Convex Relaxations for Sensor Network Localization

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(Joint work with João Gouveia and Paul Tseng)

Outline

- Sensor network localization.
- Convex relaxations: Biswas-Ye SDP, ESDP, sparse-SOS.
- Comparison of relaxations.
- Noisy case and ρESDP.
- Conclusion & extensions.

Sensor Network Localization

Basic Problem:

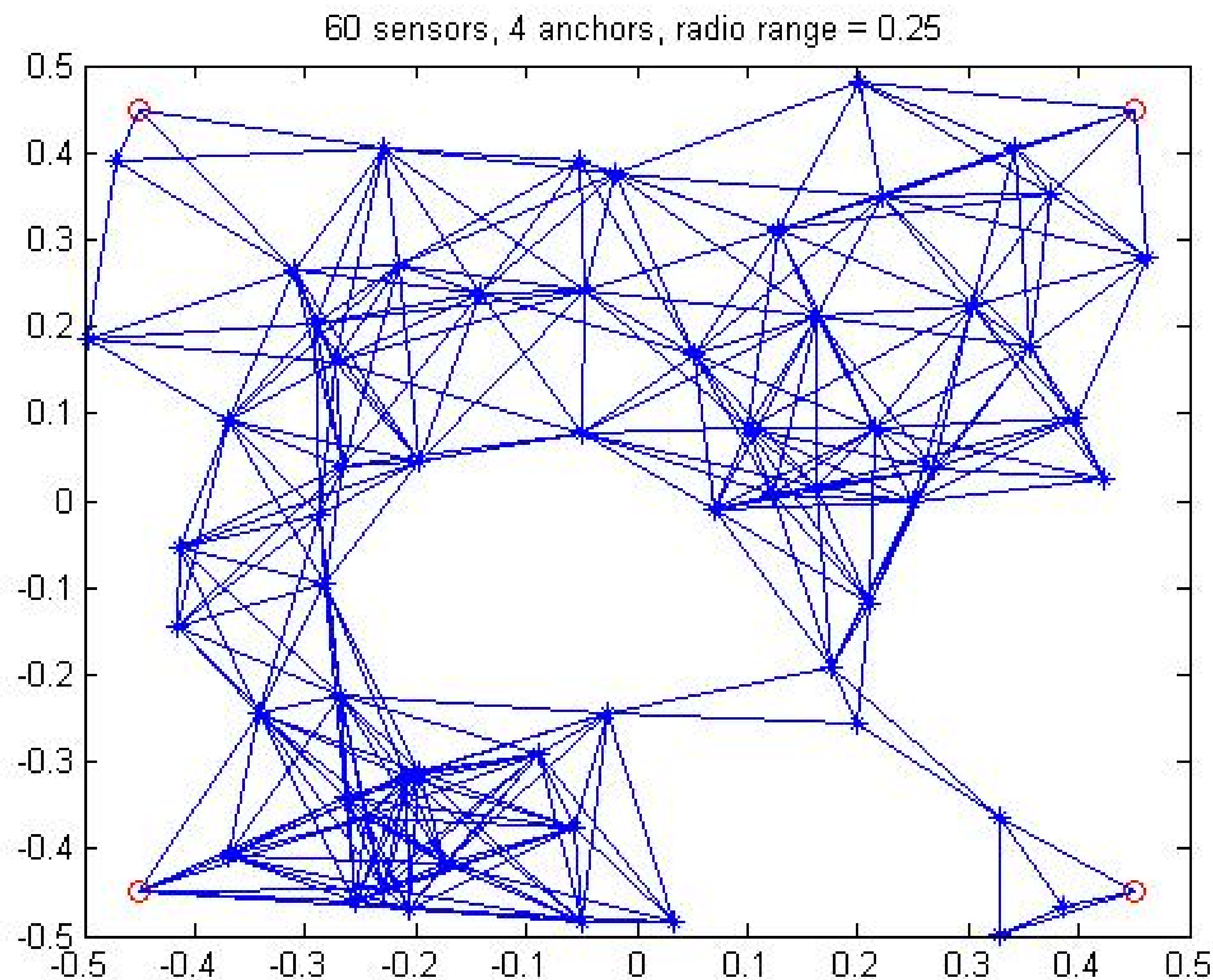
- n pts $\underbrace{x_1, ..., x_m}_{\text{sensors}}, \underbrace{x_{m+1}, ..., x_n}_{\text{anchors}}$ in $\Re^2$.

