Piecewise Multicriteria Programs with Applications in Finance

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The outline of the lecture:

1. Portfolio Selection Models

2. Multicriteria Piecewise Linear Programs

3. Bicriteria Linear Portfolio Optimization Programs

4. Conclusions

1. Portfolio Selection Models

Sharpe (1971) has remarked that “if the essence of the portfolio analysis problem could be adequately captured in a form suitable for linear programming methods, the prospect for practical application would be greatly enhanced.”

Harry Markowitz (1952) used the following variance (l_2 risk) of the random rate of return of the portfolio as the risk measure

$$\sigma_2(x) = E\left(\left[\sum_{j=1}^{n} x_j r_j - \sum_{j=1}^{n} x_j \bar{r}_j\right]^2\right) = \sum_{i,j=1}^{n} \sigma_{ij} x_i x_j,$$

where $\sigma_{ij} = E([r_i - \bar{r}_i][r_j - \bar{r}_j])$ is the correlation and formulated the **mean-variance model**:

$$\begin{aligned} \text{min} \quad & \frac{1}{2} \sum_{i,j=1}^{n} \sigma_{ij} x_i x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} \bar{r}_j x_j \geq \bar{r}, \ \sum_{j=1}^{n} x_j = M_0. \end{aligned}$$

This model has been the foundation of modern financial theory in last 60 years!

Assumption: $(r_1, \cdots, r_n)$ is multivariate normally distributed.

Konno and Yamazaki (1991)

• observed that most of the stock prices in Tokyo Stock Market are not normally nor even symmetrically distributed

• introduced the following l_1 risk (the Mean-Absolute Deviation (MAD)) of the portfolio as the risk measure

$$\sigma_1(x) = E\left(\left|\sum_{j=1}^{n} r_j x_j - \sum_{j=1}^{n} \bar{r}_j x_j\right|\right).$$

Let $\bar{r}_j = \sum_{t=1}^{T} \bar{r}_{jt}/T$. They formulated **the mean-l_1 risk model**:

$$\min_{x \geq 0} \quad \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{j=1}^{n} a_{jt} x_j\right|,$$

$$\text{subject to} \quad \sum_{j=1}^{n} \bar{r}_j x_j \geq \bar{r},\ \sum_{j=1}^{n} x_j = M_0,$$

where $a_{jt} = \bar{r}_{jt} - \bar{r}_j$ and $\bar{r}_{jt}$ is the expectation of random variable r_j during period $t, t = 1, \cdots, T$.

Using the criterion of maximizing the minimum return or minimizing the maximum loss in decision analysis, Young (1998) introduced the following **the minimax model**:

$$\begin{aligned} \max_{x \geq 0} \quad & M_p, \\ \text{subject to} \quad & \sum_{j=1}^{n} \bar{r}_{jt} x_j \geq M_p, \ t = 1, \cdots, T, \\ & \sum_{j=1}^{n} \bar{r}_j x_j \geq \bar{r}, \ \sum_{j=1}^{n} x_j = M_0, \end{aligned}$$

where $\bar{r}_j = \sum_{t=1}^{T} y_{jt}/T$ is the average return on stock j.

The method may also have logical advantages:

- when the returns are not normally distributed,
- when the investor has a strong form of risk aversion.

Motivated by H^∞ optimal control or the worst case analysis, Cai, Teo, Y. and Zhou (2000) introduced the following l_∞ risk measure:

$$\sigma_\infty(x) = \max_{1\le j\le n} E(|r_j x_j - \bar{r}_j x_j|)$$

Let $q_j = E(|r_j - \bar{r}_j|)$ and assume $x_j \ge 0$. Then

$$\sigma_\infty(x) = \max_{1\le j\le n} q_j x_j.$$

The mean-l_∞ risk model is

$$\min_{x\ge 0} \ \left(\max_{1\le j\le n} q_j x_j, -\sum_{j=1}^{n} \bar{r}_j x_j\right)$$

