Explicit estimation of KL exponent and linear convergence of 1st-order methods

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(Joint work with Guoyin Li)

Motivating applications

Sparse optimization problems:

- Logistic regression with ℓ_1 regularization:

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{m} \log(1 + \exp(Ax)_i) + \mu \sum_{i=1}^{n-1} |x_i|.$$

- Logistic regression with sparsity constraint:

$$\begin{array}{ll} \min\limits_{x \in \mathbb{R}^n} & \sum_{i=1}^{m} \log(1 + \exp(Ax)_i) \\ \text{s.t.} & \text{card}\{i : \; x_i \neq 0, 1 \leq i \leq n-1\} \leq r. \end{array}$$

- Can also consider least squares loss.

First-order method

Consider

$$f(x) := h(x) + P(x),$$

where: h is continuously differentiable with Lipschitz gradient whose continuity modulus is $L > 0$, P is proper closed.

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Proximal gradient algorithm.
Initialize x^0, set $\gamma \in (0, \frac{1}{L})$. For $k = 1, \ldots$,

$$x^{k+1} \in \operatorname{prox}_{\gamma P}\left(x^k - \gamma \nabla h(x^k)\right),$$

where

$$\operatorname{prox}_{\gamma P}(y) = \operatorname*{Arg\,min}_{x \in \mathbb{R}^n} \left\{ \frac{1}{2}\|x - y\|^2 + \gamma P(x) \right\}.$$

KL property & exponent

Definition: (Attouch et al. '10, Attouch et al. '13)
Let f be proper closed and $\alpha \in [0, 1)$.

- f is said to have the Kurdyka-Łojasiewicz (KL) property with exponent α at $\bar{x} \in \operatorname{dom} \partial f$ if there exist c, ν, $\epsilon > 0$ so that

$$c[f(x) - f(\bar{x})]^\alpha \leq \operatorname{dist}(0, \partial f(x))$$

whenever $x \in \operatorname{dom} \partial f$, $\|x - \bar{x}\| \leq \epsilon$ and $f(\bar{x}) < f(x) < f(\bar{x}) + \nu$.

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- If f has the KL property at any $\bar{x} \in \operatorname{dom} \partial f$ with the same α, then f is said to be a KL function with exponent α.

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whenever $x \in \operatorname{dom} \partial f$, $\|x - \bar{x}\| \leq \epsilon$ and $f(\bar{x}) < f(x) < f(\bar{x}) + \nu$.
- If f has the KL property at any $\bar{x} \in \operatorname{dom} \partial f$ with the same α, then f is said to be a KL function with exponent α.

Examples.

- Proper closed semialgebraic functions are KL functions with exponent $\alpha \in [0, 1)$. (Bolte et al. '07)

Prototypical local convergence results

Fact 1.
For proximal gradient algorithm and some of its variants:
Let $\{x^k\}$ be a bounded sequence generated. If f is a KL function with exponent α, then:

- if $\alpha = 0$, then $\{x^k\}$ converges finitely;
- if $\alpha \in (0, \frac{1}{2}]$, then $\{x^k\}$ converges locally linearly;
- if $\alpha \in (\frac{1}{2}, 1)$, then $\{x^k\}$ converges locally sublinearly.

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Holds also for proximal alternating minimization algorithm (Attouch et al. '10), Douglas-Rachford splitting method (Li, P. '15), etc., if f is replaced by a suitable potential function.

Existing results

For nonsmooth objectives:

- A convex piecewise linear-quadratic function is a KL function with exponent $\frac{1}{2}$. (Li '95, Bolte et al. '15)
- A convex piecewise polynomial function of degree at most d is a KL function with exponent $1 - \frac{1}{(d-1)^n+1}$. (Li '13, Bolte et al. '15)

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- If f is the maximum of m polynomials of degree at most d, then the KL exponent is $1 - \frac{1}{\max\{1,(d+1)(3d)^{n+m-2}\}}$. (Li et al. '15)
- A special quadratic minimization problem with matrix variables and orthogonality constraint has KL exponent $\frac{1}{2}$. (Liu et al. '15)

Our strategy

Aim: Explicitly estimate the KL exponent of commonly used optimization models.

Strategy:

- Relate KL property to the Luo-Tseng error bound. (Luo, Tseng '92, '92, '93)
- Develop calculus rules on KL exponents: build new KL functions from old ones with known exponents.

Luo-Tseng error bound

Denote $\mathcal{X} := \{x : \ 0 \in \partial f(x)\}$, where $f = h + P$. Assume in addition that P is convex.

