ESDP Relaxation of Sensor Network Localization

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(Ongoing work with Paul Tseng)

Talk Outline

- Sensor network localization
- SDP and ESDP relaxations: formulations and examples
- Properties of ESDP relaxations
- A robust version of ESDP for the noisy case
- Conclusion & Ongoing work

Sensor Network Localization

- n pts in $\mathbb{R}^2$.

- Know last $n-m$ pts ('anchors') $x_{m+1}, ..., x_n$ and Eucl. dist. estimate for some pairs of 'neighboring' pts (i.e. within 'radio range')

$$d_{ij} \geq 0 \quad \forall (i,j) \in \mathcal{A}$$

  with $\mathcal{A} \subseteq \{(i,j) : 1 \leq i < j \leq n\}$.

- Estimate first m pts ('sensors').

Easy cases:

- When there are lots of anchors.
- When sensors neighbor lots of anchors.

Hard cases:

- When m is large, $n - m$ is small, and sensors have few neighbors.
- Dist. measurement can have noise.

Optimization Problem Formulation

$$v_{\text{opt}} := \min_{x_1,\ldots,x_m} \sum_{(i,j)\in\mathcal{A}} \left| \|x_i - x_j\|^2 - d_{ij}^2 \right|$$

- Objective function is nonconvex. m can be large ($m > 1000$).
- Problem is NP-hard (reduction from PARTITION).
- Use a convex relaxation.

SDP Relaxation

Let $X := [x_1 \ \cdots \ x_m]$.

SDP relaxation (Biswas,Ye '03):

$$\begin{aligned} v_{\text{sdp}} := \quad & \min_{Z} \sum_{(i,j)\in\mathcal{A},\, j>m} \left| Y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2 \right| \\ & + \sum_{(i,j)\in\mathcal{A},\, j\le m} \left| Y_{ii} - 2Y_{ij} + Y_{jj} - d_{ij}^2 \right| \\ & \text{s.t.} \quad Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \succeq 0 \end{aligned}$$

Adding the nonconvex constraint $\text{rank} Z = 2$ yields original problem. Expensive to solve for m large.

Relax $Z \succeq 0$ further.

ESDP Relaxation

ESDP relaxation (Wang, Zheng, Boyd, Ye '07):

$$
\begin{aligned}
v_{\mathrm{esdp}} := \quad & \min_{Z} \sum_{(i,j)\in\mathcal{A}, j>m} \left| Y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2 \right| \\
& \quad + \sum_{(i,j)\in\mathcal{A}, j\le m} \left| Y_{ii} - 2Y_{ij} + Y_{jj} - d_{ij}^2 \right| \\
& \text{s.t.} \quad Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \\
& \begin{bmatrix} Y_{ii} & Y_{ij} & x_i^T \\ Y_{ij} & Y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix} \succeq 0 \quad \forall (i,j) \in \mathcal{A}, j \le m \\
& \begin{bmatrix} Y_{ii} & x_i^T \\ x_i & I \end{bmatrix} \succeq 0 \quad \forall i = 1, \cdots, m
\end{aligned}
$$

In simulation, ESDP is nearly as strong as SDP, and solvable much faster by IP method.

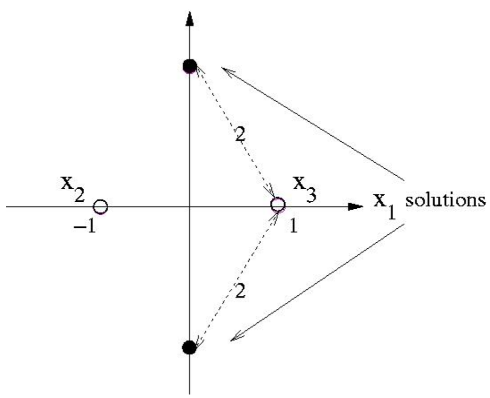
ESDP seems more amenable to solution by distributed method (compared to SDP), which is important for practical implementation.

