# On injective Jordan semi-triple maps of $M_n$

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#### Abstract

We show that every injective Jordan semi-triple map on the algebra  $M_n(\mathbb{F})$  of all  $n \times n$  matrices with entries in a field  $\mathbb{F}$  (i.e. a map  $\Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$  satisfying

 $\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A)$ 

for every A and B in  $M_n(\mathbb{F})$  ) is given by an injective multiplicative map of  $M_n(\mathbb{F})$ or by its negative. Hence, there exist  $\sigma \in \{-1, 1\} \cap \mathbb{F}$ , an injective homomorphism  $\varphi$  of  $\mathbb{F}$  and an invertible  $T \in M_n(\mathbb{F})$  such that either

$$\begin{split} \Phi(A) &= \sigma T^{-1} A_{\varphi} T \qquad \text{ for all } A \in M_n(\mathbb{F}), \text{ or } \\ \Phi(A) &= \sigma T^{-1} A_{\varphi}^t T \qquad \text{ for all } A \in M_n(\mathbb{F}). \end{split}$$

Here,  $A_{\varphi}$  is the image of A under  $\varphi$  applied entrywise.

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#### 1. Introduction and the main result

Let  $\mathcal{R}$  and  $\mathcal{R}'$  be rings and  $\phi : \mathcal{R} \to \mathcal{R}'$  be a transformation. The map  $\phi$  is called a *Jordan homomorphism* if it is additive and satisfies the condition

$$\phi(xy + yx) = \phi(x)\phi(y) + \phi(y)\phi(x)$$

for every x, y in  $\mathcal{R}$ . Rings with Jordan structure have been paid a lot of attention, moreover, Jordan operator algebras belong to the mathematical foundations of quantum mechanics [6]. Every ring homomorphism or antihomomorphism (i.e., an additive map  $\phi$ with  $\phi(xy) = \phi(y)\phi(x)$ ) is a Jordan homomorphism. If the ring  $\mathcal{R}'$  is 2-torsion free (i.e., 2x = 0 implies x = 0) then each Jordan homomorphism  $\phi : \mathcal{R} \to \mathcal{R}'$  is a Jordan triple homomorphism [3], i.e., an additive map satisfying

$$\phi(xyx) = \phi(x)\phi(y)\phi(x) \qquad (*)$$

for all x and y in  $\mathcal{R}$ . Without additivity such a map is called a Jordan (semi)triple map.

It is quite interesting how different structures of a ring, for example, the multiplicative and the additive structure, are interrelated. Molnár has proved [6] that in the case of standard operator algebras acting on infinite dimensional Banach spaces every bijective map satisfying (\*) is linear or conjugate linear and continuous. Also for n > 2 the general form of such bijective mappings on matrix algebras  $M_n$  of all  $n \times n$  complex matrices has been given there. The proof was functional-analytic in its spirit and depended on a deep result of Ovchinnikov [7]. Recently, Lu [5] presented a purely algebraic proof that works also in the dimension 2. Let us mention that a further generalization is given in [4].

In this paper we show that in the case of the algebra  $M_n(\mathbb{F})$  of all square matrices over an arbitrary field  $\mathbb{F}$  a result of this sort can be obtained for injective Jordan semi-triple maps.

We introduce some further notations that we shall use in the sequel. For a matrix  $A \in M_n(\mathbb{F})$  and a homomorphism  $\varphi$  of the underlying field let  $A_{\varphi}$  be the matrix obtained by applying  $\varphi$  entrywise, i.e.  $[A_{\varphi}]_{jk} = \varphi(a_{jk})$ . Note that this notation covers also the complex conjugation. Now we can state the main result of the paper.

**Theorem 1** Let n > 1. An injective mapping  $\Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$  is a Jordan semitriple map if and only if there exist an element  $\sigma \in \{-1, 1\} \cap \mathbb{F}$ , an invertible matrix  $T \in M_n(\mathbb{F})$  and an injective homomorphism  $\varphi$  of  $\mathbb{F}$  such that either

 $\begin{aligned} \Phi(A) &= \sigma T A_{\varphi} T^{-1} & \text{for all } A \in M_n(\mathbb{F}), \text{ or} \\ \Phi(A) &= \sigma T A_{\varphi}^t T^{-1} & \text{for all } A \in M_n(\mathbb{F}). \end{aligned}$ 

In other words, an injective Jordan semi-triple map stems either from a multiplicative map of  $M_n(\mathbb{F})$  or its negative. The result seems natural as it is easy to check that similarities, monomorphisms of underlying field applied entrywise, transposition and changing the sign are injective Jordan semi-triple maps. Before proceeding to the proof we collect some helpful and easy verifiable facts about Jordan semi-triple maps.

