ESDP Relaxation of Sensor Network Localization
Analysis, Extensions and Algorithm

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In memory of Paul Tseng
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(Joint work with Paul Tseng)

Talk Outline

- Sensor network localization.
- ESDP relaxation: properties and accuracy certificate.
- A robust version of ESDP for the noisy case.
- LPCGD algorithm and numerical simulations.
- Conclusion and extensions.

Sensor Network Localization

Basic Problem:

- n pts $\underbrace{x_1, \ldots, x_m}_{\text{sensors}}, \underbrace{x_{m+1}, \ldots, x_n}_{\text{anchors}}$ in $\mathbb{R}^2$.

- Know last $n - m$ pts ('anchors') $x_{m+1}, \ldots, x_n$ and Eucl. dist. estimate for some pairs of 'neighboring' pts (i.e. within 'radio range')

$$d_{ij} \geq 0 \quad \forall (i, j) \in \mathcal{A},$$

  with $\mathcal{A} \subseteq \{(i, j) : 1 \leq i < j \leq n\}$.

- Estimate the first m pts ('sensors') $x_1, \ldots, x_m$.

Optimization Problem Formulation

$$v_{\mathrm{p}} := \min_{x_1,\ldots,x_m} \sum_{(i,j)\in\mathcal{A}} \left| \|x_i - x_j\|^2 - d_{ij}^2 \right|.$$

Optimization Problem Formulation

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- Objective function is nonconvex. m can be large ($m > 1000$).
- Problem is NP-hard (reduction from PARTITION).
- Use a convex relaxation.

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- Problem is NP-hard (reduction from PARTITION).
- Use a convex relaxation.
- Find accuracy certificate.
- Develop fast, distributed algorithm.

SDP Relaxation

Let $X = [x_1 \cdots x_m]$. $\qquad Y = X^T X \Longleftrightarrow Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \succeq 0, \ \mathrm{rank}\, Z = 2$.

SDP Relaxation

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SDP relaxation (Biswas, Ye '03):

$$\begin{aligned} \upsilon_{\rm sdp} := \quad & \min_Z \sum_{(i,j)\in\mathcal{A}^a} \left| y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2 \right| \\ & \quad + \sum_{(i,j)\in\mathcal{A}^s} \left| y_{ii} - 2y_{ij} + y_{jj} - d_{ij}^2 \right| \\ & \text{s.t.} \quad Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \succeq 0. \end{aligned}$$

Adding nonconvex constraint $\mathrm{rank}\, Z = 2$ yields the original problem.

ESDP Relaxation

ESDP relaxation (Wang, Zheng, Boyd, Ye '07):

