Deducing Kurdyka-Łojasiewicz exponent of optimization models

Ting Kei Pong
Department of Applied Mathematics
The Hong Kong Polytechnic University
Hong Kong

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(Joint work with Guoyin Li and Peiran Yu)

Motivating applications

Structured optimization problems:

- (Overlapping) Group lasso:

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|Ax-b\|^2 + \sum_{i=1}^{m} \nu_i \|x_{J_i}\|,$$

  where $A \in \mathbb{R}^{p\times n}$, $b \in \mathbb{R}^p$, $\bigcup_{i=1}^m J_i = \{1,\ldots,n\}$, all $\nu_i \geq 0$.

- Least squares with rank constraint:

$$\begin{array}{ll} \min\limits_{X\in\mathbb{R}^{m\times n}} & \frac{1}{2}\|\mathcal{A}X-b\|^2 \\ \text{s.t.} & \mathrm{rank}(X) \leq r, \end{array}$$

  where $\mathcal{A}: \mathbb{R}^{m\times n} \to \mathbb{R}^p$ is linear, $b \in \mathbb{R}^p$, r is positive integer.

First-order method

Consider

$$f(x) := h(x) + P(x),$$

where: h is continuously differentiable with Lipschitz gradient whose continuity modulus is $L > 0$, P is proper closed.

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Proximal gradient algorithm.
Initialize x^0, set $\gamma \in (0, \frac{1}{L})$. For $k = 0, \ldots,$

$$x^{k+1} \in \mathrm{prox}_{\gamma P}\left(x^k - \gamma \nabla h(x^k)\right),$$

where

$$\mathrm{prox}_{\gamma P}(y) = \operatorname*{Arg\,min}_{x \in \mathbb{R}^n} \left\{ \frac{1}{2}\|x - y\|^2 + \gamma P(x) \right\}.$$

KL property & exponent

Definition: (Attouch et al. '10, Attouch et al. '13)
Let f be proper closed and $\alpha \in [0, 1)$.

- f is said to have the Kurdyka-Łojasiewicz (KL) property with exponent α at $\bar{x} \in \operatorname{dom} \partial f$ if there exist c, ν, $\epsilon > 0$ so that

$$c[f(x) - f(\bar{x})]^\alpha \leq \operatorname{dist}(0, \partial f(x))$$

whenever $x \in \operatorname{dom} \partial f$, $\|x - \bar{x}\| \leq \epsilon$ and $f(\bar{x}) < f(x) < f(\bar{x}) + \nu$.

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  whenever $x \in \mathrm{dom}\,\partial f$, $\|x - \bar{x}\| \leq \epsilon$ and $f(\bar{x}) < f(x) < f(\bar{x}) + \nu$.
- If f has the KL property at every $\bar{x} \in \mathrm{dom}\,\partial f$ with the same α, then f is said to be a KL function with exponent α.

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  whenever $x \in \operatorname{dom} \partial f$, $\|x - \bar{x}\| \leq \epsilon$ and $f(\bar{x}) < f(x) < f(\bar{x}) + \nu$.
- If f has the KL property at every $\bar{x} \in \operatorname{dom} \partial f$ with the same α, then f is said to be a KL function with exponent α.

Examples.

- Proper closed semialgebraic functions are KL functions with exponent $\alpha \in [0, 1)$. (Bolte et al. '07)

Prototypical local convergence results

Fact 1.
For proximal gradient algorithm and some of its variants:
Let $\{x^k\}$ be a bounded sequence generated. If f is a KL function with exponent α, then:

- if $\alpha = 0$, then $\{x^k\}$ converges finitely;
- if $\alpha \in (0, \frac{1}{2}]$, then $\{x^k\}$ converges locally linearly;
- if $\alpha \in (\frac{1}{2}, 1)$, then $\{x^k\}$ converges locally sublinearly.

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Holds also for proximal alternating minimization algorithm (Attouch et al. '10), Douglas-Rachford splitting method (Li, P. '16), etc., if f is replaced by a suitable potential function.

