Frank-Wolfe type methods for nonconvex inequality-constrained problems

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(Joint work with Guoyin Li, Liaoyuan Zeng & Yongle Zhang)

Motivating applications

- Matrix completion: (Candés, Recht '09)

$$\min_{x \in \mathbb{R}^{m \times n}} \sum_{(i,j) \in \Omega} (x_{ij} - \bar{x}_{ij})^2 \text{ subject to } \Phi(x) \leq \sigma,$$

  where $\overline{x}$ comes from observation, Ω is the index set of observed entries, $\sigma > 0$, and typical choices of $\Phi : \mathbb{R}^{m \times n} \to \mathbb{R}_+$ are:

  - ⋆ $\Phi(x) = \|x\|_*$, the nuclear norm of x;
  - ⋆ $\Phi(x) = \|x\|_* - \mu \|x\|_F$, $\mu \in (0, 1)$.

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$$\min_{x\in\mathbb{R}^{n}} \ h(\bar{x} + x) \ \text{subject to} \ \|x\|_p^p \le \sigma,$$

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- Project onto the constraint sets? Alternatives?

Frank-Wolfe method

Let $\mathbb{X}$ be a finite dimensional Hilbert space. Consider

$$\min_{x \in \mathbb{X}} \; f(x) \;\; \text{subject to} \;\; x \in D,$$

where $f \in C^1(\mathbb{X})$ and D is compact convex such that for any $v \in \mathbb{X}$, a

$$u \in \operatorname*{Arg\,min}_{x \in D} \; \langle v, x \rangle$$

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- $D = \{x \in \mathbb{R}^n : \; \|x\|_p \leq \sigma\}$ for $p \in [1, \infty]$ and some $\sigma > 0$. Then u can be computed by considering the dual norm.

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- $D = \{x \in \mathbb{R}^n : \; \|x\|_p \le \sigma\}$ for $p \in [1,\infty]$ and some $\sigma > 0$. Then u can be computed by considering the dual norm.
- $D = \{x \in \mathbb{R}^{m\times n} : \; \|x\|_* \le \sigma\}$ for some $\sigma > 0$. Then $u = -\sigma r_1 s_1^T$, where r_1 and s_1 are the left and right unit singular vectors, respectively, corresponding to the largest singular value of $-v$, obtained via Lanzcos method.
  In contrast, projecting onto D requires full SVD of v.

Frank-Wolfe method cont.

Frank-Wolfe method for convex D: (Frank, Wolfe '56)

Step 1. Choose $x^0 \in D$. Pick any $c \in (0,1)$ and set $k = 0$.

Step 2. Compute $u^k \in \operatorname{Arg\,min}_{x \in D} \langle \nabla f(x^k), x \rangle$.

Step 3. If $\langle \nabla f(x^k), u^k - x^k \rangle = 0$, terminate.

Step 4. Choose $\alpha_k \in (0,1]$ using backtracking to satisfy

$$f(x^k + \alpha_k(u^k - x^k)) \le f(x^k) + c\alpha_k \langle \nabla f(x^k), u^k - x^k \rangle.$$

Step 5. Set $x^{k+1} = x^k + \alpha_k(u^k - x^k)$, $k \leftarrow k + 1$. Go to Step 2.

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- The algorithm either terminates finitely at a stationary point $x^{\bar{k}}$, or every accumulation point of $\{x^k\}$ is stationary.

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- When f is convex with Lipschitz gradient (modulus L_f), one can choose in Step 4

$$\alpha_k = \frac{2}{k+2} \quad \text{or} \quad \alpha_k = \min\left\{1, -\frac{\langle \nabla f(x^k), u^k - x^k \rangle}{L_f \|u^k - x^k\|^2}\right\}.$$

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- Is the termination in Step 3 correct? (Termination issue)
- The convex combination in Step 5 can make $x^{k+1} \notin D$! (Feas. issue)

Existing work

D as subset of sphere: (Luss, Teboulle '13, Balashov, Polyak, Tremba '20)

- Arises naturally from sparse PCA.
- Assumes concavity of f, so that

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Our approach:

- Restrict to a different class of nonconvex D.
- Construct new linear oracles.
- Study optimality conditions.