An Example

$n = 3$, $m = 1$, $d_{12} = d_{13} = 2$

Problem:

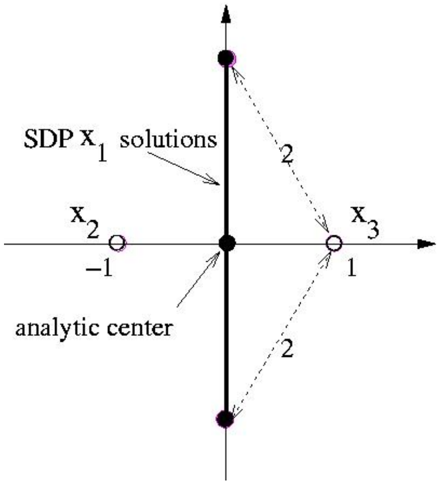
$$0 = \min_{x_1 \in \Re^2} \quad |\|x_1 - (1,0)\|^2 - 4| + |\|x_1 - (-1,0)\|^2 - 4|$$



ESDP Relaxation:

$$0 = \min_{\substack{x_1=(x_{11},x_{12})\in\Re^2 \\ Y_{11}\in\Re}} |Y_{11} - 2x_{11} - 3| + |Y_{11} + 2x_{11} - 3|$$

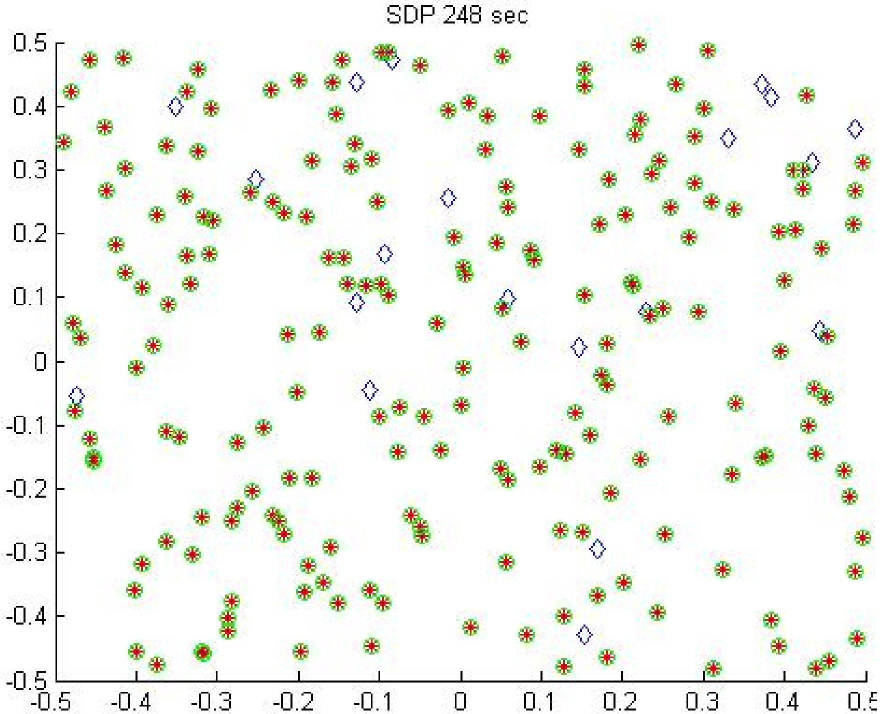
$$\text{s.t.} \quad \begin{bmatrix} Y_{11} & x_{11} & x_{12} \\ x_{11} & 1 & 0 \\ x_{12} & 0 & 1 \end{bmatrix} \succeq 0$$

