# ESDP Relaxation of Sensor Network Localization

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MOPTA 2008
August 2008

(Ongoing work with Paul Tseng)

# Talk Outline

- Sensor network localization
- ESDP relaxation: formulation and properties
- A robust version of ESDP for the noisy case
- Numerical simulations

# Sensor Network Localization

- $n$ pts $\underbrace{x_1, ..., x_m}_{\text{sensors}}, \underbrace{x_{m+1}, ..., x_n}_{\text{anchors}}$ in $\mathbb{R}^2$.

- Know last $n - m$ pts ('anchors') $x_{m+1}, ..., x_n$ and Eucl. dist. estimate for some pairs of 'neighboring' pts (i.e. within 'radio range')

$$d_{ij} \geq 0 \quad \forall (i, j) \in \mathcal{A}$$

  with $\mathcal{A} \subseteq \{(i, j) : 1 \leq i < j \leq n\}$.

- Estimate the first $m$ pts ('sensors') $x_1, ..., x_m$.

# Optimization Problem Formulation

$$v_{\mathrm{p}} := \min_{x_1,\ldots,x_m} \sum_{(i,j)\in\mathcal{A}} \left| \|x_i - x_j\|^2 - d_{ij}^2 \right|$$

- Objective function is nonconvex. $m$ can be large ($m > 1000$).
- Problem is NP-hard (reduction from PARTITION).
- Use a convex relaxation.

# SDP Relaxation

$X = [x_1 \cdots x_m]$;

SDP Relaxation (Biswas, Ye '03):

$$
\begin{aligned}
v_{\mathrm{sdp}} := \quad & \min_{Z} \sum_{(i,j)\in\mathcal{A}, j>m} \left| y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2 \right| \\
& + \sum_{(i,j)\in\mathcal{A}, j\le m} \left| y_{ii} - 2y_{ij} + y_{jj} - d_{ij}^2 \right| \\
& \text{s.t.} \quad Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \succeq 0
\end{aligned}
$$

Adding nonconvex constraint $\operatorname{rank} Z = 2$ yields the original problem.

# ESDP Relaxation

ESDP relaxation (Wang, Zheng, Boyd, Ye '07):

$$
\begin{aligned}
v_{\text{esdp}} := \quad & \min_{Z} \sum_{(i,j)\in\mathcal{A}, j>m} \left| y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2 \right| \\
& \quad + \sum_{(i,j)\in\mathcal{A}, j\le m} \left| y_{ii} - 2y_{ij} + y_{jj} - d_{ij}^2 \right| \\
& \text{s.t.} \quad Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \\
& \qquad \begin{bmatrix} y_{ii} & y_{ij} & x_i^T \\ y_{ij} & y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix} \succeq 0 \quad \forall (i,j)\in\mathcal{A}, j\le m \\
& \qquad \begin{bmatrix} y_{ii} & x_i^T \\ x_i & I \end{bmatrix} \succeq 0 \quad \forall i = 1,\cdots,m
\end{aligned}
$$

# ESDP: Testing Solution Accuracy

Aim: Determine which sensors are accurately positioned by ESDP.

Consider

$$d_{ij}^2 = \|x_i^{\text{true}} - x_j^{\text{true}}\|^2 + \delta_{ij}$$

Two cases:

- Noiseless case, i.e. $\delta_{ij} = 0, \ \forall (i,j) \in \mathcal{A}$;
- Noisy case, i.e. $\delta_{ij} \neq 0$ for some $(i,j) \in \mathcal{A}$.

**Defn**: $x_i$ is *invariant* over $\mathrm{Sol}\,(\mathrm{ESDP})$ if $\forall Z \in \mathrm{Sol}\,(\mathrm{ESDP})$, its $x_i$ entry is the same.

# ESDP: Noiseless Case

- $v_{\mathrm{esdp}} = 0 \;\Rightarrow\; \mathrm{Sol}\,(\mathrm{P}) \;\subseteq \left\{ X : \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \in \mathrm{Sol}\,(\mathrm{ESDP}) \right\}$

- **Observation**: If $x_i$ is invariant over $\mathrm{Sol}\,(\mathrm{ESDP})$, then $x_i$ is invariant over $\mathrm{Sol}\,(P)$ and hence $x_i = x_i^{\mathrm{true}}$.