$$
\begin{aligned}
v_{\text{esdp}} := \quad \min_{Z} & \sum_{(i,j)\in\mathcal{A}^a} \left| y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2 \right| \\
& + \sum_{(i,j)\in\mathcal{A}^s} \left| y_{ii} - 2y_{ij} + y_{jj} - d_{ij}^2 \right| \\
\text{s.t.} \quad & Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \\
& \begin{bmatrix} y_{ii} & y_{ij} & x_i^T \\ y_{ij} & y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix} \succeq 0 \quad \forall (i,j) \in \mathcal{A}^s \\
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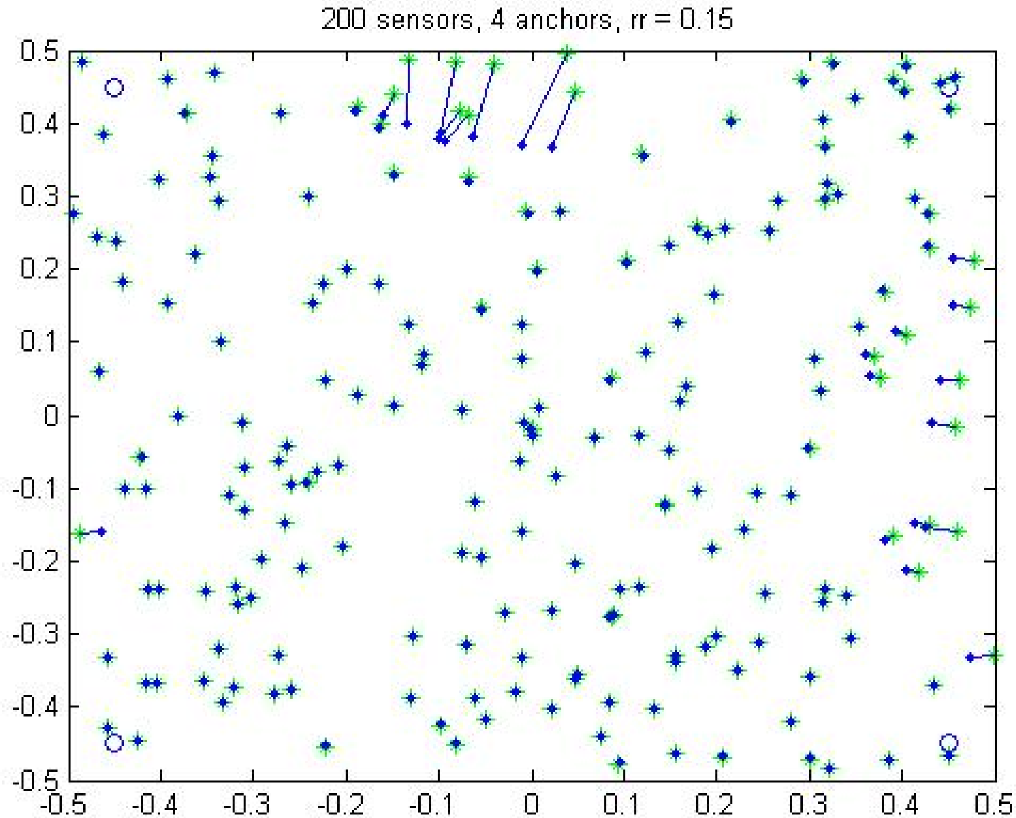
In simulations, ESDP is nearly as strong as SDP relaxation, and solvable much faster by IP method.

An Example

200 sensors, 4 anchors, $radiorange = 0.15$, solving ESDP by SeDuMi. True sensor positions denoted by green asterisk, computed sensor positions denoted by blue dots, anchors denoted by circles. True positions and computed positions joined by blue straight lines.



Properties of ESDP

Assume

$$d_{ij} = \|x_i^{\text{true}} - x_j^{\text{true}}\| \;\; \forall\, (i,j) \in \mathcal{A}.$$

"noiseless case"

($x_j^{\text{true}} = x_j \;\forall\, j > m$)

Fact 0:

$$Z^{\text{true}} := [X^{\text{true}} \;\; I]^T [X^{\text{true}} \;\; I] = \begin{bmatrix} (X^{\text{true}})^T X^{\text{true}} & (X^{\text{true}})^T \\ X^{\text{true}} & I \end{bmatrix}$$

is a soln of ESDP (i.e., $Z^{\text{true}} \in \mathrm{Sol}(\mathrm{ESDP})$).

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is a soln of ESDP (i.e., $Z^{\text{true}} \in \mathrm{Sol(ESDP)}$).

x_i is invariant over $\mathrm{Sol(ESDP)} \Rightarrow x_i = x_i^{\text{true}}$.

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Fact 2 (P, Tseng '10): For each i,

$$x_i \text{ is invariant over } \mathrm{Sol}(\mathrm{ESDP}) \quad \Longrightarrow \quad \mathrm{tr}_i[Z] = 0 \ \forall Z \in \mathrm{Sol}(\mathrm{ESDP}).$$

In practice, there are measurement noises:

$$d_{ij}^2 = \|x_i^{\text{true}} - x_j^{\text{true}}\|^2 + \delta_{ij} \quad \forall (i,j) \in \mathcal{A}.$$

When $\delta := (\delta_{ij})_{(i,j)\in\mathcal{A}} \approx 0$, does $\mathrm{tr}_i[Z] = 0$ (with $Z \in \mathrm{ri}(\mathrm{Sol}(\mathrm{ESDP}))$) imply x_i is near the true position of sensor i?