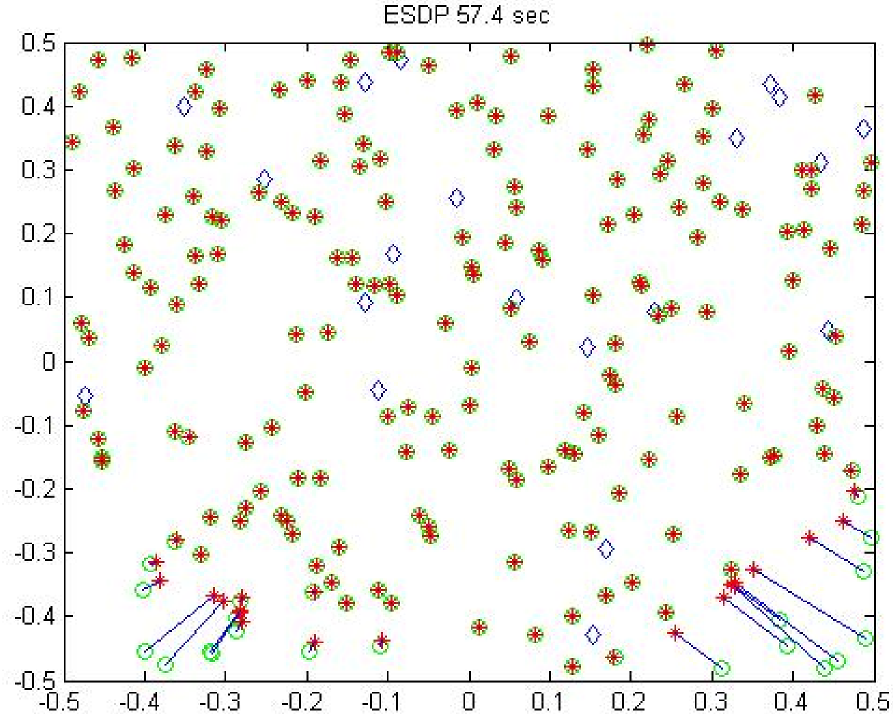


If solve ESDP by IP method, then likely get analy. center.

SDP & ESDP Relaxation: a larger example

- $n = 220$, $m = 200$. Anchors ("$\diamondsuit$") and sensors ("$\circ$") uniformly distributed in $[-.5, .5]^2$.
- $(i, j) \in \mathcal{A}$ whenever $\|x_i^{\text{true}} - x_j^{\text{true}}\| \leq 0.15$
- The soln found by SeDuMi 1.05 (implementation of a (primal-dual) IP method in C with a Matlab interface) is shown ("$*$") joined to its true position ("$\circ$") by a line.
- Run on CPU Intel Celeron 1.6 GHz, 448Mb RAM.


SDP 248 sec
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.5 -0.4 -0.3 -0.2 -0.1 0 0.1 0.2 0.3 0.4 0.5


ESDP 57.4 sec
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.5 -0.4 -0.3 -0.2 -0.1 0 0.1 0.2 0.3 0.4 0.5

ESDP: Testing Solution Accuracy

Aim: Determine which sensors are accurately positioned by ESDP.

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Two cases:

- $\exists x_1, \cdots, x_m$ such that $\|x_i - x_j\| = d_{ij}$ for $(i,j) \in \mathcal{A}$ (Noiseless);
- $d_{ij}^2 = ||x_i^{\mathrm{true}} - x_j^{\mathrm{true}}||^2 + \delta_{ij}$ (Noisy).

Defn: x_i is *invariant* over $\mathrm{Sol}\,(\mathrm{ESDP})$ if $\forall Z \in \mathrm{Sol}\,(\mathrm{ESDP})$, its x_i entry is the same.

ESDP: Noiseless Case

- $v_{\mathrm{esdp}} = 0$.
- $\mathrm{Sol}\,(P) \subseteq \left\{ X : \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \in \mathrm{Sol}\,(\mathrm{ESDP}) \right\}$
- Observation: If x_i is invariant over $\mathrm{Sol}\,(\mathrm{ESDP})$, then x_i is invariant over $\mathrm{Sol}\,(P)$ and hence $x_i = x_i^{\mathrm{true}}$.

ESDP: Noiseless Case

- Consider the individual trace for each sensor location i:
  $\mathrm{tr}_i[Z] := Y_{ii} - \|x_i\|^2, \quad i = 1, ..., m.$ (Biswas, Ye '03)

Note:

- $\mathrm{tr}_i[Z] \geq 0$ for all Z feasible for ESDP;
- $\mathrm{tr}_i[Z] = 0$ for some $Z \in \mathrm{ri}\,(\mathrm{Sol}\,(\mathrm{ESDP})) \Leftrightarrow \mathrm{tr}_i[Z] = 0$ for all $Z \in \mathrm{Sol}\,(\mathrm{ESDP})$.

ESDP: Noiseless Case

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- (Wang et al '07) If $\mathrm{tr}_i(Z) = 0$ for some $Z \in \mathrm{ri}\,(\mathrm{Sol}\,(\mathrm{ESDP}))$, then x_i is invariant.

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- (P, Tseng '08) The converse holds: if x_i is invariant, then $\mathrm{tr}_i(Z) = 0$ for all $Z \in \mathrm{Sol}\,(\mathrm{ESDP})$.