- Know last $n-m$ pts ('anchors') $x_{m+1}, ..., x_n$ and Eucl. dist. estimate for some pairs of 'neighboring' pts (i.e. within 'radio range')

$$d_{ij} \geq 0 \quad \forall (i,j) \in \mathcal{A},$$

with $\mathcal{A} \subseteq \{(i,j) : 1 \leq i < j \leq n\}$.

- Estimate the position of the first m pts ('sensors') $x_1, ..., x_m$.


60 sensors, 4 anchors, radio range = 0.25
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.5
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0.3
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0.5

Optimization Problem Formulation

$$v_{\mathrm{p}} := \min_{x_1,\ldots,x_m} \sum_{(i,j)\in\mathcal{A}} \left| \|x_i - x_j\|^2 - d_{ij}^2 \right|.$$

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- Objective function is nonconvex. m can be large ($m > 1000$).
- Problem is NP-hard (reduction from INTEGER PARTITION).

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- Objective function is nonconvex. m can be large ($m > 1000$).
- Problem is NP-hard (reduction from INTEGER PARTITION).
- Aim 1: Tractability – use a convex relaxation.
- Aim 2: Identify sensors correctly positioned by relaxation.

SDP Relaxation

Idea: Linearization.

Let $X = [x_1 \cdots x_m]$ and $Y = X^T X$. Then $y_{ij} = x_i^T x_j$ for all i, j.

Observation:

SDP Relaxation

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Let $X = [x_1 \cdots x_m]$ and $Y = X^T X$. Then $y_{ij} = x_i^T x_j$ for all i, j.

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- $$Y = X^T X \Longleftrightarrow Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \succeq 0, \ \text{rank}(Z) = 2.$$

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Observation:

- $$Y = X^T X \iff Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \succeq 0,\ \text{rank}(Z) = 2.$$

- For $(i, j) \in \mathcal{A}^s$,

  $$||x_i - x_j||^2 = \|x_i\|^2 - 2x_i^T x_j + \|x_j\|^2 = y_{ii} - 2y_{ij} + y_{jj};$$

  For $(i, j) \in \mathcal{A}^a$,

  $$||x_i - x_j||^2 = \|x_i\|^2 - 2x_i^T x_j + \|x_j\|^2 = y_{ii} - 2x_i^T x_j + \|x_j\|^2.$$

We can now reformulate the original problem as follows:

$$
\begin{aligned}
v_p := \quad & \min_{Z} \sum_{(i,j)\in\mathcal{A}^a} \left| y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2 \right| \\
& \quad + \sum_{(i,j)\in\mathcal{A}^s} \left| y_{ii} - 2y_{ij} + y_{jj} - d_{ij}^2 \right| \\
& \text{s.t.} \quad Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \succeq 0, \\
& \qquad \operatorname{rank}(Z) = 2.
\end{aligned}
$$

Dropping the rank constraint yields the SDP relaxation.

SDP Relaxation

SDP relaxation (Biswas, Ye '03):

$$\begin{aligned} v_{\text{sdp}} := \quad & \min_Z \sum_{(i,j)\in\mathcal{A}^a} \left| y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2 \right| \\ & + \sum_{(i,j)\in\mathcal{A}^s} \left| y_{ii} - 2y_{ij} + y_{jj} - d_{ij}^2 \right| \\ & \text{s.t.} \quad Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \succeq 0. \end{aligned}$$

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Although convex, solving SDP by IP method can be hard when m is large.

ESDP Relaxation

ESDP relaxation (Wang, Zheng, Boyd, Ye '08):

$$
\begin{aligned}
v_{\text{esdp}} := \quad & \min_{Z} \sum_{(i,j)\in\mathcal{A}^a} \left| y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2 \right| \\
& \quad + \sum_{(i,j)\in\mathcal{A}^s} \left| y_{ii} - 2y_{ij} + y_{jj} - d_{ij}^2 \right| \\
& \text{s.t.} \quad Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \\
& \qquad \begin{bmatrix} y_{ii} & y_{ij} & x_i^T \\ y_{ij} & y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix} \succeq 0 \quad \forall (i,j) \in \mathcal{A}^s, \\
& \qquad \begin{bmatrix} y_{ii} & x_i^T \\ x_i & I \end{bmatrix} \succeq 0 \quad \forall i = 1, \cdots, m.
\end{aligned}
$$

Alternative Problem Formulation

$$\min_{x_1,\ldots,x_m} \quad \sum_{(i,j)\in\mathcal{A}} \left(\|x_i - x_j\|^2 - d_{ij}^2\right)^2 .$$

- Objective is a nonconvex degree 4 polynomial;
- Use convex relaxation – sum of squares technique.