Existing results I

For nonsmooth objectives:

- If f is the maximum of m polynomials of degree at most d, then the KL exponent is $1 - \frac{1}{\max\{1,(d+1)(3d)^{n+m-2}\}}$. (Li et al. '15)
- A special quadratic minimization problem with matrix variables and orthogonality constraint has KL exponent $\frac{1}{2}$. (Liu et al. '15)

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- Relationship with Hölder growth condition: (Bolte et al. '17)
  Let f be proper closed convex with $\operatorname{Arg\,min} f \neq \emptyset$. Then f has KL exponent $\alpha \in [0, 1)$ if and only if $\forall\ \bar{x} \in \operatorname{Arg\,min} f$, $\exists\ c, \epsilon > 0$ so that

  $$\operatorname{dist}(x, \operatorname{Arg\,min} f) \leq c(f(x) - f(\bar{x}))^{1-\alpha}$$

  whenever $\|x - \bar{x}\| \leq \epsilon$ and $f(\bar{x}) < f(x) < f(\bar{x}) + \epsilon$.

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  whenever $\|x - \bar{x}\| \leq \epsilon$ and $f(\bar{x}) < f(x) < f(\bar{x}) + \epsilon$.
- A convex piecewise polynomial function of degree at most d is a KL function with exponent $1 - \frac{1}{(d-1)^n+1}$. (Li '13)

Existing results II

For nonsmooth objectives:

- Relationship with Luo-Tseng error bound: (Li, P. '18)
  Let $f = h + P$, where h has locally Lipschitz gradient and P is proper closed convex. Suppose $\mathcal{X} := \{x : \ 0 \in \partial f(x)\} \neq \emptyset$, and
  1. $\forall\, \zeta \geq \inf f$, $\exists\, c, \epsilon > 0$ so that
  $$\mathrm{dist}(x, \mathcal{X}) \leq c\|\mathrm{prox}_P(x - \nabla h(x)) - x\|$$
  whenever $\|\mathrm{prox}_P(x - \nabla h(x)) - x\| < \epsilon$ and $f(x) \leq \zeta$.
  2. $\exists\, \delta > 0$ so that if $x, y \in \mathcal{X}$ and $\|x - y\| \leq \delta$, then $f(x) = f(y)$.

  Then f is a KL function with exponent $\frac{1}{2}$.

Existing results III

For nonsmooth objectives:
Consequently: If $f = h + P$ and $\mathcal{X} \neq \emptyset$, then f satisfies the KL property with exponent $\frac{1}{2}$ at $\bar{x} \in \mathcal{X}$ in each of the following cases:

- h is a quadratic (not necessarily convex) and P is proper polyhedral. (Luo, Tseng '92, Tseng, Yun '09)
- $h(x) = \ell(Ax)$, where $\ell \in C^2$ is strongly convex on any compact convex set, and
  1. P is proper polyhedral; (Luo, Tseng '92, Tseng, Yun '09)
  2. $P(x) = \sum_{i=1}^m w_i \|x_{J_i}\|_p$, $w_i \geq 0$, $\{J_1, \ldots, J_m\}$ form a partition of $\{1, \ldots, n\}$, $p \in [1, 2] \cup \{\infty\}$; (Tseng '10, Zhou et al. '15)

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  1. P is proper polyhedral; (Luo, Tseng '92, Tseng, Yun '09)
  2. $P(x) = \sum_{i=1}^{m} w_i \|x_{J_i}\|_p$, $w_i \geq 0$, $\{J_1, \ldots, J_m\}$ form a partition of $\{1, \ldots, n\}$, $p \in [1, 2] \cup \{\infty\}$; (Tseng '10, Zhou et al. '15)
  3. P is the nuclear norm if in addition $0 \in \mathrm{ri}\,\partial f(\bar{x})$; (Zhou, So '17)
  4. $P(x) = g(\sigma(x))$, g is polyhedral symmetric, under some ri conditions. (Cui et al. '17)

Aim: Estimate KL exponent

Aim: Explicitly estimate the KL exponent of optimization models.

