SOS and SDP Relaxations of Sensor Network Localization

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(Joint work with João Gouveia and Paul Tseng)

Talk Outline

- Sensor network localization.
- Convex relaxations: SDP, ESDP, sparse-SOS.
- Comparison between relaxations.
- Conclusion and open problems.

Sensor Network Localization

Basic Problem:

- n pts $\underbrace{x_1, ..., x_m}_{\text{sensors}}, \underbrace{x_{m+1}, ..., x_n}_{\text{anchors}}$ in $\Re^2$.
- Know last $n-m$ pts ('anchors') $x_{m+1}, ..., x_n$ and Eucl. dist. estimate for some pairs of 'neighboring' pts (i.e. within 'radio range')

$$d_{ij} \geq 0 \quad \forall (i,j) \in \mathcal{A},$$

  with $\mathcal{A} \subseteq \{(i,j) : 1 \leq i < j \leq n\}$.

- Estimate the first m pts ('sensors') $x_1, ..., x_m$.

Optimization Problem Formulation

$$v_{\mathrm{p}} \; := \; \min_{x_1,\ldots,x_m} \;\; \sum_{(i,j)\in\mathcal{A}} \left| \|x_i - x_j\|^2 - d_{ij}^2 \right| .$$

Optimization Problem Formulation

$$v_{\mathrm{p}} \;:=\; \min_{x_1,\ldots,x_m} \quad \sum_{(i,j)\in\mathcal{A}} \left| \|x_i - x_j\|^2 - d_{ij}^2 \right| .$$

- Objective function is nonconvex. m can be large ($m > 1000$).
- Problem is NP-hard (reduction from PARTITION).

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- Objective function is nonconvex. m can be large ($m > 1000$).
- Problem is NP-hard (reduction from PARTITION).
- Aim 1: Tractability (use a convex relaxation).
- Aim 2: Identify correct sensor position.

SDP Relaxation

Idea: Linearization.

Let $X = [x_1 \cdots x_m]$ and $Y = X^T X$. Then $y_{ij} = x_i^T x_j$ for all i, j.

Observation:

SDP Relaxation

Idea: Linearization.

Let $X = [x_1 \cdots x_m]$ and $Y = X^T X$. Then $y_{ij} = x_i^T x_j$ for all i, j.

Observation:

- $$Y = X^T X \Longleftrightarrow Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \succeq 0, \ \mathrm{rank}(Z) = 2.$$

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Observation:

- $$Y = X^T X \Longleftrightarrow Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \succeq 0, \ \mathrm{rank}(Z) = 2.$$

- For $(i, j) \in \mathcal{A}^s$,

  $$||x_i - x_j||^2 = \|x_i\|^2 - 2x_i^T x_j + \|x_j\|^2 = y_{ii} - 2y_{ij} + y_{jj};$$

  For $(i, j) \in \mathcal{A}^a$,

  $$||x_i - x_j||^2 = \|x_i\|^2 - 2x_i^T x_j + \|x_j\|^2 = y_{ii} - 2x_i^T x_j + \|x_j\|^2.$$

We can now reformulate the original problem as follows:

$$\begin{aligned} v_p := \quad & \min_{Z} \sum_{(i,j)\in\mathcal{A}^a} \left| y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2 \right| \\ & + \sum_{(i,j)\in\mathcal{A}^s} \left| y_{ii} - 2y_{ij} + y_{jj} - d_{ij}^2 \right| \\ \text{s.t.} \quad & Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \succeq 0, \\ & \operatorname{rank}(Z) = 2. \end{aligned}$$

Dropping the rank constraint yields the SDP relaxation.

SDP Relaxation

SDP relaxation (Biswas, Ye '03):

$$
\begin{aligned}
v_{\text{sdp}} := \quad & \min_{Z} \sum_{(i,j)\in\mathcal{A}^a} \left| y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2 \right| \\
& + \sum_{(i,j)\in\mathcal{A}^s} \left| y_{ii} - 2y_{ij} + y_{jj} - d_{ij}^2 \right| \\
& \text{s.t.} \quad Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \succeq 0.
\end{aligned}
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& \text{s.t.} \quad Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \succeq 0.
\end{aligned}
$$

Although convex, solving SDP by IP method can be hard when m is large.

