The Lipschitz-like Property Relative to a Set with Applications

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Outline

Recovery bounds: In many applications in compressive sensing, machine learning, pattern analysis and graphical modeling, the underlying data usually can be represented approximately by an under-determined linear measurement

$$Ax = b + e.$$

where $m \ll n$ and e is an unknown noise vector.

A sparse vector $\bar{x} \in R^n$ satisfying $A\bar{x} = b$ and $\|\bar{x}\|_0 \leq s$ is required to be recovered (the ℓ_0 norm $\|x\|_0$ being the number of nonzero components of x).

The sparse optimization problem can be modeled as

$$\begin{array}{ll} \min & \|x\|_0 \\ \text{s.t.} & \|Ax - b\|_2 \leq \epsilon. \end{array}$$

where ϵ is the size of the noise vector e in the linear measurement.

To recover $\bar{x}$, one instead solves a better ℓ_1-norm (convex) constrained optimization problem

$$\begin{array}{ll} \min & \|x\|_1 \\ \text{s.t.} & \|Ax - b\|_2 \leq \epsilon. \end{array}$$

Under
- the mutual incoherence property [Candès et al (2006)] or
- the restricted isometry property [Donoho et al (2006)],

it has been shown that

$$\|x^*(\epsilon) - \bar{x}\|_2 \leq C\epsilon, \quad \forall \text{ small } \epsilon \geq 0. \tag{1}$$

The uniqueness of the optimal solution of the ℓ_1-norm problem has also been extensively investigated under the full column rank condition on the support index set $\text{supp}(x^*(\epsilon)) := \{i | x_i^*(\epsilon) \neq 0\}$, see [Zhang et al (2015)] and references therein.

The ℓ_1 regularization model or Lasso is formulated as follows:

$$\min_{x\in\mathbb{R}^n} \|Ax - b\|_2^2 + \lambda\|x\|_1.$$

The recovery bound is

$$||x_1^*(\lambda) - \bar{x}||_2 = O(\lambda^2 s) \tag{2}$$

See

- [Zhang (2009)] under the restricted isometry property;
- [Meinshausen & Yu (2009)] under the incoherent design condition and
- [Bickel et al (2009)] under the restricted eigenvalue condition.

Consider the ℓ_q regularization problem $(0 < q < 1)$:

$$\min_{x\in\mathbb{R}^n} \|Ax - b\|_2^2 + \lambda\|x\|_q^q, \text{ where } \|x\|_q = \left(\sum_{i=1}^n |x_i|^q\right)^{1/q}.$$

In [Hu et al (2017)], we obtained the following results:

Global recovery bound: if $x^*(\lambda)$ is a global solution of the ℓ_q regularization problem, then

$$\|x^*(\lambda) - \bar{x}\|_2 \le O(\lambda^{\frac{1}{2-q}} s). \tag{3}$$

Local recovery bound: There exist $\kappa > 0$ and a path of local minima of the ℓ_q regularization problem, $x^*(\lambda)$, such that, for $\lambda < \kappa$,

$$\|x^*(\lambda) - \bar{x}\|_2 \le O(\lambda s). \tag{4}$$

Properties (1), (2), (3) and (4) are known as a calmness property of the solution path $x^*(\epsilon)$ or $x^*(\lambda)$ at $\epsilon = 0$ or $\lambda = 0$ relative to the set $\mathbb{R}_+$.

Weakened constraint qualification:

In order to guarantee some stationarity conditions one may only need a regular behavior of the constraint systems with respect to one single critical direction, not on the whole space. See [Gfrerer (2013)].

Stability analysis is to determine intuitively verifiable conditions to guarantee the accuracy of the solutions with respect to the degree of approximation of the initial data.

The literature on the subject is vast when the study is of global nature.

However, the study for stability properties relative to a set is limited.

Lipschitz-like property is one of the properties at the center of stability theory. It is also called Aubin property, pseudo-Lipschitz property, introduced by [Aubin (1994)].