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No.

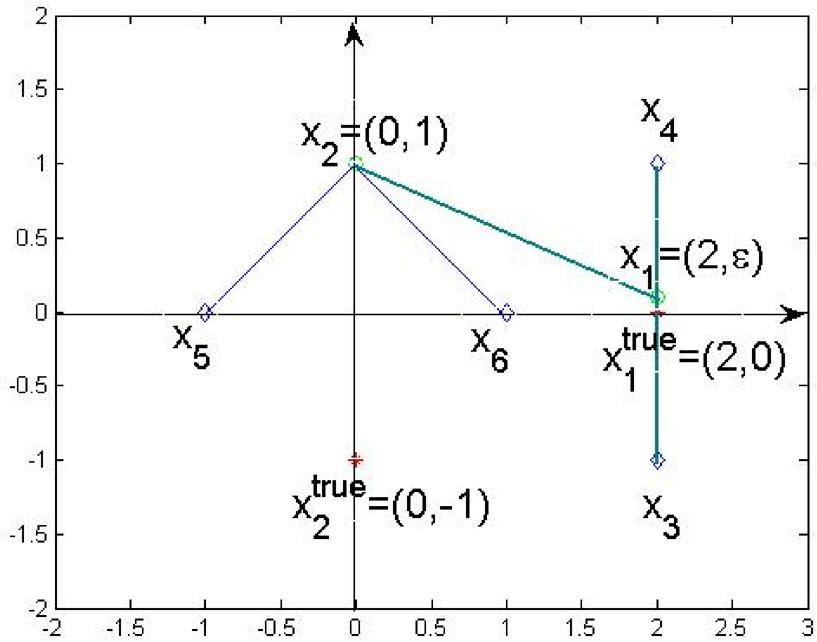
Fact 3 (P, Tseng '10): For $|\delta_{ij}| \approx 0$,

$$\mathrm{tr}_i[Z] = 0 \quad \text{for some } Z \in \mathrm{ri}(\mathrm{Sol}(\mathrm{ESDP})) \not\Longrightarrow \quad \|x_i - x_i^{\mathrm{true}}\| \approx 0.$$

Proof is by counterexample.

An example of sensitivity of ESDP solns to measurement noise:

Input distance data: $\epsilon > 0$
$d_{12} = \sqrt{4 + (1-\epsilon)^2}, d_{13} = 1 + \epsilon, d_{14} = 1 - \epsilon, d_{25} = d_{26} = \sqrt{2}$; $m = 2$, $n = 6$.



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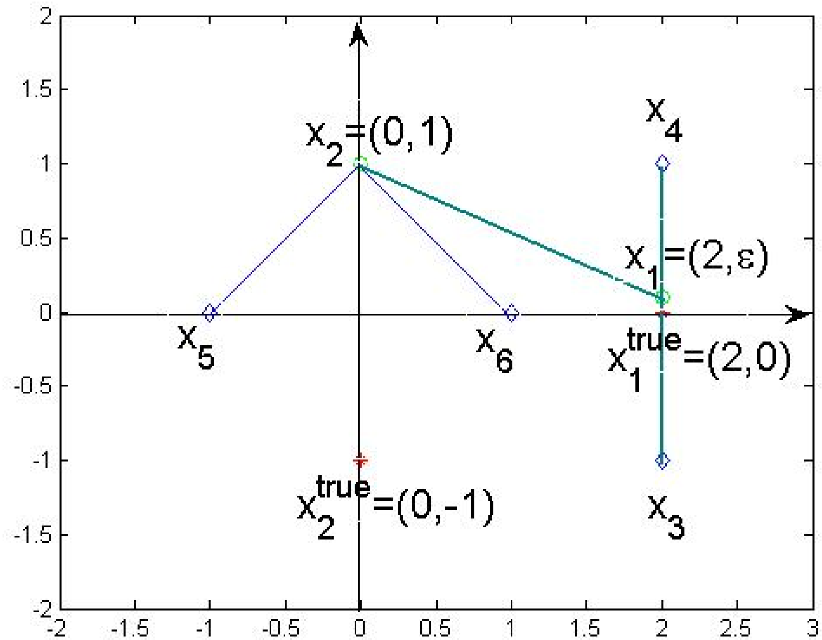