# ESDP: Noiseless Case

Consider the individual trace for each sensor location $i$:
$\mathrm{tr}_i[Z] := y_{ii} - \|x_i\|^2, \quad i = 1, ..., m.$ (Biswas, Ye '03)

- (Wang et al '07) If $\mathrm{tr}_i[Z] = 0$ for some $Z \in \mathrm{ri}\,(\mathrm{Sol}\,(\mathrm{ESDP}))$, then $x_i$ is invariant over $\mathrm{Sol}\,(\mathrm{ESDP})$.

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How necessary is "$\mathrm{tr}_i(Z) = 0$ for some $Z \in \mathrm{ri}\,(\mathrm{Sol}\,(\mathrm{ESDP}))$" for invariance?

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How necessary is "$\mathrm{tr}_i(Z) = 0$ for some $Z \in \mathrm{ri}\,(\mathrm{Sol}\,(\mathrm{ESDP}))$" for invariance?

**New**: (P, Tseng '08) If $x_i$ is invariant over $\mathrm{Sol}\,(\mathrm{ESDP})$, then $\mathrm{tr}_i[Z] = 0$ for all $Z \in \mathrm{Sol}\,(\mathrm{ESDP})$.

# ESDP: Noisy Case

Suppose $\delta_{ij} \neq 0$ for some $(i, j) \in \mathcal{A}$.

Does $\mathrm{tr}_i[Z] = 0$ (with $Z \in \mathrm{ri}(\mathrm{Sol}(\mathrm{ESDP}))$) imply $x_i$ is near the true position of sensor $i$?

# ESDP: Noisy Case

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Does $\mathrm{tr}_i[Z] = 0$ (with $Z \in \mathrm{ri}(\mathrm{Sol}(\mathrm{ESDP}))$) imply $x_i$ is near the true position of sensor $i$?

No.

**New**: (P, Tseng '08): For $|\delta_{ij}| \approx 0$,

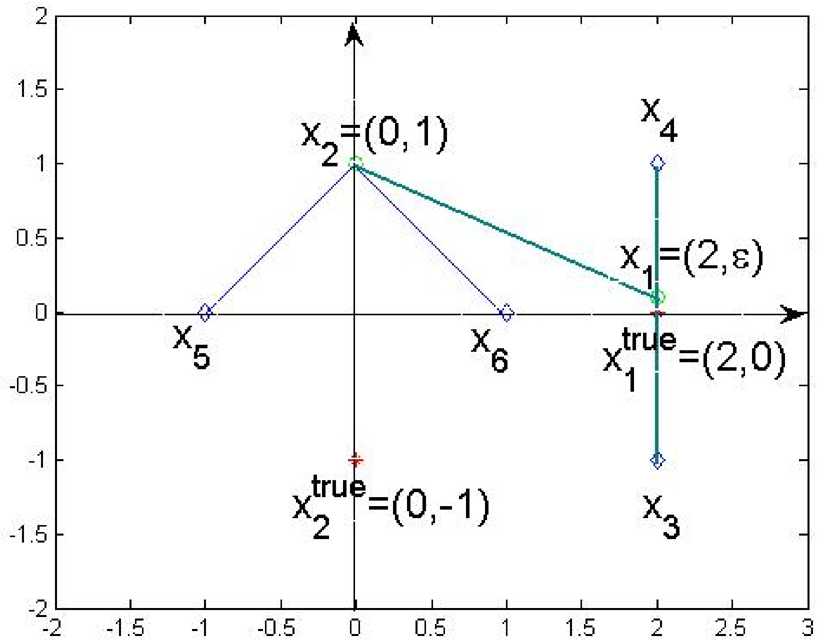
$$\mathrm{tr}_i[Z] = 0 \quad \text{for some } Z \in \mathrm{ri}(\mathrm{Sol}(\mathrm{ESDP})) \not\Longrightarrow \quad \|x_i - x_i^{\mathrm{true}}\| \approx 0.$$

Proof is by counterexample.

An example of sensitivity of ESDP solns to measurement noise:

Input distance data: $\epsilon > 0$

$d_{12} = \sqrt{4 + (1-\epsilon)^2}, d_{13} = 1 + \epsilon, d_{14} = 1 - \epsilon, d_{25} = d_{26} = \sqrt{2}$; $m = 2$, $n = 6$.