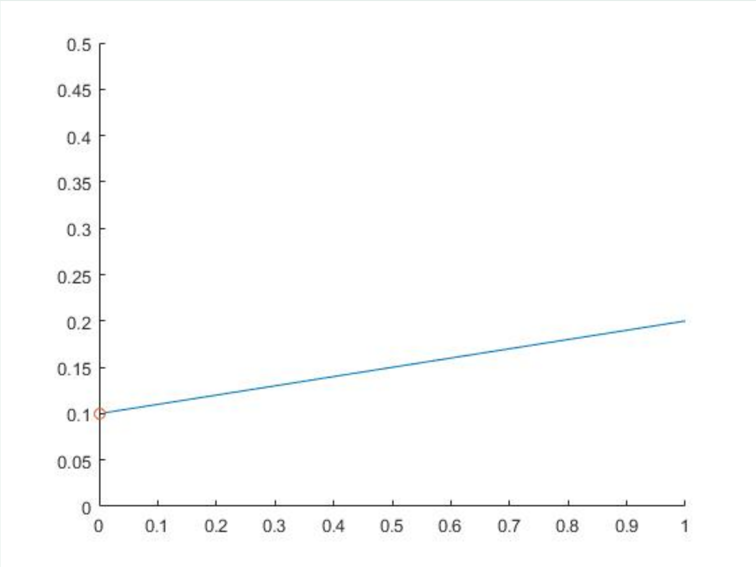
$$\text{subject to} \quad \textstyle\sum_{j=1}^{n} x_j = M_0.$$

When the transaction cost is considered, Fang, Meng and Y. (2012) studied the problem

$$\min_{x\geq 0} \; (\max_{1\leq j\leq n} q_j x_j, -\sum_{j=1}^{n} r_j x_j + \sum_{j=1}^{n} c_j^0(x_j))$$

$$\text{subject to} \quad \sum_{j=1}^{n}[x_j + c_j^0(x_j)] = M_0.$$

where the transaction cost $c_j^0(x_j)$ is plotted as:



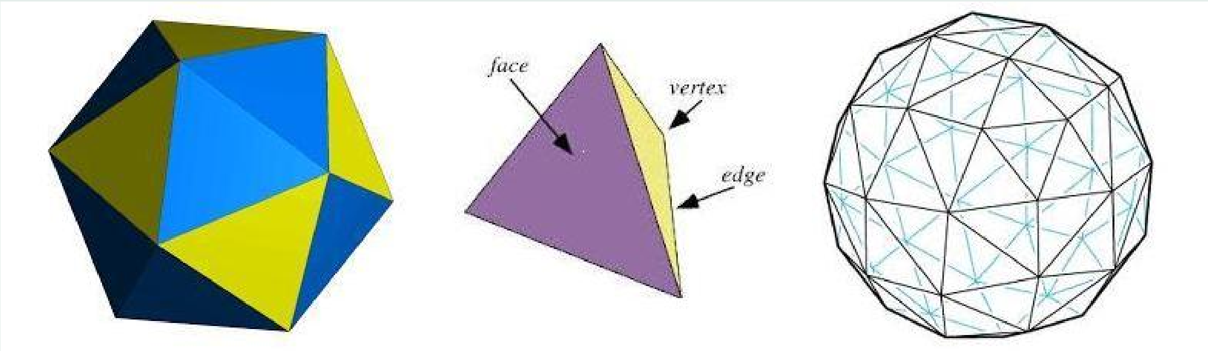
2. Multicriteria Piecewise Linear Programs

Definition 2.1 *A subset* P *of* R^n *is called a polyhedron if it is the intersection of finitely many closed half-spaces, i.e.,* $\exists\{x_1^*, x_2^*, \cdots, x_p^*\} \subset R^n$, $\{c_1, c_2, \cdots, c_p\} \subset R$ *such that*

$$P = \{x \in R^n : \langle x_i^*, x\rangle \leq c_i, 1 \leq i \leq p\}.$$

Definition 2.2 *A subset* C *of* R^n *is called a semi-closed polyhedron if it is the intersection of finitely many closed and/or open half-spaces, i.e.,* $\exists\{x_1^*, x_2^*, \cdots, x_q^*\} \subset R^n$, $\{c_1, c_2, \cdots, c_q\} \subset R$ *and* $0 \leq p \leq q$ *such that*

$$C = \{x \in R^n : \langle x_i^*, x\rangle \leq c_i, 1 \leq i \leq p\} \cap \{x \in R^n : \langle x_i^*, x\rangle < c_i, p < i \leq q\}.$$