**Proposition 2** Let  $\mathcal{A}$  and  $\mathcal{B}$  be rings and  $\Phi : \mathcal{A} \to \mathcal{B}$  be a Jordan semi-triple map. Then  $\Phi$  sends idempotents to tripotents and moreover, if  $1 \in \mathcal{A}$  and  $j = \Phi(1) \in \mathcal{B}$ , then

1.  $j^2$  is an idempotent in  $\mathcal{B}$  satisfying  $\Phi(a) = j\Phi(a)j = j^3\Phi(a)j = j^2\Phi(a) = \Phi(a)j^2$  for all  $a \in \mathcal{A}$  (in particular,  $j^2\Phi(a)j^2 = \Phi(a)$ ),

2. j commutes with  $\Phi(a)$  for every  $a \in \mathcal{A}$ ,

3.  $\Phi^2(p) = j\Phi(p)$  is an idempotent in  $\mathcal{B}$  for each idempotent  $p \in \mathcal{A}$ , and

4. a map  $\Psi : \mathcal{A} \to \mathcal{B}$ , defined for all  $a \in \mathcal{A}$  by  $\Psi(a) = j\Phi(a)$ , is a Jordan semi-triple map, which is injective if and only if  $\Phi$  is injective.

If  $z = \Phi(0)$ , then  $\Phi(a)z = z\Phi(a) = z^2$  is an idempotent for all  $a \in \mathcal{A}$  and results analogous to those above hold for the map  $a \longmapsto j(\Phi(a) - z) = \Psi(a) - z^2$ .

**Proof.** The first assertion is evident from the definition of j. Use  $j\Phi(a) = j^3\Phi(a)j^2 = j\Phi(a)j^2 = \Phi(a)j$  to see the second. The third follows from  $\Phi^2(p) = \Phi(1p1)\Phi(p) = j\Phi(p)j\Phi(p) = j\Phi(p1p) = j\Phi(p)$ . A hint about z:  $\Phi(a)z = \Phi(a)\Phi(0a0) = (\Phi(a)z)^2 = \Phi(a0a)z = z^2$ .

Next we present a useful fact valid for injective mappings of the above form. Let p and q be idempotents in a ring  $\mathcal{A}$ . We write  $p \prec q$  if and only if  $qpq = p \neq q$ . It follows that if  $\Phi$  is injective, then for any strictly increasing chain  $p_1 \prec p_2 \prec \ldots \prec p_k$  of idempotents in a ring  $\mathcal{A}$  their corresponding images  $\Psi(p_j)$ , where  $\Psi$  is defined in Proposition 2, form a strictly increasing chain of idempotents in  $\mathcal{B}$ . In the case of (complex or real) matrix algebras we deduce the following corollary that is formulated here for complex matrices only.

**Corollary 3** Let n > 1,  $m \in \mathbb{N}$  and  $\Phi : M_n \to M_m$  be an injective Jordan semi-triple map. Then  $m \ge n$ . In the case m = n, for each idempotent  $p \in M_n$  the rank of the idempotent  $\Psi(p)$  is equal to the rank of p. In particular,  $\Psi(0) = \Phi(0) = 0$  and  $\Psi(I) = I$ .

The description of Jordan semi-triple maps in the case when  $\mathbb{F}$  is either  $\mathbb{C}$  or  $\mathbb{R}$  reads as follows.

**Proposition 4** A mapping  $\phi : \mathbb{F} \to \mathbb{F}$  satisfies the condition  $\phi(a^2b) = \phi^2(a)\phi(b)$  for all a and b in  $\mathbb{F}$  if and only if  $\phi(ab) = \phi(1)\phi(a)\phi(b)$  for all a and b in  $\mathbb{F}$ . This means that either  $\phi$  is multiplicative (including the cases  $\phi \equiv 0$  and  $\phi \equiv 1$ ) or  $-\phi$  is a multiplicative map (including the case  $\phi \equiv -1$ ).

