Gauge Optimization and Duality

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(Joint work with Michael Friedlander and Ives Macêdo)

Motivating Example

Minimum norm solutions:

- In sparse optimization:

$$\begin{array}{rl} \min & \|x\|_1 \\ \text{s.t.} & \|b - Ax\| \leq \sigma. \end{array}$$

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- More generally, minimization of atomic norm (Chandrasekaran et al. ’12)

$$\|x\|_{\mathcal{A}} = \inf\{\lambda \geq 0 : \; x \in \lambda \operatorname{conv} \mathcal{A}\},$$

where $\mathcal{A}$ is a set of “atoms” characterizing the notion of sparsity:

$\star$ $\mathcal{A} = \{\pm e_i : \; i = 1, \dots, n\} \;\; \Rightarrow \;\; \|x\|_{\mathcal{A}} = \sum_{i=1}^n |x_i|$.
$\star$ $\mathcal{A} =$ unit norm rank 1 matrices $\;\; \Rightarrow \;\; \|X\|_{\mathcal{A}} = \sum_{i=1}^n \sigma_i(X)$.

Gauges

- Gauges are generalizations of norms: nonnegative convex positively homogeneous functions that are zero at the origin.

- $\kappa(x) = \inf\{\lambda \geq 0 : \; x \in \lambda U\}$ for some convex set U.

- Polar gauge generalizes dual norm:

$$\begin{aligned} \kappa^\circ(y) &= \inf\{\lambda > 0 : \; \langle x, y \rangle \leq \lambda \kappa(x) \; \forall x\} \\ &= \sup\{\langle x, y \rangle : \; \kappa(x) \leq 1\}. \end{aligned}$$

- Generalized Cauchy inequality: for all $x \in \operatorname{dom} \kappa$ and $y \in \operatorname{dom} \kappa^\circ$,

$$\langle x, y \rangle \leq \kappa(x) \kappa^\circ(y).$$

Gauge Optimization

$$\begin{aligned} v_p := \min \quad & \kappa(x) \\ \text{s.t.} \quad & \rho(b - Ax) \le \sigma. \end{aligned} \qquad (\mathrm{P}_\rho)$$

- κ is a gauge.
- ρ is a closed gauge with $\rho^{-1}(0) = \{0\}$, $0 \le \sigma < \rho(b)$.
- Lagrange and gauge dual problems:

$$\begin{aligned} v_\ell := \max \quad & \langle b, y\rangle - \sigma\rho^\circ(y) \\ \text{s.t.} \quad & \kappa^\circ(A^*y) \le 1. \end{aligned} \qquad \begin{aligned} v_g := \min \quad & \kappa^\circ(A^*y) \\ \text{s.t.} \quad & \langle b, y\rangle - \sigma\rho^\circ(y) \ge 1. \end{aligned}$$

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- The role of objective and constraint is reversed in the gauge dual.

Outline

- Further examples on gauge optimization problems
- Gauge duality theory: general framework
- Gauge duality theory: structured problem
- Conic gauge optimization and its dual

Example 2

Conic gauge optimization:

- In conic optimization:

$$\begin{array}{ll} \min & \langle c, x\rangle \\ \text{s.t.} & Ax = b,\ x \in \mathcal{K}. \end{array}$$

  If $c \in \mathcal{K}^*$, then $\langle c, \cdot\rangle + \delta_{\mathcal{K}}(\cdot)$ is a gauge.

- Examples: SDP relaxation of max-cut, phase retrieval...

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- More generally, let $\hat{y}$ be feasible for the dual, i.e., $c - A^*\hat{y} \in \mathcal{K}^*$, then $\hat{c} := c - A^*\hat{y} \in \mathcal{K}^*$ and

$$\begin{array}{ll} \min & \langle \hat{c}, x \rangle + \delta_{\mathcal{K}}(x) \\ \text{s.t.} & Ax = b \end{array}$$

  is a gauge optimization problem.

Example 3

Submodular functions:

Let $V = \{1, \ldots, n\}$ and $f : 2^V \to \mathbb{R}$ with $f(\emptyset) = 0$. The Lovàsz extension (Lovàsz '83) is:

$$\widehat{f}(x) = \sum_{k=1}^{n} x_{j_k}[f(\{j_1, \ldots, j_k\}) - f(\{j_1, \ldots, j_{k-1}\})],$$

where $x_{j_1} \geq x_{j_2} \geq \cdots \geq x_{j_n}$. Then $\widehat{f} + \delta_{\mathbb{R}^n_+}$ is a gauge if:

- f is submodular (so that $\widehat{f}$ is convex):

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \;\; \forall A, B \subseteq V; \text{ and}$$

- $A \subseteq B \;\Rightarrow\; f(A) \leq f(B)$.

Gauge Duality Framework

Let $\mathcal{C}$ be a closed convex set not containing the origin, and define its anti-polar

$$\mathcal{C}' = \{u : \ \langle u, x\rangle \geq 1 \ \ \forall x \in \mathcal{C}\}.$$

Freund ('87) defined the following primal-dual gauge pairs:

$$\begin{aligned} v_p := \min \quad & \kappa(x) \\ \text{s.t.} \quad & x \in \mathcal{C}, \end{aligned} \tag{P}$$

$$\begin{aligned} v_g := \min \quad & \kappa^\circ(u) \\ \text{s.t.} \quad & u \in \mathcal{C}'. \end{aligned} \tag{D}$$

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Fact (Freund '87): [Weak duality] Suppose that $\operatorname{dom} \kappa^\circ \cap \mathcal{C}' \neq \emptyset$ and $\operatorname{dom} \kappa \cap \mathcal{C} \neq \emptyset$. Then $v_p v_g \geq 1$.