Definition: (Luo, Tseng '92, Tseng, Yun '09)
Suppose that $\mathcal{X} \neq \emptyset$. We say that the Luo-Tseng error bound holds if for any $\zeta \geq \inf f$, there exist $c, \epsilon > 0$ so that

$$\mathrm{dist}(x, \mathcal{X}) \leq c\|\mathrm{prox}_P(x - \nabla h(x)) - x\|$$

whenever $\|\mathrm{prox}_P(x - \nabla h(x)) - x\| < \epsilon$ and $f(x) \leq \zeta$.

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whenever $\|\mathrm{prox}_P(x - \nabla h(x)) - x\| < \epsilon$ and $f(x) \leq \zeta$.

Assumption 1: (Luo, Tseng '92, Tseng, Yun '09)
There exists $\delta > 0$ so that if $x, y \in \mathcal{X}$ and $\|x - y\| \leq \delta$, then $f(x) = f(y)$.

Luo-Tseng error bound

Examples: When $\mathcal{X} \neq \emptyset$ and $f = h + P$, Assumption 1 and the Luo-Tseng error bound hold for

- $h(x) = \ell(Ax)$ and P is proper polyhedral, where ℓ is strongly convex on any compact convex set and is twice continuously differentiable. (Luo, Tseng '92, Tseng, Yun '09)

- h is a quadratic (not necessarily convex) and P is proper polyhedral. (Luo, Tseng '92, Tseng, Yun '09)

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Theorem 1. (Li, P. '16)
Suppose that $\mathcal{X} \neq \emptyset$, and Assumption 1 and the Luo-Tseng error bound hold. Then f is a KL function with exponent $\frac{1}{2}$.

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Theorem 1. (Li, P. '16)
Suppose that $\mathcal{X} \neq \emptyset$, and Assumption 1 and the Luo-Tseng error bound hold. Then f is a KL function with exponent $\frac{1}{2}$.

Key inequality in the proof. For any $x \in \operatorname{dom} \partial f$,

$$\|\operatorname{prox}_P(x - \nabla h(x)) - x\| \leq \operatorname{dist}(0, \partial f(x)).$$

Known when $P = \delta_C$ for some closed convex set C.

Calculus of KL exponent I

Theorem 2. (Li, P. '16)
Suppose that g_i are KL functions with exponents α_i, $i = 1, \ldots, m$. Suppose in addition that $g := \min_{1 \le i \le m} g_i$ is continuous on $\mathrm{dom}\, \partial g$ and that $\mathrm{dom}\, \partial g_i = \mathrm{dom}\, g_i$ for all i. Then g is a KL function with exponent $\max\{\alpha_i : 1 \le i \le m\}$.

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Key fact used in the proof. For any $x \in \mathrm{dom}\,\partial g$,

$$\partial g(x) \subseteq \bigcup_{i\in I(x)} \partial g_i(x),$$

where $I(x) := \{i : \ g(x) = g_i(x)\}$. (Mordukovich, Shao '95)

Application I

Corollary 1. (Li, P. '16)
Consider functions of the form

$$f(x) = \ell(Ax) + \min_{1 \le i \le m} P_i(x)$$

where ℓ is strongly convex on any compact convex set and is twice continuously differentiable, P_i are proper polyhedral functions. If f is continuous on $\operatorname{dom} \partial f$, then f is a KL function with exponent $\frac{1}{2}$.

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Example:

$$\begin{aligned} f(x) &= \ell(Ax) + \delta_{\|\cdot\|_0 \le r}(x) \\ &= \ell(Ax) + \min_{I \in \mathcal{I}_{n-r}} \delta_{H_I}(x), \end{aligned}$$

where $\mathcal{I}_k := \{J \subseteq \{1, \ldots, n\} : \ |J| = k\}$, $H_I := \{x : \ x_i = 0 \ \forall i \in I\}$.

Application II

Corollary 2. (Li, P. '16)
Consider functions of the form

$$f(x) = \min_{1 \le i \le m} \left\{ x^T M_i x + b_i^T x + c_i + P_i(x) \right\},$$

where M_i are symmetric matrices, P_i are proper polyhedral functions.
If f is continuous on $\mathrm{dom}\,\partial f$, then f is a KL function with exponent $\frac{1}{2}$.

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Example: Least-squares with SCAD regularization: (Fan '97)

$$f(x) = \frac{1}{2}\|Ax - b\|^2 + \sum_{i=1}^{n} r_{\lambda,\theta}(x_i),$$

with $\lambda > 0$, $\theta > 2$ and

$$r_{\lambda,\theta}(t) = \begin{cases} \lambda|t| & \text{if } |t| \leq \lambda, \\ \frac{-t^2+2\theta\lambda|t|-\lambda^2}{2(\theta-1)} & \text{if } \lambda < |t| \leq \theta\lambda, \\ \frac{(\theta+1)\lambda^2}{2} & \text{if } |t| > \theta\lambda. \end{cases}$$

Calculus of KL exponent II

Theorem 3. (Li, P. '16)
Let $h(x) = \ell(Ax)$ for some continuous strictly convex function ℓ, g be a continuous convex function, D be a closed convex set, $\alpha \in (0, 1)$. Suppose also

(i) there exists $x_0 \in D$ with $g(x_0) < 0$;

(ii) $\inf_{x \in D} h(x) < \inf_{x \in D}\{h(x) : \; g(x) \leq 0\}$;

(iii) for any $\lambda > 0$, $h + \lambda g + \delta_D$ is KL with exponent α.

