A conditional-gradient-based single-loop augmented Lagrangian method for inequality constrained problems

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(Joint work with Xiaozhou Wang and Zev Woodstock)

Our problem

$$\begin{array}{ll} \underset{\boldsymbol{x}\in\mathbb{R}^n}{\text{Minimize}} & f(\boldsymbol{x}) + h(\boldsymbol{x}) \\ \text{Subject to} & \boldsymbol{g}(\boldsymbol{x}) := \begin{bmatrix} g_1(\boldsymbol{x}) & \cdots & g_m(\boldsymbol{x}) \end{bmatrix}^T \leq \mathbf{0}, \end{array}$$

where

- $f : \mathbb{R}^n \to \mathbb{R}$ is convex and L_f-smooth for some $L_f > 0$.
- Each $g_i : \mathbb{R}^n \to \mathbb{R}$ is convex and L_{g_i}-smooth for some $L_{g_i} > 0$.
- $h : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is proper closed convex with bounded dom h.

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- $h : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is proper closed convex with bounded dom h. Moreover, for any $\boldsymbol{u} \in \mathbb{R}^n$, a minimizer of $\min_{\boldsymbol{v}\in\mathbb{R}^n}\{\langle \boldsymbol{v}, \boldsymbol{u}\rangle + h(\boldsymbol{v})\}$ exists and can be computed efficiently. (Efficient LMO)

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- A KKT point exists, i.e., there exists $(\boldsymbol{x}^*, \boldsymbol{z}^*)$ such that

$$0 \in \nabla f(\boldsymbol{x}^*) + \partial h(\boldsymbol{x}^*) + \sum_{i=1}^{m} z_i^* \nabla g_i(\boldsymbol{x}^*),$$

$$\boldsymbol{g}(\boldsymbol{x}^*) \leq 0, \quad \boldsymbol{z}^* \geq 0, \quad \langle \boldsymbol{z}^*, \boldsymbol{g}(\boldsymbol{x}^*)\rangle = 0.$$

Motivation

Natural idea:

- Apply the Augmented Lagrangian (AL) framework, with subproblems approx. solved by Frank-Wolfe method.

Algorithm A conceptual AL method based on LMO

Step 1. Choose $(\boldsymbol{x}^0, \boldsymbol{z}^0) \in \operatorname{dom} h \times \mathbb{R}^m_+$, a positive increasing sequence $\{\lambda_k\}$ with $\lim_{k\to\infty} \lambda_k = \infty$.

Step 2. For $k = 0, \ldots,$

Apply FW method to approx. minimize $F_{\lambda_k}(\cdot, \boldsymbol{z}^k) + h(\cdot)$, where

$$F_{\lambda_k}(\boldsymbol{x}, \boldsymbol{z}^k) := f(\boldsymbol{x}) + \frac{\lambda_k}{2} \sum_{i=1}^{m} \left(\left[g_i(\boldsymbol{x}) + \frac{z_i^k}{\lambda_k} \right]_+^2 - \left(\frac{z_i^k}{\lambda_k} \right)^2 \right)$$

Set $\boldsymbol{z}^{k+1} = \max\{\boldsymbol{0}, \boldsymbol{z}^k + \lambda_k \boldsymbol{g}(\boldsymbol{x}^k)\}$.

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Complicated "Double loop" structure. ☹

Existing work

Existing Single-loop (i.e., 1 LMO per iteration) approaches:

- Most existing works focus on equality constraints, e.g., (Gidel, Pedregosa, Lacoste-Julien '18, Yurtsever, Fercoq, Cevher '19).

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Questions:

- Develop an AL-based single-loop algorithm?
- Derive improved rates under uniform convexity?

Algorithmic framework

Algorithm A CG-based single-loop AL method

Step 1. Choose $(\boldsymbol{x}^0, \boldsymbol{z}^0) \in \operatorname{dom} h \times \mathbb{R}^m_+$, $\alpha_0 \in [0,1]$, a positive increasing sequence $\{\lambda_k\}$ with $\lim_{k\to\infty} \lambda_k = \infty$, and a positive summable sequence $\{\sigma_k\}$ with $\sigma_k \le \lambda_k$. Set $k = 0$.

