Error bound for conic feasibility problems: Case study on perspective cones of some nonnegative Legendre functions

Ting Kei Pong
Department of Applied Mathematics
The Hong Kong Polytechnic University
Hong Kong

Peking University Workshop
on Optimization Theory and Methods
July 2026

(Joint work with Bruno F. Lourenço and Xiaozhou Wang)

Conic feasibility problem

Conic feasibility problem: Let $\mathcal{K}$ be a closed convex cone, $\mathcal{L}$ be a subspace of a Euclidean space $\mathcal{E}$ and $a \in \mathcal{E}$.

$$\text{Find} \quad x \in \mathcal{K} \cap (\mathcal{L} + a).$$

Conic feasibility problem

Conic feasibility problem: Let $\mathcal{K}$ be a closed convex cone, $\mathcal{L}$ be a subspace of a Euclidean space $\mathcal{E}$ and $a \in \mathcal{E}$.

$$\text{Find} \quad x \in \mathcal{K} \cap (\mathcal{L} + a).$$

- Focus on the feasible case, i.e., $\mathcal{K} \cap (\mathcal{L} + a) \neq \emptyset$.
- Arises from optimality conditions of conic programs.

Conic feasibility problem

Conic feasibility problem: Let $\mathcal{K}$ be a closed convex cone, $\mathcal{L}$ be a subspace of a Euclidean space $\mathcal{E}$ and $a \in \mathcal{E}$.

$$\text{Find} \quad x \in \mathcal{K} \cap (\mathcal{L} + a).$$

- Focus on the feasible case, i.e., $\mathcal{K} \cap (\mathcal{L} + a) \neq \emptyset$.
- Arises from optimality conditions of conic programs.

When is a feasibility problem approximately solved?

Conic feasibility problem

Conic feasibility problem: Let $\mathcal{K}$ be a closed convex cone, $\mathcal{L}$ be a subspace of a Euclidean space $\mathcal{E}$ and $a \in \mathcal{E}$.

$$\text{Find} \quad x \in \mathcal{K} \cap (\mathcal{L} + a).$$

- Focus on the feasible case, i.e., $\mathcal{K} \cap (\mathcal{L} + a) \neq \emptyset$.
- Arises from optimality conditions of conic programs.

When is a feasibility problem approximately solved?

- $\mathrm{d}(x, \mathcal{K} \cap (\mathcal{L} + a))$ is a measure on how "feasible" x is.

Conic feasibility problem

Conic feasibility problem: Let $\mathcal{K}$ be a closed convex cone, $\mathcal{L}$ be a subspace of a Euclidean space $\mathcal{E}$ and $a \in \mathcal{E}$.

$$\text{Find} \quad x \in \mathcal{K} \cap (\mathcal{L} + a).$$

- Focus on the feasible case, i.e., $\mathcal{K} \cap (\mathcal{L} + a) \neq \emptyset$.
- Arises from optimality conditions of conic programs.

When is a feasibility problem approximately solved?

- $\mathrm{d}(x, \mathcal{K} \cap (\mathcal{L} + a))$ is a measure on how "feasible" x is. Hard to compute!
- Typically, $\mathrm{d}(x, \mathcal{K})$ and $\mathrm{d}(x, \mathcal{L} + a)$ are relatively easier to compute.
- Is x "a good soln." when $\max\{\mathrm{d}(x, \mathcal{K}), \mathrm{d}(x, \mathcal{L} + a)\}$ is small?

Conic feasibility problem

Conic feasibility problem: Let $\mathcal{K}$ be a closed convex cone, $\mathcal{L}$ be a subspace of a Euclidean space $\mathcal{E}$ and $a \in \mathcal{E}$.

$$\text{Find} \quad x \in \mathcal{K} \cap (\mathcal{L} + a).$$

- Focus on the feasible case, i.e., $\mathcal{K} \cap (\mathcal{L} + a) \neq \emptyset$.
- Arises from optimality conditions of conic programs.

When is a feasibility problem approximately solved?