ESDP Relaxation

ESDP relaxation (Wang, Zheng, Boyd, Ye '08):

$$
\begin{aligned}
v_{\text{esdp}} := \quad & \min_{Z} \sum_{(i,j)\in\mathcal{A}^a} \left| y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2 \right| \\
& + \sum_{(i,j)\in\mathcal{A}^s} \left| y_{ii} - 2y_{ij} + y_{jj} - d_{ij}^2 \right| \\
& \text{s.t.} \quad Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \\
& \begin{bmatrix} y_{ii} & y_{ij} & x_i^T \\ y_{ij} & y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix} \succeq 0 \quad \forall (i,j) \in \mathcal{A}^s.
\end{aligned}
$$

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& \qquad \begin{bmatrix} y_{ii} & y_{ij} & x_i^T \\ y_{ij} & y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix} \succeq 0 \quad \forall (i,j) \in \mathcal{A}^s.
\end{aligned}
$$

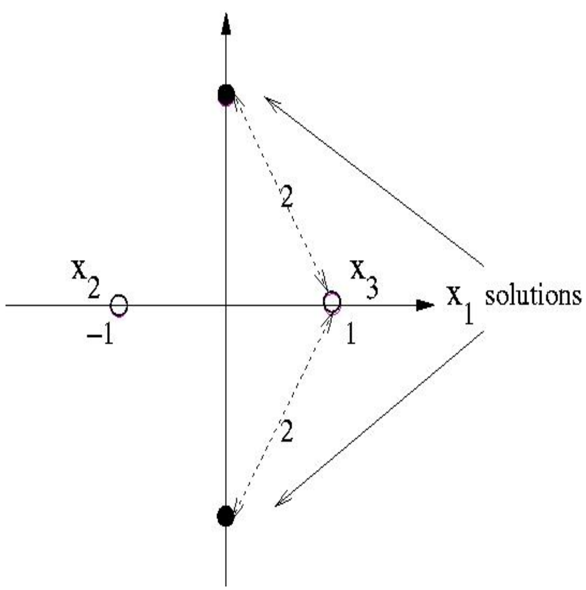
In simulations, ESDP is nearly as strong as SDP relaxation, and solvable much faster by IP method.

An Example

$n = 3$, $m = 1$, $d_{12} = d_{13} = 2$

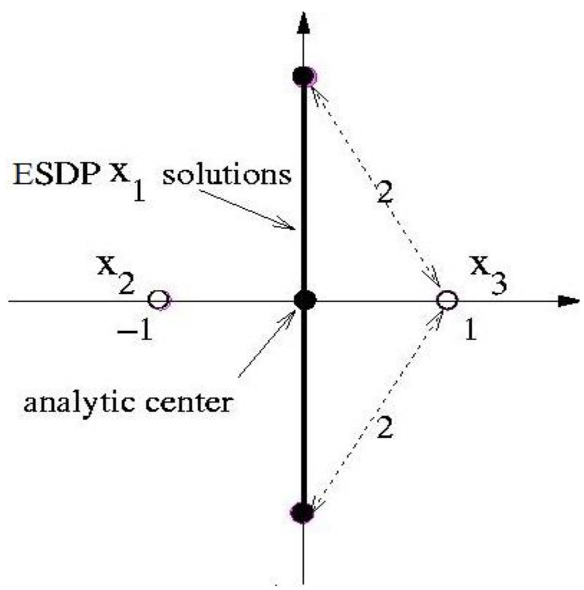
Problem:

$$0 = \min_{x_1 \in \Re^2} \quad \left| \|x_1 - (1,0)\|^2 - 4 \right| + \left| \|x_1 - (-1,0)\|^2 - 4 \right|$$



ESDP/SDP Relaxation:

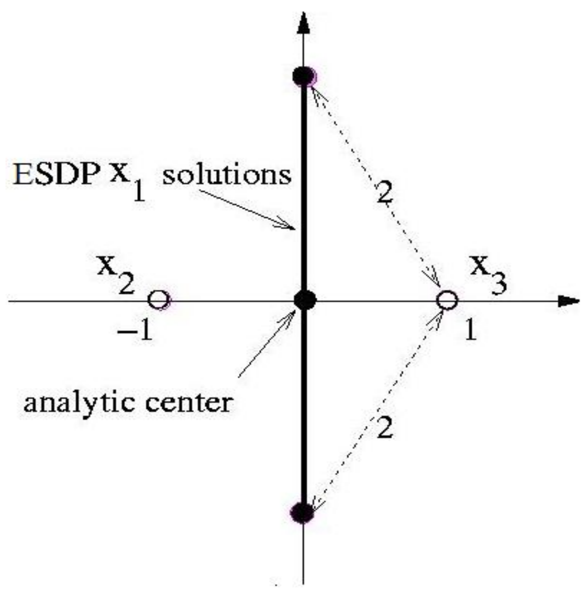
$$
\begin{aligned}
0 = \min_{x_1=(x_1^1,x_1^2)\in\Re^2,\, y_{11}\in\Re} \quad & |y_{11} - 2x_1^1 - 3| + |y_{11} + 2x_1^1 - 3| \\
\text{s.t.} \quad & \begin{bmatrix} y_{11} & x_1^T \\ x_1 & I \end{bmatrix} \succeq 0.
\end{aligned}
$$



ESDP/SDP Relaxation:

$$0 = \min_{x_1=(x_1^1,x_1^2)\in\Re^2,\, y_{11}\in\Re} |y_{11} - 2x_1^1 - 3| + |y_{11} + 2x_1^1 - 3|$$

$$\text{s.t.} \quad \begin{bmatrix} y_{11} & x_1^T \\ x_1 & I \end{bmatrix} \succeq 0.$$



If we solve ESDP by IP method, then we likely get an analytic center.