Thus, even when $Z \in \mathrm{Sol(ESDP)}$ is unique, $\mathrm{tr}_i[Z] = 0$ fails to certify accuracy of x_i in the noisy case!

Robust ESDP

For each $(i,j) \in \mathcal{A}$, fix $\rho_{ij} > |\delta_{ij}|$ $(\rho > |\delta|)$.

$\mathrm{Sol}(\rho\mathrm{ESDP})$ denotes the set of $Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix}$ satisfying

$$
\begin{aligned}
\begin{bmatrix} y_{ii} & y_{ij} & x_i^T \\ y_{ij} & y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix} &\succeq 0 \quad \forall (i,j) \in \mathcal{A}^s \\
\begin{bmatrix} y_{ii} & x_i^T \\ x_i & I \end{bmatrix} &\succeq 0 \quad \forall i = 1, \cdots, m \\
|y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2| &\leq \rho_{ij} \quad \forall (i,j) \in \mathcal{A}^a \\
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|y_{ii} - 2y_{ij} + y_{jj} - d_{ij}^2| &\le \rho_{ij} \quad \forall (i,j) \in \mathcal{A}^s.
\end{aligned}$$

Note: $Z^{\mathrm{true}} = [X^{\mathrm{true}} \;\; I]^T [X^{\mathrm{true}} \;\; I] \in \mathrm{Sol}(\rho\mathrm{ESDP})$.

Let

$$Z^\rho := \underset{Z\in \mathrm{Sol}(\rho\mathrm{ESDP})}{\arg\min} \quad - \sum_{(i,j)\in\mathcal{A}^s} \ln\det\left(\begin{bmatrix} y_{ii} & y_{ij} & x_i^T \\ y_{ij} & y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix}\right) - \sum_{i\le m} \ln\det\left(\begin{bmatrix} y_{ii} & x_i^T \\ x_i & I \end{bmatrix}\right).$$

Fact 4 (P, Tseng '10): $\exists \eta > 0,\ \bar\rho > 0$ such that for each i,

$$\mathrm{tr}_i(Z^\rho) < \eta, \text{ for some } |\delta| < \rho < \bar\rho e \implies \lim_{|\delta|<\rho\to 0} x_i^\rho = x_i^{\mathrm{true}}.$$

$$\mathrm{tr}_i(Z^\rho) \ge 0.1\eta, \text{ for some } |\delta| < \rho < \bar\rho e \implies x_i \text{ not invar. when } \delta = 0.$$

Moreover,

$$\|x_i^\rho - x_i^{\mathrm{true}}\| \le \sqrt{2|\mathcal{A}^s| + m}\,(\mathrm{tr}_i[Z^\rho])^{\frac{1}{2}} \quad |\delta| < \rho.$$

LPCGD Algorithm

Let $h_a(t) := \frac{1}{2}(t-a)_+^2 + \frac{1}{2}(-t-a)_+^2$ $(|t| \leq a \iff h_a(t) = 0)$.

