Splitting methods for nonconvex feasibility problems

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Feasibility Problem

- Given closed sets D_i, $i = 1, \ldots, m$, find a point

$$x \in \bigcap_{i=1}^{m} D_i.$$

- Example: Finding a solution of $Ax = b$ with $\|x\|_0 \leq r$.

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$$x \in \bigcap_{i=1}^{m} D_i.$$

- Example: Finding a solution of $Ax = b$ with $\|x\|_0 \leq r$.
- The general problem can be reformulated as finding a point in

$$\{(x_1, \ldots, x_m) : \; x_1 = \cdots = x_m\} \cap (D_1 \times D_2 \times \cdots \times D_m).$$

- Only need to consider the intersection of a closed *convex* set C and a closed set D.

When D is convex

- Alternating projection:

$$x^{t+1} = P_D(P_C(x^t)).$$

- Splitting methods ($0 < \alpha \leq 2$):

$$\begin{cases} y^{t+1} = \arg\min_{y \in C} \left\{ \|y - x^t\| \right\}, \\ z^{t+1} = \arg\min_{z \in D} \left\{ \|2y^{t+1} - x^t - z\| \right\}, \\ x^{t+1} = x^t + \alpha(z^{t+1} - y^{t+1}). \end{cases}$$

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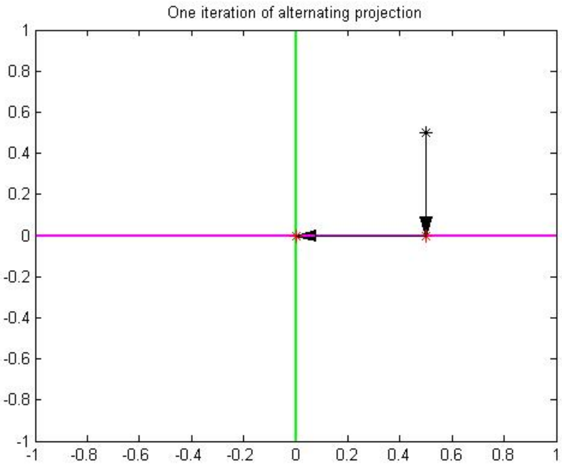
- Douglas-Rachford (DR): $\alpha = 1$.
- Peaceman-Rachford (PR): $\alpha = 2$.

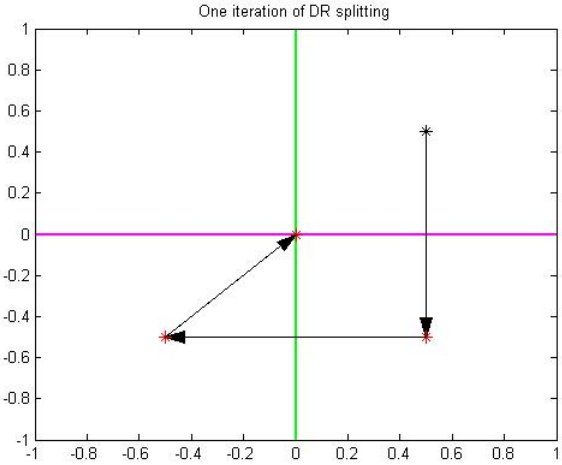
Behavior in convex case: AP

Finding the intersection of the axes, starting from $(0.5, 0.5)$.



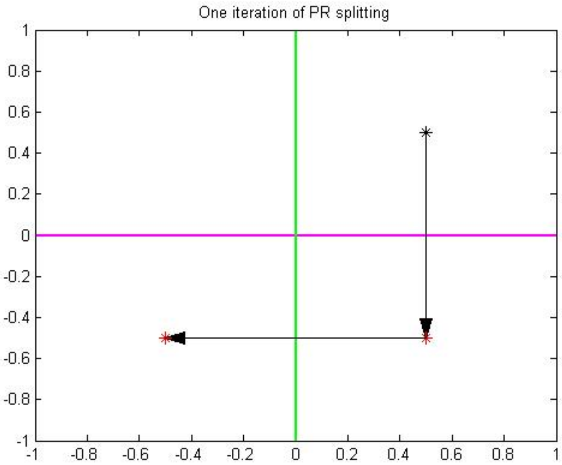
Behavior in convex case: DR

Showing the x-iterates: average after two successive reflections.
$P_C(x^t)$ will converge to the intersection.