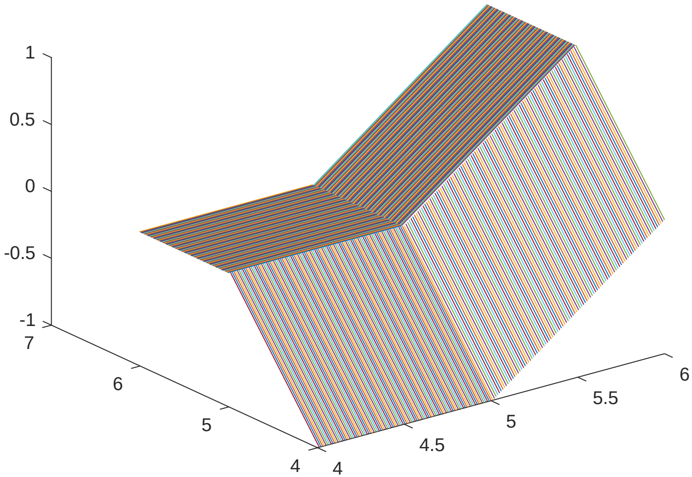


Definition 2.3 *(i) A function* $F : R^n \to R^m$ *is said to be piecewise linear if* $\exists$ *semi-closed polyhedra* $C_1, C_2, \cdots, C_l$ *in* R^n*, matrices* $T_1, T_2, \cdots, T_l$ *in* $R^{m\times n}$ *and vectors* $b_1, b_2, \cdots, b_l$ *in* R^m *such that*

$$R^n = \cup_{i=1}^{l} C_i \quad \textit{and} \quad F(x) = T_i x + b_i, \quad \forall x \in C_i \textit{ and } 1 \le i \le l.$$

(ii) If furthermore F *is continuous, then* F *is called a continuous piecewise linear function.*

(iii) Otherwise F *is called a discontinuous piecewise linear function.*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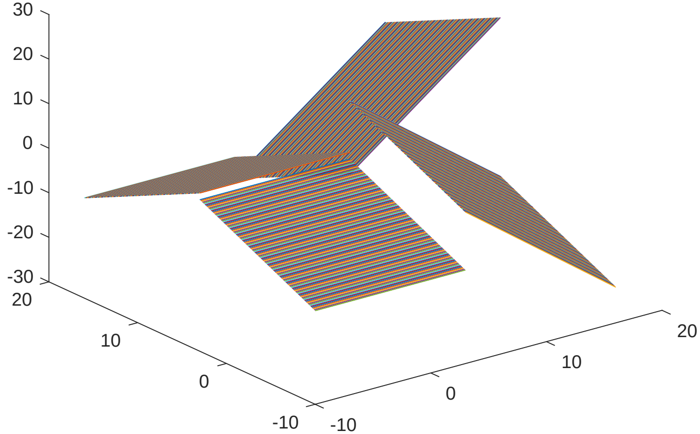
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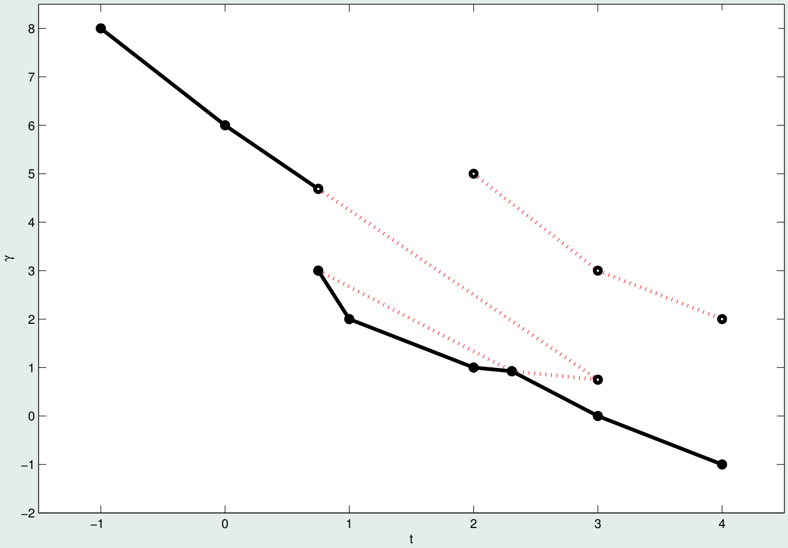
Consider the following multicriteria piecewise linear program