Sparse-SOS Relaxation

Idea: Linearization.

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For $(i,j) \in \mathcal{A}^s$,

$$\beta_{ij}^4 := \{\ 1 \quad x_i^1 \quad x_i^2 \quad x_j^1 \quad x_j^2 \quad (x_i^1)^2 \quad \cdots \quad (x_j^2)^2 \quad x_i^1 x_i^2 \quad \cdots \quad x_i^1 x_i^2 x_j^1 x_j^2\ \}$$

$$\{\ 1 \quad u_{x_i^1} \quad u_{x_i^2} \quad u_{x_j^1} \quad u_{x_j^2} \quad u_{(x_i^1)^2} \quad \cdots \quad u_{(x_j^2)^2} \quad u_{x_i^1 x_i^2} \quad \cdots \quad u_{x_i^1 x_i^2 x_j^1 x_j^2}\ \}$$

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Moment Matrix

Idea: Linearization of the outer product matrix by monomials up to degree 2, β_{ij}^2. Here shows $M_{\beta_{ij}^2}(u)$ for $(i,j) \in \mathcal{A}^a$:

$$
\begin{array}{c c}
 & \begin{array}{cccccc} 1 & x_i^1 & x_i^2 & (x_i^1)^2 & x_i^1 x_i^2 & (x_i^2)^2 \end{array} \\
\begin{array}{c} 1 \\ x_i^1 \\ x_i^2 \\ (x_i^1)^2 \\ x_i^1 x_i^2 \\ (x_i^2)^2 \end{array} &
\begin{bmatrix}
1 & u_{x_i^1} & u_{x_i^2} & u_{(x_i^1)^2} & u_{x_i^1 x_i^2} & u_{(x_i^2)^2} \\
u_{x_i^1} & u_{(x_i^1)^2} & u_{x_i^1 x_i^2} & u_{(x_i^1)^3} & u_{(x_i^1)^2 x_i^2} & u_{x_i^1 (x_i^2)^2} \\
u_{x_i^2} & u_{x_i^1 x_i^2} & u_{(x_i^2)^2} & u_{(x_i^1)^2 x_i^2} & u_{x_i^1 (x_i^2)^2} & u_{(x_i^2)^3} \\
u_{(x_i^1)^2} & u_{(x_i^1)^3} & u_{(x_i^1)^2 x_i^2} & u_{(x_i^1)^4} & u_{(x_i^1)^3 x_i^2} & u_{(x_i^1)^2 (x_i^2)^2} \\
u_{x_i^1 x_i^2} & u_{(x_i^1)^2 x_i^2} & u_{x_i^1 (x_i^2)^2} & u_{(x_i^1)^3 x_i^2} & u_{(x_i^1)^2 (x_i^2)^2} & u_{x_i^1 (x_i^2)^3} \\
u_{(x_i^2)^2} & u_{x_i^1 (x_i^2)^2} & u_{(x_i^2)^3} & u_{(x_i^1)^2 (x_i^2)^2} & u_{x_i^1 (x_i^2)^3} & u_{(x_i^2)^4}
\end{bmatrix}
\end{array}
$$

Sparse-SOS Relaxation

Sparse-SOS relaxation (Nie '09):

$$\begin{aligned} v_{\text{spsos}} := \quad & \min_{u} \sum_{(i,j)\in\mathcal{A}} \sum_{\sigma\in\beta_{ij}^4} p_\sigma^{ij} u_\sigma \\ & \text{s.t.} \;\; M_{\beta_{ij}^2}(u) \succeq 0 \quad \forall (i,j) \in \mathcal{A}, \end{aligned}$$

where

$$(\|x_i - x_j\|^2 - d_{ij}^2)^2 =: \sum_{\sigma\in\beta_{ij}^4} p_\sigma^{ij} \sigma(x) \quad \forall (i,j) \in \mathcal{A}.$$

Properties of Relaxations

Assume that every connected component contains an anchor. Let $\mathrm{pos}(\cdot)$ denote the set of sensor positions ($\subseteq \Re^2$) obtained by solving the relaxation $(\cdot)$.