- $\mathrm{d}(x, \mathcal{K} \cap (\mathcal{L} + a))$ is a measure on how "feasible" x is. Hard to compute!
- Typically, $\mathrm{d}(x, \mathcal{K})$ and $\mathrm{d}(x, \mathcal{L} + a)$ are relatively easier to compute.
- Is x "a good soln." when $\max\{\mathrm{d}(x, \mathcal{K}), \mathrm{d}(x, \mathcal{L} + a)\}$ is small?

Key: Compare the orders of magnitude of $\mathrm{d}(x, \mathcal{K} \cap (\mathcal{L} + a))$ and $\max\{\mathrm{d}(x, \mathcal{K}), \mathrm{d}(x, \mathcal{L} + a)\}$.

Conic feasibility problem

Conic feasibility problem: Let $\mathcal{K}$ be a closed convex cone, $\mathcal{L}$ be a subspace of a Euclidean space $\mathcal{E}$ and $a \in \mathcal{E}$.

$$\text{Find} \quad x \in \mathcal{K} \cap (\mathcal{L} + a).$$

- Focus on the feasible case, i.e., $\mathcal{K} \cap (\mathcal{L} + a) \neq \emptyset$.
- Arises from optimality conditions of conic programs.

When is a feasibility problem approximately solved?

- $\mathrm{d}(x, \mathcal{K} \cap (\mathcal{L} + a))$ is a measure on how "feasible" x is. Hard to compute!
- Typically, $\mathrm{d}(x, \mathcal{K})$ and $\mathrm{d}(x, \mathcal{L} + a)$ are relatively easier to compute.
- Is x "a good soln." when $\max\{\mathrm{d}(x, \mathcal{K}), \mathrm{d}(x, \mathcal{L} + a)\}$ is small?

Key: Compare the orders of magnitude of $\mathrm{d}(x, \mathcal{K} \cap (\mathcal{L} + a))$ and $\max\{\mathrm{d}(x, \mathcal{K}), \mathrm{d}(x, \mathcal{L} + a)\}$.

Note: Typically, $(\mathcal{L} + a) \cap \operatorname{ri} \mathcal{K} = \emptyset$.

Error bounds

Definition: Let $\theta \in (0, 1]$. We say that $\{\mathcal{K}, \mathcal{L} + a\}$ satisfies a (uniform) Hölderian error bound with exponent θ if for every bounded set B, there exists $c_B > 0$ such that

$$\mathrm{d}(x, \mathcal{K} \cap (\mathcal{L} + a)) \leq c_B \left(\max\{\mathrm{d}(x, \mathcal{K}), \mathrm{d}(x, \mathcal{L} + a)\}\right)^\theta \quad \forall x \in B.$$

If $\theta = 1$, we say that a Lipschitz error bound holds.

Error bounds

Definition: Let $\theta \in (0, 1]$. We say that $\{\mathcal{K}, \mathcal{L} + a\}$ satisfies a (uniform) Hölderian error bound with exponent θ if for every bounded set B, there exists $c_B > 0$ such that

$$\mathrm{d}(x, \mathcal{K} \cap (\mathcal{L} + a)) \leq c_B \left(\max\{\mathrm{d}(x, \mathcal{K}), \mathrm{d}(x, \mathcal{L} + a)\}\right)^{\theta} \quad \forall x \in B.$$

If $\theta = 1$, we say that a Lipschitz error bound holds.

Examples:

- If $\mathcal{K}$ is polyhedral, Lipschitz error bound holds. (Hoffman '57)
- If $(\mathcal{L} + a) \cap \mathrm{ri}\,\mathcal{K} \neq \emptyset$, Lipschitz error bound holds. (Bauschke, Borwein '96)

Error bounds

Definition: Let $\theta \in (0,1]$. We say that $\{\mathcal{K}, \mathcal{L}+a\}$ satisfies a (uniform) Hölderian error bound with exponent θ if for every bounded set B, there exists $c_B > 0$ such that

$$\mathrm{d}(x, \mathcal{K} \cap (\mathcal{L}+a)) \leq c_B \left(\max\{\mathrm{d}(x,\mathcal{K}), \mathrm{d}(x, \mathcal{L}+a)\}\right)^{\theta} \quad \forall x \in B.$$

If $\theta = 1$, we say that a Lipschitz error bound holds.