Step 2. Compute

$$\boldsymbol{v}^k \in \operatorname*{Arg\,min}_{\boldsymbol{v}\in\mathbb{R}^n} \langle \nabla_{\boldsymbol{x}} F_{\lambda_k}(\boldsymbol{x}^k, \boldsymbol{z}^k), \boldsymbol{v}\rangle + h(\boldsymbol{v}),$$

$$\boldsymbol{x}^{k+1} = \boldsymbol{x}^k + \alpha_k(\boldsymbol{v}^k - \boldsymbol{x}^k),$$

$$\boldsymbol{z}^{k+1} = \boldsymbol{z}^k + \sigma_k \max\left\{-\frac{\boldsymbol{z}^k}{\lambda_k}, \boldsymbol{g}(\boldsymbol{x}^{k+1})\right\}.$$

Step 3. Choose $\alpha_{k+1} \in [0,1]$ and update $k \leftarrow k+1$. Go to **Step 2**.

Here, $F_\lambda(\boldsymbol{x}, \boldsymbol{z}) := f(\boldsymbol{x}) + \frac{\lambda}{2}\sum_{i=1}^m \left(\left[g_i(\boldsymbol{x}) + \frac{z_i}{\lambda}\right]_+^2 - \left(\frac{z_i}{\lambda}\right)^2\right)$.

General convergence result

Define $\mathcal{L}_\lambda(\boldsymbol{x},\boldsymbol{z}) := f(\boldsymbol{x}) + \frac{\lambda}{2}\sum_{i=1}^m \left(\left[g_i(\boldsymbol{x}) + \frac{z_i}{\lambda}\right]_+^2 - \left(\frac{z_i}{\lambda}\right)^2 \right) + h(\boldsymbol{x})$.

Theorem 1. (Wang, P., Woodstock '26)
Let $\{(\boldsymbol{x}^k,\boldsymbol{z}^k)\}$ be generated by Algorithm 1 and define

$$T_k := [\mathcal{L}_{\lambda_k}(\boldsymbol{x}^k,\boldsymbol{z}^k) - L^*]_+ \quad \forall k \geq 0.$$

Then it holds that

$$\max\{\|[\boldsymbol{g}(\boldsymbol{x}^k)]_+\|, |f(\boldsymbol{x}^k) + h(\boldsymbol{x}^k) - L^*|\} = \mathcal{O}(\max\{T_k, 1/\lambda_k\}),$$

where L^* is the optimal value. Consequently, if

$$\lim_{k\to\infty} T_k = 0,$$

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$$\lim_{k\to\infty} \|[\boldsymbol{g}(\boldsymbol{x}^k)]_+\| = \lim_{k\to\infty} |f(\boldsymbol{x}^k) + h(\boldsymbol{x}^k) - L^*| = 0.$$

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Question: How can we guarantee $\lim_{k\to\infty} T_k = 0$?

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Question: How can we guarantee $\lim_{k \to \infty} T_k = 0$? Suitably choose α_k, λ_k and σ_k.

Open-loop stepsize

Open-loop stepsize assumptions:

(i) $\{\lambda_k\}$ is increasing, $\lambda_k = \Theta(k^\tau)$, $\lambda_{k+1} - \lambda_k = \mathcal{O}(k^{-(1-\tau)})$ for some $\tau \in (0, 1)$.

(ii) $\{\alpha_k\} \subset (0, 1]$ is nonincreasing, $\alpha_k = \Theta(k^{-p})$ for some $p \in (\tau, 1)$.

(iii) $\{\sigma_k\}$ satisfies $\sigma_k \leq \lambda_k$ and $\sigma_k = \Theta(k^{-(1+\gamma)})$ for some $\gamma > 0$.