Alternative Problem Formulation

$$\min_{x_1,\ldots,x_m} \quad \sum_{(i,j)\in\mathcal{A}} \left(\|x_i - x_j\|^2 - d_{ij}^2\right)^2.$$

- Objective is a nonconvex degree 4 polynomial;
- Use convex relaxation.

Sparse-SOS Relaxation

Idea: Linearization.

Sparse-SOS Relaxation

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For $(i,j) \in \mathcal{A}^s$,

$$\beta_{ij}^4 := \{ \; 1 \quad x_i^1 \quad x_i^2 \quad x_j^1 \quad x_j^2 \quad (x_i^1)^2 \quad \cdots \quad (x_j^2)^2 \quad x_i^1 x_i^2 \quad \cdots \quad x_i^1 x_i^2 x_j^1 x_j^2 \; \}$$

$$\{ \; 1 \quad u_{x_i^1} \quad u_{x_i^2} \quad u_{x_j^1} \quad u_{x_j^2} \quad u_{(x_i^1)^2} \quad \cdots \quad u_{(x_j^2)^2} \quad u_{x_i^1 x_i^2} \quad \cdots \quad u_{x_i^1 x_i^2 x_j^1 x_j^2} \; \}$$

Sparse-SOS Relaxation

Idea: Linearization.

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$$\beta^4_{ij} := \{\ 1 \quad x^1_i \quad x^2_i \quad x^1_j \quad x^2_j \quad (x^1_i)^2 \quad \cdots \quad (x^2_j)^2 \quad x^1_i x^2_i \quad \cdots \quad x^1_i x^2_i x^1_j x^2_j\ \}$$

$$\{\ 1 \quad u_{x^1_i} \quad u_{x^2_i} \quad u_{x^1_j} \quad u_{x^2_j} \quad u_{(x^1_i)^2} \quad \cdots \quad u_{(x^2_j)^2} \quad u_{x^1_i x^2_i} \quad \cdots \quad u_{x^1_i x^2_i x^1_j x^2_j}\ \}$$

For $(i,j) \in \mathcal{A}^a$,

$$\beta^4_{ij} := \{\ 1 \quad x^1_i \quad x^2_i \quad (x^1_i)^2 \quad (x^2_i)^2 \quad x^1_i x^2_i \quad \cdots \quad (x^1_i)^2 (x^2_i)^2\ \}$$

$$\{\ 1 \quad u_{x^1_i} \quad u_{x^2_i} \quad u_{(x^1_i)^2} \quad u_{(x^2_i)^2} \quad u_{x^1_i x^2_i} \quad \cdots \quad u_{(x^1_i)^2 (x^2_i)^2}\ \}$$

Moment Matrix

Idea: Linearization of the inner product matrix by monomials up to degree 2, β^2_{ij}. Here shows $M_{\beta^2_{ij}}(u)$ for $(i,j) \in \mathcal{A}^a$:

$$
\begin{array}{cc}
 & \begin{array}{cccccc} 1 & x_i^1 & x_i^2 & (x_i^1)^2 & x_i^1 x_i^2 & (x_i^2)^2 \end{array} \\
\begin{array}{c} 1 \\ x_i^1 \\ x_i^2 \\ (x_i^1)^2 \\ x_i^1 x_i^2 \\ (x_i^2)^2 \end{array} &
\begin{bmatrix}
1 & u_{x_i^1} & u_{x_i^2} & u_{(x_i^1)^2} & u_{x_i^1 x_i^2} & u_{(x_i^2)^2} \\
u_{x_i^1} & u_{(x_i^1)^2} & u_{x_i^1 x_i^2} & u_{(x_i^1)^3} & u_{(x_i^1)^2 x_i^2} & u_{x_i^1 (x_i^2)^2} \\
u_{x_i^2} & u_{x_i^1 x_i^2} & u_{(x_i^2)^2} & u_{(x_i^1)^2 x_i^2} & u_{x_i^1 (x_i^2)^2} & u_{(x_i^2)^3} \\
u_{(x_i^1)^2} & u_{(x_i^1)^3} & u_{(x_i^1)^2 x_i^2} & u_{(x_i^1)^4} & u_{(x_i^1)^3 x_i^2} & u_{(x_i^1)^2 (x_i^2)^2} \\
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\end{bmatrix}
\end{array}
$$

Sparse-SOS Relaxation

Observation: At $(i,j) \in \mathcal{A}^s$,

$$u_s = s(x) \ \ \forall \deg(s) \le 4 \Leftrightarrow M_{\beta_{ij}^2}(u) \succeq 0 \text{ and } \operatorname{rank}(M_{\beta_{ij}^2}(u)) = 1.$$

