Gauge optimization: Duality and polar envelope

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April 2019
(Joint work with Michael Friedlander and Ives Macêdo)

Motivating example

Minimum norm solutions:

- In sparse optimization:

$$\begin{array}{rl} \min & \|x\|_1 \\ \text{s.t.} & \|b - Ax\| \leq \sigma, \end{array}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $\sigma < \|b\|$.

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  where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $\sigma < \|b\|$.

- More generally, minimization of atomic norm (Chandrasekaran et al. '12)

$$\|x\|_{\mathcal{A}} = \inf\{\lambda \geq 0 : \ x \in \lambda \operatorname{conv} \mathcal{A}\},$$

  where $\mathcal{A}$ is a set of "atoms" characterizing the notion of sparsity:
  - ⋆ $\mathcal{A} = \{\pm e_i : \ i = 1, ..., n\} \quad \Rightarrow \quad \|x\|_{\mathcal{A}} = \sum_{i=1}^n |x_i|$.
  - ⋆ $\mathcal{A} =$ unit norm rank 1 matrices $\quad \Rightarrow \quad \|X\|_{\mathcal{A}} = \sum_{i=1}^n \sigma_i(X)$.

Gauges

- Gauges are generalizations of norms: nonnegative convex positively homogeneous functions that are zero at the origin.
- $\kappa(x) = \inf\{\lambda \geq 0 : \; x \in \lambda U\}$ for some convex set U.

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- $\kappa(x) = \inf\{\lambda \geq 0 : \; x \in \lambda U\}$ for some convex set U.
- Polar gauge generalizes dual norm:

$$\begin{aligned}\kappa^\circ(y) &= \inf\{\lambda > 0 : \; \langle x, y\rangle \leq \lambda\kappa(x) \; \forall x\} \\ &= \sup\{\langle x, y\rangle : \; \kappa(x) \leq 1\}.\end{aligned}$$

- Generalized Cauchy inequality: for all $x \in \operatorname{dom}\kappa$ and $y \in \operatorname{dom}\kappa^\circ$,

$$\langle x, y\rangle \leq \kappa(x)\kappa^\circ(y).$$

Gauge optimization

$$v_p := \min \quad \kappa(x) \qquad \text{s.t.} \quad \rho(b - Ax) \le \sigma. \tag{P_ρ}$$

- κ is a gauge.
- ρ is a closed gauge with $\rho^{-1}(0) = \{0\}$, $0 \le \sigma < \rho(b)$.

Gauge optimization

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- κ is a gauge.
- ρ is a closed gauge with $\rho^{-1}(0) = \{0\}$, $0 \le \sigma < \rho(b)$.
- Lagrange and gauge dual problems:

$$v_\ell := \max \quad \langle b, y\rangle - \sigma\rho^\circ(y) \quad \text{s.t.} \quad \kappa^\circ(A^* y) \le 1. \qquad v_g := \min \quad \kappa^\circ(A^* y) \quad \text{s.t.} \quad \langle b, y\rangle - \sigma\rho^\circ(y) \ge 1.$$

- The role of objective and constraint is reversed in the gauge dual.

Outline

- Gauge duality: general framework.
- Gauge duality: structured problem.
- Smoothing technique: polar envelope.
- Projected polar proximal point algorithm.

Gauge duality framework

Let $\mathcal{C}$ be a nonempty closed convex set not containing the origin, and define its anti-polar

$$\mathcal{C}' = \{u : \langle u, x \rangle \geq 1 \ \ \forall x \in \mathcal{C}\}.$$

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Freund ('87) defined the following primal-dual gauge pairs:

$$\begin{aligned} v_p := \min \quad & \kappa(x) \\ \text{s.t.} \quad & x \in \mathcal{C}, \end{aligned} \tag{P}$$

$$\begin{aligned} v_g := \min \quad & \kappa^\circ(u) \\ \text{s.t.} \quad & u \in \mathcal{C}'. \end{aligned} \tag{D}$$

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Theorem 1. [Strong duality] (Freund '87)
Suppose that κ is closed, $\operatorname{ri}\operatorname{dom}\kappa^\circ \cap \operatorname{ri}\mathcal{C}' \neq \emptyset$ and $\operatorname{ri}\operatorname{dom}\kappa \cap \operatorname{ri}\mathcal{C} \neq \emptyset$.
Then $v_p v_g = 1$ and both values are attained.