Fact 1:

- pos(ESDP), pos(sSOS) and pos(SDP) are bounded convex sets.
- When $d_{ij} = \|x_i^{\mathrm{true}} - x_j^{\mathrm{true}}\|$ for all $(i,j) \in \mathcal{A}$ (noiseless case),

$$\mathrm{pos}(\mathrm{SDP}) \subseteq \mathrm{pos}(\mathrm{ESDP}),$$ (Wang et al. ’08)

$$\mathrm{pos}(\mathrm{sSOS}) \subseteq \mathrm{pos}(\mathrm{ESDP}).$$ (Gouveia, P ’10)

Define $\mathrm{tr}_i(Z) := y_{ii} - \|x_i\|^2$ for SDP and ESDP relaxations, and $\mathrm{Tr}_i(u) := u_{(x_i^1)^2} + u_{(x_i^2)^2} - (u_{x_i^1})^2 - (u_{x_i^2})^2$ for the sSOS relaxation.

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In the noiseless case

Fact 2 (Tseng '07, Wang et al. '08):

$$\mathrm{tr}_i(Z) = 0 \; \exists Z \in \mathrm{ri}(\mathrm{Sol}(\mathrm{SDP})) \text{ (resp., } \mathrm{ri}(\mathrm{Sol}(\mathrm{ESDP}))) \Rightarrow \quad x_i = x_i^{\mathrm{true}}.$$

Fact 3 (Gouveia, P '10):

$$\mathrm{Tr}_i(u) = 0 \; \exists u \in \mathrm{ri}(\mathrm{Sol}(\mathrm{sSOS})) \Rightarrow (u_{x_i^1}, u_{x_i^2})^T = x_i^{\mathrm{true}}.$$

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Converse?

The converse holds for ESDP, in the noiseless case.

Fact 4 (P, Tseng '10):

$$x_i \text{ is invariant over } \mathrm{pos}(\text{ESDP}) \Rightarrow \ \mathrm{tr}_i(Z) = 0 \ \ \forall \ Z \in \mathrm{ri}(\mathrm{Sol}(\mathrm{ESDP})).$$

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Fact 5 (P, Tseng '10):

$$\begin{aligned} & x_i \text{ is invariant over } \mathrm{pos}(\text{ESDP}) \\ \Rightarrow \quad & \text{it is connected to an anchor via a path of invariant sensors} \end{aligned}$$

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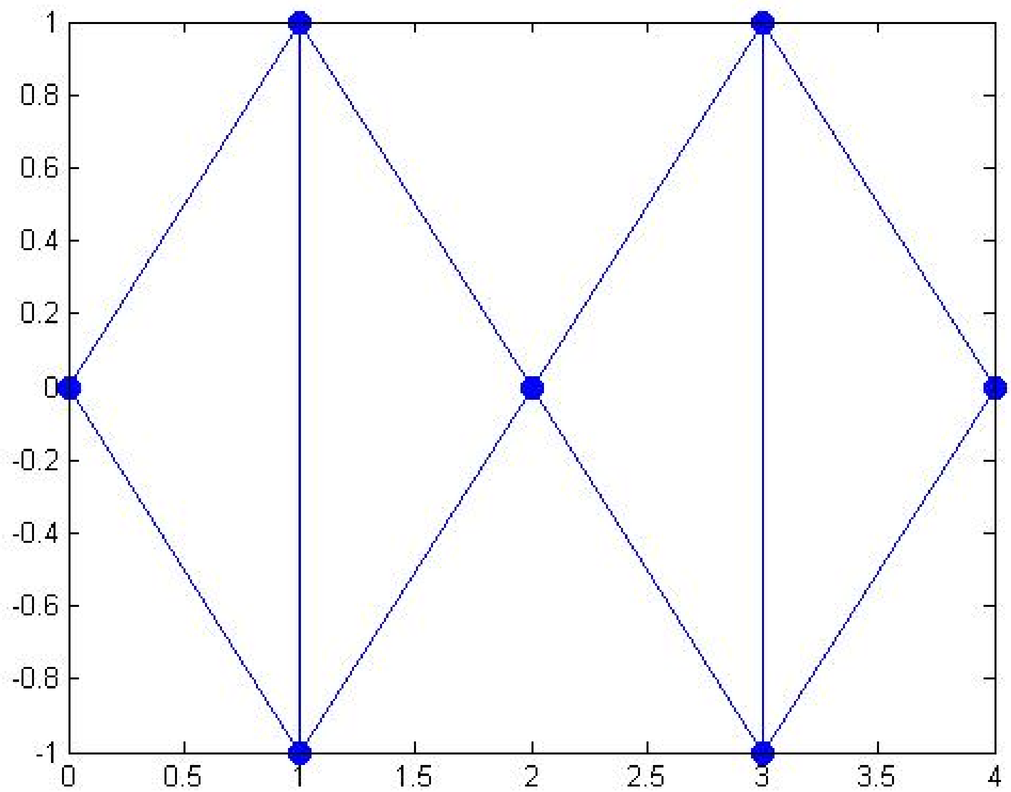
Extension to SDP / sSOS?