Behavior in convex case: PR

Showing the x-iterate; not convergent.



When D is nonconvex

For the convergence of DR splitting:

- Mainly local convergence results.
- Require various regularity conditions on the sets.
- Local convergence for finding intersection of $Ax = b$ and $\|x\|_0 \leq r$. (Hesse, Luke, Neumann '13).

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- Mainly local convergence results.
- Require various regularity conditions on the sets.
- Local convergence for finding intersection of $Ax = b$ and $\|x\|_0 \leq r$. (Hesse, Luke, Neumann '13).
- Global convergence shown for the intersection of a circle and a straight line in $\mathbb{R}^2$. (Artacho, Borwein '12)

Our DR splitting

- DR splitting: ($\gamma > 0$)

$$\begin{cases} y^{t+1} = \arg\min_{y} \left\{ \frac{1}{2} d_C^2(y) + \frac{1}{2\gamma}\|y - x^t\|^2 \right\}, \\ z^{t+1} \in \operatorname*{Arg\,min}_{z \in D} \left\{ \|2y^{t+1} - x^t - z\|^2 \right\}, \\ x^{t+1} = x^t + (z^{t+1} - y^{t+1}). \end{cases}$$

- The y-update is $\frac{1}{1+\gamma}(x^t + \gamma P_C(x^t))$.

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- The y-update is $\frac{1}{1+\gamma}(x^t + \gamma P_C(x^t))$.
- DR splitting applied to minimizing $\frac{1}{2}d_C^2 + \delta_D$.

DR Convergence result I

Fact 1 (Li, P '14): [Global convergence]
Suppose that $0 < \gamma < \sqrt{\frac{3}{2}} - 1$, and either C or D is compact.

Then the sequence $\{(y^t, z^t, x^t)\}$ generated from DR splitting is bounded, and any cluster point (y^*, z^*, x^*) satisfies $z^* = y^*$. Moreover, y^* is a stationary point of

$$\min_{u \in D} \frac{1}{2} d_C^2(u),$$

i.e., $0 \in y^* - P_C(y^*) + N_D(y^*)$.

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i.e., $0 \in y^* - P_C(y^*) + N_D(y^*)$.

- Clearly, if $d_C(y^*) = 0$, then y^* solves the feasibility problem.

DR Convergence result II

Fact 2 (Li, P '14): [Convergence of the whole sequence]

Suppose that $0 < \gamma < \sqrt{\frac{3}{2}} - 1$, C and D are semi-algebraic, and one of them is compact.

Then the sequence $\{(y^t, z^t, x^t)\}$ generated from DR splitting is bounded, and is convergent to some (y^*, z^*, x^*) satisfying $z^* = y^*$, with y^* being a stationary point of the problem $\min_{u \in D} \frac{1}{2} d_C^2(u)$. Furthermore,

$$\sum_{t=1}^{\infty} \|y^{t+1} - y^t\| < \infty.$$

DR Convergence result III

Fact 3 (Li, P '14): [Local convergence]
Let $C = \{x : \; Ax = b\}$ and D be a closed semi-algebraic set, $0 < \gamma < \sqrt{\frac{3}{2}} - 1$ and $\lim(y^t, z^t, x^t) = (y^*, z^*, x^*)$.
Suppose that $z^* \in C \cap D$ with

$$N_C(z^*) \cap -N_D(z^*) = \{0\}.$$

Then there exist $\eta \in (0, 1)$ and $\kappa > 0$ such that for all large t,

$$\mathrm{dist}\big(0, z^t - P_C(z^t) + N_D(z^t)\big) \leq \kappa\, \eta^t.$$

Our DR vs classical DR

- Example that the classical DR diverges: for $\eta \in (0, 1]$ (Bauschke, Noll '14)

$$C = \{x \in \mathbb{R}^2 : \ x_2 = 0\}$$
$$D = \{(0, 0), (7 + \eta, \eta), (7, -\eta)\}$$

Initialized at $x^0 = (7, \eta)$, the classical DR exhibits a discrete limit cycle.

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- For our DR, with $\gamma \in (0, \sqrt{\frac{3}{2}} - 1)$, explicit computation shows that $y^* = z^* = (7 + \eta, \eta)$ and $x^* = (7 + \eta, (1 + \gamma)\eta)$ for this starting point.

