sigma(z - \bar{z}) \in C$ for some $x_1 \in X_1$. It follows from (5.10) and the convexity of C that

$$\begin{aligned}\hat{T}\Big(\frac{x_1 + \sigma\bar{x}_1}{1+\sigma}\Big) + z + \frac{\sigma r B_{\hat{T}(X_1)\oplus Z}}{1+\sigma} &= \frac{\hat{T}(x_1) + z + \sigma(z - \bar{z})}{1+\sigma} + \frac{\sigma(\hat{T}(\bar{x}_1) + \bar{z} + rB_{\hat{T}(X_1)\oplus Z})}{1+\sigma} \\ &\subset (\hat{T}(X_1) \oplus Z) \cap C.\end{aligned}$$

Hence $z + \frac{\sigma r B_Z}{1+\sigma} \subset C_Z$ (thanks to (5.8)). This shows that $z \in \mathrm{int}_Z(C_Z)$. □

Define $\hat{f} : X_2 \to Z$ as follows

$$\hat{f}(x_2) := (\Pi_Z \circ f)(x_2) = \Pi_Z(f(x_2)) \quad \forall x_2 \in X_2.$$

Then, $\hat{f}$ is a piecewise linear function between the two finite dimensional spaces X_2 and Z. To solve the original piecewise linear vector optimization problem (PLP), consider the following piecewise linear problem in the framework of finite dimensional spaces:

$$(\widehat{\mathrm{PLP}}) \qquad C_Z - \min \hat{f}(x_2) \quad \text{subject to } x_2 \in X_2 \text{ and } \varphi_1(x_2) \leq 0, \cdots, \varphi_l(x_2) \leq 0.$$

Then the feasible set of $(\widehat{\text{PLP}})$ is $\hat{A}$ and the feasible set A of (PLP) is equal to $X_1 + \hat{A}$.

Next we establish the relationship between the weak Pareto optimal value set and weak Pareto solution set (resp. the Pareto solution set) of (PLP) and that of $(\widehat{\text{PLP}})$.

THEOREM 5.1. *Let S^w and $\hat{S}^w$ denote the weak Pareto solution sets of piecewise linear problems* (PLP) *and* $(\widehat{\text{PLP}})$*, respectively. The following statements hold:*
(i) If $(\hat{T}(X_1) \oplus Z) \cap \text{int}(C) = \emptyset$ then $\text{WE}(f(A), C) = f(A)$ and $S^w = A$.
(ii) If $(\hat{T}(X_1) \oplus Z) \cap \text{int}(C) \neq \emptyset$ then

$$\text{WE}(f(A), C) = \hat{T}(X_1) + \text{WE}(\hat{f}(\hat{A}), C_Z) \text{ and } S^w = X_1 + \hat{S}^w. \tag{5.11}$$

Proof. First suppose that $(\hat{T}(X_1) \oplus Z) \cap \text{int}(C) = \emptyset$. Then, since $\hat{T}(X_1) \oplus Z$ is a linear subspace of Y, $(\hat{T}(X_1) \oplus Z) \cap ((\hat{T}(X_1) \oplus Z) - \text{int}(C)) = \emptyset$. Noting that

$$f(x_2) \in \hat{T}(X_1) + \hat{f}(x_2) \text{ and } f(x_1 + x_2) = \hat{T}(x_1) + f(x_2) \quad \forall (x_1, x_2) \in X_1 \times X_2 \tag{5.12}$$