Strategy:

- Develop calculus rules on KL exponents:

Deduce exponent of functions from ones with known exponents

Calculus of KL exponent I

Theorem 1. (Li, P. '18)
Let $h(x) = \ell(Ax)$ for some continuous strictly convex function ℓ, g be a continuous convex function, D be a closed convex set, $\alpha \in (0, 1)$. Suppose also

(i) there exists $x_0 \in D$ with $g(x_0) < 0$;

(ii) $\inf_{x \in D} h(x) < \inf_{x \in D}\{h(x) : \ g(x) \leq 0\}$;

(iii) for any $\lambda > 0$, $h + \lambda g + \delta_D$ is KL with exponent α.

Then $h + \delta_{g(\cdot) \leq 0} + \delta_D$ is KL with exponent α.

Application I

Consider functions of the form

$$f(x) = \ell(Ax) + \delta_\Omega(x),$$

where $\ell \in C^2$ is strongly convex on any compact convex set, and

$$\Omega := \left\{ x : \sum_{i=1}^{m} w_i \|x_i\|_p \le \sigma \right\},$$

with $x_i \in \mathbb{R}^{n_i}$, $\sum_{i=1}^{m} n_i = n$, $w_i > 0$, $\sigma > 0$ and $p \in [1, 2]$.

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Corollary 1. (Li, P. '18)
Suppose that $\inf f(x) > \inf \ell(Ax)$. Then f is KL with exponent $\frac{1}{2}$.

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Corollary 1. (Li, P. '18)
Suppose that $\inf f(x) > \inf \ell(Ax)$. Then f is KL with exponent $\frac{1}{2}$.

Proof: When $\mathcal{X} \neq \emptyset$, Luo-Tseng error bound holds for the regularized version. (Zhou et al. '15)

Calculus of KL exponent II

Theorem 2. (Li, P. '18, Yu, Li, P. '19)
Suppose that g_i are KL functions with exponents α_i, $1 \leq i \leq m$, and that $\operatorname{dom} \partial g_i = \operatorname{dom} g_i$ for all i. Then $g := \min_{1 \leq i \leq m} g_i$ is a KL function with exponent $\max\{\alpha_i : \ 1 \leq i \leq m\}$.

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Key fact used in the proof: For any $x \in \mathrm{dom}\,\partial g$,

$$\partial g(x) \subseteq \bigcup_{i\in I(x)} \partial g_i(x),$$

where $I(x) := \{i : g(x) = g_i(x)\}$. (Mordukovich, Shao '95)

Application II

Corollary 2. (Li, P. '18, Yu, Li, P. '19)
Consider functions of the form

$$f(x) = \min_{1 \le i \le m} \left\{ x^T M_i x + b_i^T x + c_i + P_i(x) \right\},$$

where M_i are symmetric matrices, P_i are proper polyhedral functions. Then f is a KL function with exponent $\frac{1}{2}$.

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Example: Least-squares with SCAD regularization: (Fan '97)

$$f(x) = \frac{1}{2}\|Ax - b\|^2 + \sum_{i=1}^{n} r_{\lambda,\theta}(x_i),$$

with $\lambda > 0$, $\theta > 2$ and

$$r_{\lambda,\theta}(t) = \begin{cases} \lambda|t| & \text{if } |t| \le \lambda, \\ \frac{-t^2+2\theta\lambda|t|-\lambda^2}{2(\theta-1)} & \text{if } \lambda < |t| \le \theta\lambda, \\ \frac{(\theta+1)\lambda^2}{2} & \text{if } |t| > \theta\lambda. \end{cases}$$

Calculus of KL exponent III

Theorem 3. (Yu, Li, P. '19)
Let $F : \mathbb{X} \times \mathbb{Y} \to \bar{\mathbb{R}}$ be proper closed and define $f(x) := \inf_{y \in \mathbb{Y}} F(x, y)$ and $Y(x) := \mathrm{Arg\,min}_{y \in \mathbb{Y}} F(x, y)$ for $x \in \mathbb{X}$. Suppose $\bar{x} \in \mathrm{dom}\,\partial f$, $\alpha \in [0, 1)$ and the following conditions hold:

(i) F is level-bounded in y locally uniformly in x.