Mordukhovich criterion

Let $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a multifunction. The graph of multifunction $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is defined by

$$\text{gph}S = \{(x, y) | y \in S(x), x \in \text{dom}S\}$$

The coderivative of S at $\bar{x}$ for any $\bar{u} \in S(\bar{x})$ is the multifunction $D^*S(\bar{x}|\bar{u}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ defined by

$$v \in D^*S(\bar{x}|\bar{u})(y) \Longleftrightarrow (v, -y) \in N_{\text{gph}S}(\bar{x}, \bar{u}), \ \forall y \in \mathbb{R}^m$$

The well-known Mordukhovich criterion says that S has the Lipschitz-like property at $\bar{x}$ for $\bar{u}$ if and only if

$$D^*S(\bar{x}|\bar{u})(0) = \{0\} \tag{5}$$

Mordukhovich criterion was initially developed by [Mordukhovich (1993)]. A more direct proof by using the basic variational analysis tools was given in [Rockafellar and Wets (1998)].

Applications of Mordukhovich criterion:

(i) The solution mapping of generalized equations [Levy and Mordukhovich (2004)].

(ii) The solution mapping of linear semi-infinite and infinite systems in [Cánovas et al (2009)].

(iii) The stationary set of minimizing a quadratic function with a ball constraint in [Lee and Yen (2014)].

(iv) The solution mapping of a parametric linear constraint system in [Huyen and Yen (2016)].

Lipschitz-like property via other tools:

Critical face condition: The solution mapping of a linear variational inequality and nonlinear variational inequality (via linearization) over a polyhedral set in [Dontchev and Rockafellar (1996)].

Directional limiting coderivative: Implicit set-valued mappings in [Gfrerer and Outrata (2016)].

Determinantal condition: The solution mapping of linear variational inequality over perturbed polyhedral covex cones [Lu and Robinson (2008)].

Lipschitz-like property relative to a closed set:

Sufficient conditions for the solution map for a class of parameterized variational systems with constraints depending, apart from the parameter, also on the solution itself, including the quasi-variational inequality, were obtained in [Benko et al (2019)].

Other stability properties relative to a set:

- Metric regularity relative to a set, see Arutyunov and Izmailov (2006), Huynh and Thera (2015), Ioffe (2010), by using strong slopes.

- Directional metric (sub)regularity and isolated calmness relative to a set, see Gfrerer (2013) and Benko et al (2019), by using directional limiting coderivative.

Definition

A mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ has the **Lipschitz-like property relative to** X at $\bar{x}$ for $\bar{u}$, where $\bar{x} \in X$ and $\bar{u} \in S(\bar{x})$, if gph$S$ is locally closed at $(\bar{x}, \bar{u})$ and there are neighborhoods $V \in \mathcal{N}(\bar{x})$, $W \in \mathcal{N}(\bar{u})$, and a constant $\kappa \in \mathbb{R}_+$ such that

$$S(x') \cap W \subset S(x) + \kappa \|x' - x\| \mathbf{B} \quad \forall x, x' \in X \cap V.$$

If $X = \mathbb{R}^n$, then S has the Aubin property at $\bar{x}$ for $\bar{u}$.

The graphical modulus of S relative to X at $\bar{x}$ for $\bar{u}$ is then

$$\begin{aligned} \mathrm{lip}_X S(\bar{x} \mid \bar{u}) := \inf \ \{ \ \kappa \geq 0 \mid \exists V, W, \ &\text{such that} \\ &S(x') \cap W \subset S(x) + \kappa \|x' - x\| \mathbf{B} \quad \forall x, x' \in X \cap V \ \}. \end{aligned}$$

For a set $X \subset \mathbb{R}^n$, we denote the restricted mapping of S on X by

$$S|_X := \begin{cases} S(x) & \text{if } x \in X, \\ \emptyset & \text{if } x \notin X. \end{cases}$$