(thanks to (5.1), (5.2) and (5.6)), one has

$$f(A) = \hat{T}(X_1) + f(\hat{A}) = \hat{T}(X_1) + \hat{f}(\hat{A}) \subset \hat{T}(X_1) \oplus Z,$$

and so $f(A) \cap (f(A) - \text{int}(C)) = \emptyset$. This shows that $\text{WE}(f(A), C) = f(A)$ and $S^w = A$. Next suppose that $(\hat{T}(X_1) \oplus Z) \cap \text{int}(C) \neq \emptyset$. Then, by Lemma 5.1, $\text{int}_Z(C_Z) = \Pi_Z(\hat{T}(X_1) \oplus Z) \cap \text{int}(C))$. Since Π_Z is the projection from $\hat{T}(X_1) \oplus Z$ to Z,

$$\begin{aligned} \hat{T}(X_1) + (\hat{T}(X_1) \oplus Z) \cap \text{int}(C) &= \hat{T}(X_1) + \Pi_Z((\hat{T}(X_1) \oplus Z) \cap \text{int}(C)) \\ &= \hat{T}(X_1) + \text{int}_Z(C_Z). \end{aligned}$$

Hence

$$\begin{aligned} \text{WE}(f(A), C) &= f(A) \setminus (f(A) + \text{int}(C)) \\ &= (\hat{T}(X_1) + \hat{f}(\hat{A})) \setminus (\hat{T}(X_1) + \hat{f}(\hat{A}) + \text{int}(C)) \\ &= (\hat{T}(X_1) + \hat{f}(\hat{A})) \setminus (\hat{T}(X_1) + \hat{f}(\hat{A}) + (\hat{T}(X_1) \oplus Z) \cap \text{int}(C)) \\ &= (\hat{T}(X_1) + \hat{f}(\hat{A})) \setminus (\hat{T}(X_1) + \hat{f}(\hat{A}) + \text{int}_Z(C_Z)). \end{aligned}$$

Noting that $\hat{f}(\hat{A}) \subset Z$ and $\hat{T}(X_1) \cap Z = \{0\}$, it follows that

$$\text{WE}(f(A), C) = \hat{T}(X_1) + \hat{f}(\hat{A}) \setminus (\hat{f}(\hat{A}) + \text{int}_Z(C_Z)) = \hat{T}(X_1) + \text{WE}(\hat{f}(\hat{A}), C_Z).$$

This shows the first equality of (5.11). To prove the second equality of (5.11), let $x_2 \in \hat{S}^w$. Then $x_2 \in \hat{A}$ and $\hat{f}(x_2) \in \text{WE}(\hat{f}(\hat{A}), C_Z)$. Hence,

$$X_1 + x_2 \subset X_1 + \hat{A} = A \text{ and } f(X_1 + x_2) = \hat{T}(X_1) + \hat{f}(x_2) \subset \text{WE}(f(A), C)$$

(thanks to (5.12) and the first equality of (5.11)). It follows that $X_1 + x_2 \subset S^w$ and so $X_1 + \hat{S}^w \subset S^w$. Conversely, let $x \in S^w$. Then there exists $(x_1, x_2) \in X_1 \times \hat{A}$ such that $x = x_1 + x_2$ and $f(x_1 + x_2) \in \mathrm{WE}(f(A), C) = \hat{T}(X_1) + \mathrm{WE}(\hat{f}(\hat{A}), C_Z)$. Noting that $f(x_1 + x_2) \in f(X_1 + x_2) = \hat{T}(X_1) + \hat{f}(x_2)$, one has $\hat{f}(x_2) \in \mathrm{WE}(\hat{f}(\hat{A}), C_Z)$. It follows that $x_2 \in \hat{S}^w$ and $x = x_1 + x_2 \in X_1 + \hat{S}^w$. This shows that $S^w \subset X_1 + \hat{S}^w$. Hence the second equality of (5.11) holds. The proof is complete. □

THEOREM 5.2. *Let $(x_1, x_2) \in X_1 \times \hat{A}$. Then $f(x_1 + x_2) \in \mathrm{E}(f(A), C)$ if and only if $\hat{f}(x_2) \in \mathrm{E}(\hat{f}(\hat{A}), C_Z)$ and $C_Z = C \cap (\hat{T}(X_1) \oplus Z)$.*