$$\text{(MPLP)} \qquad \min \; F(x) \quad \text{subject to} \quad x \in X,$$

$F : \mathbb{R}^n \to \mathbb{R}^m$ is a piecewise linear function, and $X \subset \mathbb{R}^n$ is a polyhedron.



$S := \{$ Pareto solutions$\}$, $\quad E := F(S) = \{$ Pareto points$\}$ (lower envelope)

Solution Set Structures of (MPLP)

Consider the following multicriteria linear program

$$\min\,(c_1^\top x, \cdots, c_l^\top x)^\top \quad \text{subject to } Ax \leq b, x \geq 0.$$

Arrow, Barankin and Blackwell (1953):

Then the set S of all Pareto solutions of (MPLP) is the union of finitely many polyhedra and connected by line segments.

Zheng and Y. (2008), Y. and Yen (2010) and Fang, Meng and Y. (2012):

Let $F(x)$ be piecewise linear. Then the set S of all Pareto solutions of (MPLP) is the union of finitely many semi-closed polyhedra.

3. Bicriteria Portfolio Optimization with the l_∞ Risk and Transaction Cost

Let R_j be the random return rate of the stock S_j. Define

$$\bar{r}_j = E(r_j), \; q_j = E(|r_j - \bar{r}_j|),$$

as the expected rate of return of the stock S_j and the expected absolute deviation of R_j from its mean, respectively.

Let $x_j \geq 0$ be the allocation to S_j from the initial wealth M_0, $j = 1, \cdots, n$. The l_∞ risk measure is defined as

$$l_\infty(x) = \max_{1 \leq j \leq n} E(|r_j x_j - \bar{r}_j x_j|) = \max_{1 \leq j \leq n} q_j x_j.$$

3.1. The mean-l_∞ risk model without transaction cost

The model is

$$\begin{array}{ll} \min & (\max_{1\le j\le n} q_j x_j, -\sum_{j=1}^{n} \bar{r}_j x_j) \\ \text{subject to} & \sum_{j=1}^{n} x_j = M_0, \\ & x_j \ge 0,\ j = 1, \cdots, n. \end{array}$$

Let $0 \le \lambda < 1$ be the investor's weight on the risk. Then we have

$$\begin{array}{ll} \min & \lambda \max_{1\le j\le n} q_j x_j - (1-\lambda) \sum_{j=1}^{n} \bar{r}_j x_j \\ \text{subject to} & \sum_{j=1}^{n} x_j = M_0, \\ & x_j \ge 0,\ j = 1, \cdots, n. \end{array}$$

Assume that

$$\bar{r}_1 \le \bar{r}_2 \le \cdots \le \bar{r}_n.$$

Let $\beta_k := \frac{\bar{r}_n - \bar{r}_{n-k}}{q_n} + \frac{\bar{r}_{n-1} - \bar{r}_{n-k}}{q_{n-1}} + \cdots + \frac{\bar{r}_{n-k+1} - \bar{r}_{n-k}}{q_{n-k+1}}, \quad (k = 1, 2, \cdots, n-1)$, an increasing sequence.