It is clear to see that

$$\operatorname{gph} S|_X = \operatorname{gph} S \cap (X \times \mathbb{R}^m) \quad \text{and} \quad \operatorname{dom} S|_X = X \cap \operatorname{dom} S.$$

Theorem (Necessity)

Let $X \subset \mathbb{R}^n$ be a closed set. Consider $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, $\bar{x} \in X \subset \mathbb{R}^n$, $\bar{u} \in S(\bar{x})$ and $\kappa \geq 0$. If S has the Lipschitz-like property relative to X at $\bar{x}$ for $\bar{u}$ with constant κ, then the inequality

$$\max_{w \in T_X(x) \cap \mathcal{S}} \langle x^*, w \rangle \leq \kappa \|u^*\|$$

holds for all (x, u) close enough to $(\bar{x}, \bar{u})$ in $\operatorname{gph} S|_X$ and $x^* \in \widehat{D}^* S|_X(x \mid u)(u^*)$, where $\mathcal{S}$ is the unit sphere.

Theorem (Sufficiency)

Let $X \subset \mathbb{R}^n$ be a closed and convex set. Consider $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, $\bar{x} \in X \subset \mathbb{R}^n$, $\bar{u} \in S(\bar{x})$ and $\tilde{\kappa} > \kappa > 0$. Suppose that $\operatorname{gph} S$ is locally closed at $(\bar{x}, \bar{u})$ and that X is closed. If the inequality

$$\max_{w \in \operatorname{cl} \operatorname{pos}(X - x) \cap \mathcal{S}} \langle x^*, w \rangle \leq \kappa \|u^*\|$$

holds for all (x, u) close enough to $(\bar{x}, \bar{u})$ in $\operatorname{gph} S|_X$ and $x^* \in D^* S|_X(x \mid u)(u^*)$, then S has the Lipschitz-like property relative to X at $\bar{x}$ for $\bar{u}$ with constant $\tilde{\kappa}$.

Theorem (Lipschitz-like property relative to closed convex sets)

Consider $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, $\bar{x} \in X \subset \mathbb{R}^n$ *and* $\bar{u} \in S(\bar{x})$. *Suppose that* $\operatorname{gph} S$ *is locally closed at* $(\bar{x}, \bar{u})$ *and that* X *is closed and convex. The following properties are equivalent:*

(a) S *has the Lipschitz-like property relative to* X *at* $\bar{x}$ *for* $\bar{u}$.

(b) *There is some* $\kappa \geq 0$ *such that the following inequality holds for all* (x, u) *close enough to* $(\bar{x}, \bar{u})$ *in* $\operatorname{gph} S|_X$:

$$\|\operatorname{proj}_{T_X(x)}(x^*)\| \leq \kappa \|u^*\| \quad \forall x^* \in D^* S|_X(x \mid u)(u^*).$$

Moreover, we have

$$lip_X S(\bar{x} \mid \bar{u}) = \limsup_{(x,u) \xrightarrow{\operatorname{gph} S|_X} (\bar{x},\bar{u})} \sup_{u^* \in \mathbb{B}} \sup_{x^* \in D^* S|_X(x|u)(u^*)} \|proj_{T_X(x)}(x^*)\|.$$

Definition (Projectional coderivatives)

Consider a mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and a point $\bar{x} \in X \subset \mathbb{R}^n$. The projectional coderivative of S at $\bar{x}$ for any $\bar{u} \in S(\bar{x})$ with respect to X is the mapping $D^*_X S(\bar{x} \mid \bar{u}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ defined by

$$x^* \in D^*_X S(\bar{x} \mid \bar{u})(u^*)$$
$$\Longleftrightarrow (x^*, -u^*) \in \limsup_{\substack{(x,u) \longrightarrow (\bar{x},\bar{u}) \\ (x,u) \in \operatorname{gph} S|_X}} \operatorname{proj}_{T_X(x)\times\mathbb{R}^m} N_{\operatorname{gph} S|_X)}(x,u).$$

That is, $x^* \in D^*_X S(\bar{x} \mid \bar{u})(u^*)$ if and only if there are some $(x_k, u_k) \xrightarrow{\operatorname{gph} S|_X} (\bar{x}, \bar{u})$ and $x_k^* \in D^* S|_X(x_k \mid u_k)(u_k^*)$ such that $u_k^* \to u^*$ and $\operatorname{proj}_{T_X(x_k)}(x_k^*) \to x^*$.