Proof. By (5.12), $f(x_1 + x_2) \in \hat{T}(X_1) + \hat{f}(x_2)$ and $f(A) = \hat{T}(X_1) + \hat{f}(\hat{A})$. Hence

$$f(A) - f(x_1 + x_2) = \hat{T}(X_1) + \hat{f}(\hat{A}) - \hat{f}(x_2).$$

Noting that $\hat{f}(\hat{A}) - \hat{f}(x_2) \subset Z$, it follows that

$$(f(A) - f(x_1 + x_2)) \cap -C = (\hat{T}(X_1) + \hat{f}(\hat{A}) - \hat{f}(x_2)) \cap -(C \cap (\hat{T}(X_1) \oplus Z)).$$

Thus, from the definitions of the projection $\Pi_Z : \hat{T}(X_1) \oplus Z \to Z$ (see (5.7)), it is easy to verify that

$$(f(A) - f(x_1 + x_2)) \cap -C = \Pi_1(C \cap (\hat{T}(X_1) \oplus Z)) + (\hat{f}(\hat{A}) - \hat{f}(x_2)) \cap -C_Z,$$

where $\Pi_1(y + z) = y$ for all $(y, z) \in \hat{T}(X_1) \oplus Z$. Therefore, $f(x_1 + x_2) \in \mathrm{E}(f(A), C)$ is equivalent to

$$\Pi_1(C \cap (\hat{T}(X_1) \oplus Z)) + (\hat{f}(\hat{A}) - \hat{f}(x_2)) \cap -C_Z = \{0\}.$$

Since $\hat{T}(X_1) \cap Z = \{0\}$, it follows that $f(x_1 + x_2) \in \mathrm{E}(f(A), C)$ if and only if

$$\Pi_1(C \cap (\hat{T}(X_1) \oplus Z)) = (\hat{f}(\hat{A}) - \hat{f}(x_2)) \cap -C_Z = \{0\},$$

namely $C_Z = C \cap (\hat{T}(X_1) \oplus Z)$ and $\hat{f}(x_2) \in E(\hat{f}(\hat{A}), C_Z)$. The proof is complete. □

The following corollary is a consequence of Theorems 3.1 and 5.2.

COROLLARY 5.1. *Let $\hat{S}$ denote the Pareto solution set of piecewise linear problem $(\widehat{\mathrm{PLP}})$. The following statements hold:*
(i) If $C_Z \neq C \cap (\hat{T}(X_1) \oplus Z)$ then $S = \emptyset$.
(ii) If $C_Z = C \cap (\hat{T}(X_1) \oplus Z)$ then

$$S = X_1 + \hat{S} \ \text{ and } \ \mathrm{E}(f(A), C) = \hat{T}(X_1) + \mathrm{E}(\hat{f}(\hat{A}), C_Z).$$

Remark. *By Corollary 5.1(i) and Theorem 5.1(i), piecewise linear problem (PLP) has no Pareto solution when $C_Z \neq C \cap (\hat{T}(X_1) \oplus Z)$, and the weak Pareto solution set of (PLP) is just the entire feasible set A of (PLP) when $(\hat{T}(X_1) \oplus Z) \cap \mathrm{int}(C) = \emptyset$. Therefore, we only need to consider the Pareto solution set and the weak Pareto solution of (PLP) when $C_Z = C \cap (\hat{T}(X_1) \oplus Z)$ and $(\hat{T}(X_1) \oplus Z) \cap \mathrm{int}(C) \neq \emptyset$, respectively.*

In the framework of finite dimensional spaces, for $i \in \overline{1m}$, we consider the following linear subproblem

$$(\widehat{\mathrm{LP}})_i \qquad C_Z - \min \Pi_Z(T_i x + b_i) \ \text{ subject to } x \in \hat{A}_i.$$

By Theorem 5.1 and Corollary 5.1 (with linear problems $(\text{LP})_i$ and $(\widehat{\text{LP}})_i$ replacing respectively piecewise linear problems (PLP) and $(\widehat{\text{PLP}})$), we have the following result (thanks to (4.10)).