Proof sketch for the converse implication:

1. For $(i,j) \in \mathcal{A}$, $j > m$, if x_i is invariant over $\mathrm{Sol(ESDP)}$, then $\mathrm{tr}_i(Z) = 0$ for all $Z \in \mathrm{Sol(ESDP)}$.

Why: $v_{\mathrm{opt}} = 0$ and x_i invariant over $\mathrm{Sol(ESDP)}$ imply, for any $Z \in \mathrm{Sol(ESDP)}$,

$$Y_{ii} - 2x_j^T x_i + \|x_j\|^2 = d_{ij}^2, \qquad \|x_i - x_j\|^2 = d_{ij}^2$$

So $$\mathrm{tr}_i(Z) = Y_{ii} - \|x_i\|^2 = d_{ij}^2 - \|x_i - x_j\|^2 = 0.$$

2. For $(i,j) \in \mathcal{A}$, $j \le m$, if x_i, x_j are invariant over $\mathrm{Sol(ESDP)}$, then $\mathrm{tr}_i(Z) = \mathrm{tr}_j(Z)$ for all $Z \in \mathrm{Sol(ESDP)}$.

Why? $v_{\mathrm{opt}} = 0$ and x_i, x_j invariant over $\mathrm{Sol(ESDP)}$ imply, for any $Z \in \mathrm{Sol(ESDP)}$,

$$Y_{ii} - 2Y_{ij} + Y_{jj} = d_{ij}^2, \qquad \|x_i - x_j\|^2 = d_{ij}^2$$

So $Y_{ij} - x_i^T x_j = \frac{1}{2}(\mathrm{tr}_i(Z) + \mathrm{tr}_j(Z))$.

Then

$$\begin{bmatrix} Y_{ii} & Y_{ij} & x_i^T \\ Y_{ij} & Y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix} = \mathrm{tr}_i(Z) \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \mathrm{tr}_j(Z) \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} x_i^T \\ x_j^T \\ I \end{bmatrix} [\, x_i \quad x_j \quad I \,]$$

This is psd, which implies ...that $\mathrm{tr}_i(Z) = \mathrm{tr}_j(Z)$.

Noisy Case

Suppose dist. measurement has errors:

$$d_{ij}^2 = {d_{ij}^{\mathrm{true}}}^2 + \delta_{ij},$$

where $\delta_{ij} \in \mathbb{R}$ and $d_{ij}^{\mathrm{true}} := \|x_i^{\mathrm{true}} - x_j^{\mathrm{true}}\|$.

Does $\mathrm{tr}_i[Z] = 0$ (with $Z \in \mathrm{ri}(\mathrm{Sol}(\mathrm{ESDP}))$) imply x_i is near the true position of sensor i?

Noisy Case

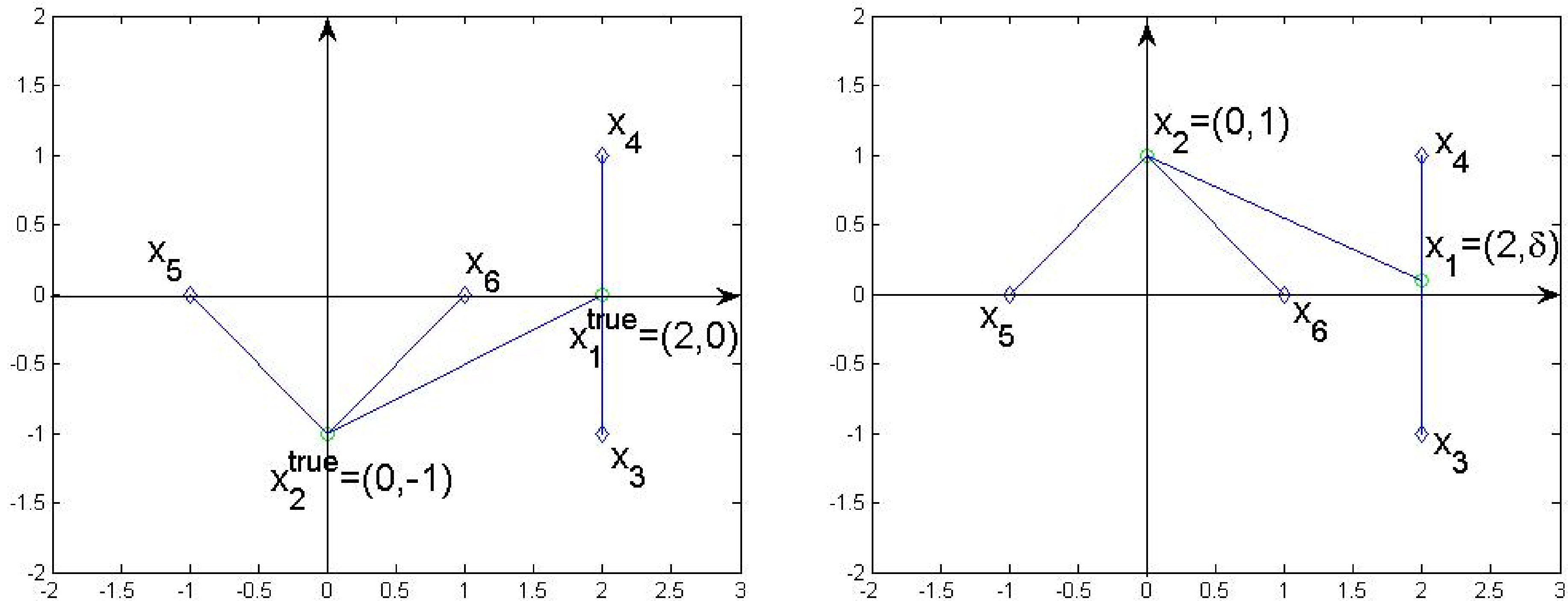
Unfortunately, no.