Examples:

- If $\mathcal{K}$ is polyhedral, Lipschitz error bound holds. (Hoffman '57)
- If $(\mathcal{L}+a) \cap \mathrm{ri}\,\mathcal{K} \neq \emptyset$, Lipschitz error bound holds. (Bauschke, Borwein '96)
- If $\mathcal{K} = \mathcal{S}^n_+$, Hölderian error bound with exponent $2^{-(\ell-1)}$ holds (Sturm '00); here ℓ has to do with facial reduction. (Borwein, Wolkowicz '81)

Faces and facial reduction

Definition: A sub-cone $\mathcal{F} \subseteq \mathcal{K}$ is called

- a face if $x,\ y \in \mathcal{K}$ and $x + y \in \mathcal{F}$ implies $x,\ y \in \mathcal{F}$;
- an exposed face if $\exists\ z \in \mathcal{K}^*$ such that $\mathcal{F} = \mathcal{K} \cap \{z\}^\perp$.

Note: Recall that $\mathcal{K}^* := \{x \mid \langle x, y\rangle \geq 0 \ \ \forall y \in \mathcal{K}\}$.

Faces and facial reduction

Definition: A sub-cone $\mathcal{F} \subseteq \mathcal{K}$ is called

- a face if $x,\ y \in \mathcal{K}$ and $x + y \in \mathcal{F}$ implies $x,\ y \in \mathcal{F}$;
- an exposed face if $\exists\ z \in \mathcal{K}^*$ such that $\mathcal{F} = \mathcal{K} \cap \{z\}^\perp$.

Note: Recall that $\mathcal{K}^* := \{x \mid \langle x, y\rangle \geq 0 \ \ \forall y \in \mathcal{K}\}$.

Theorem 1. (Borwein, Wolkowicz '81, Lourenço, Muramatsu, Tsuchiya '18)
Suppose $\mathcal{K} \cap (\mathcal{L} + a) \neq \emptyset$. Then there exists a chain of faces

$$\mathcal{F}_\ell \subsetneq \cdots \subsetneq \mathcal{F}_1 = \mathcal{K}$$

and vectors $\{z_1, \ldots, z_{\ell-1}\}$ satisfying

- For all $i \in \{1, \ldots, \ell - 1\}$,

$$z_i \in \mathcal{F}_i^* \cap \mathcal{L}^\perp \cap \{a\}^\perp \quad \text{and} \quad \mathcal{F}_{i+1} = \mathcal{F}_i \cap \{z_i\}^\perp.$$

- $\mathcal{F}_\ell \cap (\mathcal{L} + a) = \mathcal{K} \cap (\mathcal{L} + a)$ and $\{\mathcal{F}_\ell, \mathcal{L} + a\}$ satisfies a Lipschitz error bound.

Sturm's error bounds and facial reduction

Key observation: (Sturm '00, Lourenço '21)
Let $\mathcal{F} \trianglelefteq \mathcal{S}^n_+$ and $z \in \mathcal{F}^*$. Then $\exists\ \kappa > 0$ such that for all x,

$$\mathrm{d}(x, \mathcal{F} \cap \{z\}^\perp) \leq \kappa\epsilon + \kappa\sqrt{\epsilon\|x\|},$$

where $\epsilon = \max\{\mathrm{d}(x, \mathcal{F}), \mathrm{d}(x, \{z\}^\perp)\}$.

Sturm's error bounds and facial reduction

Key observation: (Sturm '00, Lourenço '21)
Let $\mathcal{F} \trianglelefteq \mathcal{S}^n_+$ and $z \in \mathcal{F}^*$. Then $\exists\, \kappa > 0$ such that for all x,

$$\mathrm{d}(x, \mathcal{F} \cap \{z\}^\perp) \leq \kappa\epsilon + \kappa\sqrt{\epsilon\|x\|},$$

where $\epsilon = \max\{\mathrm{d}(x, \mathcal{F}), \mathrm{d}(x, \{z\}^\perp)\}$.