Choose k such that

$$\beta_k < \frac{\lambda}{1-\lambda} \le \beta_{k+1}.$$

Then an optimal investment strategy is:

$$x_j^* = \begin{cases} \frac{M_0}{q_j} \left(\sum_{n-k}^{n} \frac{1}{q_l} \right)^{-1}, & \text{if } j \ge n-k, \\ 0, & \text{otherwise.} \end{cases}$$

Comparison between analytic aolutions:

$$\begin{array}{rl} \min & \lambda(\sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + 2\sigma_{12} x_1 x_2) - (1-\lambda)(\bar{r}_1 x_1 + \bar{r}_2 x_2) \\ \text{subject to} & x_1 + x_2 = M_0, x_1 \geq 0, x_2 \geq 0. \end{array}$$

Let $\sigma_{12} = 0$, namely, assume that the two assets are not correlated. Then

$$\hat{x}_1 = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} M_0 + \Big(\frac{1-\lambda}{2\lambda}\Big)(\frac{\bar{r}_1 - \bar{r}_2}{\sigma_1^2 + \sigma_2^2}),$$

$$\hat{x}_2 = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} M_0 + \Big(\frac{1-\lambda}{2\lambda}\Big)(\frac{\bar{r}_2 - \bar{r}_1}{\sigma_1^2 + \sigma_2^2}).$$

On the other hand, it is easy to see from the analytic solution of the mean-l_∞ risk model that

$$x_1^* = \frac{q_2}{q_1 + q_2} M_0,$$

$$x_2^* = \frac{q_1}{q_1 + q_2} M_0.$$

- The role of q_i is similar to that of σ_i^2.

- The term $\left(\frac{1-\lambda}{2\lambda}\right)\left(\frac{r_1 - r_2}{\sigma_1^2 + \sigma_2^2}\right)$ for $\hat{x}_1$ can be regarded as a compensative term.

3.2. The mean-l_∞ risk model with transaction cost

For each stock S_j, the transaction cost is defined as

$$c_j^0(x_j) = \begin{cases} c_j x_j + d_j, & \text{if } x_j > 0, \\ 0, & \text{if } x_j = 0, \end{cases}$$

where $c_j > 0$ is the ratio of the transaction cost, and $d_j > 0$ is the minimum charge.

The bicriteria portfolio optimization problem with the l_∞ risk measure and transaction cost is formulated:

$$\begin{array}{ll} \min & \left(\max_{1\le j\le n} q_j x_j, -\sum_{j=1}^{n} \bar{r}_j x_j + \sum_{j=1}^{n} c_j^0(x_j)\right) \\ \text{subject to} & \sum_{j=1}^{n} [x_j + c_j^0(x_j)] = M_0, \\ & x_j \ge 0,\ j = 1, \cdots, n. \end{array}$$

Denote by E the set of Pareto points of (P).

Let $J \subset I = \{1, 2, \cdots, n\}$. There are $2^n - 1$ such indexes of $J's$.

Let $x_j > 0$ if $j \in J$ and $x_j = 0$ if $j \notin J$.

Consider the following subproblems:

$$
\begin{array}{lll}
(P)^J & \min & (y, -\sum_{j\in J}(\bar{r}_j - c_j)x_j) + (0, \sum_{j\in J} d_j) \\
 & \text{subject to} & q_j x_j \leq y,\ j \in J \\
 & & x_j > 0,\ j \in J,\ \sum_{j\in J}(1 + c_j)x_j = M_0 - \sum_{j\in J} d_j,
\end{array}
$$