Anti-polar calculus

Let $\mathcal{D} := \{u : \ \rho(b-u) \le \sigma\}$. Then

$$\mathcal{C} = \{x : \ \rho(b-Ax) \le \sigma\} = A^{-1}\mathcal{D}.$$

How do we compute $\mathcal{C}'$?

Fact 2.

$$\mathcal{D}' = \{y : \ \langle b, y\rangle - \sigma\rho^\circ(y) \ge 1\}.$$

Proposition 1. (Friedlander, Macêdo, P. '14)

$$(A^{-1}\mathcal{D})' = \mathrm{cl}(A^*\mathcal{D}').$$

If, in addition, $\mathrm{ri}\,\mathcal{D} \cap \mathrm{Range}\,A \ne \emptyset$, then

$$(A^{-1}\mathcal{D})' = A^*\mathcal{D}' = \{A^*y : \ \langle b, y\rangle - \sigma\rho^\circ(y) \ge 1\}.$$

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Let $\mathcal{D} := \{u : \ \rho(b - u) \leq \sigma\}$ so that primal feasible set is $A^{-1}\mathcal{D}$.

Theorem 2. (Friedlander, Macêdo, P. '14)
Suppose that κ is closed, $\operatorname{ri}\operatorname{dom}\kappa^\circ \cap \operatorname{ri} A^*\mathcal{D}' \neq \emptyset$ and $\operatorname{ri}\operatorname{dom}\kappa \cap A^{-1}\operatorname{ri}\mathcal{D} \neq \emptyset$. Then $v_p v_g = 1$ and both values are attained.

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Unlike the Lagrange dual, the gauge dual (D_ρ) has a complicated objective and simple constraint.

Solution method: Smoothing

For proper closed convex functions f_1 and f_2, with suitable CQ:

$$
\begin{tikzcd}[column sep=large, row sep=huge]
\underset{x}{\text{minimize}}\ f_1(x) + f_2(x) \arrow[r, "\text{addition}"] \arrow[d, leftrightarrow, "x \in \partial f_1^*(y) \cap \partial f_2^*(-y)", "\substack{\text{Fenchel}\\ \text{duality}}"'] & \underset{x}{\text{minimize}}\ (f_1 + \frac{\alpha}{2}\|\cdot\|_2^2)(x) + f_2(x) \arrow[d, leftrightarrow, "x = \nabla(f_1^* \,\square\, \frac{1}{2\alpha}\|\cdot\|_2^2)(y)", "\substack{\text{Fenchel}\\ \text{duality}}"'] \\
\underset{y}{\text{minimize}}\ f_1^*(y) + f_2^*(-y) \arrow[r, "\substack{\text{sum}\\ \text{convolution}}"] & \underset{y}{\text{minimize}}\ (f_1^* \,\square\, \frac{1}{2\alpha}\|\cdot\|_2^2)(y) + f_2^*(-y)
\end{tikzcd}
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Here, $f_1^* \square \frac{1}{2\alpha}\|\cdot\|^2$ is the Moreau envelope of αf_1^*:

$$(f_1^* \square \frac{1}{2\alpha}\|\cdot\|^2)(x) := \inf_{y \in \mathbb{R}^n} \left\{ f_1^*(y) + \frac{1}{2\alpha}\|x - y\|^2 \right\}$$

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\underset{y}{\text{minimize}}\ \ f_1^*(y)+f_2^*(-y) \arrow[r, "\text{sum convolution}"] & \underset{y}{\text{minimize}}\ \ (f_1^*\,\Box\,\tfrac{1}{2\alpha}\|\cdot\|_2^2)(y)+f_2^*(-y)
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which is smooth. Does not preserve gauge structure!

Polar envelope

Definition. (Friedlander, Macêdo, P. '18)
Let κ be a gauge, $\alpha > 0$. The polar envelope and polar proximal mapping are

$$\kappa_\alpha(x) := \inf_z \max\left\{\kappa(z), \frac{1}{\alpha}\|x - z\|\right\},$$

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- For gauges κ_1 and κ_2: (Friedlander, Macêdo, P. '18)

$$(\kappa_1 \diamond \kappa_2)^\circ = \kappa_1^\circ + \kappa_2^\circ.$$

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$$(\kappa_1 \diamond \kappa_2)^\circ = \kappa_1^\circ + \kappa_2^\circ.$$

Recall that for proper convex functions f_1 and f_2, $(f_1 \Box f_2)^* = f_1^* + f_2^*$.