Error bound for $\{\mathcal{S}^n_+, (\mathcal{L} + a)\}$ follows from induction: For $x \in B$,

$$\mathrm{d}(x, \mathcal{K} \cap (\mathcal{L} + a)) = \mathrm{d}(x, \mathcal{F}_\ell \cap (\mathcal{L} + a)) \leq c_\ell \max\{\mathrm{d}(x, \mathcal{F}_\ell), \mathrm{d}(x, \mathcal{L} + a)\}$$

Sturm's error bounds and facial reduction

Key observation: (Sturm '00, Lourenço '21)
Let $\mathcal{F} \trianglelefteq \mathcal{S}^n_+$ and $z \in \mathcal{F}^*$. Then $\exists\, \kappa > 0$ such that for all x,

$$\mathrm{d}(x, \mathcal{F} \cap \{z\}^\perp) \leq \kappa\epsilon + \kappa\sqrt{\epsilon\|x\|},$$

where $\epsilon = \max\{\mathrm{d}(x, \mathcal{F}), \mathrm{d}(x, \{z\}^\perp)\}$.

Error bound for $\{\mathcal{S}^n_+, (\mathcal{L} + a)\}$ follows from induction: For $x \in B$,

$$\begin{aligned}
&\mathrm{d}(x, \mathcal{K} \cap (\mathcal{L} + a)) = \mathrm{d}(x, \mathcal{F}_\ell \cap (\mathcal{L} + a)) \leq c_\ell \max\{\mathrm{d}(x, \mathcal{F}_\ell), \mathrm{d}(x, \mathcal{L} + a)\} \\
&\leq c_\ell[\mathrm{d}(x, \mathcal{L} + a) + \mathrm{d}(x, \mathcal{F}_{\ell-1} \cap \{z_{\ell-1}\}^\perp)] \\
&\leq c_\ell[\mathrm{d}(x, \mathcal{L} + a) + c_{\ell-1}\big(\max\{\mathrm{d}(x, \mathcal{F}_{\ell-1}), \mathrm{d}(x, \{z_{\ell-1}\}^\perp)\} \\
&\qquad + \sqrt{\max\{\mathrm{d}(x, \mathcal{F}_{\ell-1}), \mathrm{d}(x, \{z_{\ell-1}\}^\perp)\}\|x\|}\big)]
\end{aligned}$$

Sturm's error bounds and facial reduction

Key observation: (Sturm '00, Lourenço '21)
Let $\mathcal{F} \trianglelefteq \mathcal{S}^n_+$ and $z \in \mathcal{F}^*$. Then $\exists\ \kappa > 0$ such that for all x,

$$\mathrm{d}(x, \mathcal{F} \cap \{z\}^\perp) \leq \kappa\epsilon + \kappa\sqrt{\epsilon\|x\|},$$

where $\epsilon = \max\{\mathrm{d}(x, \mathcal{F}), \mathrm{d}(x, \{z\}^\perp)\}$.

Error bound for $\{\mathcal{S}^n_+, (\mathcal{L} + a)\}$ follows from induction: For $x \in B$,

$$\begin{aligned}
&\mathrm{d}(x, \mathcal{K} \cap (\mathcal{L} + a)) = \mathrm{d}(x, \mathcal{F}_\ell \cap (\mathcal{L} + a)) \leq c_\ell \max\{\mathrm{d}(x, \mathcal{F}_\ell), \mathrm{d}(x, \mathcal{L} + a)\} \\
&\leq c_\ell[\mathrm{d}(x, \mathcal{L} + a) + \mathrm{d}(x, \mathcal{F}_{\ell-1} \cap \{z_{\ell-1}\}^\perp)] \\
&\leq c_\ell[\mathrm{d}(x, \mathcal{L} + a) + c_{\ell-1}\big(\max\{\mathrm{d}(x, \mathcal{F}_{\ell-1}), \mathrm{d}(x, \{z_{\ell-1}\}^\perp)\} \\
&\qquad + \sqrt{\max\{\mathrm{d}(x, \mathcal{F}_{\ell-1}), \mathrm{d}(x, \{z_{\ell-1}\}^\perp)\}\|x\|}\big)] \\
&= O([\max\{\mathrm{d}(x, \mathcal{F}_{\ell-1}), \mathrm{d}(x, \mathcal{L} + a)\}]^{\frac{1}{2}})
\end{aligned}$$