PROPOSITION 5.1. *For each $i \in \overline{1m}$, let S_i (resp. S_i^w) and $\hat{S}_i$ (resp. $\hat{S}_i^w$) denote the Pareto solution sets (resp. weak Pareto solution sets) of linear problem $(LP)_i$ and $(\widehat{\text{LP}})_i$, respectively. The following statements hold:*
(i) $S_i = \emptyset$ if $C_Z \neq C \cap (\hat{T}(X_1) \oplus Z)$.
(ii) $S_i = X_1 + \hat{S}_i$ if $C_Z = C \cap (\hat{T}(X_1) \oplus Z)$.
(iii) $S_i^w = A_i$ if $(\hat{T}(X_1) \oplus Z) \cap \text{int}(C) = \emptyset$.
(iv) $S_i^w = X_1 + \hat{S}_i^w$ if $(\hat{T}(X_1) \oplus Z) \cap \text{int}(C) \neq \emptyset$.

The following theorem establishes the structure of the weak Pareto solution set for piecewise linear problem $(\widehat{\text{PLP}})$.

THEOREM 5.3. *For each $i \in \overline{1m}$, let*

$$\hat{V}_i^w := \Pi_Z\big(T_i(\hat{A}_i) + b_i\big) \setminus (\hat{f}(\hat{A}) + \text{int}_Z(C_Z)), \tag{5.13}$$

$$\breve{S}_i := \hat{A}_i \cap (\Pi_Z \circ T_i)^{-1}(\hat{V}_i^w - \Pi_Z(b_i)) \tag{5.14}$$

and suppose that $(\hat{T}(X_1) \oplus Z) \cap \text{int}(C) \neq \emptyset$. Then the following statements hold:
(i) $\hat{S}^w = \bigcup_{i \in \bar{I}} \breve{S}_i$ and $\text{WE}(\hat{f}(\hat{A}), C_Z) = \bigcup_{i \in \bar{I}} \hat{V}_i^w$, where $I := \{i \in \overline{1m} : \ \hat{A}_i \neq \emptyset\}$.
(ii) If, in addition, the ordering cone C in Y is assumed to be polyhedral, then for each $i \in \bar{I}$ there exist finitely many polyhedra $\hat{P}_{i1}, \cdots, \hat{P}_{iq_i}$ in X_2 and faces $\hat{F}_{i1}, \cdots, \hat{F}_{iq_i}$ of $\hat{A}_i$ such that $\breve{S}_i = \bigcup_{j=1}^{q_i} \hat{P}_{ij}$ and $\hat{P}_{ij} \subset \hat{F}_{ij} \subset \hat{S}_i^w$ for all $j \in \overline{1q_i}$. Consequently, $\hat{S}^w$ is the union of finitely many polyhedra in X_2, each one of which is contained in a weak Pareto face of some linear subproblem $(\widehat{\text{LP}})_i$.

Proof. Let i be an arbitrary element in $\bar{I}$. Since $\hat{f}(\hat{x}) = \Pi_Z(T_i(\hat{x})) + \Pi_Z(b_i)$ for all $\hat{x} \in \hat{A}_i$, $(\Pi_Z \circ T_i)^{-1}(\hat{V}_i^w - \Pi_Z(b_i)) = \hat{f}^{-1}(\hat{V}_i^w)$. Hence, by (5.13) and (5.14),

$$\breve{S}_i = \hat{A}_i \cap \hat{f}^{-1}(\hat{V}_i^w) \quad \text{and} \quad \hat{V}_i^w = \hat{f}(\breve{S}_i). \tag{5.15}$$