(ii) It holds that $\partial F(\bar{x}, \bar{y}) \neq \emptyset$ for all $\bar{y} \in Y(\bar{x})$.

(iii) F satisfies the KL property with exponent α in $\{\bar{x}\} \times Y(\bar{x})$.

Then f satisfies the KL property at $\bar{x}$ with exponent α.

Remark: F is level-bounded in y locally uniformly in x if for any x and $\beta \in \mathbb{R}$, there exists $\rho > 0$ so that

$$\{(u, y) : \; \|u - x\| \leq \rho, F(u, y) \leq \beta\}$$

is bounded.

Application III

Corollary 3. (Yu, Li, P. '19)
Let $f = \sum_{i=1}^{m} f_i$, each $f_i : \mathbb{R}^n \to \bar{\mathbb{R}}$ be proper closed.
Suppose that each f_i is LMI-representable, i.e., there exist $d_i > 0$ and matrices $\{A_{00}^i, A_0^i, A_1^i, \ldots, A_n^i\} \subset \mathcal{S}^{d_i}$ such that

$$\mathrm{epi} f_i = \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} : \; A_{00}^i + \sum_{j=1}^{n} A_j^i x_j + A_0^i t \succeq 0 \right\}.$$

Suppose also that $\exists\ x^s \in \mathbb{R}^n$ and $s^s \in \mathbb{R}^m$ such that for $i = 1, \ldots, m$,

$$A_{00}^i + \sum_{j=1}^{n} A_j^i x_j^s + A_0^i s_i^s \succ 0.$$

If $0 \in \mathrm{ri}\,\partial f(\bar{x})$, then f satisfies the KL property at $\bar{x}$ with exponent $\frac{1}{2}$.

Application III cont.

Each of the following functions satisfies the KL property with exponent $\frac{1}{2}$ at an $\bar{x}$ satisfying $0 \in \mathrm{ri}\,\partial f(\bar{x})$:

(i) Group Lasso with *overlapping* blocks of variables:

$$f(x) = \frac{1}{2}\|Ax - b\|^2 + \sum_{i=1}^{s} w_i \|x_{J_i}\|,$$

where $b \in \mathbb{R}^p$, $A \in \mathbb{R}^{p\times n}$, $\bigcup_{i=1}^{s} J_i = \{1, \ldots, n\}$, all $w_i \geq 0$, $i = 1, \ldots, s$.

(ii) Group fused Lasso: (Alaíz et al. '13)

$$f(x) = \frac{1}{2}\|Ax - b\|^2 + \sum_{i=1}^{s} w_i \|x_{J_i}\| + \sum_{i=2}^{s} \nu_i \|x_{J_i} - x_{J_{i-1}}\|,$$

where $b \in \mathbb{R}^p$, $A \in \mathbb{R}^{p\times sr}$, $\bigcup_{i=1}^{s} J_i = \{1, \ldots, n\}$, $J_i \cap J_{i'} = \emptyset$ for $i \neq i'$, all $w_i, \nu_i \geq 0$ and $|J_i| = r$.