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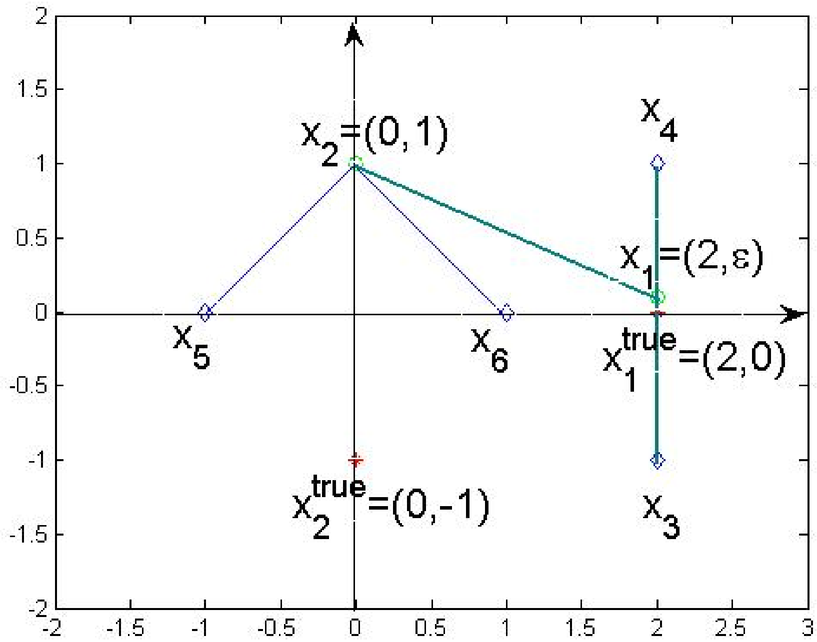



Thus, even when $Z \in \mathrm{Sol}(\mathrm{ESDP})$ is unique, $\mathrm{tr}_i[Z] = 0$ fails to certify accuracy of $x_i$ in the noisy case!

# Robust ESDP

For each $(i, j) \in \mathcal{A}$, fix $\rho_{ij} > |\delta_{ij}|$. ($\rho > |\delta|$)

$\mathrm{Sol}(\rho\mathrm{ESDP})$ denotes the set of $Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix}$ satisfying

$$
\begin{aligned}
\begin{bmatrix} y_{ii} & y_{ij} & x_i^T \\ y_{ij} & y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix} &\succeq 0 \quad \forall (i, j) \in \mathcal{A}, j \leq m \\
\begin{bmatrix} y_{ii} & x_i^T \\ x_i & I \end{bmatrix} &\succeq 0 \qquad \forall i = 1, \cdots, m \\
|y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2| &\leq \rho_{ij} \quad \forall (i, j) \in \mathcal{A}, j > m \\
|y_{ii} - 2y_{ij} + y_{jj} - d_{ij}^2| &\leq \rho_{ij} \quad \forall (i, j) \in \mathcal{A}, j \leq m
\end{aligned}
$$

**Note**: $[X^{\mathrm{true}} \;\; I][X^{\mathrm{true}} \;\; I]^T \in \mathrm{Sol}(\rho\mathrm{ESDP})$.

Let

$$Z^{\rho,\delta} := \arg\min_{Z \in \mathrm{Sol}(\rho\mathrm{ESDP})} \quad - \sum_{(i,j)\in\mathcal{A}, j\le m} \ln\det\left(\begin{bmatrix} y_{ii} & y_{ij} & x_i^T \\ y_{ij} & y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix}\right) - \sum_{i\le m} \ln\det\left(\begin{bmatrix} y_{ii} & x_i^T \\ x_i & I \end{bmatrix}\right)$$

**New**: (P, Tseng '08) $\exists\, \bar\tau > 0$, $\forall \tau \le \bar\tau$, $\exists \bar\rho > 0$ such that for each $i$,

$$\mathrm{tr}_i[Z^{\rho,\delta}] < \tau \text{ for some } 0 \le |\delta| < \rho \le \bar\rho e \implies \lim_{|\delta|<\rho\to 0} x_i^{\rho,\delta} = x_i^{\mathrm{true}}$$

Moreover,

$$\|x_i^{\rho,\delta} - x_i^{\mathrm{true}}\| \le \sqrt{2|\mathcal{A}| + m}\left(\mathrm{tr}_i[Z^{\rho,\delta}]\right)^{\frac{1}{2}} \quad 0 \le |\delta| < \rho.$$

# Algorithm: CGD-barrier

Let $h_a(t) := \frac{1}{2}(t-a)_+^2 + \frac{1}{2}(-t-a)_+^2$.