Sturm's error bounds and facial reduction

Key observation: (Sturm '00, Lourenço '21)
Let $\mathcal{F} \trianglelefteq \mathcal{S}^n_+$ and $z \in \mathcal{F}^*$. Then $\exists\ \kappa > 0$ such that for all x,

$$\mathrm{d}(x, \mathcal{F} \cap \{z\}^\perp) \leq \kappa\epsilon + \kappa\sqrt{\epsilon\|x\|},$$

where $\epsilon = \max\{\mathrm{d}(x,\mathcal{F}), \mathrm{d}(x, \{z\}^\perp)\}$.

Error bound for $\{\mathcal{S}^n_+, (\mathcal{L}+a)\}$ follows from induction: For $x \in B$,

$$\begin{aligned}
&\mathrm{d}(x, \mathcal{K} \cap (\mathcal{L}+a)) = \mathrm{d}(x, \mathcal{F}_\ell \cap (\mathcal{L}+a)) \leq c_\ell \max\{\mathrm{d}(x,\mathcal{F}_\ell), \mathrm{d}(x, \mathcal{L}+a)\}\\
&\leq c_\ell[\mathrm{d}(x,\mathcal{L}+a) + \mathrm{d}(x, \mathcal{F}_{\ell-1} \cap \{z_{\ell-1}\}^\perp)]\\
&\leq c_\ell[\mathrm{d}(x,\mathcal{L}+a) + c_{\ell-1}\Big(\max\{\mathrm{d}(x,\mathcal{F}_{\ell-1}), \mathrm{d}(x,\{z_{\ell-1}\}^\perp)\}\\
&\qquad + \sqrt{\max\{\mathrm{d}(x,\mathcal{F}_{\ell-1}), \mathrm{d}(x,\{z_{\ell-1}\}^\perp)\}\|x\|}\Big)]\\
&= O([\max\{\mathrm{d}(x,\mathcal{F}_{\ell-1}), \mathrm{d}(x,\mathcal{L}+a)\}]^{\frac{1}{2}}) = O([\max\{\mathrm{d}(x,\mathcal{K}), \mathrm{d}(x,\mathcal{L}+a)\}]^{\frac{1}{2^{\ell-1}}}).
\end{aligned}$$

Facial residual function

Definition: (Lourenço '21, Lindstrom, Lourenço, P. '23)
Let $\mathcal{F} \trianglelefteq \mathcal{K}$ and $z \in \mathcal{F}^*$. Suppose $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

- ψ is nondecreasing in each argument and $\psi(0, t) = 0 \; \forall \, t \in \mathbb{R}_+$;
- It holds that

$$\mathrm{d}(x, \mathcal{F} \cap \{z\}^\perp) \leq \psi(\max\{\mathrm{d}(x, \mathcal{F}), \mathrm{d}(x, \{z\}^\perp)\}, \|x\|) \quad \forall x \in \operatorname{span} \mathcal{F}.$$

Then ψ is called a 1-step facial residual function for $\mathcal{F}$ and z.

Facial residual function

Definition: (Lourenço '21, Lindstrom, Lourenço, P. '23)
Let $\mathcal{F} \trianglelefteq \mathcal{K}$ and $z \in \mathcal{F}^*$. Suppose $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

- ψ is nondecreasing in each argument and $\psi(0, t) = 0 \ \forall\ t \in \mathbb{R}_+$;
- It holds that

$$\mathrm{d}(x, \mathcal{F} \cap \{z\}^\perp) \le \psi(\max\{\mathrm{d}(x, \mathcal{F}), \mathrm{d}(x, \{z\}^\perp)\}, \|x\|) \quad \forall x \in \operatorname{span} \mathcal{F}.$$

Then ψ is called a 1-step facial residual function for $\mathcal{F}$ and z.