- (P, Tseng '08): For $|\delta_{ij}| \approx 0$,

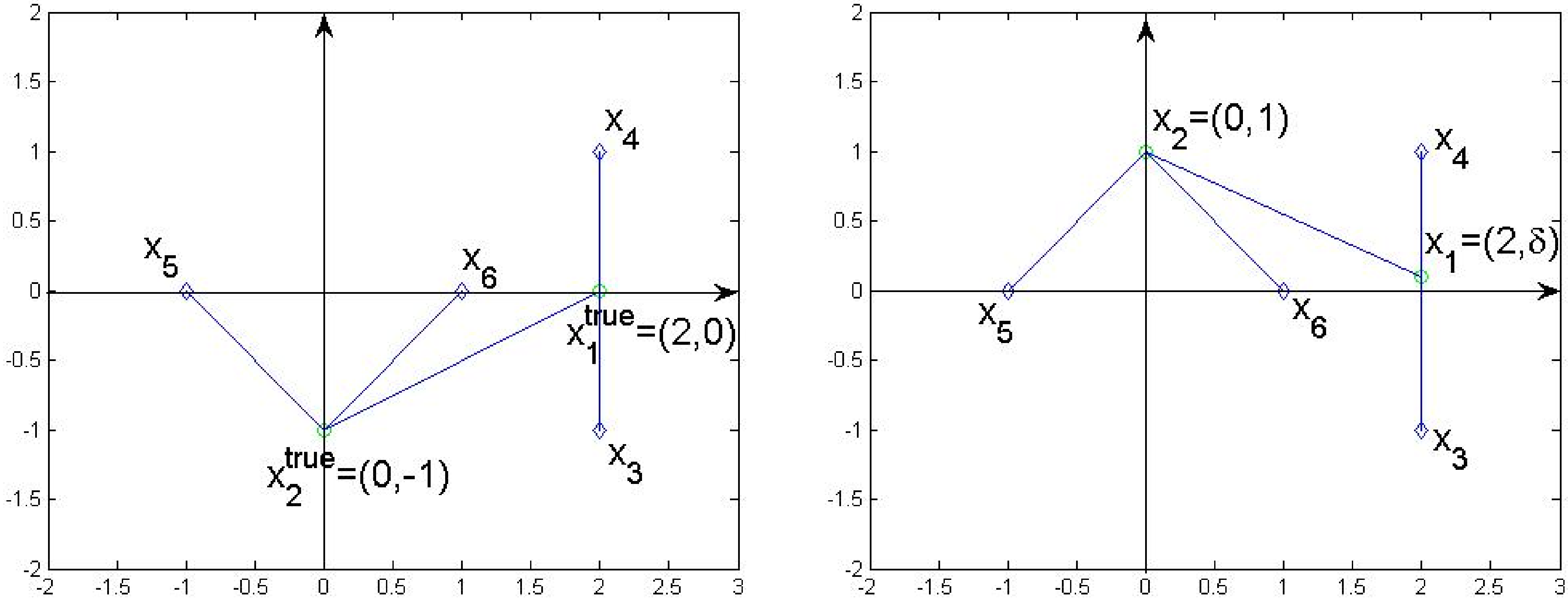
$$\mathrm{tr}_i[Z] = 0 \quad \text{for some } Z \in \mathrm{ri}(\mathrm{Sol}(\mathrm{ESDP})) \not\Longrightarrow \quad \|x_i - x_i^{\mathrm{true}}\| \approx 0.$$

Proof is by counter-example.