Thus, to prove (i), it suffices to show that $\breve{S}_i = \hat{S}^w \cap \hat{A}_i$ (because $\hat{A} = \bigcup_{i \in \bar{I}} \hat{A}_i$ and $\hat{f}(\hat{S}^w) = \text{WE}(\hat{f}(\hat{A}), C_Z)$). To do this, let $\hat{a}_i \in \hat{A}_i \cap \hat{S}^w$. Then $\hat{f}(\hat{a}_i) \in \text{WE}(\hat{f}(\hat{A}), C_Z)$, that is, $\hat{f}(\hat{a}_i) \notin \hat{f}(\hat{A}) + \text{int}_Z(C_Z)$. Since

$$\hat{f}(\hat{a}_i) = \Pi_Z(T_i(\hat{a}_i) + b_i) \in \Pi_Z(T_i(\hat{A}_i) + b_i),$$

this and (5.13) imply that $\hat{f}(\hat{a}_i) \in \hat{V}_i^w$. Hence $\hat{a}_i \in \breve{S}_i$ (thanks to (5.15)). This shows that $\hat{A}_i \cap \hat{S}^w \subset \breve{S}_i$. Conversely, let $\hat{a}_i \in \breve{S}_i$. Then, by (5.14), $\Pi_Z(T_i\hat{a}_i) \in \hat{V}_i^w - \Pi_Z(b_i)$, namely, $\hat{f}(\hat{a}_i) \in \hat{V}_i^w$. Hence, by (5.13), $\hat{f}(\hat{a}_i) \notin \hat{f}(\hat{A}) + \text{int}_Z(C_Z)$. Noting that $\hat{a}_i \in \hat{A}_i \subset \hat{A}$, it follows that $\hat{f}(\hat{a}_i) \in \text{WE}(\hat{f}(\hat{A}), C_Z)$, and so $\hat{a}_i \in \hat{A}_i \cap \hat{f}^{-1}(\text{WE}(\hat{f}(\hat{A}), C_Z)) = \hat{A}_i \cap \hat{S}^w$. This shows that $\breve{S}_i \subset \hat{A}_i \cap \hat{S}^w$. Therefore, $\breve{S}_i = \hat{A}_i \cap \hat{S}^w$. The proof of (i) is complete.

To prove (ii), suppose that the ordering cone C is polyhedral. Then, since the projection mapping $\Pi_Z : \hat{T}(X_1) \oplus Z \to Z$ is a linear operator and Z is finite dimensional, $C_Z = \Pi_Z((\hat{T}(X_1) \oplus Z) \cap C)$ is a polyhedral cone in Z (thanks to [16, Theorem 19.3] and Proposition 2.1). On the other hand, by the assumption that $(\hat{T}(X_1) \oplus Z) \cap \mathrm{int}(C) \neq \emptyset$, Lemma 5.1 implies that $\mathrm{int}_Z(C_Z) = \Pi_Z((\hat{T}(X_1) \oplus Z) \cap \mathrm{int}(C)) \neq \emptyset$. Since $\Pi_Z(T_j(\hat{A}_j) + b_j)$ and C_Z are polyhedra in the finite dimensional space Z, their sum $\Pi_Z(T_j(\hat{A}_j) + b_j) + C_Z$ is a polyhedron in Z and so is closed. Hence

$$\Pi_Z(T_j(\hat{A}_j) + b_j) + C_Z = \mathrm{cl}(\Pi_Z(T_j(\hat{A}_j) + b_j) + \mathrm{int}_Z(C_Z)).$$

Noting that $\Pi_Z(T_j(\hat{A}_j) + b_j) + \mathrm{int}_Z(C_Z)$ is open in Z, it follows that

$$\mathrm{int}_Z(\Pi_Z(T_j(\hat{A}_j) + b_j) + C_Z) = \Pi_Z(T_j(\hat{A}_j) + b_j) + \mathrm{int}_Z(C_Z).$$

Thus, by Proposition 2.1, there exist $(z^*_{j1}, r_{j1}), \cdots, (z^*_{jq_j}, r_{jq_j})$ in $Z^* \times \mathbb{R}$ such that

$$\Pi_Z(T_j(\hat{A}_j) + b_j) + \mathrm{int}_Z(C_Z) = \{z \in Z : \ \langle z^*_{jk}, z\rangle < r_{jk}, \ k = 1 \cdots, q_j\}. \tag{5.16}$$