Remarks:

- For $\mathcal{K} = \mathcal{S}^n_+$, we have $\psi(s, t) = \kappa \cdot (s + \sqrt{st})$ for some $\kappa > 0$.
- Induction arguments show that error bound can be derived given the face chain from facial reduction and by composing 1-step facial residual functions. (Lindstrom, Lourenço, P. '23)

Facial residual function

Definition: (Lourenço '21, Lindstrom, Lourenço, P. '23)
Let $\mathcal{F} \trianglelefteq \mathcal{K}$ and $z \in \mathcal{F}^*$. Suppose $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

- ψ is nondecreasing in each argument and $\psi(0, t) = 0\ \forall\ t \in \mathbb{R}_+$;
- It holds that

$$\mathrm{d}(x, \mathcal{F} \cap \{z\}^\perp) \leq \psi(\max\{\mathrm{d}(x, \mathcal{F}), \mathrm{d}(x, \{z\}^\perp)\}, \|x\|) \quad \forall x \in \operatorname{span} \mathcal{F}.$$

Then ψ is called a 1-step facial residual function for $\mathcal{F}$ and z.

Remarks:

- For $\mathcal{K} = \mathcal{S}^n_+$, we have $\psi(s, t) = \kappa \cdot (s + \sqrt{st})$ for some $\kappa > 0$.
- Induction arguments show that error bound can be derived given the face chain from facial reduction and by composing 1-step facial residual functions. (Lindstrom, Lourenço, P. '23)
- Two key ingredients: For each nonpolyhedral $\mathcal{F}$,
  - ⋆ Identify all its exposed faces.
  - ⋆ Obtain all 1-step facial residual functions: Depends on $\mathcal{F}$ and z.

Summary of known $\mathbb{1}$-FRFs

$\mathcal{K}$	$z = (z_x, z_t, z_r)$ $z \in \mathrm{bdry}(\mathcal{K}^*) \setminus \{0\}$	$\mathbb{1}$-FRFs for $\mathcal{K}$ and z	dominated by exponent 1/2
power cone $p \in [2, \infty)$ $1/p + 1/q = 1$ (Lin, Lindstrom, Lourenço, P. '24)	$z_x \neq 0$	$(\epsilon, t) \mapsto \rho_1(t)\epsilon + \rho_2(t)\epsilon^{\frac{1}{2}}$	✓
	$z_x = z_r = 0$; or $z_x = z_t = 0, p = 2$	$(\epsilon, t) \mapsto \rho_1(t)\epsilon + \rho_2(t)\epsilon^{\frac{1}{q}}$	✓
	$z_x = z_t = 0, p > 2$	$(\epsilon, t) \mapsto \rho_1(t)\epsilon + \rho_2(t)\epsilon^{\frac{1}{p}}$	✗
$K_{\exp}$ (Lindstrom, Lourenço, P. '23)	$z_x < 0$	$(\epsilon, t) \mapsto \rho_1(t)\epsilon + \rho_2(t)\epsilon^{\frac{1}{2}}$	✓
	$z_x = 0, z_t > 0, z_r > 0$	$(\epsilon, t) \mapsto \rho_1(t)\epsilon$	✓
	$z_x = 0, z_t = 0, z_r > 0$	log-type function	✗
	$z_x = 0, z_t > 0, z_r = 0$	entropy-type function	✓
p-cone $p \in (1, \infty)$ (Lindstrom, Lourenço, P. '25)	$z_t \neq 0, z_x \neq 0$	$(\epsilon, t) \mapsto \rho_1(t)\epsilon + \rho_2(t)\epsilon^{\frac{1}{2}}$	✓
	$z_t \neq 0, z_x = 0, p \in (1, 2]$; or $z_t = 0, z_x \neq 0, p \in (1, 2]$	$(\epsilon, t) \mapsto \rho_1(t)\epsilon + \rho_2(t)\epsilon^{\frac{1}{p}}$	✓
	$z_t \neq 0, z_x = 0, p > 2$; or $z_t = 0, z_x \neq 0, p > 2$	$(\epsilon, t) \mapsto \rho_1(t)\epsilon + \rho_2(t)\epsilon^{\frac{1}{p}}$	✗

