Condition for the higher rank numerical range to be non-empty

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Abstract
It is shown that the rank-\( k \) numerical range of every \( n \times n \) complex matrix is non-empty if \( k < n/3 + 1 \). The proof is based on a recent characterization of the rank-\( k \) numerical range by Li and Sze, the Helly’s theorem on compact convex sets, and some eigenvalue inequalities. In particular, the result implies that \( \Lambda_2(A) \) is non-empty if \( n \geq 4 \). This confirms a conjecture of Choi et al. If \( k \geq n/3 + 1 \), an \( n \times n \) complex matrix is given for which the rank-\( k \) numerical range is empty. Extension of the result to bounded linear operators acting on an infinite dimensional Hilbert space is also discussed.

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1 Introduction
Let \( M_n \) be the algebra of \( n \times n \) complex matrices. In [3], the authors introduced the notion of the rank-\( k \) numerical range of \( A \in M_n \) defined and denoted by
\[
\Lambda_k(A) = \{ \lambda \in \mathbb{C} : X^*AX = \lambda I_k, \ X \text{ is } n \times k \text{ such that } X^*X = I_k \}
\]
in connection to the study of quantum error correction; see [4]. Evidently, \( \lambda \in \Lambda_k(A) \) if and only if there is a unitary matrix \( U \in M_n \) such that \( U^*AU \) has \( \lambda I_k \) as the leading principal submatrix. When \( k = 1 \), this concept reduces to the classical numerical range, which is well known to be convex by the Toeplitz-Hausdorff theorem; for example, see [7] for a simple proof. In [1] the authors conjectured that \( \Lambda_k(A) \) is convex, and reduced the convexity problem to the problem of showing that \( 0 \in \Lambda_k(A) \) for
\[
A = \begin{pmatrix} I_k & X \\ Y & -I_k \end{pmatrix}
\]
for arbitrary \( X, Y \in M_k \). They further reduced this problem to the existence of a Hermitian matrix \( H \) satisfying the matrix equation
\[
I_k + MH + HM^* - HPH = H \tag{1}
\]
for arbitrary \( M \in M_k \) and positive definite \( P \in M_k \). In [10], the author observed that equation (1) can be rewritten as the continuous Riccati equation
\[
HPH - H(M^* - I_k/2) - (M - I_k/2)H - I_k = 0_k, \tag{2}
\]
and existing results on Riccati equation will ensure its solvability; for example, see [5, Theorem 4]. This establishes the convexity of \( \Lambda_k(A) \).

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For a Hermitian $X \in M_n$, let $\lambda_1(X) \geq \cdots \geq \lambda_n(X)$ be the eigenvalues of $X$. In [8], it was shown that
\begin{equation}
\Lambda_k(A) = \{ \mu \in \mathbb{C} : e^{it}\mu + e^{-it}\mathfrak{f} \leq \lambda_k(e^{it}A + e^{-it}A^*) \text{ for all } t \in [0, 2\pi] \} .
\end{equation}
In particular, $\Lambda_k(A)$ is the intersection of closed half planes on $\mathbb{C}$, and therefore is always convex. Moreover, if $A \in M_n$ is normal with eigenvalues $\lambda_1, \ldots, \lambda_n$, then
\[ \Lambda_k(A) = \bigcap_{1 \leq j_1 < \cdots < j_{k+1} \leq n} \text{conv} \{ \lambda_{j_1}, \ldots, \lambda_{j_{k+1}} \} . \]
This confirms a conjecture in [2].

While many interesting results have been obtained for $\Lambda_k(A)$, see [1, 2, 3, 4] for example, there are some basic questions whose answers are unknown. The purpose of this paper is to answer the following.

**Problem** Determine $n$ and $k$ such that $\Lambda_k(A)$ is non-empty for every $A \in M_n$.

It is well-known that the classical numerical range $\Lambda_1(A)$ is non-empty. For $k > n/2$, $\Lambda_k(A)$ has at most one element and one can easily construct $A \in M_n$ such that $\Lambda_k(A) = \emptyset$; see Proposition 2.2 and Corollary 2.3 in [3]. The situation for $\Lambda_k(A)$ with $n/2 \geq k > 1$ is not so clear. In [2], the authors conjectured that $\Lambda_2(A) \neq \emptyset$ for $n \geq 4$.

In the next section, we show that $\Lambda_k(A)$ is non-empty for every $A \in M_n$ if and only if $k < n/3 + 1$. In particular, it confirms the conjecture in [2] that $\Lambda_2(A) \neq \emptyset$ if $n \geq 4$. We also consider extension of the result to infinite dimensional bounded linear operators.

## 2 Results and proofs

**Theorem 1** Let $A \in M_n$, and let $k$ be a positive integer such that $k < n/3 + 1$. Then $\Lambda_k(A)$ is non-empty.