**Proof.** From  $\phi(1) = \phi^3(1)$  it follows that  $\phi(1) \in \{-1, 0, 1\}$ . In the case  $\phi(1) = 0$  the conclusion follows immediately: for each  $a \in \mathbb{F}$  one has  $\phi(a) = \phi(1)^2 \phi(a) = 0$ . On the other hand, if  $\phi(1) \neq 0$  then  $\phi(ab) = \phi(c^2b) = \phi^2(c)\phi(b)$  for any c with  $c^2 = a$  and a in  $\mathbb{C}$  or  $\mathbb{R}^+$ . Also,  $\phi(a) = \phi(c^21) = \phi^2(c)\phi(1) = \phi^2(c)\phi^{-1}(1)$  and the result follows for complex scalars or in the real case when at least one of factors is positive. If both factors are negative then we use the fact that for a negative d there exists  $c \in \mathbb{R}$  such that  $d = -c^2$  and hence,  $\phi(d) = \phi(-1)\phi^2(c) = \phi(-1)\phi(1)\phi(-d)$ . A simple identity  $\phi^2(-1)\phi(1) = \phi(1)$ 

implies  $\phi^2(-1) = 1$  and the chain of equalities  $\phi(ab) = \phi((-a)(-b)) = \phi(1)\phi(-a)\phi(-b) = \phi(1)\phi^2(-1)\phi^2(1)\phi(a)\phi(b)$  finishing the proof

Remark. If  $\phi$  is also additive, then it is a (ring) homomorphism of  $\mathbb{F}$ , and hence, it is trivial in the real case [1], p. 57. Even in the case when  $\phi$  is injective there are a lot of such nontrivial monomorphisms in the complex case, see [1], p. 59. Considering the algebra  $M_n(\mathbb{R})$  of all real  $n \times n$  matrices, the characterization of injective Jordan semi-triple maps reads as follows.

**Theorem 5** An injective mapping  $\Phi : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ , n > 1, is a Jordan semi-triple map if and only if there exist a number  $\sigma \in \{-1, 1\}$  and an invertible matrix  $T \in M_n(\mathbb{R})$ such that either

$$\Phi(A) = \sigma T A T^{-1} \qquad \text{for all } A \in M_n(\mathbb{R}), \text{ or} \\ \Phi(A) = \sigma T A^t T^{-1} \qquad \text{for all } A \in M_n(\mathbb{R}).$$

## 2. The proof

The main idea is to use the induction on n after proving the result for  $2 \times 2$  matrices. For any  $1 \leq j, k \leq n$  we write  $E_{jk}$  for the matrix having 1 as its (j, k)-th entry and zeros elsewhere.

Let  $\Psi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$  be a map associated to  $\Phi$  as before. From Proposition 2 we know that it is an injective Jordan semi-triple map if  $\Phi$  is. Moreover, the following observation is helpful:

**Lemma 6** If there exist an invertible matrix T in  $M_n(\mathbb{F})$  and a map  $\varphi$  of  $\mathbb{F}$  with  $\varphi(0) = 0$  and  $\varphi(1) = 1$  such that  $\Psi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$  has the form

 $A \mapsto TA_{\varphi}T^{-1}$  or  $A \mapsto TA_{\varphi}^{t}T^{-1}$ ,

then  $\Phi(I_n) = \sigma I_n$ , where  $\sigma \in \{-1, 1\} \cap \mathbb{F}$ . Thus,  $\Phi$  has either the form  $A \mapsto \sigma T A_{\varphi} T^{-1}$ or  $A \mapsto \sigma T A_{\omega}^t T^{-1}$ .

**Proof.** Let  $X = \Phi(I_n)$ . Since  $X^2 = \Phi(I_n)\Phi(I_n) = \Psi(I_n) = I_n$ , X is invertible and  $X = X^{-1}$ . By Proposition 2, X commutes with  $\Phi(A)$  for all  $A \in M_n(\mathbb{F})$ . It follows that  $X^{-1} = X$  commutes with  $\Psi(A)$  for all  $A \in M_n(\mathbb{F})$ . If  $\Psi$  has the form described, X must be a scalar matrix. As  $X^2 = I_n$ ,  $X = \sigma I_n$  and hence,  $\Phi(.) = X\Psi(.)$  has the asserted form.