Numerical Example: Fact 5 fails for SDP



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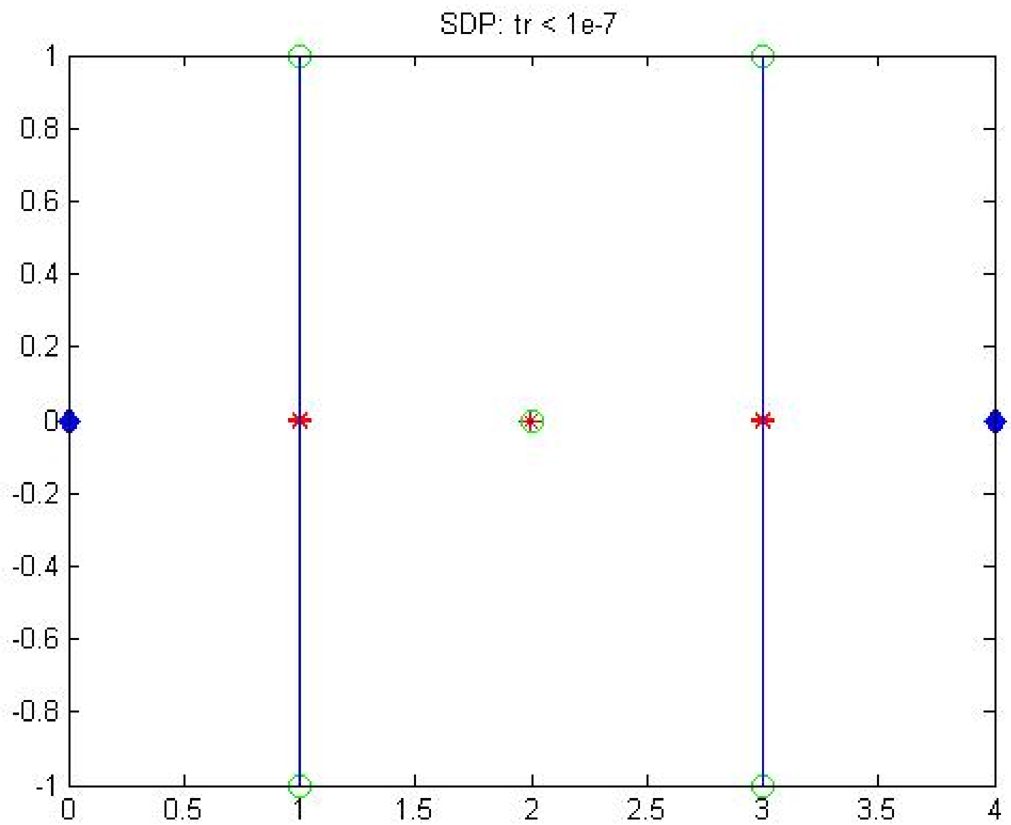
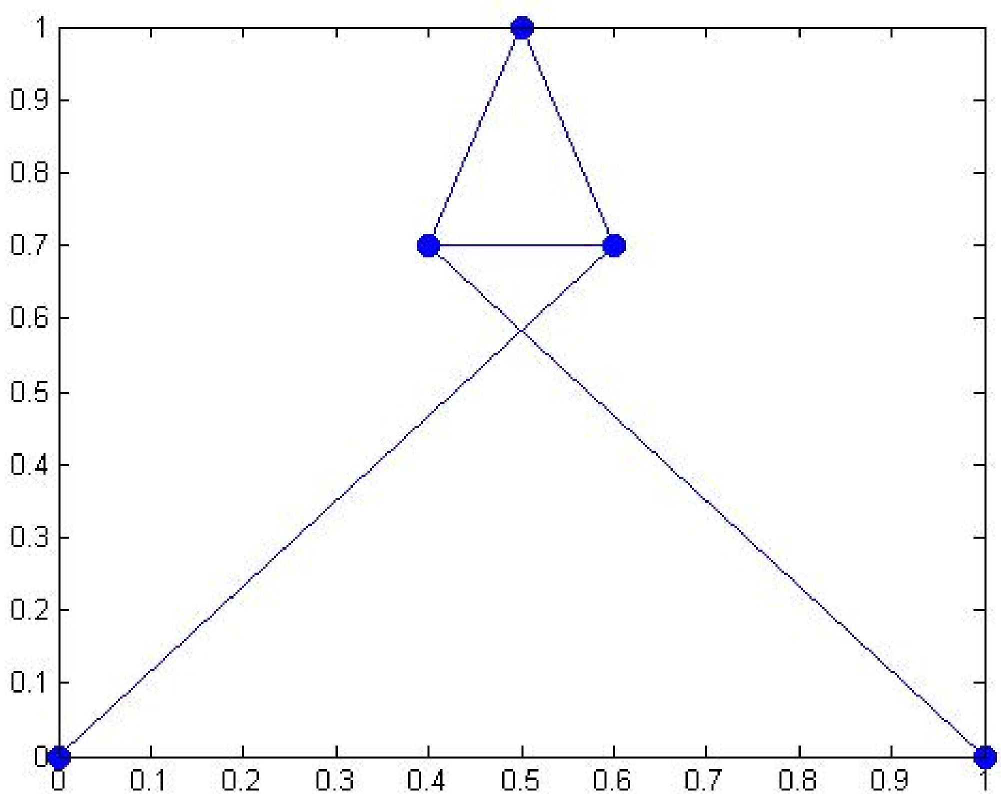