Sparse-SOS relaxation (Nie '09):

$$\begin{aligned} v_{\text{spsos}} := \quad & \min_{y} \sum_{(i,j)\in\mathcal{A}} \sum_{s\in\beta_{ij}^4} p_s^{ij} u_s \\ & \text{s.t. } \ M_{\beta_{ij}^2}(u) \succeq 0 \quad \forall (i,j) \in \mathcal{A}^s, \end{aligned}$$

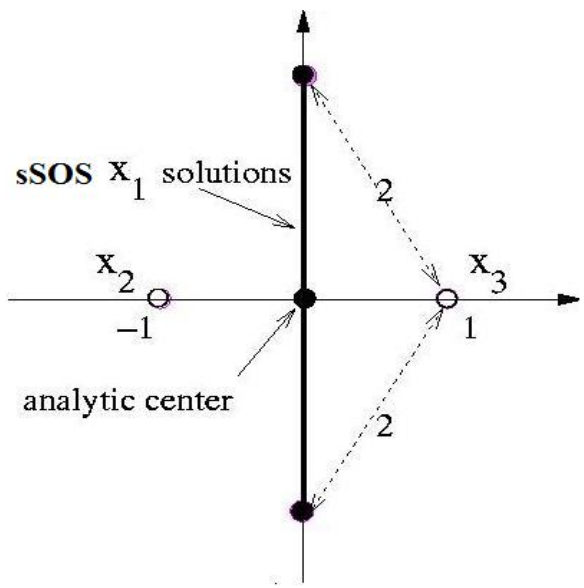
where

$$(\|x_i - x_j\|^2 - d_{ij}^2)^2 =: \sum_{s\in\beta_{ij}^4} p_s^{ij} s(x) \quad \forall (i,j) \in \mathcal{A}.$$

An Example: cont'd

Sparse-SOS Relaxation:

$$
\begin{aligned}
0 = \quad & \min_u \quad 2u_{(x_1^1)^4} + 4u_{(x_1^1 x_1^2)^2} - 4u_{(x_1^1)^2} + 2u_{(x_1^2)^4} - 12u_{(x_1^2)^2} + 18 \\
& \text{s.t.} \quad M_{\beta_{12}^2}(u) \succeq 0.
\end{aligned}
$$



An Example: cont'd

Sparse-SOS Relaxation:

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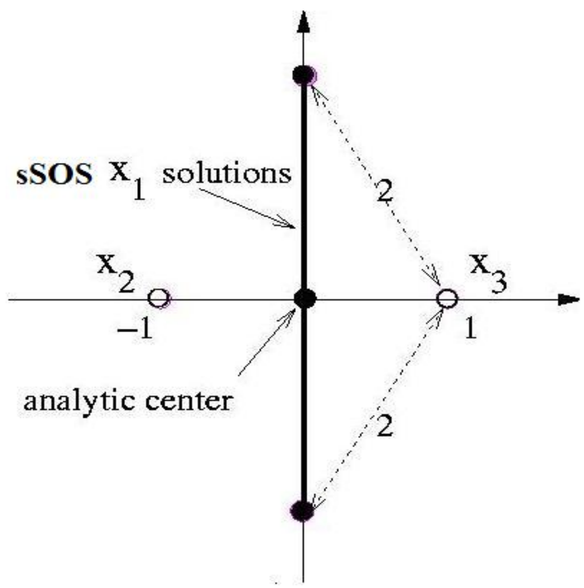

If we solve sparse-SOS by IP method, then we likely get the analytic center.

Properties of Relaxations

Assume that every connected component contains an anchor. Let $\mathrm{pos}(\cdot)$ denote the set of sensor positions ($\subseteq \Re^2$) obtained by solving the relaxation $(\cdot)$.

Fact 1:

- pos(ESDP), pos(sSOS) and pos(SDP) are compact convex sets.

- When $d_{ij} = \|x_i^{\mathrm{true}} - x_j^{\mathrm{true}}\|$ for all $(i,j) \in \mathcal{A}$ (noiseless case),

$$\mathrm{pos}(\mathrm{SDP}) \subseteq \mathrm{pos}(\mathrm{ESDP}),$$ (Wang et al. '08)

$$\mathrm{pos}(\mathrm{sSOS}) \subseteq \mathrm{pos}(\mathrm{ESDP}).$$ (Gouveia, P '10)

Fact 2 (local exactness):

- Define $\mathrm{tr}_i(Z) := y_{ii} - \|x_i\|^2$ for SDP and ESDP relaxation, and $\mathrm{Tr}_i(u) := u_{(x_i^1)^2} + u_{(x_i^2)^2} - (u_{x_i^1})^2 - (u_{x_i^2})^2$ for sparse-SOS relaxation.