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Theorem 2. (Wang, P., Woodstock '26)
Under the Open-loop stepsize assumptions, let $\{(\boldsymbol{x}^k, \boldsymbol{z}^k)\}$ be generated by Algorithm 1 and L^* be the optimal value. Then

$$\max\{|f(\boldsymbol{x}^k) + h(\boldsymbol{x}^k) - L^*|, \|[\boldsymbol{g}(\boldsymbol{x}^k)]_+\|\} = \mathcal{O}(\max\{k^{-(p-\tau)}, k^{-\tau}\}).$$

If, in addition, our problem has a Slater point, i.e., $\exists \hat{\boldsymbol{x}} \in \operatorname{dom} h$ with $\boldsymbol{g}(\boldsymbol{x}) < \boldsymbol{0}$, then there exists a subsequence of $\{(\boldsymbol{x}^k, [\lambda_k \boldsymbol{g}(\boldsymbol{x}^k) + \boldsymbol{z}^k]_+)\}$ such that all of its cluster points are KKT points.

Short stepsize

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(i) $\{\lambda_k\}$ is increasing, $\lambda_k = \Theta(k^\tau)$ and $\lambda_{k+1} - \lambda_k = \mathcal{O}(k^{-(1-\tau)})$ for some $\tau \in (0,1)$.

(ii) The α_k is the short stepsize, i.e.,

$$\alpha_k = \begin{cases} 0 & \text{if } G_{\boldsymbol{z}^k,\lambda_k}(\boldsymbol{x}^k) = \|\boldsymbol{v}^k - \boldsymbol{x}^k\| = 0, \\ \min\left\{1, \frac{G_{\boldsymbol{z}^k,\lambda_k}(\boldsymbol{x}^k)}{(L_\Psi(\boldsymbol{x}^k,\boldsymbol{z}^k)+L_f)\|\boldsymbol{v}^k-\boldsymbol{x}^k\|^2}\right\} & \text{else,} \end{cases}$$

where

$$G_{\boldsymbol{z},\lambda}(\boldsymbol{x}) := \langle \nabla_{\boldsymbol{x}} F_\lambda(\boldsymbol{x},\boldsymbol{z}), \boldsymbol{x} - \boldsymbol{v}^+\rangle + h(\boldsymbol{x}) - h(\boldsymbol{v}^+),$$

where $\boldsymbol{v}^+ \in \operatorname{Arg\,min}_{\boldsymbol{v}} \langle \nabla_{\boldsymbol{x}} F_\lambda(\boldsymbol{x},\boldsymbol{z}), \boldsymbol{v}\rangle + h(\boldsymbol{v})$, and $L_\Psi(\boldsymbol{x},\boldsymbol{z}) + L_f$ is a Lipschitz constant for $\nabla_{\boldsymbol{x}} F_\lambda(\boldsymbol{x},\boldsymbol{z})$ on $\operatorname{dom} h$.

(iii) $\{\sigma_k\}$ satisfies $\sigma_k \le \lambda_k$ and $\sigma_k = \Theta(k^{-(1+\gamma)})$ for some $\gamma > 0$.

Recall that $F_\lambda(\boldsymbol{x},\boldsymbol{z}) := f(\boldsymbol{x}) + \frac{\lambda}{2}\sum_{i=1}^m \left([g_i(\boldsymbol{x}) + \frac{z_i}{\lambda}]_+^2 - (\frac{z_i}{\lambda})^2 \right)$.

Short stepsize cont.

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Remark: Taking $\tau = 0.5$, we see that Algorithm 1 achieves the rate

$$\max\{|f(\boldsymbol{x}^k) + h(\boldsymbol{x}^k) - L^*|, \|[\boldsymbol{g}(\boldsymbol{x}^k)]_+\|\} = \mathcal{O}(1/\sqrt{k}).$$

Uniform convexity

Uniform convexity assumptions:

(i) Function h is the indicator function of a set $\mathcal{C}$ that is compact and (ν, q)-uniformly convex for some $q \geq 2$ and $\nu > 0$.

(ν, q)-uniformly convexity means $\forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathcal{C}$, $t \in [0, 1]$ and $\|\boldsymbol{z}\| \leq 1$, it holds that $t\boldsymbol{x} + (1 - t)\boldsymbol{y} + t(1 - t)\nu\|\boldsymbol{x} - \boldsymbol{y}\|^q \boldsymbol{z} \in \mathcal{C}$.