Since $\hat{A} = \bigcup_{j \in \bar{I}} \hat{A}_j$, it follows from (5.13) that

$$\begin{aligned}
\hat{V}^w_i &= \Pi_Z(T_i(\hat{A}_i) + b_i) \setminus \left(\bigcup_{j\in\bar{I}} (\hat{f}(\hat{A}_j) + \mathrm{int}_Z(C_Z) \right) \\
&= \Pi_Z(T_i(\hat{A}_i) + b_i) \setminus \left(\bigcup_{j\in\bar{I}} (\Pi_Z(T_j(\hat{A}_j) + b_j) + \mathrm{int}_Z(C_Z) \right) \\
&= \Pi_Z(T_i(\hat{A}_i) + b_i) \setminus \left(\bigcup_{j\in\bar{I}} \bigcap_{k=1}^{q_j} \{z \in Z : \ \langle z^*_{jk}, z\rangle < r_{jk}\} \right) \\
&= \bigcap_{j\in\bar{I}} \bigcup_{k=1}^{q_j} \Big(\Pi_Z(T_i(\hat{A}_i) + b_i) \setminus \{z \in Z : \ \langle z^*_{jk}, z\rangle < r_{jk}\} \Big) \\
&= \bigcap_{j\in\bar{I}} \bigcup_{k=1}^{q_j} \Big(\Pi_Z(T_i(\hat{A}_i) + b_i) \cap \{z \in Z : \ \langle z^*_{jk}, z\rangle \geq r_{jk}\} \Big).
\end{aligned}$$

Since $\bar{I}$ is a subset of $\overline{1m}$, we assume without loss of generality that there exists $n \in \overline{1m}$ such that $\bar{I} = \overline{1n}$. For any $(k_1, \cdots, k_n) \in \overline{1q_1} \times \cdots \times \overline{1q_n}$, let

$$Q^i_{(k_1,\cdots,k_n)} := \bigcap_{j=1}^{n} \Big(\Pi_Z(T_i(\hat{A}_i) + b_i) \cap \{z \in Z : \ \langle z^*_{jk_j}, z\rangle \geq r_{jk_j}\} \Big).$$

Then, each $Q^i_{(k_1,\cdots,k_n)}$ is a polyhedron in Z and

$$\hat{V}^w_i = \bigcup_{(k_1,\cdots,k_n)\in\Pi_i} Q^i_{(k_1,\cdots,k_n)}, \tag{5.17}$$

where $\Pi_i := \{(k_1, \cdots, k_n) \in \overline{1q_1} \times \cdots \times \overline{1q_n} : Q^i_{(k_1,\cdots,k_n)} \neq \emptyset\}$. Let

$$\hat{P}^i_{(k_1,\cdots,k_n)} := \hat{A}_i \cap (\Pi_Z \circ T_i)^{-1}(Q^i_{(k_1,\cdots,k_n)} - \Pi_Z(b_i)) \quad \forall (k_1, \cdots, k_n) \in \Pi_i.$$

Then each $\hat{P}^i_{(k_1,\cdots,k_n)}$ is a polyhedron in the finite dimensional space X_2 and

$$\breve{S}_i = \hat{A}_i \cap (\Pi_Z \circ T_i)^{-1}(\hat{V}^w_i - \Pi_Z(b_i)) = \bigcup_{(k_1,\cdots,k_n)\in\Pi_i} \hat{P}^i_{(k_1,\cdots,k_n)}. \tag{5.18}$$