Generalized LO

Consider compact sets of the form

$$D := \{x \in \mathbb{X} : \; P_1(x) - P_2(x) \leq \sigma\},$$

where $P_1 : \mathbb{X} \to \mathbb{R}$ and $P_2 : \mathbb{X} \to \mathbb{R}$ are convex, $\sigma > 0$.

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Definition: For P_1, P_2 and σ as above, $y \in D$ and $\xi \in \partial P_2(y)$, define

$$D(y, \xi) := \{x \in \mathbb{X} : \; P_1(x) - P_2(y) - \langle \xi, x - y \rangle \le \sigma\}.$$

For any $v \in \mathbb{X}$, a linear-optimization oracle for (v, y, ξ) (denoted by $\mathcal{LO}(v, y, \xi)$) computes a solution of

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Remarks:

- It holds that $y \in D(y, \xi) \subseteq D$. Thus, $\mathcal{LO}(v, y, \xi)$ is well-defined.
- For any output u of $\mathcal{LO}(v, y, \xi)$ and any $\alpha \in (0, 1)$, we have

$$\alpha y + (1 - \alpha) u \in D(y, \xi)$$

Generalized LO: Example

Matrix completion: Let $\mathbb{X} = \mathbb{R}^{m\times n}$, $P_1(x) := \|x\|_*$, $P_2(x) := \mu\|x\|_F$ for some $\mu \in (0,1)$ and $\sigma > 0$ so that $D := \{x : \ \|x\|_* - \mu\|x\|_F \le \sigma\}$.

Now, for any $v \in \mathbb{R}^{m\times n}$, $y \in D$ and $\xi \in \partial P_2(y)$, the $\mathcal{LO}(v, y, \xi)$ solves

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where $\|\xi\|_F \le \mu < 1$.

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Theorem 1. (Zeng, Zhang, Li, P. '21)

Suppose that $v \ne 0$. Let $z = \begin{bmatrix} z_1^T & z_2^T \end{bmatrix}^T$ with $z_1 \in \mathbb{R}^m$ and $z_2 \in \mathbb{R}^n$ be a generalized eigenvector of the smallest generalized eigenvalue of the matrix pencil $(\widetilde{v}, I - \widetilde{\xi})$, and satisfy $z^T(I - \widetilde{\xi})z = 1$, where

$$\widetilde{v} = \begin{bmatrix} 0 & v \\ v^T & 0 \end{bmatrix} \quad \text{and} \quad \widetilde{\xi} = \begin{bmatrix} 0 & \xi \\ \xi^T & 0 \end{bmatrix}.$$

Then $u^* = 2\sigma z_1 z_2^T$ is an output of $\mathcal{LO}(v, y, \xi)$.

Remark: Since $I - \widetilde{\xi} \succ 0$, the above z can be computed using eigifp.

CQ & Optimality conditions

Consider

$$\min_{x \in \mathbb{X}} \; f(x) \;\; \text{subject to} \;\; D := \{x \in \mathbb{X} : \; P_1(x) - P_2(x) \le \sigma\}, \qquad (\spadesuit)$$

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where

- D is compact, $P_1, P_2 : \mathbb{X} \to \mathbb{R}$ are convex, $\sigma > 0$; and
- the generalized Slater's condition holds: For any $y \in D$ and $\xi \in \partial P_2(y)$, there exists $\hat{x} \in \mathbb{X}$ such that

$$P_1(\hat{x}) - P_2(y) - \langle \xi, \hat{x} - y \rangle < \sigma.$$

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Theorem 2. (Zeng, Zhang, Li, P. '21)

Assume the generalized Slater's condition. Then TFAE:

- x^* is a stationary point of ($\spadesuit$), i.e., $\exists \lambda \ge 0$ such that
$$0 \in \nabla f(x^*) + \lambda \partial P_1(x^*) - \lambda \partial P_2(x^*).$$
- $\exists\, \xi^* \in \partial P_2(x^*)$ and $u^* \in \operatorname{Arg\,min}_{x \in D(x^*, \xi^*)} \langle \nabla f(x^*), x \rangle$ such that
$$\langle \nabla f(x^*), u^* - x^* \rangle = 0.$$

Nonconvex FW method

FW$_{\rm ncvx}$: Frank-Wolfe method for (♠)

Step 1. Choose $x^0 \in D$ and set $k = 0$.