- In noiseless case,

  ⋆ If $\mathrm{tr}_i(Z) = 0$ for some $Z \in \mathrm{ri}(\mathrm{Sol}(\mathrm{SDP}))$, then x_i is invariant over pos(SDP) (Tseng '07).

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  - ⋆ If $\mathrm{tr}_i(Z) = 0$ for some $Z \in \mathrm{ri}(\mathrm{Sol}(\mathrm{ESDP}))$, then x_i is invariant over pos(ESDP) (Wang et al. '08); the converse also holds (P, Tseng '10).

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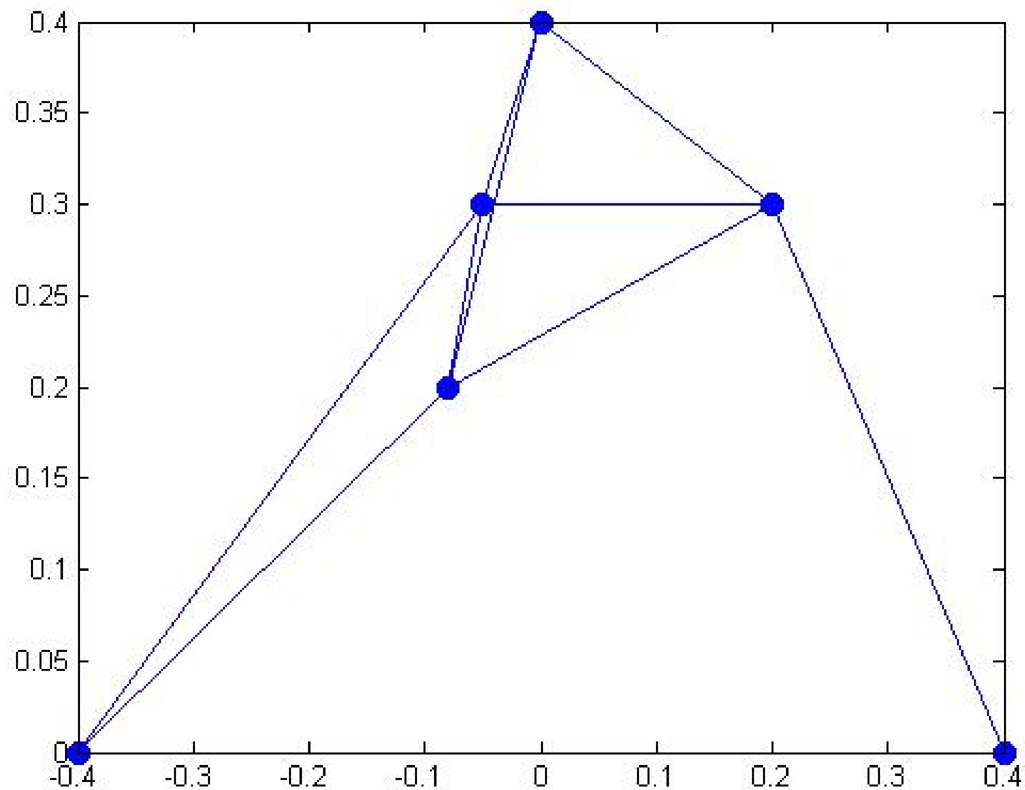
  - ⋆ If $\mathrm{Tr}_i(u) = 0$ for some $u \in \mathrm{ri}(\mathrm{Sol}(\mathrm{sSOS}))$, then $(u_{x_i^1}, u_{x_i^2})^T$ is invariant over pos(sSOS) (Gouveia, P '10).