$$
\begin{aligned}
f_\mu(Z) := & \sum_{(i,j)\in\mathcal{A}, j>m} h_{\rho_{ij}}(y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2) \\
& + \sum_{(i,j)\in\mathcal{A}, j\le m} h_{\rho_{ij}}(y_{ii} - 2y_{ij} + y_{jj} - d_{ij}^2) \\
& - \mu \sum_{(i,j)\in\mathcal{A}, j\le m} \ln\det\left(\begin{bmatrix} y_{ii} & y_{ij} & x_i^T \\ y_{ij} & y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix}\right) \\
& - \mu \sum_{i\le m} \ln\det\left(\begin{bmatrix} y_{ii} & x_i^T \\ x_i & I \end{bmatrix}\right)
\end{aligned}
$$

# Algorithm: CGD-barrier

- For each $(i, j) \in \mathcal{A}$,

$$h_{\rho_{ij}}(y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2) = 0 \iff |y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2| \leq \rho_{ij}$$

$$h_{\rho_{ij}}(y_{ii} - 2y_{ij} + y_{jj} - d_{ij}^2) = 0 \iff |y_{ii} - 2y_{ij} + y_{jj} - d_{ij}^2| \leq \rho_{ij}.$$

- For each $\rho > |\delta|$, $\operatorname{argmin} f_\mu \to Z^{\rho,\delta}$ as $\mu \to 0$.

- In the noiseless case ($\delta = 0$), setting $\rho = 0$, then, as $\mu \to 0$, $\operatorname{argmin} f_\mu \to Z$ for some $Z \in \operatorname{ri}(\operatorname{Sol}(\text{ESDP}))$.

# Algorithm: CGD-barrier

Use Block Coordinate Gradient Descent:

For each $i$, compute gradient $\nabla_{Z_i} f_\mu$ of $f_\mu$ w.r.t. the subset of variables $Z_i := \{x_i, y_{ii}, y_{ij} : (i,j) \in \mathcal{A}\}$.

- If $\|\nabla_{Z_i} f_\mu\| \geq \max\{\mu, 1e-6\}$ for some $i$, update $Z_i$ by moving along the Newton direction $-\left(\nabla^2_{Z_i Z_i} f_\mu\right)^{-1} \nabla_{Z_i} f_\mu$ with Armijo stepsize rule.
- Decrease $\mu$ when $\|\nabla_{Z_i} f_\mu\| < \max\{\mu, 1e-6\}$ for all $i$.

$\mu_{\text{initial}} = 100$, $\mu_{\text{final}} = 1e-9$. Decrease $\mu$ by a factor of $10$.

Coded in Fortran. Computation easily distributes.

# Simulation Results

- Compare $\rho$ESDP as solved by CGD-barrier and ESDP as solved by Sedumi (with the interface to Sedumi coded by Wang et al.).

- Uniformly generated $\{x_1^{\text{true}}, ..., x_n^{\text{true}}\}$ in $[-.5, .5]^2$, $m = .9n$. Two pts are neighbors iff $\text{dist} < rr$. Set

$$d_{ij} = \|x_i^{\text{true}} - x_j^{\text{true}}\| \cdot \max\{0, 1 + \epsilon_{ij} \cdot nf\},$$

  where $\epsilon_{ij} \sim N(0,1)$.