Theorem (generalized Mordukhovich criterion)

Consider $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, $\bar{x} \in X \subset \mathbb{R}^n$ *and* $\bar{u} \in S(\bar{x})$. *Suppose that* $\operatorname{gph} S$ *is locally closed at* $(\bar{x}, \bar{u})$ *and that* X *is closed and convex. The following properties are equivalent:*

(a) S *has the Lipschitz-like property relative to* X *at* $\bar{x}$ *for* $\bar{u}$.

(b) $D^*_X S(\bar{x} \mid \bar{u})(0) = \{0\}$.

(c) $|D^*_X S(\bar{x} \mid \bar{u})|^+ < +\infty$.

Furthermore, we have

$$\operatorname{lip}_X S(\bar{x} \mid \bar{u}) = |D^*_X S(\bar{x} \mid \bar{u})|^+.$$

Remark

In the case of $\bar{x} \in \operatorname{int} X$, *we have for all* $\bar{u} \in S(\bar{x})$,

$$D_X^* S(\bar{x} \mid \bar{u}) = D^* S(\bar{x} \mid \bar{u}),$$

the results in Theorem 6 as well as in Theorem 4 recover the Mordukhovich criterion for the 'classical' Lipschitz-like property with no restriction on any set, see [Rockafellar and Wets (1998), Theorem 9.40].

Remark (projectional coderivatives of smooth mappings with respect to sets with simple structures)

Consider a smooth, single-valued mapping $F : \mathbb{R}^n \to \mathbb{R}^m$.
Case 1. X *is an affine set:* $X := \{x \in \mathbb{R}^n \mid Bx = b\}$, *where* B *is an* $m \times n$ *matrix and* $b \in \mathbb{R}^m$, *we have for all* $\bar{x} \in \operatorname{bdry} X$,

$$D_X^* F(\bar{x})(y) = \operatorname{proj}_{\ker B}(\nabla F(\bar{x})^* y), \textit{where } \ker B := \{x \in \mathbb{R}^n \mid Bx = 0\}.$$

Case 2. X *is a closed half-space:* $X := \{x \mid \langle a, x\rangle \le \beta\}$, *we have for all* $\bar{x} \in \operatorname{bdry} X$,

$$D_X^* F(\bar{x})(y) = \begin{cases} \left[\nabla F(\bar{x})^* y, \ \ \operatorname{proj}_{[a]^\perp}(\nabla F(\bar{x})^* y)\right] & \textit{if } \langle \nabla F(\bar{x})^* y, a\rangle \le 0, \\ \left\{\nabla F(\bar{x})^* y, \ \ \operatorname{proj}_{[a]^\perp}(\nabla F(\bar{x})^* y)\right\} & \textit{if } \langle \nabla F(\bar{x})^* y, a\rangle > 0. \end{cases}$$

The directional limiting normal cone to Ω in direction u at $\bar{x}$ is defined by

$$N_\Omega(\bar{x}; u) := \limsup_{t \downarrow 0,\ u' \to u} \widehat{N}_\Omega(\bar{x} + tu').$$

The directional limiting coderivative of S in the direction (x, u) at $(\bar{x}, \bar{u})$.

$$D^*S((\bar{x}, \bar{u}); (x, u))(v^*) := \{u^* \in \mathbb{R}^n \mid (u^*, -v^*) \in N_{\mathrm{gph}\, S}((\bar{x}, \bar{u}); (x, u))\}.$$

Theorem ([Benko et al (2019), Theorem 3.5] in an explicit form)