Numerical Example: ESDP Vs Sparse-SOS



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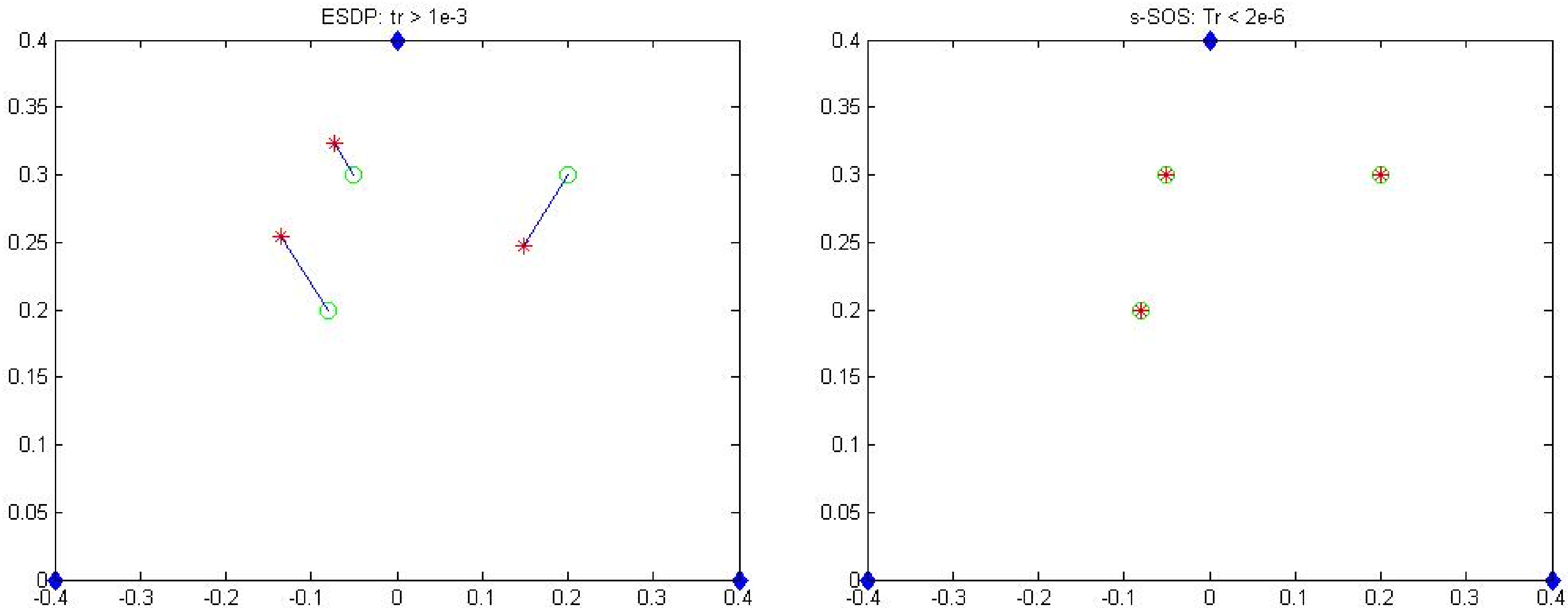
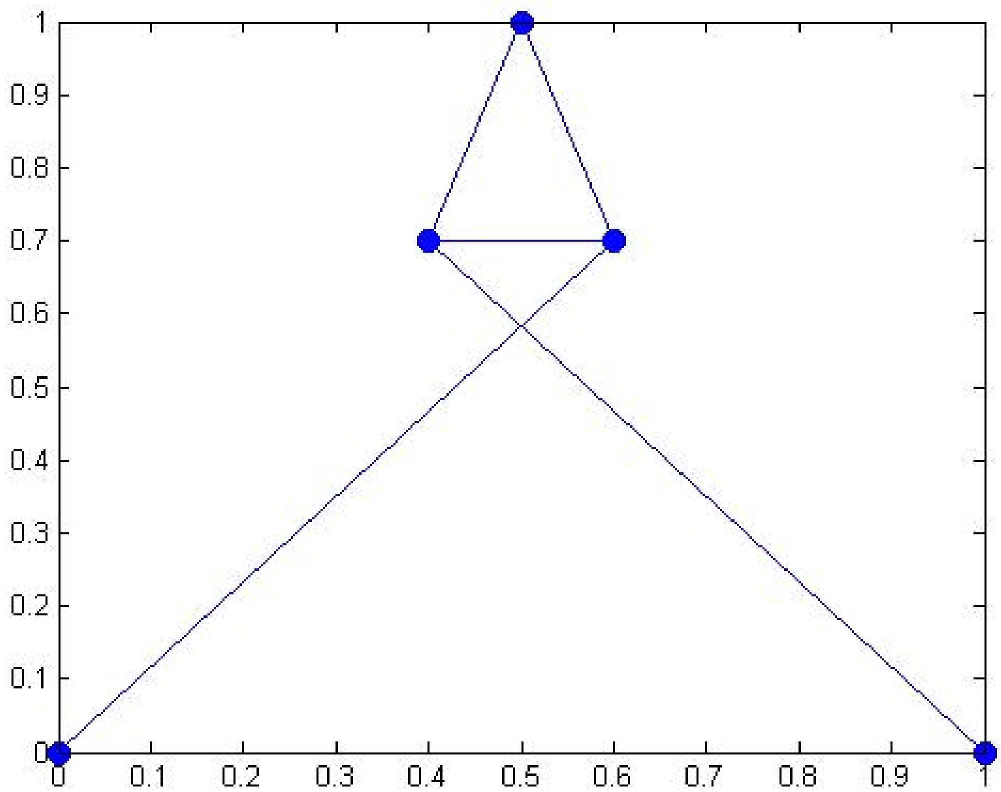