Then $h + \delta_{g(\cdot) \leq 0} + \delta_D$ is KL with exponent α.

Application III

Consider functions of the form

$$f(x) = \ell(Ax) + \delta_C(x),$$

where ℓ is strongly convex on any compact convex set and is twice continuously differentiable, and

$$C := \left\{ x : \sum_{i=1}^{m} w_i \|x_i\|_p \leq \sigma \right\},$$

with $x_i \in \mathbb{R}^{n_i}$, $\sum_{i=1}^{m} n_i = n$, $w_i > 0$, $\sigma > 0$ and $p \in [1, 2]$.

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Corollary 3. (Li, P. '16)
Suppose that $\inf f(x) > \inf \ell(Ax)$. Then f is KL with exponent $\frac{1}{2}$.

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Corollary 3. (Li, P. '16)
Suppose that $\inf f(x) > \inf \ell(Ax)$. Then f is KL with exponent $\frac{1}{2}$.

Proof: When $\mathcal{X} \neq \emptyset$, Luo-Tseng error bound holds for the regularized version. (Zhou et al. '15)

Open questions

- What is the KL exponent of logistic regression with SCAD regularization?
- Analyzing optimization problems with matrix variables, e.g., nuclear norm regularization, rank constraints, etc.
- Deducing the KL exponent of the *potential* function used in prototypical convergence results, based on the exponent of the original objective.
  Done for inertial proximal gradient algorithm:

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- Deducing the KL exponent of the *potential* function used in prototypical convergence results, based on the exponent of the original objective.
  Done for inertial proximal gradient algorithm:

  ⋆ **Theorem 4.** (Li, P. '16)
  If f has the KL property at $\bar{x} \in \mathrm{dom}\,\partial f$ with exponent $\alpha \in [0, 1)$, then for any $\beta > 0$, $F(x, y) := f(x) + \frac{\beta}{2}\|x - y\|^2$ has the KL property at $(\bar{x}, \bar{x})$ with exponent $\max\{\alpha, \frac{1}{2}\}$.

Open questions

Forward-backward envelope (Patrinos, Bemporad '13, Stella et al. '16):
When P is convex and h is C^2, take any $\gamma \in (0, \frac{1}{L})$ and define

$$F_\gamma(x) := \inf_y \left\{ h(x) + \langle \nabla h(x), y - x \rangle + \frac{1}{2\gamma}\|y - x\|^2 + P(y) \right\}.$$

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$$F_\gamma(x) := \inf_y \left\{ h(x) + \langle \nabla h(x), y - x \rangle + \frac{1}{2\gamma}\|y - x\|^2 + P(y) \right\}.$$

Basic facts:

- F_γ is smooth.
- $\mathcal{X} = \{x : \nabla F_\gamma(x) = 0\}$.
- $\nabla F_\gamma(x) = \gamma^{-1}(I - \gamma \nabla^2 h(x))(x - \mathrm{prox}_{\gamma P}(x - \gamma \nabla h(x)))$.

Open questions

Theorem 5. (Liu, P. '16)
Suppose that $\gamma \in (0, \frac{1}{L})$, h is analytic, P is continuous on $\mathrm{dom}\, \partial P$ and is subanalytic with $\inf P > -\infty$. Moreover, the Luo-Tseng error bound holds for $h + P$.

Then F_γ is a KL function with exponent $\frac{1}{2}$.

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Then F_γ is a KL function with exponent $\frac{1}{2}$.

Question: Can the error bound condition be replaced by KL property?

Conclusion

- The Luo-Tseng error bound together with an assumption on the separation of stationary values implies that the KL exponent is $\frac{1}{2}$.
- Based on this and some calculus rules for KL exponents, the KL exponent for a large class of convex/nonconvex optimization models is obtained, including
  - ⋆ logistic regression with ℓ_1 regularization/sparsity constraints;
  - ⋆ least squares problem with SCAD regularization.

Reference:

- G. Li and T. K. Pong.
  Calculus of the exponent of Kurdyka-Łojasiewicz inequality and its applications to linear convergence of first-order methods.
  Available at *http://arxiv.org/abs/1602.02915*.

Thanks for coming! ☺