For PR splitting

- Does not converge in general even if D is convex.
- Modifying as follows also cannot guarantee convergence even if both sets are convex:

$$
\begin{cases}
y^{t+1} = \arg\min_{y} \left\{ \frac{1}{2} d_C^2(y) + \frac{1}{2\gamma} \|y - x^t\|^2 \right\}, \\
z^{t+1} \in \operatorname{Arg\,min}_{z \in D} \left\{ \|2y^{t+1} - x^t - z\|^2 \right\}, \\
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- Indeed, PR splitting applied to minimizing sum of convex functions $f + g$ converges when f is continuous and strictly convex. (Lions, Mercier '79)

Our PR splitting

- PR splitting: ($\gamma > 0$)

$$\begin{cases} y^{t+1} = \arg\min\limits_{y} \left\{ \frac{1}{2} d_C^2(y) + \frac{5}{2}\|y\|^2 + \frac{1}{2\gamma}\|y - x^t\|^2 \right\}, \\ z^{t+1} \in \operatorname*{Arg\,min}\limits_{z \in D} \left\{ -\frac{5}{2}\|z\|^2 + \frac{1}{2\gamma}\|2y^{t+1} - x^t - z\|^2 \right\}, \\ x^{t+1} = x^t + 2(z^{t+1} - y^{t+1}). \end{cases}$$

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- Closed form updates for $\gamma \in (0, \frac{1}{5})$:

$$y^{t+1} = \frac{1}{6\gamma + 1}\left[x^t + \gamma P_C\left(\frac{x^t}{5\gamma + 1}\right)\right], \quad z^{t+1} \in P_D\left(\frac{2y^{t+1} - x^t}{1 - 5\gamma}\right).$$

PR Convergence result

Fact 5 (Li, P '15): [Global convergence]
Suppose that $0 < \gamma < \frac{1}{12}$, and D is compact.

Then the sequence $\{(y^t, z^t, x^t)\}$ generated from PR splitting is bounded, and any cluster point (y^*, z^*, x^*) satisfies $z^* = y^*$. Moreover, y^* is a stationary point of

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If both sets are in addition semi-algebraic, then the whole sequence is convergent.

More general settings

- Splitting methods for $\min_u \; f(u) + g(u)$, $\alpha = 1$ or 2:

$$\begin{cases} y^{t+1} \in \operatorname*{Arg\,min}_y \left\{ f(y) + \frac{1}{2\gamma}\|y - x^t\|^2 \right\}, \\ z^{t+1} \in \operatorname*{Arg\,min}_z \left\{ g(z) + \frac{1}{2\gamma}\|2y^{t+1} - x^t - z\|^2 \right\}, \\ x^{t+1} = x^t + \alpha(z^{t+1} - y^{t+1}). \end{cases}$$

- f has Lipschitz gradient whose continuity modulus is L, g is proper closed; $f + \frac{l}{2}\|\cdot\|^2$ is convex.

Convergence results for DR: $\alpha = 1$

Fact 6 (Li, P '14): [Global convergence]
Suppose that $(1+\gamma L)^2 + \frac{5\gamma l}{2} - \frac{3}{2} < 0$, f and g are bounded below and at least one of them is coercive.
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If, in addition, f and g are semi-algebraic, then the whole sequence is convergent.

Convergence proof?

- KEY: Makes use of

$$\mathfrak{D}_\gamma(y,z,x) := f(y) + g(z) + \frac{1}{2\gamma}\|x-y\|^2 - \frac{1}{2\gamma}\|x-z\|^2.$$

- Can show that for some $k_1, k_2 > 0$:

$$\begin{aligned}\mathfrak{D}_\gamma(y^t,z^t,x^t) - \mathfrak{D}_\gamma(y^{t+1},z^{t+1},x^{t+1}) &\geq k_1\|y^{t+1}-y^t\|^2;\\ \mathrm{dist}(0,\partial\mathfrak{D}_\gamma(y^t,z^t,x^t)) &\leq k_2\|y^{t+1}-y^t\|.\end{aligned}$$

Convergence results for PR: $\alpha = 2$

Fact 7 (Li, P '15): [Global convergence]
Suppose that f is strongly convex with strong convexity modulus $\sigma > 0$ and $3\sigma > 2L$, g is bounded below, and $\gamma \in (0, \frac{3\sigma - 2L}{L^2})$.
Then the sequence generated is bounded, and any cluster point (y^*, z^*, x^*) satisfies $z^* = y^*$. Moreover, y^* is a stationary point of

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If, in addition, f and g are semi-algebraic, then the whole sequence is convergent.