Numerical Example: SDP Vs Sparse-SOS



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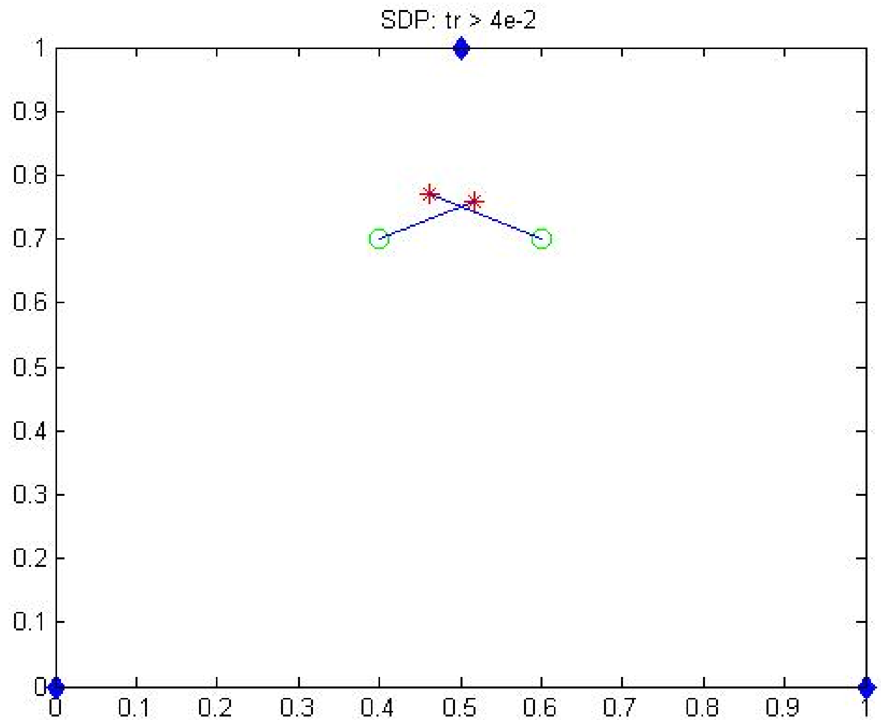
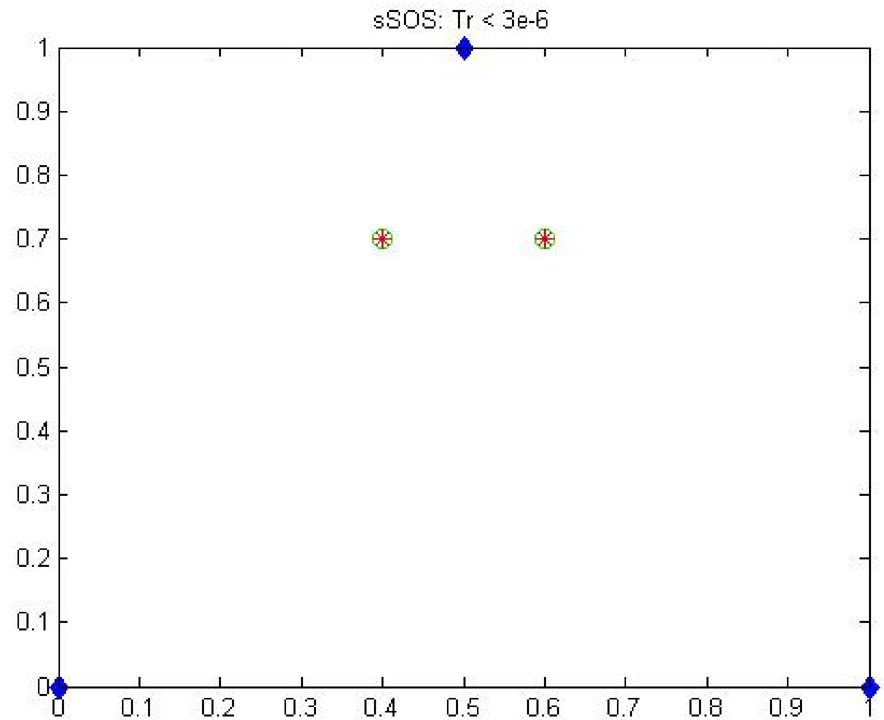




In practice, there are measurement noises:

$$d_{ij}^2 = \|x_i^{\mathrm{true}} - x_j^{\mathrm{true}}\|^2 + \delta_{ij} \quad \forall (i,j) \in \mathcal{A}.$$

How can one certify solution accuracy when $\delta_{ij} \approx 0$?

In practice, there are measurement noises:

$$d_{ij}^2 = \|x_i^{\text{true}} - x_j^{\text{true}}\|^2 + \delta_{ij} \quad \forall (i,j) \in \mathcal{A}.$$

How can one certify solution accuracy when $\delta_{ij} \approx 0$?

Individual trace test fails for ESDP relaxation.