(ii) For some $\epsilon > 0$, it holds that

$$\zeta := \inf\left\{\left\|\nabla f(\boldsymbol{x}) + \textstyle\sum_{i=1}^m z_i \nabla g_i(\boldsymbol{x})\right\| : (\boldsymbol{x}, \boldsymbol{z}) \in \mathcal{A}_\epsilon, \boldsymbol{x} \in \mathcal{C}\right\} > 0,$$

where

$$\mathcal{A}_\epsilon := \{(\boldsymbol{x}, \boldsymbol{z}) \in \boldsymbol{g}^{-1}((-\infty, \epsilon]^m) \times \mathbb{R}_+^m : \forall i, \text{ if } g_i(\boldsymbol{x}) < -\epsilon, \text{ then } z_i = 0\}.$$

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Proposition 1. (Wang, P., Woodstock '26)
Suppose Uniform convexity assumption (i) holds. Suppose that $\exists \epsilon > 0$ and $\tilde{\boldsymbol{x}}$ such that $g_i(\tilde{\boldsymbol{x}}) < -\epsilon$ for all i, and it holds that

$$\operatorname*{Arg\,min}_{\boldsymbol{x} \in \mathbb{R}^n}\{f(\boldsymbol{x}) : \boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{\eta}\} \cap \mathcal{C} = \emptyset \quad \forall \boldsymbol{\eta} \in [-\epsilon, \epsilon]^m.$$

Then Uniform convexity assumption (ii) holds with the above ϵ.

Uniform convexity: Open-loop stepsize

Theorem 4. (Wang, P., Woodstock '26)
Suppose that the Open-loop stepsize assumptions and Uniform convexity assumptions hold. Let $\{(\boldsymbol{x}^k, \boldsymbol{z}^k)\}$ be generated by Algorithm 1. Then it holds that

$$\max\{|f(\boldsymbol{x}^k) + h(\boldsymbol{x}^k) - L^*|, \|[\boldsymbol{g}(\boldsymbol{x}^k)]_+\|\}$$
$$= \begin{cases} \mathcal{O}(k^{-\tau}) & \text{if } \mu = 0, \\ \mathcal{O}(\max\{\frac{1}{k^{(1-\mu)\omega+(p-\tau)}}, \frac{1}{k^\tau}\}) & \text{if } \mu \in (0,1), \end{cases}$$

where $\omega := \min\{p-\tau, 1-p+\tau\}$ with $0 < \tau < p < 1$ and $\mu := 1 - 2/q$.

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Remark:

- The above rates are no worse than the previously proved rate of $\mathcal{O}(\max\{k^{-(p-\tau)}, k^{-\tau}\})$.
- When $\mathcal{C}$ is strongly convex, the above rate can be made arbitrarily close to $\mathcal{O}(1/k)$ by choosing $p > \tau$ with $\tau \in (0,1)$ being arbitrarily close to 1.

Uniform convexity: Short stepsize

Theorem 5. (Wang, P., Woodstock '26)
Suppose that the Short stepsize assumptions and Uniform convexity assumptions hold. Let $\{(\boldsymbol{x}^k, \boldsymbol{z}^k)\}$ be generated by Algorithm 1. Then it holds that

$$\max\{|f(\boldsymbol{x}^k)+h(\boldsymbol{x}^k)-L^*|, \|[\boldsymbol{g}(\boldsymbol{x}^k)]_+\|\} = \mathcal{O}(\max\{1/k^{\frac{1}{1+\mu}}, 1/k^{\frac{2-2\tau}{1+\mu}}, 1/k^{\tau}\}),$$

where $\mu := 1 - 2/q \in [0, 1)$.

Uniform convexity: Short stepsize

Theorem 5. (Wang, P., Woodstock '26)
Suppose that the Short stepsize assumptions and Uniform convexity assumptions hold. Let $\{(\boldsymbol{x}^k, \boldsymbol{z}^k)\}$ be generated by Algorithm 1. Then it holds that

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Remark:

- The above rates are no worse than the previously proved rate of $\mathcal{O}(\max\{k^{-1/2}, k^{-(1-\tau)}, k^{-\tau}\})$.
- When $\mathcal{C}$ is strongly convex, the best rate is $\mathcal{O}(k^{-2/3})$ by choosing $\tau = 2/3$.