Therefore, in order to prove Theorem 1, it suffices to prove that  $\Psi$  has the form given in the lemma above. From Proposition 2 and Corollary 3,  $\Psi$  has the following useful properties:

- (a) For any idempotent  $A \in M_n(\mathbb{F})$ ,  $\Psi(A)$  is an idempotent of the same rank as A.
- (b)  $\Psi(0_n) = 0_n$  and  $\Psi(I_n) = I_n$ .
- (c) for any  $A \in M_n(\mathbb{F})$  one has  $\Psi(A^2) = \Psi(A)^2$ .

Now, the proof of Theorem 1 is given in two steps.

**Step 1**. The proof for  $M_2(\mathbb{F})$ .

The matrix  $E_{11}$  is an idempotent of rank 1. By the above property (a),  $\Psi(E_{11})$  is a rank one idempotent. There exists an invertible  $2 \times 2$  matrix S such that  $\Psi(E_{11}) = SE_{11}S^{-1}$ . By replacing  $\Psi$  with the map  $A \mapsto S^{-1}\Psi(A)S$ , we may assume that  $\Psi(E_{11}) = E_{11}$ .

Using the properties (b) and (c) we see that  $\Psi(E_{12})^2 = \Psi(E_{12}^2) = \Psi(0) = 0$ . Thus,  $\Psi(E_{12})$  is a nonzero nilpotent because of injectivity, and hence its rank is 1. From this and the following fact

$$E_{11}\Psi(E_{12})E_{11} = \Psi(E_{11})\Psi(E_{12})\Psi(E_{11}) = \Psi(E_{11}E_{12}E_{11}) = \Psi(0) = 0$$

we conclude that either  $\Psi(E_{12}) = aE_{12}$  or  $\Psi(E_{12}) = aE_{21}$  for some nonzero  $a \in \mathbb{F}$ . Let D = diag(a, 1). Replacing  $\Psi$  with the map  $A \mapsto D^{-1}\Psi(A)D$  (or  $A \mapsto D^{-1}\Psi(A)^t D$ , respectively) we may further assume that  $\Psi(E_{12}) = E_{12}$ .

Similarly, we check that  $\Psi(E_{21})$  is a rank one nilpotent. Then we use

$$\Psi(E_{11})\Psi(E_{21})\Psi(E_{11}) = \Psi(0) = 0$$
 and  $\Psi(E_{12})\Psi(E_{21})\Psi(E_{12}) = \Psi(E_{12}) = E_{12}$ 

to verify that  $\Psi(E_{21}) = E_{21}$ . Next, as  $\Psi(E_{22})$  is a rank one idempotent satisfying  $\Psi(A)\Psi(E_{22})\Psi(A) = 0$  for  $A \in \{E_{11}, E_{12}, E_{21}\}$ , we deduce that  $\Psi(E_{22}) = E_{22}$ .

Now for any  $A = (a_{ij}) \in M_2(\mathbb{F})$ , let  $B = (b_{ij}) = \Psi(A)$ . Then

$$b_{ij}E_{ji} = E_{ji}BE_{ji} = \Psi(E_{ji})\Psi(A)\Psi(E_{ji}) = \Psi(E_{ji}AE_{ji}) = \Psi(a_{ij}E_{ji}).$$

Thus, the (i, j)-th entry of  $\Psi(A)$  depends on the (i, j)-th entry of A only. Therefore, we may write

$$\Psi(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}) = \begin{bmatrix} \varphi_{11}(a_{11}) & \varphi_{12}(a_{12}) \\ \varphi_{21}(a_{21}) & \varphi_{22}(a_{22}) \end{bmatrix}$$

for some maps  $\varphi_{ij}$  on  $\mathbb{F}$ . Furthermore, from  $\Psi(E_{ij}) = E_{ij}$  for all  $i, j \in \{1, 2\}$  we conclude that  $\varphi_{ij}(0) = 0$  and  $\varphi_{ij}(1) = 1$ . Let  $J = E_{11} + E_{12} + E_{21} + E_{22}$ . Then  $\Psi(J) = J$  and for any  $a \in \mathbb{F}$ 

$$\varphi_{11}(a)J = J(\varphi_{11}(a)E_{11})J = \Psi(J)\Psi(aE_{11})\Psi(J) = \Psi(aJE_{11}J) = \Psi(aJ) = \begin{bmatrix} \varphi_{11}(a) & \varphi_{12}(a) \\ \varphi_{21}(a) & \varphi_{22}(a) \end{bmatrix}$$