$$
\begin{aligned}
f_\mu(Z) := & \sum_{(i,j)\in\mathcal{A}^a} h_{\rho_{ij}}(y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2) \\
& + \sum_{(i,j)\in\mathcal{A}^s} h_{\rho_{ij}}(y_{ii} - 2y_{ij} + y_{jj} - d_{ij}^2) \\
& - \mu \sum_{(i,j)\in\mathcal{A}^s} \ln\det\left(\begin{bmatrix} y_{ii} & y_{ij} & x_i^T \\ y_{ij} & y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix}\right) \\
& - \mu \sum_{i\leq m} \ln\det\left(\begin{bmatrix} y_{ii} & x_i^T \\ x_i & I \end{bmatrix}\right).
\end{aligned}
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- f_μ is partially separable, strictly convex & diff. on its domain.
- For each $\rho > |\delta|$, $\operatorname{argmin} f_\mu \to Z^\rho$ as $\mu \to 0$.
- In the noiseless case ($\delta = 0$), if $\rho > 0$ is small, then $Z^\rho \approx \mathrm{some}\, Z \in \mathrm{ri}\,(\mathrm{Sol}\,(\mathrm{ESDP}))$.

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Use Block Coordinate Gradient Descent (Tseng, Yun '06)

LPCGD Algorithm:

Given $Z \in \mathrm{dom}\, f_\mu$, compute gradient $\nabla_{Z_i} f_\mu$ of f_μ w.r.t. $Z_i := \{x_i, y_{ii}, y_{ij} : (i,j) \in \mathcal{A}\}$ for each i.

- If $\|\nabla_{Z_i} f_\mu\| \geq \max\{\mu, 1e-7\}$ for some i, update Z_i by moving along the Newton direction $-\left(\partial^2_{Z_i Z_i} f_\mu\right)^{-1} \nabla_{Z_i} f_\mu$ with Armijo stepsize rule.
- Decrease μ when $\|\nabla_{Z_i} f_\mu\| < \max\{\mu, 1e-7\}$ for all i.

$\mu_{\text{initial}} = 1e-1$, $\mu_{\text{final}} = 1e-14$. Decrease μ by a factor of 10.

Coded in Fortran. Obtain Newton direc. using sparse Cholesky.
Computation easily distributes.

Simulation Results

- Compare ρESDP, as solved by LPCGD, and ESDP, as solved by Sedumi (with the interface to Sedumi coded by Wang et al.).

- Uniformly generate $\{x_1^{\text{true}}, ..., x_n^{\text{true}}\}$ in $[-.5, .5]^2$, $m = .9n$. Two pts are neighbors iff $\text{dist} < rr$. Set

$$d_{ij} = \|x_i^{\text{true}} - x_j^{\text{true}}\| \cdot |1 + \sigma\epsilon_{ij}|,$$

  where $\epsilon_{ij} \sim N(0, 1)$.