Fact 3 (P, Tseng '10): For $|\delta_{ij}| \approx 0$,

$$\mathrm{tr}_i[Z] = 0 \quad \text{for some } Z \in \mathrm{ri}(\mathrm{Sol}(\mathrm{ESDP})) \not\Longrightarrow \quad \|x_i - x_i^{\text{true}}\| \approx 0.$$

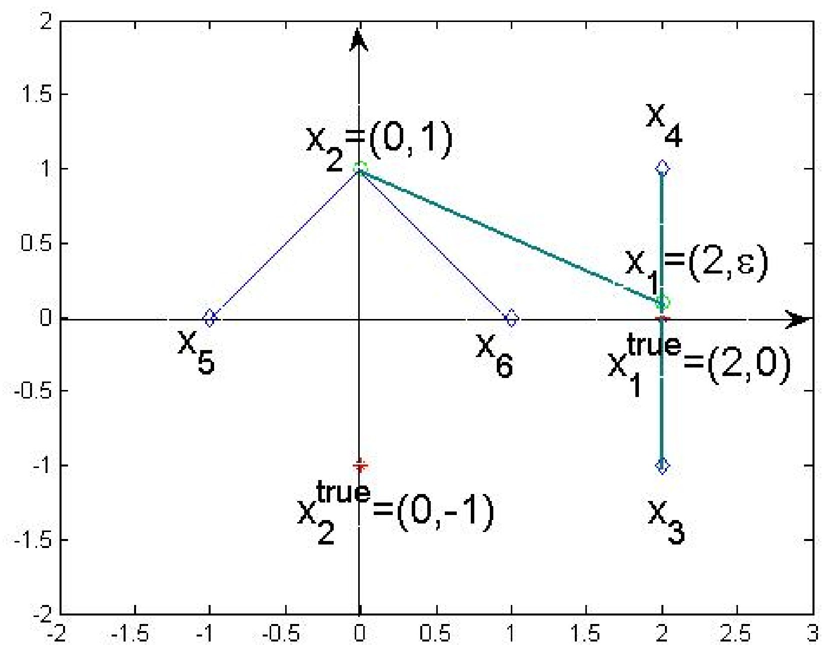
Same result holds for SDP/sSOS relaxation.

Proof is by counterexample.

An example of sensitivity of ESDP solns to measurement noise:

Input distance data: $\epsilon > 0$

$d_{12} = \sqrt{4 + (1-\epsilon)^2}, d_{13} = 1 + \epsilon, d_{14} = 1 - \epsilon, d_{25} = d_{26} = \sqrt{2}$; $m = 2$, $n = 6$.



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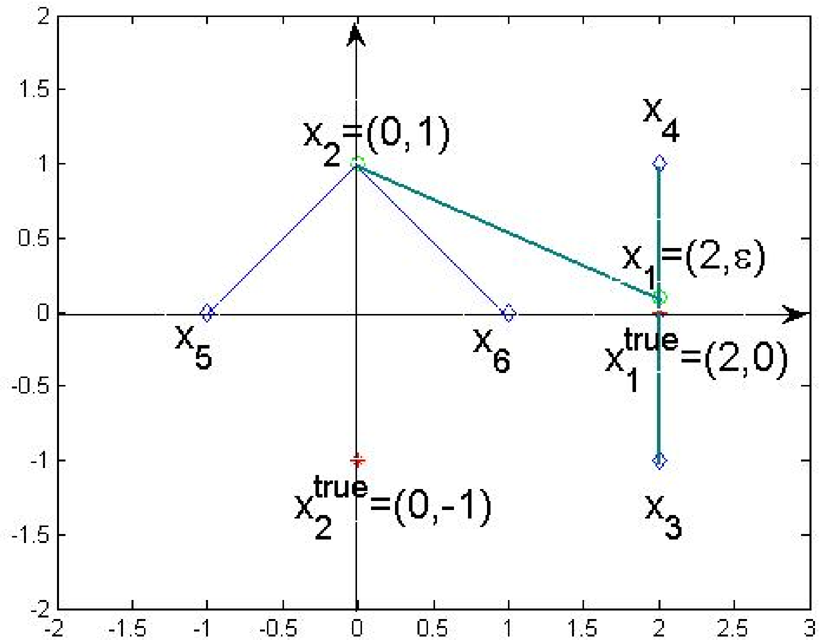

Thus, even when $Z \in \mathrm{Sol(ESDP)}$ is unique, $\mathrm{tr}_i[Z] = 0$ fails to certify accuracy of x_i in the noisy case!

Conclusion & Extension

- In the noiseless case:
  - ⋆ Sparse-SOS and SDP relaxations are stronger than ESDP relaxation;
  - ⋆ Zero trace is sufficient for accuracy for all three relaxations, and is necessary for ESDP relaxation.

- In the noisy case, the trace test fails for all three relaxations.

- Is zero trace condition necessary for accuracy for sparse-SOS relaxation?

- There is an efficient fast distributed algorithm for ESDP relaxation (P, Tseng '10). Is it possible to develop fast distributed algorithm for sparse-SOS relaxation?

Thanks for coming! ☺