Polar envelope: Differential properties

Theorem 3. (Friedlander, Macêdo, P. '18)
Let κ be a gauge and $\alpha > 0$.

(i) If $\bar{x} \in \operatorname{pprox}_{\alpha\kappa}(x)$, then $\|x - \bar{x}\| \geq \alpha\kappa(\bar{x})$. If in addition κ is continuous, then $\|x - \bar{x}\| = \alpha\kappa(\bar{x})$.

(ii) If κ is closed, then $\operatorname{pprox}_{\alpha\kappa}(x)$ is a singleton for all x, and $\operatorname{pprox}_{\alpha\kappa}$ is continuous and positively homogeneous.

(iii) Suppose κ is closed. Then κ_α is differentiable at all x such that $\kappa_\alpha(x) > 0$. Moreover, at these x, it holds that $\langle x, x - \bar{x}\rangle > 0$ and

$$\nabla\kappa_\alpha(x) = \frac{\|x - \bar{x}\|}{\alpha\langle x, x - \bar{x}\rangle}(x - \bar{x}),$$

where $\bar{x} = \operatorname{pprox}_{\alpha\kappa}(x)$.

Polar envelope: Explicit example

Proposition 2. (Friedlander, Macêdo, P. '18)
Let κ be a continuous gauge. Then for any x satisfying $\kappa_\alpha(x) > 0$, it holds that

$$\kappa_\alpha(x) = \bar{r} \;\text{ and }\; \mathrm{pprox}_{\alpha\kappa}(x) = \mathrm{Proj}_{[\kappa \le \bar{r}]}(x),$$

where $\bar{r}$ is the unique root satisfying

$$\alpha^2 \bar{r}^2 = \|x - \mathrm{Proj}_{[\kappa \le \bar{r}]}(x)\|^2. \qquad (\clubsuit)$$

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Example: Consider $\kappa = \|\cdot\|_\infty$. For $x \neq 0$, ($\clubsuit$) becomes

$$\alpha^2 \bar{r}^2 = \sum_{i=1}^{n} (|x_i| - \bar{r})_+^2 \quad \overset{\bar{r}=\bar{\gamma}^{-1}}{\Longleftrightarrow} \quad \alpha^2 = \sum_{i=1}^{n} (\bar{\gamma}|x_i| - 1)_+^2.$$

Solved by simple root-finding procedure.

Solution method: Smoothing revisited

For closed gauges κ, ρ satisfying $\kappa^{-1}(0) = \{0\}$ and $\rho^{-1}(0) = \{0\}$, $\mathcal{C} = \{x : \ \rho(b - Ax) \le \sigma\}$ and $\sigma \in [0, \rho(b))$, with suitable CQ:

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\begin{tikzcd}[column sep=huge, row sep=large]
\underset{x \in \mathcal{C}}{\text{minimize}} \ \kappa(x) \arrow[r, "\text{addition}"] \arrow[d, leftrightarrow, "\text{gauge duality}"', "x \in [\text{cl cone}\, \partial\kappa^\circ(y)] \cap \partial\delta_{\mathcal{C}}^*(-y)"] & \underset{x \in \mathcal{C}}{\text{minimize}} \ (\kappa + \alpha\|\cdot\|_2)(x) \arrow[d, leftrightarrow, "\text{gauge duality}"', "x = (\text{Theorem } 6.2)(y)"] \\
\underset{y \in \mathcal{C}'}{\text{minimize}} \ \kappa^\circ(y) \arrow[r, "\text{max convolution}"] & \underset{y \in \mathcal{C}'}{\text{minimize}} \ (\kappa^\circ \diamond \tfrac{1}{\alpha}\|\cdot\|_2)(y)
\end{tikzcd}
$$

Here, "$x = (\text{Theorem } 6.2)(y)$" refers to (Friedlander, Macêdo, P. '18)

$$\bar{x} = \frac{\bar{r}^{-1}}{\kappa(\text{prox}_{\bar{r}\kappa}(A^*\bar{u})) + \alpha\|\text{prox}_{\bar{r}\kappa}(A^*\bar{u}))\|}\text{prox}_{\bar{r}\kappa}(A^*\bar{u})),$$

where $A^*\bar{u}$ solves the smoothed gauge dual with optimal value $\bar{r}$.