Preliminary numerics

Consider random instances of

$$\begin{array}{rl} \underset{\boldsymbol{x}\in\mathcal{C}}{\text{Minimize}} & f(\boldsymbol{x}) := \langle \boldsymbol{x}-\boldsymbol{b}, \boldsymbol{A}(\boldsymbol{x}-\boldsymbol{b})\rangle \\ \text{Subject to} & g_i(\boldsymbol{x}) := \langle \boldsymbol{x}, \boldsymbol{Q}_i\boldsymbol{x}\rangle + \langle \boldsymbol{x}, \boldsymbol{r}_i\rangle + d_i \le 0, \quad i = 1, 2, \end{array}$$

where

- $\mathcal{C} := \{\boldsymbol{x} \in \mathbb{R}^{500\times 500} : \boldsymbol{x}\mathbf{1} = \mathbf{1},\ \boldsymbol{x}^T\mathbf{1} = \mathbf{1},\ \boldsymbol{x} \ge \mathbf{0}\}$ is the Birkhoff polytope.
- $\boldsymbol{A} \in \mathcal{S}^{500}$ and $\boldsymbol{Q}_i \in \mathcal{S}^{500}$ are positive definite.
- d_i is chosen to contain the barycenter $\boldsymbol{c}$ of $\mathcal{C}$ (s.t. $g_i(\boldsymbol{c}) < 0$) while ruling out one of its corners.
- $\boldsymbol{b} \notin \mathcal{C}$ is obtained by extrapolating (out of $\mathcal{C}$) from $\boldsymbol{c}$ towards a ruled-out corner.

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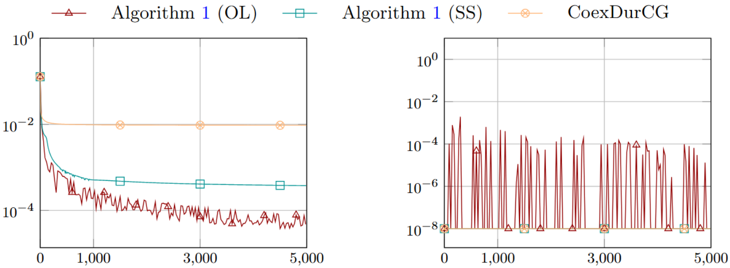
We compare our algorithm with open-loop stepsize (OL), short stepsize (SS) (suitably scaled for the first 1000 iterations), and the CoexDurCG in Lan, Romeijn, Zhou '21.
We use Gurobi as the benchmark.

Preliminary numerics cont.

$$\mathrm{val}_k := \frac{|f(\boldsymbol{x}^k) - f^*_{\mathrm{Gurobi}}|}{\max\{f^*_{\mathrm{Gurobi}}, 1\}}, \quad \mathrm{feas}_k := \max\left\{ \frac{\|[\boldsymbol{g}(\boldsymbol{x}^k)]_+\|_\infty}{\max\{\|\boldsymbol{g}(\boldsymbol{0})\|_\infty, 1\}}, 10^{-8} \right\}.$$



(a) val_k over iterations

(b) feas_k over iterations

Algorithms	iterations	objective	feasibility	val_{5000}	time (sec)
Algorithm 1 (OL)	5000	123414.30	0	4.75×10^{-5}	1260.93
Algorithm 1 (SS)	5000	123454.93	0	3.77×10^{-4}	852.82
CoexDurCG	5000	124580.37	0	9.45×10^{-3}	2835.03
Gurobi	18	123408.44	6.87×10^{-13}	0	5108.93

Conclusion

- An AL-based single-loop algorithm was developed for minimizing the sum of a smooth convex function and an LMO-friendly convex function over smooth convex inequality constraints: each iteration requires one call to the LMO.
- Iteration complexity results were obtained for both open-loop stepsize and short stepsize schedules: the latter matches the state-of-the-art complexity result.
- Improved complexity results were established under uniform convexity assumption.

Reference:

- X. Wang, T. K. Pong and Z. Woodstock.
  A conditional-gradient-based single-loop augmented Lagrangian method for inequality constrained optimization.
  Available at https://arxiv.org/abs/2605.22539.

Thanks for coming! ☺