**Proof.** Evidently, $\Lambda_k(A) \subseteq \Lambda_1(A)$. Given $A \in M_n$ and $t \in [0, 2\pi)$, let $A(t) = e^{it}A + e^{-it}A^*$. Consider the compact convex sets
\[ S(t) = \{ \mu \in \Lambda_1(A) : e^{it}\mu + e^{-it}\mathfrak{f} \leq \lambda_k(A(t)) \} , \quad t \in [0, 2\pi) . \]
By (3),
\[ \Lambda_k(A) = \bigcap_{t \in [0, 2\pi)} S(t) . \]
By Helly’s Theorem [6, Theorem 24.9], it suffices to show that $S(t_1) \cap S(t_2) \cap S(t_3) \neq \emptyset$ for all choices of $t_1, t_2, t_3$ with $0 \leq t_1 < t_2 < t_3 < 2\pi$.

For $1 \leq j \leq 3$, let $V_j$ be the subspace spanned by the eigenvectors of $A(t_j)$ corresponding to the eigenvalues $\lambda_k(A(t_j)), \ldots, \lambda_n(A(t_j))$. Then $\dim V_j \geq n - k + 1$. Hence, we have
\begin{align*}
\dim (V_1 \cap V_2 \cap V_3) &= \dim (V_1 \cap V_2) + \dim V_3 - \dim ((V_1 \cap V_2) + V_3) \\
&= \dim V_1 + \dim V_2 - \dim (V_1 + V_2) - \dim (V_1 \cap V_2 + V_3) \\
&\geq 3(n - k + 1) - 2n \\
&= n - 3k + 3 \\
&\geq 1 .
\end{align*}
Let \( v \) be a unit (column) vector in \( V_1 \cap V_2 \cap V_3 \). Then \( \mu = v^*Av \in \Lambda_1(A) \) and for \( 1 \leq j \leq 3 \), we have
\[
e^{it}\mu + e^{-it}\mu = v^*(A(t_j))v \leq \lambda_k(A(t_j)).
\]
Hence, \( \mu \in S(t_1) \cap S(t_2) \cap S(t_3) \).

The following answers a question in [2].

**Corollary 2** Let \( A \in M_n \) with \( n \geq 4 \). Then \( \Lambda_2(A) \neq \emptyset. \)

Without additional information on \( A \in M_n \), the bound on \( n \) in Theorem 1 is best possible as shown by the following result.

**Theorem 3** Suppose \( k \) is a positive integer such that \( k \geq n/3 + 1 \). There exists \( A \in M_n \) such that \( \Lambda_k(A) = \emptyset. \)

**Proof.** We first consider the case when \( 3k = n + 3 \). Let \( w = e^{i2\pi/3} \), and
\[
A = I_{k-1} \oplus w I_{k-1} \oplus w^2 I_{k-1}.
\]
Write \( A = H + iG \) with \( H = H^* \) and \( G = G^* \). Then \( H = I_{k-1} \oplus (-1/2)I_{2k-2} \). Thus, \( \Lambda_k(H) = \{-1/2\} \); see also [3, Theorem 2.4]. So,
\[
\Lambda_k(A) \subseteq \mathcal{L} = \{z : \text{Re } z = -1/2\}.
\]
By rotation of \( 2\pi/3 \) and \( 4\pi/3 \), one can show that \( \Lambda_k(A) \subseteq w\mathcal{L} \) and \( \Lambda_k(A) \subseteq w^2\mathcal{L} \). So,
\[
\Lambda_k(A) \subseteq \mathcal{L} \cap w\mathcal{L} \cap w^2\mathcal{L} = \emptyset.
\]
Now, suppose \( 3k > n + 3 \). Then we can consider a principal submatrix \( B \in M_n \) of the matrix \( A \in M_{3k-3} \) constructed in the preceding paragraph. Then \( \Lambda_k(B) \subseteq \Lambda_k(A) = \emptyset. \)

Note that we can perturb the example in the above proof to get a non-normal matrix \( A \in M_n \) such that \( \Lambda_k(A) = \emptyset \) if \( k \geq n/3 + 1 \). Also, Theorem 3 can be obtained from parts (1), (2), (3) of [2, Theorem 4.7] and the fact that \( \Lambda_k(A) \) is a subset of
\[
\bigcap_{1 \leq j_1 < \cdots < j_{n-k+1} \leq n} \text{conv} \{\lambda_{j_1}, \ldots, \lambda_{j_{n-k+1}}\}
\]
if \( A \in M_n \) is normal with eigenvalues \( \lambda_1, \ldots, \lambda_n \).

Let \( B(\mathcal{H}) \) be the algebra of bounded linear operator acting on an infinite dimensional Hilbert space \( \mathcal{H} \). One can extend the definition of \( \Lambda_k(A) \) for a bounded linear operator \( A \in B(\mathcal{H}) \) by
\[
\Lambda_k(A) = \{\gamma \in \mathbb{C} : X^*AX = \gamma I_k, \ X : \mathbb{C}^k \rightarrow \mathcal{H}, \ X^*X = I_k\}.
\]
By Theorem 1, we have the following.

**Corollary 4** Suppose \( k \) is a positive integer and \( A \in B(\mathcal{H}) \) for an infinite dimensional Hilbert space \( \mathcal{H} \). Then
\[
\Lambda_k(A) \neq \emptyset.
\]

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References


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