- Sensor i is judged as "accurately positioned" if

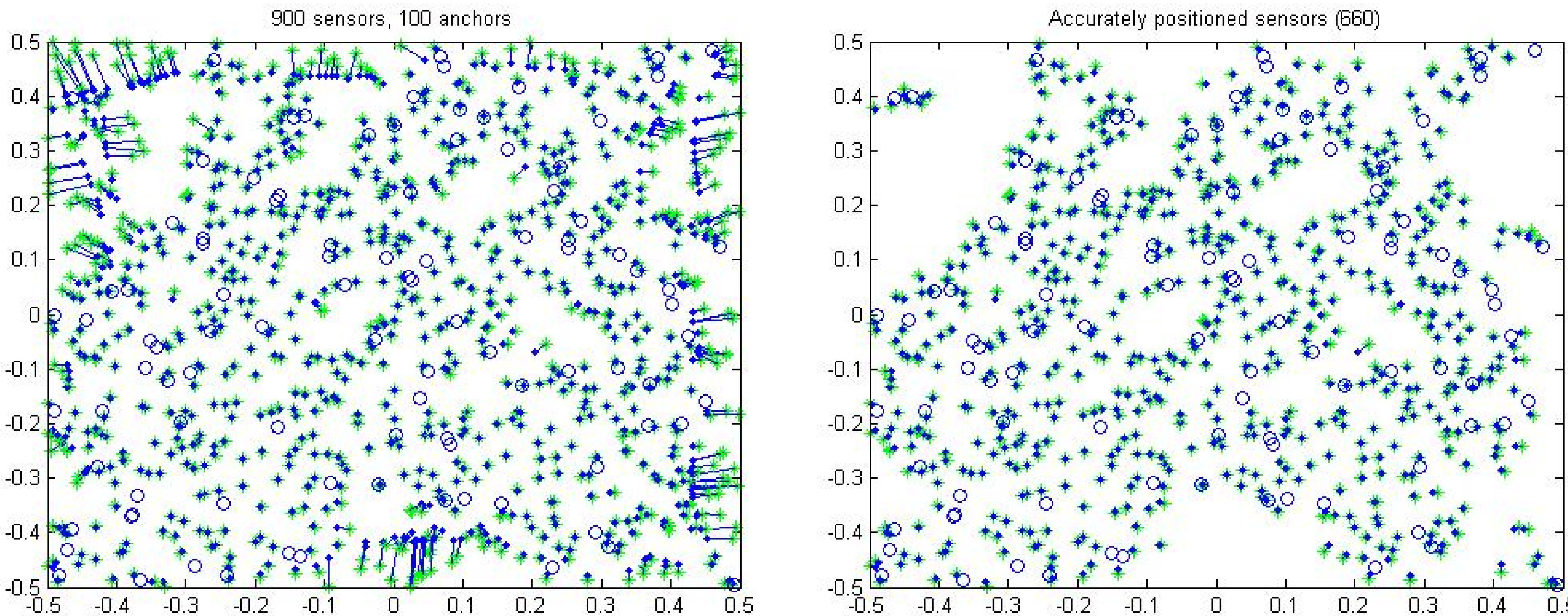
$$\text{tr}_i[Z^{\text{found}}] < (.01 + 30\sigma)\bar{d}_i^2.$$

Simulation Results

				$\rho\mathbf{ESDP}_{\text{LPCGD}}$	$\mathbf{ESDP}_{\text{Sedumi}}$
n	m	σ	rr	**cpu**/m_{ap}/err_{ap}	**cpu(cpus)**/m_{ap}/err_{ap}
1000	900	0	.06	4/666/1.6e-3	45(30)/676/2.1e-3
1000	900	.01	.06	3/660/2.2e-2	36(20)/737/4.3e-2
2000	1800	0	.06	14/1762/3.6e-4	206(101)/1759/4.9e-4
2000	1800	.01	.06	11/1699/1.4e-2	182(79)/1750/2.2e-2
10000	9000	0	.02	33/7845/2.3e-3	3148(424)/6472/2.5e-3
10000	9000	.01	.02	27/8334/9.9e-3	3130(403)/8600/8.7e-3

- cpu(sec) times are on an Dell POWEREDGE 1950 with Matlab Version 7.8. ESDP is solved by Sedumi; cpus:$=$ time taken in running Sedumi.
- Take $\rho_{ij} = d_{ij}^2 \cdot ((1 - 2\tilde{\sigma})^{-2} - 1)$; $\tilde{\sigma} = \max\{\sigma, 1e - 6\}$.
- m_{ap} := # accurately positioned sensors.
  $err_{\text{ap}} := \max_{i \text{ accurate. pos.}} \|x_i^{\text{found}} - x_i^{\text{true}}\|$.

900 sensors, 100 anchors, $rr = 0.06$, $\sigma = 0.01$, solving ρESDP by LPCGD. x_i^{true} denoted by green asterisks, x_i^{LPCGD} denoted by blue dots, anchors denoted by circles. x_i^{true} and x_i^{LPCGD} joined by blue straight line.



Solution Refinement

Update $x_i \leftarrow x_i - \alpha \nabla_{x_i} \hat{f}(X), \forall i$, where $\hat{f}(X) := \sum_{(i,j)\in\mathcal{A}} (\|x_i - x_j\| - d_{ij})^2$.