Numerical Example: SDP Vs sSOS



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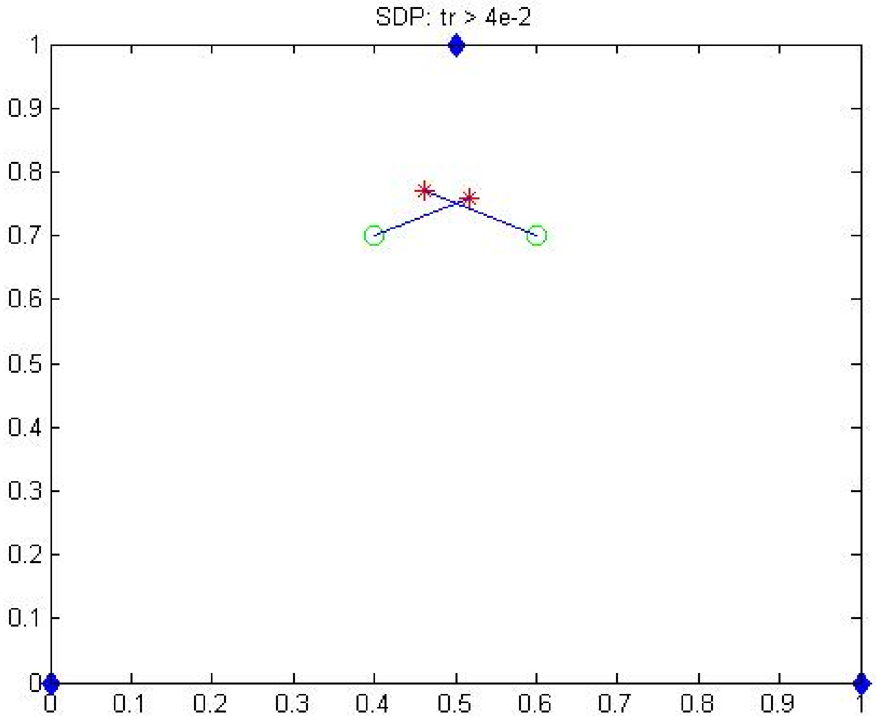
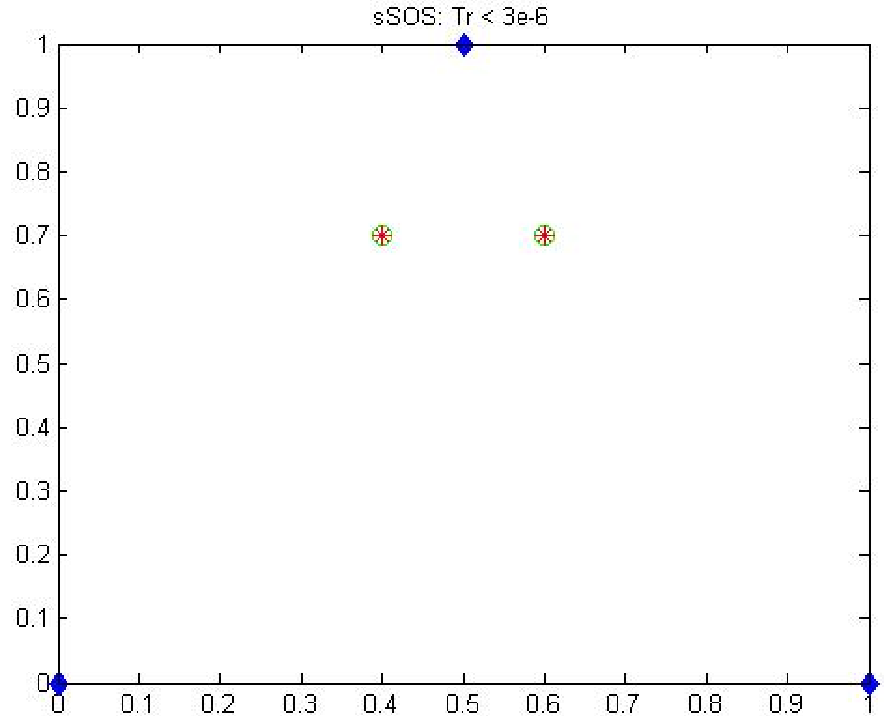