denote by E_J the set of Pareto points of $(P)^J$,

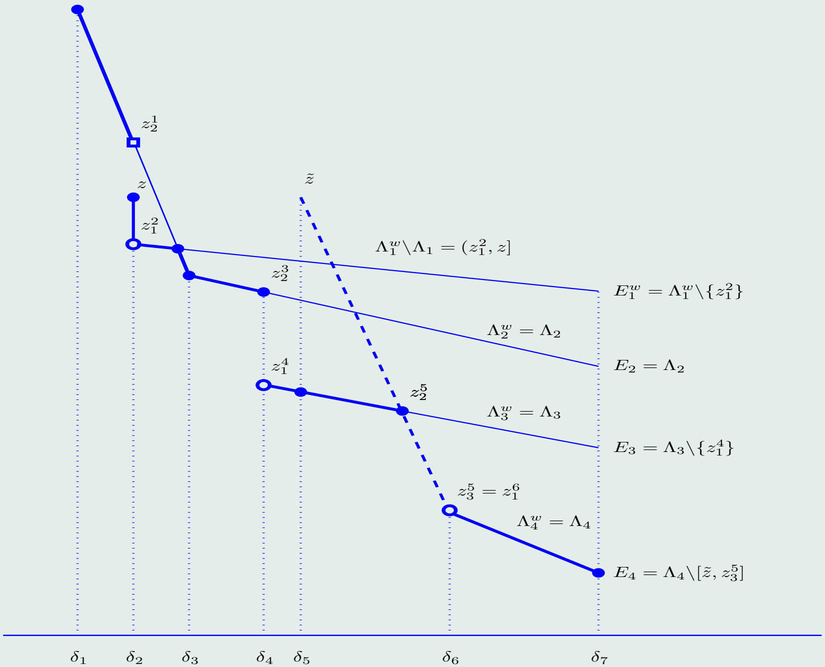
$$
\begin{array}{lll}
(AP)^J & \min & (y, -\sum_{j\in J}(\bar{r}_j - c_j)x_j) + (0, \sum_{j\in J} d_j) \\
 & \text{subject to} & q_j x_j \leq y,\ j \in J \\
 & & x_j \geq 0,\ j \in J, \sum_{j\in J}(1 + c_j)x_j = M_0 - \sum_{j\in J} d_j,
\end{array}
$$

denote by Λ_{AJ} the set of Pareto points of $(AP)^J$.

Then we have

$$E \subset \cup_{J \subset I} E_J \subset \cup_{J \subset I} \Lambda_{AJ}.$$

When there are 4 subproblems:



Then the following algorithm locates the set of all the Pareto points.

Algorithm 3.1

Procedure A. Compute all the auxiliary bicriteria linear programs $(AP)^J$.

Procedure B. Find all the break points and the lower envelopes.

Procedure C. Identify the set E of Pareto points.

Implementation of Algorithm 3.1:

- The number of subproblems with n stocks is $2^n - 1$.
- Data from Hong Kong stock market is used.
- Time in seconds.
- Computing time of Algorithm 3.1:

n	Procedure A	Procedure B	Procedure C	Total time
6	0.2891	0.0859	0.3375	0.7125
7	0.7109	0.3547	0.8547	1.9203
8	1.7500	1.7047	1.9891	5.4437
9	4.2797	8.8281	4.3750	17.4828
10	10.2828	52.2813	10.9828	73.5469
11	23.9906	399.8297	23.4812	447.3016

- Procedure B takes most of the computing time.

Improving Algorithm 3.1:

• Adapt the techniques of ideal points to exclude Pareto point sets of some subproblems from entering Procedure B.

• Computing time of improved Algorithm 3.1:

n	Procedure A	Procedure B	Procedure C	Total time
6	0	0.0563	0.2875	0.3438
7	0.0016	0.2437	0.7312	0.9766
8	0.0047	1.2625	1.7578	3.0250
9	0.0078	6.4891	3.9031	10.4000
10	0.0203	40.3000	10.0500	50.3703
11	0.0453	306.9844	21.7484	328.7781

• The total computing time is saved by 25%.

4. Conclusions

• Semi-closed polyhedron has practical applications.

• One can find the whole set of Pareto points for bicriteria piecewise linear programs.

• Application was given to a portfolio optimization problem with l_∞ risk measure and transaction cost, but not efficiently.

• Open question: How to find the set of Pareto points for tri-criteria piecewise linear programs?

• Open question: How to solve bicriteria piecewise linear programs with sparse constraints?

References

1. Cai, X.Q., Teo, K.L. Yang, X.Q. and Zhou, X.Y., Portfolio optimization under a minimax rule. *Management Sci.*, 46(7) (2000), pp. 957-972.

2. Fang, Y.P., Meng, K.W. and Yang, X.Q., Piecewise linear multi-criteria programs: the continuous case and its discontinuous generalization, *Operations Research* Vol. 60, (2012) pp. 398-409.