Let $\bar{u} \in S(\bar{x})$. Assume that the following conditions are satisfied:

(i) *For every $x \in T_X(\bar{x})$ and $t_k \downarrow 0$, there exists $u \in \mathbb{R}^n$ such that*

$$\liminf_{k \to \infty} \frac{d((\bar{x} + t_k x, \bar{u} + t_k u), \mathrm{gph}\, S)}{t_k} = 0.$$

(ii) *For all $x \in T_X(\bar{x})$ and $(x, u) \in T_{\mathrm{gph}\, S}(\bar{x}, \bar{u})$, we have*

$$D^*S\left((\bar{x}, \bar{u}); (x, u)\right)(0) = \{0\}.$$

Then S has the Lipschitz-like property relative to X at $\bar{x}$ for $\bar{u}$.

Example

Let $M = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$ and $(\bar{q}, \bar{x}) = (0,0) \in \operatorname{gph} S$, where $S : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ is the solution mapping of a linear complementarity system:

$$S(q) := \left\{ x \in \mathbb{R}^2 \mid x \geq 0,\ Mx + q \geq 0,\ \langle x,\ Mx + q \rangle = 0 \right\}.$$

$$\operatorname{lip}_{\operatorname{dom} S} S(\bar{q} \mid \bar{x}) = \sqrt{\frac{3+\sqrt{5}}{2}} > 0 \ \text{ implies that } S \text{ does have}$$

the Lipschitz-like property relative to $\operatorname{dom} S$ at $\bar{q}$ for $\bar{x}$.

Let Q be a closed subset of $\operatorname{dom} S$ such that $\bar{q} \in Q$ and $q := (0, -1)^T$ $\in T_Q(\bar{q})$. We have

$$D^* S((\bar{q}, \bar{x}); (q, x))(0) = \mathbb{R}_- \times \{0\} \neq \{(0,0)^T\},$$

suggesting that [Benko et al (2019), Theorem 3.5] is not applicable.

Reinterpretation of subgradients: Given a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, a point $\bar{x} \in \mathbb{R}^n$ where f is finite and locally lsc, and a vector $\bar{v} \in \mathbb{R}^n$, the level-set mapping

$$S : \alpha \mapsto \{x \mid f(x) - \langle \bar{v}, x - \bar{x} \rangle \leq \alpha\}$$

fails to have the Lipschitz-like property at $f(\bar{x})$ for $\bar{x}$ if and only if $\bar{v} \in \partial f(\bar{x})$. See [Rockafellar and Wets (1998)].

This is done by applying the Mordukhovich criterion via the coderivative

$$[D^* S\,(f(\bar{x}) \mid \bar{x})]^{-1}(\lambda) = \begin{cases} \lambda(\partial f(\bar{x}) - \bar{v}) & \text{if } \lambda < 0, \\ -\partial^\infty f(\bar{x}) & \text{if } \lambda = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Assume f is convex and $\bar{v} \in \partial f(\bar{x})$. Then we have

$$\bar{x} \in \arg\min_{x \in \mathbb{R}^n} \{f(x) - \langle \bar{v}, x - \bar{x} \rangle\},$$

implying that $\mathrm{dom} S = \{\alpha \in \mathbb{R} \mid \alpha \geq f(\bar{x})\}$ and hence that $f(\bar{x}) \notin \mathrm{int}(\mathrm{dom} S)$, but $f(\bar{x}) \in \mathrm{brdy}(\mathrm{dom} S)$.

Let $f(\bar{x}) \in \text{bdry}(\text{dom} S)$.

The outer limiting subdifferential set of f at $\bar{x}$ w.r.t. $\bar{v}$ is defined by

$$\begin{aligned}\partial^{>}_{\bar{v}} f(\bar{x}) := \Big\{ \lim_{k\to+\infty} v_k \mid \exists x_k \to_f \bar{x},\ \forall k : \\ f(x_k) > f(\bar{x}) + \langle \bar{v}, x_k - \bar{x}\rangle \text{ and } v_k \in \partial f(x_k)\Big\}.\end{aligned}$$

In the case of $\bar{v} = 0$, $\partial^{>}_{\bar{v}} f(\bar{x})$ reduces to the outer limiting subdifferential set $\partial^{>} f(\bar{x})$, which has been studied extensively in the literature [Kruger et al (2010), Fabian et al (2010), Ioffe (2016), Cánovas et al (2016), Li et al (2017)].