Application IV

Corollary 4. (Yu, Li, P. '19)
Let $f(X) = \sum_{k=1}^{m} f_k(X) + \|X\|_*$, each $f_k : \mathbb{R}^{m\times n} \to \bar{\mathbb{R}}$ be proper closed. Suppose that each f_k is LMI-representable, i.e., there exist $d_k > 0$ and matrices $\{A_{00}^k, A_0^k, A_{11}^k, \dots, A_{mn}^k\} \subset \mathcal{S}^{d_k}$ such that

$$\mathrm{epi} f_k = \left\{ (X,t) \in \mathbb{R}^n \times \mathbb{R} :\ A_{00}^k + \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}^k X_{ij} + A_0^k t \succeq 0 \right\}.$$

Suppose also that $\exists\ X^s \in \mathbb{R}^{m\times n}$ and $s^s \in \mathbb{R}^m$ such that for $k = 1, \dots, m$,

$$A_{00}^k + \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}^k X_{ij}^s + A_0^k s_k^s \succ 0.$$

If $0 \in \mathrm{ri}\,\partial f(\bar{X})$, then f satisfies the KL property at $\bar{X}$ with exponent $\frac{1}{2}$.

Digression: C^2-cone reducible structures

Definition: (Shapiro '03)
Let $\mathfrak{D} \subseteq \mathbb{X}$ be a nonempty closed set. We say that it is C^2-cone reducible at $\bar{w} \in \mathfrak{D}$ if there exist a closed convex pointed cone $K \subseteq \mathbb{Y}$, $\rho > 0$, and a mapping $\Theta : \mathbb{X} \to \mathbb{Y}$ such that

(i) Θ is twice continuously differentiable in $B(\bar{w}, \rho)$;

(ii) $\Theta(\bar{w}) = 0$ and $D\Theta(\bar{w}) : \mathbb{X} \to \mathbb{Y}$ is onto;

(iii) $\mathfrak{D} \cap B(\bar{w}, \rho) = \{w : \ \Theta(w) \in K\} \cap B(\bar{w}, \rho)$.

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Examples: (Shapiro '03)

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Examples: (Shapiro '03)

1. Polyhedral sets, second-order cone, positive semidefinite cone.
2. $\mathfrak{D} = \{x : \ g_i(x) \leq 0, i = 1, \ldots, m\}$, $g_i \in C^2$, LICQ holds at $\bar{x} \in \mathfrak{D}$ $\Rightarrow$ C^2-cone reducible at $\bar{x}$.

Digression: C^2-cone reducible structures

Theorem 4. (Yu, Li, P. '19)
Let $\ell : \mathbb{Y} \to \mathbb{R}$ be strongly convex on any compact convex set and have locally Lipschitz gradient, $\mathcal{A} : \mathbb{X} \to \mathbb{Y}$ be a linear map, and $v \in \mathbb{X}$. Consider the function

$$h(x) := \ell(\mathcal{A}x) + \langle v, x \rangle + \sigma_{\mathfrak{D}}(x)$$

with $\mathfrak{D}$ being a C^2-cone reducible closed convex set. Suppose that $0 \in \partial h(\bar{x})$ and

$$\mathcal{A}^{-1}\{\mathcal{A}\bar{x}\} \cap \mathrm{ri} N_{\mathfrak{D}}(-\mathcal{A}^*\nabla\ell(\mathcal{A}\bar{x}) - v) \neq \emptyset,$$

then h satisfies the KL property at $\bar{x}$ with exponent $\frac{1}{2}$.

Application V

Let $\ell : \mathbb{R}^m \to \mathbb{R}$ be strongly convex on any compact convex set and have locally Lipschitz gradient, $\mathcal{A} : \mathcal{S}^n \to \mathbb{R}^m$ be a linear map.

Each of the following functions satisfies the KL property with exponent $\frac{1}{2}$ at an $\bar{X}$ satisfying the **ri condition**:

(i) PSD cone constraint:

$$f(X) = \ell(\mathcal{A}X) + \langle V, X\rangle + \delta_{\mathcal{S}^n_+}(X).$$

(ii) Schatten p-norm regularization:

$$f(X) = \ell(\mathcal{A}X) + \langle V, X\rangle + \mu\|X\|_p,$$

where $\|X\|_p$ is the Schatten p-norm with $p \in [1, 2] \cup \{\infty\}$ and $\bar{X} \neq 0$.