Projected polar proximal point algorithm

Let f be proper closed convex with $\inf f > 0$ and $\operatorname{Arg\,min} f \neq \emptyset$. Then

$$\min_{x} \; f(x) \;\text{ if and only if }\; \min_{\lambda=1,x} f^{\pi}(x,\lambda),$$

where

$$f^{\pi}(x,\lambda) := \begin{cases} \lambda f(\lambda^{-1}x) & \text{if } \lambda > 0, \\ f_{\infty}(x) & \text{if } \lambda = 0, \\ \infty & \text{if } \lambda < 0. \end{cases}$$

It holds that $\operatorname{epi} f^{\pi} = \operatorname{cl}\operatorname{cone}(\operatorname{epi} f \times \{1\})$.

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Definition. (Friedlander, Macêdo, P. '18)
For any $\alpha > 0$, define the projected polar envelope and projected polar proximal map of f^π by

$$\mathrm{p}_{\alpha,f}(x) := (f^\pi)_\alpha(x,1) \;\text{ and }\; \mathfrak{P}_{\alpha,f}(x) := \operatorname{pprox}_{\alpha f^\pi}(x,1).$$

Projected polar proximal point algorithm

Theorem 4. (Friedlander, Macêdo, P. '18)

(i) Suppose that $x \in \operatorname{Arg\,min} \mathsf{p}_{\alpha,f}$ and let $(\bar{x}, \bar{\lambda}) = \mathfrak{P}_{\alpha,f}(x)$. Then $\bar{\lambda} > 0$, $\bar{x} = x$ and $\bar{\lambda}^{-1}x \in \operatorname{Arg\,min} f$.

(ii) If $x \in \operatorname{Arg\,min} f$, then $[1 + \alpha f(x)]^{-1}\, x \in \operatorname{Arg\,min} \mathsf{p}_{\alpha,f}$.

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(iii) If $(x, \lambda) = \mathfrak{P}_{\alpha,f}(x)$, then $\lambda > 0$ and $\lambda^{-1} x \in \operatorname{Arg\,min} f$.

(iv) If $x \in \operatorname{Arg\,min} f$, then $\exists \lambda > 0$ so that $(\tau x, \lambda) = \mathfrak{P}_{\alpha,f}(\tau x)$, where $\tau = [1 + \alpha f(x)]^{-1}$.

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Idea: Find "fixed point" so that $(x^*, \lambda_*) = \mathfrak{P}_{\alpha,f}(x^*)$!

Projected polar proximal point algorithm

P^{4}A: Fix $\alpha > 0$ and any x^0. For each $k = 0, 1, ...,$

$$(x^{k+1}, \lambda_{k+1}) = \mathfrak{P}_{\alpha,f}(x^k).$$

Projected polar proximal point algorithm

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$$(x^{k+1}, \lambda_{k+1}) = \mathfrak{P}_{\alpha,f}(x^k).$$

Theorem 5. (Friedlander, Macêdo, P. '18)
Let $\{(x^k, \lambda_k)\}$ be generated by **P⁴A**. Then the following statements hold.

(i) $\mathrm{p}_{\alpha,f}(x^{k+1}) \leq \mathrm{p}_{\alpha,f}(x^k)$.

(ii) If $\exists\gamma > 0$ so that $(x, \lambda) \mapsto [f^\pi(x, \lambda)]^2 - (\gamma/2)\|x\|^2$ is convex, then any accumulation point (x^*, λ_*) of $\{(x^k, \lambda_k)\}$ satisfies $\lambda_* > 0$ and $\lambda_*^{-1} x^* \in \operatorname{Arg\,min} f$.

Future directions

- Efficient implementation of **P^{4}A**: proximal gradient versus proximal point?
- Polar analogue of Moreau identity: For any proper closed convex f and any x, it holds that

$$\mathrm{prox}_f(x) + \mathrm{prox}_{f^*}(x) = x.$$

  What about pprox_κ and $\mathrm{pprox}_{\kappa^\circ}$?
- It is known that f is proper closed strongly convex if and only if f^* is Lipschitz differentiable. Any analogue for polar operation?

Conclusion

- Gauge optimization framework captures many applications.
- Gauge strong duality holds under conditions similar to standard CQ in Lagrange duality theory.
- Polar envelope and polar proximal mapping appear naturally in dual smoothing and primal solution recovery.

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Thanks for coming! ☺