- Sensor $i$ is judged as "accurately positioned" if

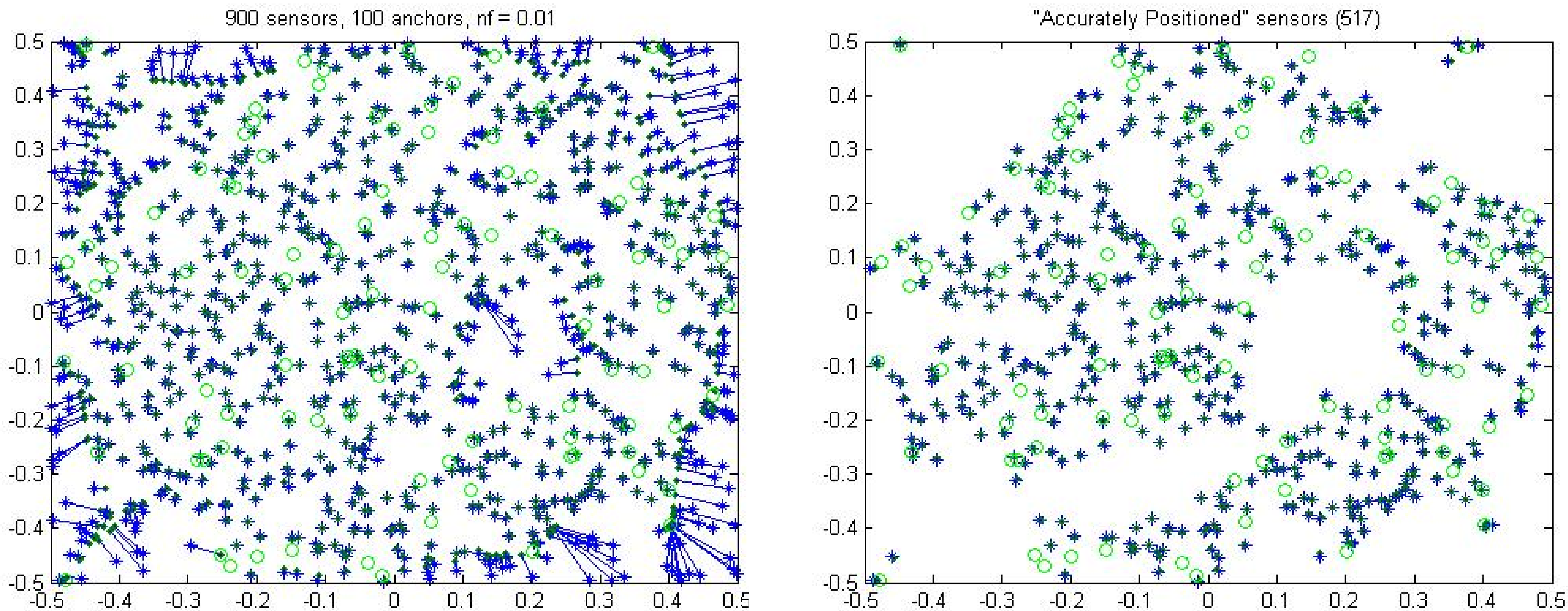
$$\text{tr}_i[Z^{\rho,\delta}] < 5 \times 10^{-6} + 0.02nf.$$

# Simulation Results

| | | | | $\rho\mathbf{ESDP}_{\mathrm{CGD-barrier}}$ | $\mathbf{ESDP}_{\mathrm{Sedumi}}$ |
|---|---|---|---|---|---|
| $n$ | $m$ | $nf$ | $rr$ | **cpu**/$m_{\mathrm{ap}}$/$err_{\mathrm{ap}}$ | **cpu(cpus)**/$m_{\mathrm{ap}}$/$err_{\mathrm{ap}}$ |
| 1000 | 900 | 0 | .06 | 31/574/3.4e-4/ | 189(106)/626/2.2e-4 |
| 1000 | 900 | .001 | .06 | 23/520/2.8e-3 | 170(89)/624/3.1e-3 |
| 1000 | 900 | .01 | .06 | 14/517/1.1e-2 | 128(48)/664/1.5e-2 |
| 2000 | 1800 | 0 | .06 | 63/1626/1.7e-4/ | 1157(397)/1689/2.7e-4 |
| 2000 | 1800 | .001 | .06 | 50/1596/8.5e-4/ | 1255(503)/1653/1.3e-3 |
| 2000 | 1800 | .01 | .06 | 52/1602/6.2e-3/ | 1374(417)/1689/1.2e-2 |

- cpu(sec) times are on a HP DL360 workstation, running Linux 3.5. ESDP is solved by Sedumi; cpus:$=$ time taken in running Sedumi.
- Take $\rho_{ij} = d_{ij}^2 \cdot ((1 - 2 \cdot nf)^{-2} - 1)$.
- $m_{ap}$ :=number of accurately positioned sensors.
  $err_{ap} := \max_{i \text{ accurate. pos.}} \|x_i - x_i^{\mathrm{true}}\|$.

$900$ sensors, $100$ anchors, $rr = 0.06$, $nf = 0.01$, solving $\rho$ESDP by CGD-barrier method. $x_i^{\rm true}$ denoted by blue asterisks, $x_i^{\rho,\delta}$ denoted by green dots, anchors denoted by circles. $x_i^{\rm true}$ and $x_i^{\rho,\delta}$ joined by blue straight line.



Thanks for coming!