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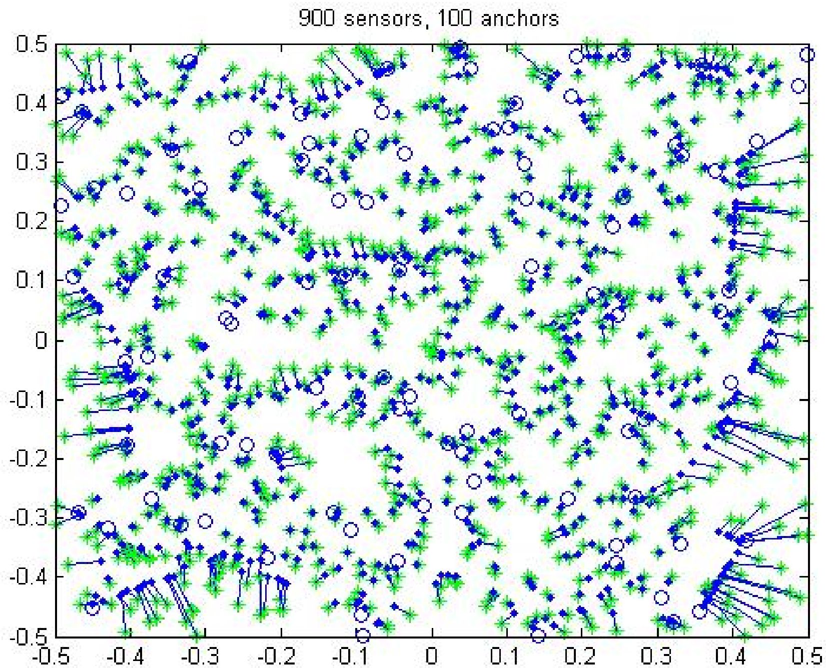

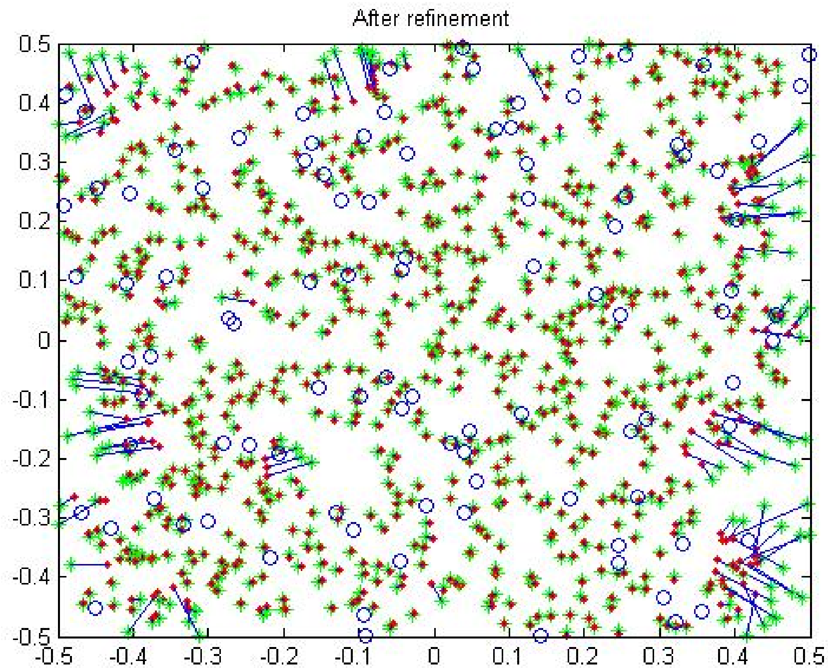

Conclusion & Extension

- ESDP is sensitive to dist. measurement noise. Lack soln accuracy certificate.

- ρESDP has soln accuracy certificate when $\rho > |\delta|$. Can $\rho > |\delta|$ be relaxed?
  seems not.

- ESDP/ρESDP solns can be refined by performing gradient descent.
  heuristics.

- Extension to sparse SOS relaxation by Nie '06 (Gouveia, P '10).

- Extension to handle lower bound on distances, i.e., $\|x_i - x_j\| > rr$ for $(i, j) \notin \mathcal{A}$ (P '10).

Thanks for coming! $\ddot{\angle}$