Table: $\mathbb{1}$-FRFs for exposed faces of some important cones in $\mathbb{R}^3$. Here, $\rho_1, \rho_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are nonnegative nondecreasing functions, and

- the power cone is given by $\{(v_x, v_t, v_r) : \ v_r^{1/p} v_t^{1/q} \geq |v_x|, \ v_r \geq 0, \ v_t \geq 0\}$;
- $K_{\exp}$ is given by $\{(v_x, v_t, v_r) : v_r \geq v_t \exp(v_x/v_t), v_t > 0\} \cup \{(v_x, 0, v_s) : v_x \leq 0, v_s \geq 0\}$;
- the p-cone is given by $\{(v_x, v_t, v_r) : v_r \geq \sqrt[p]{|v_x|^p + |v_t|^p}\}$.

Summary of known $\mathbb{1}$-FRFs

$\mathcal{K}$	$z = (z_x, z_t, z_r)$ $z \in \mathrm{bdry}(\mathcal{K}^*) \setminus \{0\}$	$\mathbb{1}$-FRFs for $\mathcal{K}$ and z	dominated by exponent 1/2
power cone $p \in [2, \infty)$ $1/p + 1/q = 1$ (Lin, Lindstrom, Lourenço, P. '24)	$z_x \neq 0$	$(\epsilon, t) \mapsto \rho_1(t)\epsilon + \rho_2(t)\epsilon^{\frac{1}{2}}$	✓
	$z_x = z_r = 0$; or $z_x = z_t = 0, p = 2$	$(\epsilon, t) \mapsto \rho_1(t)\epsilon + \rho_2(t)\epsilon^{\frac{1}{q}}$	✓
	$z_x = z_t = 0, p > 2$	$(\epsilon, t) \mapsto \rho_1(t)\epsilon + \rho_2(t)\epsilon^{\frac{1}{p}}$	✗
$K_{\exp}$ (Lindstrom, Lourenço, P. '23)	$z_x < 0$	$(\epsilon, t) \mapsto \rho_1(t)\epsilon + \rho_2(t)\epsilon^{\frac{1}{2}}$	✓
	$z_x = 0, z_t > 0, z_r > 0$	$(\epsilon, t) \mapsto \rho_1(t)\epsilon$	✓
	$z_x = 0, z_t = 0, z_r > 0$	log-type function	✗
	$z_x = 0, z_t > 0, z_r = 0$	entropy-type function	✓
p-cone $p \in (1, \infty)$ (Lindstrom, Lourenço, P. '25)	$z_t \neq 0, z_x \neq 0$	$(\epsilon, t) \mapsto \rho_1(t)\epsilon + \rho_2(t)\epsilon^{\frac{1}{2}}$	✓
	$z_t \neq 0, z_x = 0, p \in (1, 2]$; or $z_t = 0, z_x \neq 0, p \in (1, 2]$	$(\epsilon, t) \mapsto \rho_1(t)\epsilon + \rho_2(t)\epsilon^{\frac{1}{p}}$	✓
	$z_t \neq 0, z_x = 0, p > 2$; or $z_t = 0, z_x \neq 0, p > 2$	$(\epsilon, t) \mapsto \rho_1(t)\epsilon + \rho_2(t)\epsilon^{\frac{1}{p}}$	✗

Table: $\mathbb{1}$-FRFs for exposed faces of some important cones in $\mathbb{R}^3$. Here, $\rho_1, \rho_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are nonnegative nondecreasing functions, and

- the power cone is given by $\{(v_x, v_t, v_r) : \ v_r^{1/p} v_t^{1/q} \geq |v_x|, \ v_r \geq 0, \ v_t \geq 0\}$;
- $K_{\exp}$ is given by $\{(v_x, v_t, v_r) : v_r \geq v_t \exp(v_x/v_t), v_t > 0\} \cup \{(v_x, 0, v_s) : v_x \leq 0, v_s \geq 0\}$;
- the p-cone is given by $\{(v_x, v_t, v_r) : v_r \geq \sqrt[p]{|v_x|^p + |v_t|^p}\}$.

Question: Why is $1/2$ appearing so often?