Thus, to prove (ii), it suffices to show that for each $(k_1, \cdots, k_n) \in \Pi_i$ there exists a face $\hat{F}$ of $\hat{A}_i$ such that $\hat{P}^i_{(k_1,\cdots,k_n)} \subset \hat{F} \subset \hat{S}^w_i$. By Theorem ABB (applied to linear problem $(\widehat{\mathrm{LP}})_i$), there exist finitely many faces $\hat{F}_{i1} \cdots, \hat{F}_{i\nu_i}$ of $\hat{A}_i$ such that $\hat{S}^w_i = \bigcup_{j=1}^{\nu_i} \hat{F}_{ij}$. Noting that each $\hat{P}^i_{(k_1,\cdots,k_n)}$ is contained in $\hat{S}^w_i$ (thanks to (i) and (5.18)), it follows from Proposition 2.4 that $\hat{P}^i_{(k_1,\cdots,k_n)} \subset \hat{F}_{ij'}$ for some $j' \in \overline{1\nu_i}$. The proof is complete. □

Theorem 4.2 is immediate from Theorem 5.1 and 5.3. To prove the structure theorem (Theorem 4.3) of the Pareto solution set of (PLP), we need the following lemma, which is a variant of a formula appearing in the proof of [22, Theorem 3.4].

Lemma 5.2. *Let $B_1, \cdots, B_m$ be subsets of Y. Then*

$$\mathrm{E}(\bigcup_{i\in\overline{1m}} B_i, C) = \bigcup_{i\in\overline{1m}} \bigcap_{j\in\overline{1m}} (\mathrm{E}(B_i, C) \setminus ((B_j + C) \setminus \mathrm{E}(B_j, C))).$$

Proof. Let $B := \bigcup_{i\in\overline{1m}} B_i$ and $E_i := \bigcap_{j\in\overline{1m}} (\mathrm{E}(B_i, C) \setminus ((B_j + C) \setminus \mathrm{E}(B_j, C)))$ for all $i \in \overline{1m}$. We need to show $\mathrm{E}(B, C) = \bigcup_{i=1}^{m} E_i$. For each $y' \in \mathrm{E}(B, C)$, there exists $i' \in \overline{1m}$ such that $y' \in B_{i'}$ and so $y' \in E(B_{i'}, C)$. Since $(B_j + C) \cap \mathrm{E}(B, C) \subset \mathrm{E}(B_j, C)$ for all $j \in \overline{1m}$, $y' \in \mathrm{E}(B_j, C)$ for all $j \in \overline{1m}$ with $y' \in B_j + C$. It follows that $y' \notin (B_j + C) \setminus \mathrm{E}(B_j, C)$ for all $j \in \overline{1m}$. Hence $y' \in E(B_{i'}, C) \setminus ((B_j + C) \setminus \mathrm{E}(B_j, C)))$ for all $j \in \overline{1m}$, that is, $y' \in E_{i'}$. This shows that $\mathrm{E}(B, C) \subset \bigcup_{i\in\overline{1m}} E_i$. Conversely, let $y \in \bigcup_{i=1}^{m} E_i$. Then there exists $i_0 \in \overline{1m}$ such that $y \in E_{i_0}$. Let $z \in B \cap (y - C)$. We only need to show $z = y$. Take $j \in \overline{1m}$ such that $z \in B_j$. It follows that $z \in B_j \cap (y - C)$. Noting that $E_{i_0} \subset \mathrm{E}(B_{i_0}, C)$, it is clear that $z = y$ if $j = i_0$. Now suppose that $j \neq i_0$. By the definition of E_{i_0}, one has $y \in E(B_{i_0}, C) \setminus ((B_j + C) \setminus \mathrm{E}(B_j, C))$, and so $y \notin (B_j + C) \setminus \mathrm{E}(B_j, C)$. Since $y \in z + C \subset B_j + C$, $y \in \mathrm{E}(B_j, C)$, and so $\{y\} = B_j \cap (y - C) \ni z$. This shows that $y = z$. The proof is complete. □