Gauge Duality Framework

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Consider the bi-dual:

$$\begin{array}{ll} \min & \kappa^{\circ\circ}(x) \\ \text{s.t.} & x \in \mathcal{C}'', \end{array} \qquad \text{(bi-D)}$$

and observe that $\mathcal{C}'' = \bigcup_{\lambda \geq 1} \lambda\mathcal{C}$.

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Fact (Freund '87, Friedlander, Macêdo, P. '13): [Strong duality II] Suppose that κ is closed, $\operatorname{ri}\operatorname{dom}\kappa^\circ \cap \operatorname{ri}\mathcal{C}' \neq \emptyset$ and $\operatorname{ri}\operatorname{dom}\kappa \cap \operatorname{ri}\mathcal{C} \neq \emptyset$. Then $v_p v_g = 1$ and both values are attained.

Anti-polar Calculus

Let $\mathcal{D} := \{u : \ \rho(b-u) \le \sigma\}$. Then

$$\mathcal{C} = \{x : \ \rho(b - Ax) \le \sigma\} = A^{-1}\mathcal{D}.$$

Fact:

$$\mathcal{D}' = \{y : \ \langle b, y\rangle - \sigma\rho^\circ(y) \ge 1\}.$$

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Fact:

$$(A^{-1}\mathcal{D})' = \mathrm{cl}(A^*\mathcal{D}').$$

If, in addition, $\mathrm{ri}\,\mathcal{D} \cap \mathrm{Range}\,A \ne \emptyset$, then

$$(A^{-1}\mathcal{D})' = A^*\mathcal{D}' = \{A^*y : \ \langle b, y\rangle - \sigma\rho^\circ(y) \ge 1\}.$$

Strong Duality

Consider the following primal-dual gauge pairs:

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Let $\mathcal{D} := \{u : \ \rho(b - u) \le \sigma\}$ so that primal feasible set is $A^{-1}\mathcal{D}$.

Fact: Suppose that $\operatorname{dom} \kappa^\circ \cap A^* \mathcal{D}' \neq \emptyset$ and $\operatorname{ri} \operatorname{dom} \kappa \cap A^{-1} \operatorname{ri} \mathcal{D} \neq \emptyset$. Then $v_p v_g = 1$ and v_g is attained.

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Unlike the Lagrange dual, the gauge dual (D$_\rho$) has a complicated objective and simple constraint.

Conic Gauge Optimization Revisited

Nonnegative SDP

Let $C \succeq 0$ and consider

$$\begin{array}{rl} \min & \operatorname{tr}(CX) + \delta_{\cdot \succeq 0}(X) \\ \text{s.t.} & \mathcal{A}(X) = b. \end{array}$$

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The polar of this objective is

$$\kappa^\circ(U) = \inf\{\alpha \geq 0 : \ \alpha C - U \in \mathcal{S}^n_+\} = \max\{0, \lambda_{\max}(U, C)\}.$$

Aside, $\operatorname{dom} \kappa^\circ = \mathbb{R}_+ C - \mathcal{S}^n_+$: not closed for any nonzero C (Ramana, Tunçel, Wolkowicz ’97).

Conic Gauge Optimization Revisited

Nonnegative SDP (cont.)

If $C = I$, then $v_p v_g = 1$ and v_p is attained:

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- The dual feasible set is easy to project onto. Can even be eliminated.
- The gauge dual is an eigenvalue optimization problem.

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Let $c \in \operatorname{int} \mathcal{K}^*$ and consider

$$\begin{array}{rl} \min & \langle c, x \rangle + \delta_{\mathcal{K}}(x) \\ \text{s.t.} & Ax = b. \end{array}$$

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$$\kappa^\circ(u) = \inf\{\alpha \geq 0 : \ \alpha c - u \in \mathcal{K}^*\}.$$

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Example: Second-order cone $\mathcal{L} = \left\{ x = \begin{pmatrix} x_0 & \bar{x}^T \end{pmatrix}^T \in \mathbb{R}^{n+1} : \ x_0 \geq \|\bar{x}\| \right\}$

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Idea: Find a linear map that maps c to something simple and keeps $\mathcal{L}$.

Conic Gauge Optimization Revisited

Nonnegative SOCP

A classical result for symmetric cones: There exists (explicit formula) $d \in \operatorname{int} \mathcal{L}$ with $Q_d c = e_0$, where

$$Q_d = \begin{bmatrix} \|d\|^2 & 2d_0\bar{d}^T \\ 2d_0\bar{d} & (d_0^2 - \|\bar{d}\|^2)I + 2\bar{d}\bar{d}^T \end{bmatrix}$$

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$$\alpha c - u \in \mathcal{L} \Leftrightarrow \alpha e_0 - Q_d u \in \mathcal{L} \Leftrightarrow \alpha - (Q_d u)_0 \geq \|\overline{Q_d u}\|.$$

Gauge Dual:

$$\begin{array}{ll} \min & (Q_d A^* y)_0 + \|\overline{Q_d A^* y}\| \\ \text{s.t.} & \langle b, y \rangle = 1. \end{array}$$

Future Directions

- Develop algorithms for solving gauge dual, exploiting the “simplicity” of constraints
- Sensitivity analysis

Conclusion

- Gauge optimization framework captures a wide range of applications.
- The gauge dual leads to a nonsmooth problem over a simple set.
- Strong duality holds under conditions similar to standard CQ in Lagrange duality theory.

Reference:

M. Friedlander, I. Macêdo and T. K. Pong.
Gauge Optimization and Duality.
Available at `http://arxiv.org/abs/1310.2639`.

Thanks for coming!