The projectional coderivative formula of level-set mappings:

$$[D_X^* S(f(\bar{x}) \mid \bar{x})]^{-1}(\lambda) = \begin{cases} \lambda(\partial_{\bar{v}}^{\geq} f(\bar{x}) - \bar{v}) & \text{if } \lambda < 0, \\ -\partial^\infty \tilde{f}(\bar{x}) \cup -\operatorname{pos}(\partial \tilde{f}(\bar{x})) & \text{if } \lambda = 0, \\ \emptyset & \text{otherwise,} \end{cases}$$

and

$$|D_X^* S(f(\bar{x}) \mid \bar{x})|^+ = \frac{1}{d(\bar{v}, \partial_{\bar{v}}^{\geq} f(\bar{x}))},$$

where $\tilde{f}(x) := \max\{f(x) - \langle \bar{v}, x - \bar{x} \rangle, f(\bar{x})\}$.

Theorem (reinterpretation of structural subgradients)

The following properties are equivalent:

(a) *The level-set mapping S fails to have the Lipschitz-like property relative to X at $f(\bar{x})$ for $\bar{x}$.*

(b) $\bar{v} \in \partial^{\gtrless}_{\bar{v}} f(\bar{x})$.

Moreover,

$$\mathrm{lip}_X S(f(\bar{x}) \mid \bar{x}) = \frac{1}{d(\bar{v}, \partial^{\gtrless}_{\bar{v}} f(\bar{x}))}.$$

Example

Consider the absolute value function $f(x) = |x|$. By some direct calculation, we have $\partial f(0) = [-1, 1]$ and

$$\partial^{\gtrless}_{\bar{v}} f(0) = \begin{cases} \{-1, 1\} & \text{if } \bar{v} \in (-1, 1), \\ \{-1\} & \text{if } \bar{v} \in [1, +\infty), \\ \{1\} & \text{if } \bar{v} \in (-\infty, -1]. \end{cases}$$

So we have $\bar{v} \notin \partial^{\gtrless}_{\bar{v}} f(0)$ for all $\bar{v} \in \mathbb{R}$. Then, we assert that for all $\bar{v} \in \mathbb{R}$, the level-set mapping

$$S : \alpha \mapsto \{x \in \mathbb{R} \mid |x| - \bar{v}x \leq \alpha\}$$

has the Lipschitz-like property relative to $\mathbb{R}_+$ at 0 for 0.

We have

$$\operatorname{lip}_{\mathbb{R}_+} S(0 \mid 0) = \begin{cases} \frac{1}{\min\{1-\bar{v}, 1+\bar{v}\}} & \text{if } \bar{v} \in (-1, 1), \\ \frac{1}{1+\bar{v}} & \text{if } \bar{v} \in [1, +\infty), \\ \frac{1}{1-\bar{v}} & \text{if } \bar{v} \in (-\infty, -1]. \end{cases}$$

In contrast, S has the Lipschitz-like property (without relative to a set) at 0 for 0 if and only if $\bar{v} \notin [-1, 1]$. Moreover, we have

$$\operatorname{lip}_{\mathbb{R}_+} S(0 \mid 0) < \operatorname{lip} S(0 \mid 0) = \begin{cases} +\infty & \text{if } \bar{v} \in [-1, 1], \\ \frac{1}{\bar{v}-1} & \text{if } \bar{v} \in (1, +\infty), \\ \frac{1}{-1-\bar{v}} & \text{if } \bar{v} \in (-\infty, -1). \end{cases}$$

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Thank You for Your Attention.