Proposition 5.2. *Let $\hat{S}$ and $\hat{S}_i$ $(i \in \bar{I} := \{i \in \overline{1m} : \hat{A}_i \neq \emptyset\})$ denote the Pareto solution set of piecewise linear problem $(\widehat{\mathrm{PLP}})$ and linear subproblem $(\widehat{\mathrm{LP}})_i$, respectively. Suppose that the ordering cone C is polyhedral. Then there exist finitely many generalized polyhedra $\hat{F}_1, \cdots, \hat{F}_p$ in X_2 such that the following statements hold:*
(i) $\hat{S} = \bigcup_{k=1}^{p} \hat{F}_k$.
(ii) For each $k \in \overline{1p}$ there exist $i \in \bar{I}$ and a face $\hat{F}$ of $\hat{A}_i$ such that $\hat{F}_k \subset \hat{F} \subset \hat{S}_i$.

Proof. For each $i \in \bar{I}$, let $\tilde{S}_i := \hat{A}_i \cap \hat{S}$. Then $\hat{S} = \bigcup_{i\in\bar{I}} \tilde{S}_i$, and $\tilde{S}_i$ is clearly contained in the Pareto solution set $\hat{S}_i$ of linear subproblem $(\widehat{\mathrm{LP}})_i$. Thus, by Theorem ABB and Proposition 2.4, it suffices to show that there exist finitely many generalized polyhedra $\hat{G}_{i1}, \cdots, \hat{G}_{i\nu_i}$ in X_2 such that $\tilde{S}_i = \bigcup_{k=1}^{\nu_i} \hat{G}_{ik}$. Noting that $\hat{f}|_{\hat{A}_i} = \Pi_Z \circ f|_{\hat{A}_i} = \Pi_Z \circ T_i|_{\hat{A}_i} + \Pi_Z(b_i)$, one has

$$\text{(5.19)}\quad \tilde{S}_i = \hat{A}_i \cap \hat{f}^{-1}(\mathrm{E}(\hat{f}(\hat{A}), C_Z)) = \hat{A}_i \cap (\Pi_Z \circ T_i)^{-1}(\mathrm{E}(\hat{f}(\hat{A}), C_Z) - \Pi_Z(b_i)).$$

Since C is a polyhedral cone in Y, $C \cap (\hat{T}(X_1) \oplus Z)$ is a polyhedral cone in $\hat{T}(X_1) \oplus Z$. Hence $C_Z = \Pi_Z(C \cap (\hat{T}(X_1) \oplus Z))$ is a polyhedral cone in the finite dimensional space Z. It follows that $B_j + C_Z$ is a polyhedron in Z and $\mathrm{E}(B_j, C_Z) = \mathrm{E}(B_j + C_Z, C_Z)$ is the union of finitely many polyhedra in Z for each $j \in \bar{I}$ (thanks to Theorem ABB), where $B_j := \Pi_Z(T_j(\hat{A}_j) + b_j)$. Hence $E_i := \bigcap_{j\in\bar{I}} \mathrm{E}(B_i, C_Z) \setminus \big(B_j + C_Z) \setminus \mathrm{E}(B_j, C_Z)\big)$ is the union of finitely many generalized polehedra in Z for all $i \in \bar{I}$. Since

$$\hat{f}(\hat{A}) = \bigcup_{i\in\bar{I}} \hat{f}(\hat{A}_i) = \bigcup_{i\in\bar{I}} B_i,$$

This and Lemma 5.2 imply that $\mathrm{E}(\hat{f}(\hat{A}), C_Z) = \bigcup_{i\in\bar{I}} E_i$ and so $\mathrm{E}(\hat{f}(\hat{A}), C_Z)$ is the union of finitely many generalized polyhedra in Z. Thus, by (5.19), for each $i \in \bar{I}$ there exist finitely many generalized polyhedra $\hat{G}_{i1}, \cdots, \hat{G}_{i\nu_i}$ in X_2 such that $\tilde{S}_i = \bigcup_{k=1}^{\nu_i} \hat{G}_{ik}$. The proof is complete. □

Clearly, Theorem 4.3 follows from Corollary 5.1 and Propositions 5.2 and 2.2.

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