Application VI

Corollary 5. (Yu, Li, P. '19)
Let f be proper closed with $\inf f > -\infty$ and $\phi \in C^2$ is strongly convex.
If f is KL with exponent $\alpha \in [\frac{1}{2}, 1)$, then so is the envelope function

$$F_\phi(x) := \inf_y \{f(y) + \mathfrak{B}_\phi(y, x)\},$$

where $\mathfrak{B}_\phi$ is the Bregman distance:

$$\mathfrak{B}_\phi(y, x) := \phi(y) - \phi(x) - \langle \nabla\phi(x), y - x\rangle.$$

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$$\mathfrak{B}_\phi(y, x) := \phi(y) - \phi(x) - \langle \nabla \phi(x), y - x \rangle.$$

Remark:

- When $\phi(\cdot) = \frac{1}{2\gamma}\|\cdot\|^2$, $\gamma > 0$, F_ϕ is the Moreau envelope.
- When $f = h + P$, where $h \in C^2$ has Lipschitz gradient with modulus L, and $\phi(\cdot) = \frac{1}{2\gamma}\|\cdot\|^2 - h(\cdot)$, $\gamma \in (0, \frac{1}{L})$, F_ϕ is the forward-backward envelope. (Stella et al. '17)

Calculus of KL exponent IV

Theorem 5. (Yu, Li, P. '19)
Let $h : \mathbb{X} \to \mathbb{R}$ and $G : \mathbb{X} \to \mathbb{Y}$ be continuously differentiable. Assume that $G^{-1}\{0\} \neq \emptyset$ and define the functions g and g_1 by

$$g(x) := h(x) + \delta_{G^{-1}\{0\}}(x), \quad g_1(x, \lambda) := h(x) + \langle \lambda, G(x) \rangle.$$

If $\nabla G(\bar{x}) : \mathbb{Y} \to \mathbb{X}$ is injective and g_1 is a KL function with exponent α, then so is g.

Application VII

Corollary 6. (Yu, Li, P. '19)
Consider the function

$$f(X) := \frac{1}{2}\|\mathcal{A}X - b\|^2 + \delta_{\mathrm{rank}(\cdot)\leq r}(X)$$

for $X \in \mathbb{R}^{m\times n}$, where $\mathcal{A} : \mathbb{R}^{m\times n} \to \mathbb{R}^p$ is a linear map, $b \in \mathbb{R}^p$. Then f is KL with exponent $1 - \frac{1}{4\cdot 9^{\kappa}}$, where $\kappa = mn + m(m-r) + n(m-r) - 1$.

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Key proof idea:

$$f(X) = \inf_U \left\{ \frac{1}{2}\|\mathcal{A}X - b\|^2 + \frac{1}{2}\|U^T U - I_{m-r}\|_F^2 + \delta_{\mathfrak{D}}(X,U) + \delta_{\mathfrak{B}}(X,U) \right\},$$

where

$$\mathfrak{D} := \{(X,U) \in \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times(m-r)} : \ U^T X = 0\},$$

$$\mathfrak{B} := \{(X,U) \in \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times(m-r)} : 0.5 I_{m-r} \preceq U^T U \preceq 2 I_{m-r}\},$$

Conclusion

- KL exponent is an important quantity for determining the qualitative convergence behavior of first-order methods.
- We presented some rules for deducing KL exponents:
  - ⋆ Lagrangian relaxation.
  - ⋆ Min of finitely many functions.
  - ⋆ Inf-projection.

References:

- G. Li and T. K. Pong.
  Calculus of the exponent of Kurdyka-Łojasiewicz inequality and its applications to linear convergence of first-order methods.
  Found. Comput. Math. 18:1199–1232, 2018.
- P. Yu, G. Li and T. K. Pong.
  Deducing Kurdyka-Łojasiewicz exponent via inf-projection.
  Available at https://arxiv.org/abs/1902.03635.

Thanks for coming! :-)