In practice, there are measurement noises:

$$d_{ij}^2 = \|x_i^{\text{true}} - x_j^{\text{true}}\|^2 + \delta_{ij} \quad \forall (i,j) \in \mathcal{A}.$$

How can one certify solution accuracy when $\delta_{ij} \approx 0$?

In practice, there are measurement noises:

$$d_{ij}^2 = \|x_i^{\text{true}} - x_j^{\text{true}}\|^2 + \delta_{ij} \quad \forall (i,j) \in \mathcal{A}.$$

How can one certify solution accuracy when $\delta_{ij} \approx 0$?

Individual trace test fails for ESDP relaxation.

Fact 6 (P, Tseng '10): For $|\delta_{ij}| \approx 0$,

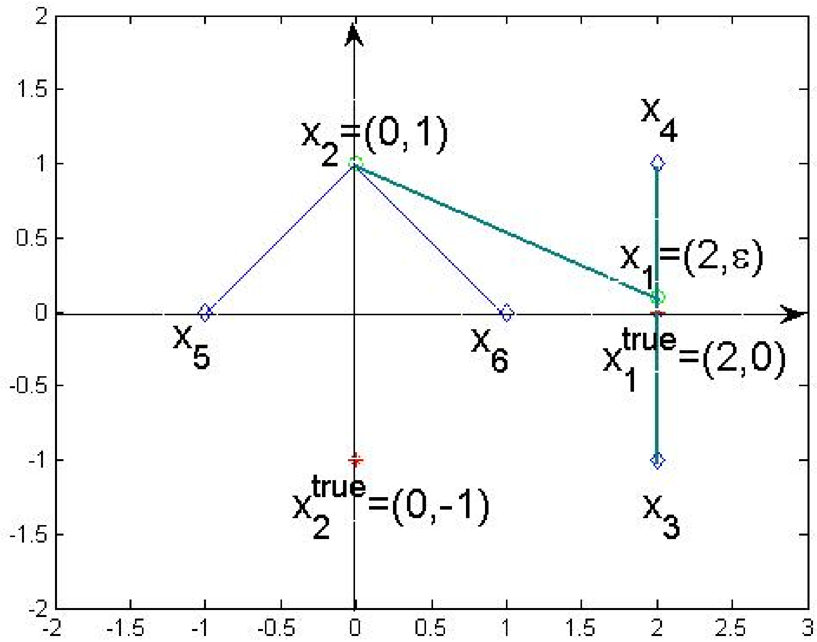
$$\mathrm{tr}_i[Z] = 0 \quad \text{for some } Z \in \mathrm{ri}(\mathrm{Sol}(\mathrm{ESDP})) \not\Longrightarrow \quad \|x_i - x_i^{\text{true}}\| \approx 0.$$

Same for SDP/sSOS relaxation.

Proof is by counterexample.

An example of sensitivity of ESDP solns to measurement noise:

Input distance data: $\epsilon > 0$
$d_{12} = \sqrt{4 + (1-\epsilon)^2}, d_{13} = 1 + \epsilon, d_{14} = 1 - \epsilon, d_{25} = d_{26} = \sqrt{2}$; $m = 2$, $n = 6$.