Therefore,  $\varphi_{11} = \varphi_{12} = \varphi_{21} = \varphi_{22}$ . We label this common map by  $\varphi$  and it follows that  $\Psi(A) = A_{\varphi}$  for every  $A \in M_2(\mathbb{F})$ . It remains to prove that  $\varphi$  is an endomorphism of the underlying field  $\mathbb{F}$ .

For any  $a, b \in \mathbb{F}$ , let  $A = aE_{11} + bE_{12}$ . Then  $\Psi(A) = \varphi(a)E_{11} + \varphi(b)E_{12}$ . Since

$$\varphi(a)^2 E_{11} + \varphi(a)\varphi(b)E_{12} = \Psi(A)^2 = \Psi(A^2) = \varphi(a^2)E_{11} + \varphi(ab)E_{12}$$

and

$$(\varphi(a) + \varphi(b))J = \Psi(J)\Psi(A)\Psi(J) = \Psi(JAJ) = \Psi((a+b)J) = \varphi(a+b)J,$$

we have  $\varphi(ab) = \varphi(a)\varphi(b)$  and  $\varphi(a+b) = \varphi(a) + \varphi(b)$ .

Step 2. The induction

Let  $P = I_{n-1} \oplus [0]$ . Then P is a rank n-1 idempotent, so as  $\Psi(P)$  by the property (a). There exists an invertible matrix  $S \in M_n(\mathbb{F})$  such that  $\Psi(P) = SPS^{-1}$ . Replacing  $\Psi$  by the map  $A \longmapsto S^{-1}\Psi(A)S$  we may assume that  $\Psi(P) = P$ . For any  $\widehat{A} \in M_{n-1}(\mathbb{F})$  let  $A = \widehat{A} \oplus [0]$ . Then PAP = A implies

$$P\Psi(A)P = \Psi(P)\Psi(A)\Psi(P) = \Psi(PAP) = \Psi(A).$$

It follows that  $\Psi(\widehat{A} \oplus [0]) = \Psi(A) = \widehat{X} \oplus [0]$  for some matrix  $\widehat{X} \in M_{n-1}(\mathbb{F})$ . Define the map  $\widehat{\Psi}$  on  $M_{n-1}(\mathbb{F})$  by  $\widehat{\Psi}(\widehat{A}) = \widehat{X}$ . It is easy to check that  $\widehat{\Psi}$  is an injective Jordan semi-triple map on  $M_{n-1}(\mathbb{F})$ . Furthermore,  $\Psi(P) = P$  implies  $\widehat{\Psi}(I_{n-1}) = I_{n-1}$ . By the induction hypothesis there is an invertible  $\widehat{T} \in M_{n-1}(\mathbb{F})$  and a nonzero endomorphism  $\varphi$ on  $\mathbb{F}$  such that  $\widehat{\Psi}$  has either the form

$$\widehat{A}\longmapsto \widehat{T}\widehat{A}_{\varphi}\widehat{T}^{-1} \qquad \text{or} \qquad \widehat{A}\longmapsto \widehat{T}\widehat{A}_{\varphi}^t\widehat{T}^{-1}.$$

Let T be the matrix  $\widehat{T} \oplus [1]$ . Replacing  $\Psi$  by either the map  $A \mapsto T^{-1}\Psi(A)T$  or  $A \mapsto (T^{-1}\Psi(A)T)^t$ , we may further assume that  $\widehat{\Psi}(\widehat{A}) = \widehat{A}_{\varphi}$  for all  $\widehat{A} \in M_{n-1}(\mathbb{F})$ . This is equivalent to  $\Psi(\widehat{A} \oplus [0]) = \widehat{A}_{\varphi} \oplus [0]$ . Also, for any  $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in M_n(\mathbb{F})$  with  $A_{11} \in M_{n-1}(\mathbb{F})$  we have  $PAP = A_{11} \oplus [0]$ . Thus,

$$P\Psi(A)P = \Psi(P)\Psi(A)\Psi(P) = \Psi(PAP) = (A_{11})_{\varphi} \oplus [0]. \quad (**)$$