Numerical simulations

- Find a point in $Ax = b$ with $\|x\|_0 \leq r$ and $\|x\|_\infty \leq 10^6$.
- Consider random instances: generate an r-sparse vector $\tilde{x}$, an $m \times n$ matrix A, and set $b = A\tilde{x}$.
- Compare with alternating projection. Initialize all three algorithms at $x^0 = 0$.
- Terminate when successive changes are less than 10^{-8}.

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- Compare with alternating projection. Initialize all three algorithms at $x^0 = 0$.
- Terminate when successive changes are less than 10^{-8}.
- For the splitting methods, start with a γ larger than the threshold, decrease γ if $\|y^{t+1} - y^t\|$ does not deteriorate quickly enough or $\|y^t\|$ becomes too large.

Numerical simulations

Over 50 trials for each m, n; sparsity is $\lceil \frac{m}{5} \rceil$; succ means $\mathrm{fval} < 10^{-12}$.

Data	DR: $fval = \frac{1}{2}d_C^2(z^t)$			PR: $fval = \frac{1}{2}d_C^2(z^t)$			Alt Proj: $fval = \frac{1}{2}d_C^2(x^t)$		
m, n	iter	$\mathrm{fval}_{\max}$	succ	iter	$\mathrm{fval}_{\max}$	succ	iter	$\mathrm{fval}_{\max}$	succ
100, 4000	1967	3e-02	30	491	7e-2	0	1694	8e-2	0
100, 5000	2599	2e-02	18	586	7e-2	0	1978	7e-2	0
100, 6000	2046	1e-02	12	684	5e-2	0	2350	5e-2	0
200, 4000	836	2e-15	50	310	2e-1	14	1076	3e-1	0
200, 5000	1080	3e-15	50	364	1e-1	2	1223	2e-1	0
200, 6000	1279	7e-02	43	431	1e-1	5	1510	2e-1	1
300, 4000	600	3e-15	50	223	2e-1	35	872	4e-1	3
300, 5000	710	4e-15	50	295	2e-1	25	1068	3e-1	3
300, 6000	812	3e-15	50	350	2e-1	21	1252	3e-1	1
400, 4000	520	2e-15	50	156	3e-1	47	818	6e-1	30
400, 5000	579	3e-15	50	213	3e-1	42	946	4e-1	12
400, 6000	646	4e-15	50	288	2e-1	38	1108	3e-1	4

Open questions

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$\mathrm{AP} + \mathrm{ls}$: Alternating projection with nonmonotone line search using Barzalai-Borwein stepsize.

Data		DR: $fval = \frac{1}{2}d_C^2(z^t)$			AP + ls: $fval = \frac{1}{2}d_C^2(x^t)$		
m	n	iter	$\mathrm{fval}_{\max}$	succ	iter	$\mathrm{fval}_{\max}$	succ
100	4000	1967	3e-02	30	81	2e-02	8
100	5000	2599	2e-02	18	78	2e-02	5
200	4000	836	2e-15	50	75	6e-02	28
200	5000	1080	3e-15	50	86	7e-02	15
300	4000	600	3e-15	50	60	7e-02	48
300	5000	710	4e-15	50	67	1e-01	36

Conclusion

- The DR splitting applied to $\min_{u \in D} \frac{1}{2} d_C^2(u)$ with a compact C or D generates a sequence that clusters at a stationary point.
- The PR splitting *suitably* applied to $\min_{u \in D} \frac{1}{2} d_C^2(u)$ with a compact D generates a sequence that clusters at a stationary point.
- Under semi-algebraicity assumption, the whole sequence converges.

Reference:

- G. Li and T. K. Pong.
  Douglas-Rachford splitting for nonconvex optimization with application to nonconvex feasibility problems.
  Available at *http://arxiv.org/abs/1409.8444*.
- G. Li and T. K. Pong.
  Peaceman-Rachford splitting for a class of nonconvex optimization problems.
  Available at *http://arxiv.org/abs/1507.00887*.

Thanks for coming! $\because$