Step 2. Pick any $\xi^k \in \partial P_2(x^k)$ and compute

$$u^k \in \operatorname*{Arg\,min}_{x \in D(x^k, \xi^k)} \langle \nabla f(x^k), x \rangle.$$

Step 3. If $\langle \nabla f(x^k), u^k - x^k \rangle = 0$, terminate.

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Theorem 3. (Zeng, Zhang, Li, P. '21)
Assume the generalized Slater's condition. Then:

- Finite termination returns a stationary point $x^{\bar{k}}$.
- Line-search loop in Step 4 terminates finitely.
- $\{x^k\} \subseteq D$ and each accumulation point is stationary.

Convergence proof idea

Define a gap function $G : D \to \mathbb{R}$ by

$$G(x) = \inf_{\xi \in \partial P_2(x)} \max_{y \in D(x,\xi)} \langle \nabla f(x), x - y \rangle.$$

Theorem 4. (Zeng, Zhang, Li, P. '21)
Assume the generalized Slater's condition. Then $G(x) \geq 0$ for all $x \in D$. Moreover, if $\{w^k\} \subseteq D$ is such that

$$G(w^k) \to 0 \ \text{ and } \ w^k \to x^*$$

for some x^*, then $x^* \in D$ and is a stationary point of (♠).

Convergence of $\mathrm{FW}_{\mathrm{ncvx}}$: Let $\{x^k\}$ be generated by $\mathrm{FW}_{\mathrm{ncvx}}$.

- Direct computation shows that $0 \leq G(x^k) \leq -\langle \nabla f(x^k), u^k - x^k \rangle$.
- Backtracking + Armijo rule give $\langle \nabla f(x^k), u^k - x^k \rangle \to 0$.
- Convergence follows from these and **Theorem 4**.

Numerical experiments

- Matrix completion:

$$\min_{x \in \mathbb{R}^{m\times n}} \sum_{(i,j)\in\Omega} (x_{ij} - \bar{x}_{ij})^2 \ \text{ subject to } \ \|x\|_* - 0.5\|x\|_F \le \sigma,$$

where

- ⋆ Ω collects the indices of observed entries;
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  - ⋆ **KEY**: Since

$$x^{k+1} = (1 - \alpha_k)x^k + \alpha_k u^k,$$

    one can obtain $(R^{k+1}, \Sigma^{k+1}, T^{k+1})$ using SVD rank-one update.

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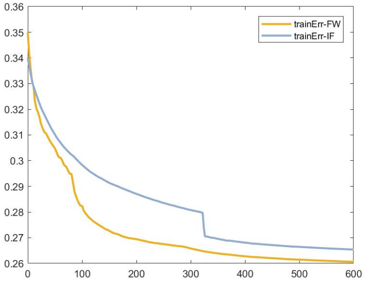
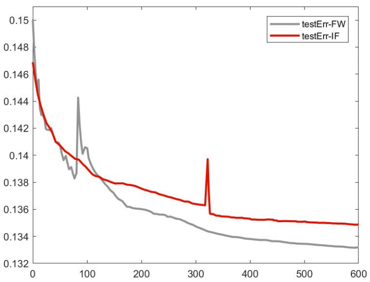
  In contrast, GP will need to form x^k, and perform full SVD (for projection).

Numerical experiments cont.

- MovieLens10M: $n = 10677$ movie ratings from $m = 69878$ users.
- Randomly choose 70% as training dataset (i.e.,Ω). Training and testing errors as the algorithm progresses are shown below.
- For simplicity, we used the same Ω and the same σ (determined via CV on nuc. norm model) as in (Freund, Grigas, Mazumder '17).





Matlab 2017b on a 64-bit PC with an Intel(R) Core(TM) i5-7200 CPU (2.50GHz) and 8GB of RAM

Conclusion and future work

Conclusion:

- Extended FW method for special nonconvex sets: Level set of DC functions satisfying some regularity conditions.
- Introduced generalized LO: Efficient implementation for applications such as matrix completion.
- Established subsequential convergence.

Reference:

- L. Zeng, Y. Zhang, G. Li and T. K. Pong.
  Frank-Wolfe type methods for nonconvex inequality-constrained problems.
  Preprint. Available at https://arxiv.org/abs/2112.14404.

Thanks for coming!