An example of sensitivity of ESDP solns to measurement noise:



An example of sensitivity of ESDP solns to measurement noise:



Thus, even when $Z \in \mathrm{Sol}(\mathrm{ESDP})$ is unique, $\mathrm{tr}_i[Z] = 0$ certifies accuracy of x_i only in the noiseless case! (Still true when changed to SDP)

Robust ESDP

Fix $\rho > \delta_{\max} := \max_{(i,j)\in\mathcal{A}} |\delta_{ij}|$.

$\mathrm{Sol}(\rho\mathrm{ESDP})$ denotes the set of $Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix}$ satisfying

$$
\begin{aligned}
\begin{bmatrix} Y_{ii} & Y_{ij} & x_i^T \\ Y_{ij} & Y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix} &\succeq 0 \quad \forall (i,j) \in \mathcal{A}, j \le m \\
\begin{bmatrix} Y_{ii} & x_i^T \\ x_i & I \end{bmatrix} &\succeq 0 \qquad \forall i = 1, \cdots, m \\
|Y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2| &\le \rho \quad \forall (i,j) \in \mathcal{A}, j > m \\
|Y_{ii} - 2Y_{ij} + Y_{jj} - d_{ij}^2| &\le \rho \quad \forall (i,j) \in \mathcal{A}, j \le m
\end{aligned}
$$

Note: $\begin{bmatrix} X^{\text{true}} & I \end{bmatrix}^T \begin{bmatrix} X^{\text{true}} & I \end{bmatrix} \in \mathrm{Sol}(\rho\mathrm{ESDP})$.

Let

$$
\begin{aligned}
Z^\rho := \operatorname*{arg\,min}_{Z \in \mathrm{Sol}(\rho\mathrm{ESDP})} \quad & - \sum_{(i,j)\in\mathcal{A},\, j\le m} \ln\det\left(\begin{bmatrix} Y_{ii} & Y_{ij} & x_i^T \\ Y_{ij} & Y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix}\right) \\
& - \sum_{i\le m} \ln\det\left(\begin{bmatrix} Y_{ii} & x_i^T \\ x_i & I \end{bmatrix}\right)
\end{aligned}
$$

Note: $\begin{bmatrix} X^{\text{true}} & I \end{bmatrix}^T \begin{bmatrix} X^{\text{true}} & I \end{bmatrix} \in \text{Sol}(\rho\text{ESDP})$.

Let

$$
\begin{aligned}
Z^\rho := \underset{Z \in \text{Sol}(\rho\text{ESDP})}{\arg\min} \quad & - \sum_{(i,j)\in\mathcal{A}, j \le m} \ln\det\left(\begin{bmatrix} Y_{ii} & Y_{ij} & x_i^T \\ Y_{ij} & Y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix}\right) \\
& - \sum_{i \le m} \ln\det\left(\begin{bmatrix} Y_{ii} & x_i^T \\ x_i & I \end{bmatrix}\right)
\end{aligned}
$$

- (P, Tseng '08) $\exists\, \tau, \bar{\rho} > 0$ such that for each i,

$$
\begin{aligned}
& \text{tr}_i[Z^\rho] < \tau \quad \forall 0 \le \delta_{\max} < \rho \le \bar{\rho} \\
\Longrightarrow \quad & \|x_i^\rho - x_i^{\text{true}}\| \le \sqrt{2|\mathcal{A}| + n}\, (\text{tr}_i(Z^\rho))^{\frac{1}{2}}
\end{aligned}
$$

Conclusion & Ongoing work

SDP and ESDP are stronger relaxations, but the soln is not stable relative to measurement noise. Lack soln accuracy certificate.

ρESDP is a weaker relaxation, but has more stable solns. Has soln accuracy certificate.

- Distributed method to compute Z^ρ?
- Simulation and numerical testing?

Thanks for coming!