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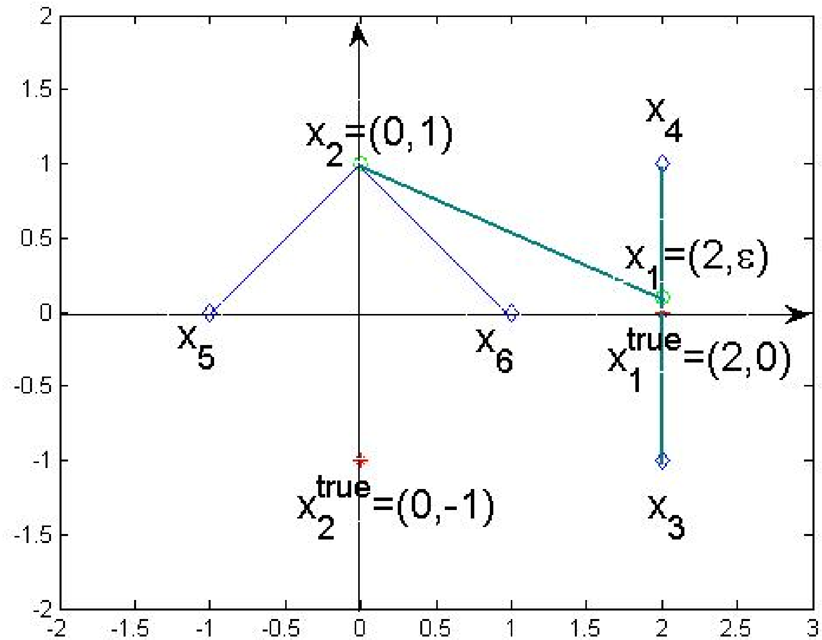

Thus, even when $Z \in \mathrm{Sol(ESDP)}$ is unique, $\mathrm{tr}_i[Z] = 0$ fails to certify accuracy of x_i in the noisy case!

Robust ESDP

For each $(i,j) \in \mathcal{A}$, fix $\rho_{ij} > |\delta_{ij}|$ ($\rho > |\delta|$).

$\mathrm{Sol}(\rho\mathrm{ESDP})$ denotes the set of $Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix}$ satisfying

$$\begin{aligned}
\begin{bmatrix} y_{ii} & y_{ij} & x_i^T \\ y_{ij} & y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix} &\succeq 0 \quad \forall (i,j) \in \mathcal{A}^s, \\
\begin{bmatrix} y_{ii} & x_i^T \\ x_i & I \end{bmatrix} &\succeq 0 \quad \forall i = 1, \cdots, m, \\
|y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2| &\le \rho_{ij} \quad \forall (i,j) \in \mathcal{A}^a, \\
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|y_{ii} - 2y_{ij} + y_{jj} - d_{ij}^2| &\leq \rho_{ij} \quad \forall (i,j) \in \mathcal{A}^s.
\end{aligned}
$$

Note: $X^{\mathrm{true}} \in \mathrm{pos}(\rho\mathrm{ESDP})$.

Let

$$Z^{\rho,\delta} := \underset{Z \in \mathrm{Sol}(\rho\mathrm{ESDP})}{\arg\min} \quad - \sum_{(i,j)\in\mathcal{A}^s} \ln\det\left(\begin{bmatrix} y_{ii} & y_{ij} & x_i^T \\ y_{ij} & y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix}\right) - \sum_{i\le m} \ln\det\left(\begin{bmatrix} y_{ii} & x_i^T \\ x_i & I \end{bmatrix}\right).$$

Fact 7 (P, Tseng '10): $\exists \eta > 0,\ \bar{\rho} > 0$ such that for each i,

$$\mathrm{tr}_i(Z^{\rho,\delta}) < \eta, \text{ for some } 0 < \rho < \bar{\rho}e \quad \Longrightarrow \quad \lim_{|\delta|<\rho\to 0} x_i^{\rho,\delta} = x_i^{\mathrm{true}}.$$

Moreover,

$$\|x_i^{\rho,\delta} - x_i^{\mathrm{true}}\| \le \sqrt{2|\mathcal{A}^s| + m}\,\left(\mathrm{tr}_i[Z^{\rho,\delta}]\right)^{\frac{1}{2}} \quad 0 \le |\delta| < \rho.$$

Conclusion

- In the noiseless case:
  - ⋆ sSOS and SDP relaxations are stronger than ESDP relaxation;
  - ⋆ Zero trace is sufficient for accuracy for all three relaxations, and is necessary for ESDP relaxation.

- In the noisy case, the trace test fails for all three relaxations.

- ρESDP is proposed. For a particular solution Z^*,

$$x_i^* \approx x_i^{\mathrm{true}} \Leftrightarrow \mathrm{tr}_i(Z^*) \approx 0,$$

  when noise is small.

Extensions

- Is zero trace condition necessary for accuracy for sSOS relaxation?
- There is an fast distributed algorithm for ρESDP (P, Tseng '10). Is it possible to develop fast distributed algorithm for sSOS relaxation?
- Is it possible to develop a lower bound on number of sensors with small traces?

Thanks for coming!