Let us define matrices  $R_i$  for each  $i \in \{1, 2, ..., n-1\}$  by  $R_1 = I_n - E_{ii} - E_{nn} + E_{in} + E_{ni}$ . Let i be arbitrary, but fixed. From (\*\*) we have  $P\Psi(R_i)P = (I_{n-1} - E_{ii}) \oplus [0]$ . Then there exist  $x, y \in \mathbb{F}^{n-1}$  and  $z \in \mathbb{F}$  such that  $\Psi(R_i) = \begin{bmatrix} I_{n-1} - E_{ii} & x \\ y^t & z \end{bmatrix}$ . From the equality  $\Psi(R_i)^2 = \Psi(R_i^2) = \Psi(I_n) = I_n$  we get  $I_{n-1} - E_{ii} + xy^t = I_{n-1}$  and  $y^tx + z^2 = 1$ . These equalities imply that  $xy^t = E_{ii}$  and  $z^2 = 1 - y^tx = 1 - \operatorname{tr}(xy^t) = 0$ . Hence, only the *i*-th entries of x and y are nonzero and their product is 1. Denote x by  $a_i$ . It follows that  $\Psi(R_i) = I_n - E_{ii} - E_{nn} + a_i E_{in} + a_i^{-1} E_{ni}$ . Next, take any two distinct  $i, j \in \{1, 2, \ldots, n-1\}$ . From  $R_i R_j R_i = I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}$  we get using (\*\*)

$$\Psi(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}) = \Psi(R_i)\Psi(R_j)\Psi(R_i) = I_n - E_{ii} - E_{jj} + a_i^{-1}a_jE_{ij} + a_ia_j^{-1}E_{ji},$$

which implies that  $a_i = a_j$ . Let  $D = I_{n-1} \oplus [a_1]$ . Replacing  $\Psi$  by the map  $A \mapsto D\Psi(A)D^{-1}$  we may further assume that  $\Psi(R_i) = R_i$  for all  $i \in \{1, 2, ..., n-1\}$ .

Let us fix some  $i \in \{1, 2, ..., n-1\}$  again. As n > 2, there is another  $j \in \{1, 2, ..., n-1\}$ such that  $E_{in} = R_j E_{ij} R_j$  and  $E_{ni} = R_j E_{ji} R_j$ . Then for any  $a \in \mathbb{F}$ ,

$$\Psi(aE_{in}) = \Psi(R_j)\Psi(aE_{ij})\Psi(R_j) = R_j\varphi(a)E_{ij}R_j = \varphi(a)E_{in}$$

and

$$\Psi(aE_{ni}) = \Psi(R_j)\Psi(aE_{ji})\Psi(R_j) = R_j\varphi(a)E_{ji}R_j = \varphi(a)E_{ni}$$

Also we have

$$\Psi(aE_{nn}) = \Psi(R_1)\Psi(aE_{11})\Psi(R_1) = R_1\varphi(a)E_{11}R_1 = \varphi(a)E_{nn}$$

Together with (\*\*) these equalities imply that  $\Psi(aE_{ij}) = \varphi(a)E_{ij}$  for all  $i, j \in \{1, 2, ..., n-1\}$ and  $a \in \mathbb{F}$ .

Finally, for any  $A = (a_{ij}) \in M_n(\mathbb{F})$  let  $B = (b_{ij}) = \Psi(A)$ . Then

$$b_{ij}E_{ji} = E_{ji}BE_{ji} = \Psi(E_{ji})\Psi(A)\Psi(E_{ji}) = \Psi(E_{ji}AE_{ji}) = \Psi(a_{ij}E_{ji}) = \varphi(a_{ij})E_{ji},$$

i.e.,  $b_{ij} = \varphi(a_{ij})$ . Thus,  $\Psi(A) = A_{\varphi}$ , and hence the proof is complete.

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