A special perspective cone

Consider the perspective cone

$$\operatorname{epi} f^{\pi} = \overline{\bigcup_{t>0} \{(x, t, r) : \ (x, r) \in t \cdot \operatorname{epi} f\}},$$

where f satisfies the following assumption:

Assumption: $f : \mathbb{R} \to \mathbb{R}_+ \cup \{\infty\}$ is proper, closed, convex and essentially smooth, and $\exists$ a finite set $W \subset \operatorname{int}(\operatorname{dom} f)$ such that f is strongly convex on every compact convex subset of $\operatorname{int}(\operatorname{dom} f) \backslash W$.

A special perspective cone

Consider the perspective cone

$$\operatorname{epi} f^{\pi} = \overline{\bigcup_{t>0} \{(x,t,r) : \ (x,r) \in t \cdot \operatorname{epi} f\}},$$

where f satisfies the following assumption:

Assumption: $f : \mathbb{R} \to \mathbb{R}_+ \cup \{\infty\}$ is proper, closed, convex and essentially smooth, and $\exists$ a finite set $W \subset \operatorname{int}(\operatorname{dom} f)$ such that f is strongly convex on every compact convex subset of $\operatorname{int}(\operatorname{dom} f) \setminus W$.

Remark: Such an f must be Legendre, i.e., essentially smooth and essentially convex.

A special perspective cone

Consider the perspective cone

$$\operatorname{epi} f^{\pi} = \overline{\bigcup_{t>0}\{(x,t,r):\ (x,r)\in t\cdot \operatorname{epi} f\}},$$

where f satisfies the following assumption:

Assumption: $f : \mathbb{R} \to \mathbb{R}_+ \cup \{\infty\}$ is proper, closed, convex and essentially smooth, and $\exists$ a finite set $W \subset \operatorname{int}(\operatorname{dom} f)$ such that f is strongly convex on every compact convex subset of $\operatorname{int}(\operatorname{dom} f) \backslash W$.

Remark: Such an f must be Legendre, i.e., essentially smooth and essentially convex.

Examples: $f_1(x) = \exp(x)$,

A special perspective cone

Consider the perspective cone

$$\operatorname{epi} f^{\pi} = \overline{\bigcup_{t>0} \{(x,t,r) : \ (x,r) \in t \cdot \operatorname{epi} f\}},$$

where f satisfies the following assumption:

Assumption: $f : \mathbb{R} \to \mathbb{R}_+ \cup \{\infty\}$ is proper, closed, convex and essentially smooth, and $\exists$ a finite set $W \subset \operatorname{int}(\operatorname{dom} f)$ such that f is strongly convex on every compact convex subset of $\operatorname{int}(\operatorname{dom} f) \setminus W$.

Remark: Such an f must be Legendre, i.e., essentially smooth and essentially convex.

Examples: $f_1(x) = \exp(x)$, $f_2(x) = \exp(\exp(x))$,

A special perspective cone

Consider the perspective cone

$$\operatorname{epi} f^{\pi} = \overline{\bigcup_{t>0} \{(x,t,r) :\ (x,r) \in t \cdot \operatorname{epi} f\}},$$

where f satisfies the following assumption:

Assumption: $f : \mathbb{R} \to \mathbb{R}_+ \cup \{\infty\}$ is proper, closed, convex and essentially smooth, and $\exists$ a finite set $W \subset \operatorname{int}(\operatorname{dom} f)$ such that f is strongly convex on every compact convex subset of $\operatorname{int}(\operatorname{dom} f) \backslash W$.

Remark: Such an f must be Legendre, i.e., essentially smooth and essentially convex.

Examples: $f_1(x) = \exp(x)$, $f_2(x) = \exp(\exp(x))$,

$$f_3(x) = \begin{cases} \alpha^{-p}|x|^p & \text{if } x < 0, \\ x^p & \text{if } x \geq 0, \end{cases}$$

where $p \in [2, \infty)$ and $\alpha \in (0, 1]$.

How restrictive is our assumption?

According to Hildebrand '14, every 3D pointed, solid, and closed convex cone with automorphism group of dimension at least 2 is linearly isomorphic to exactly one of the five cones below :

1. $K_{\exp}$;
2. $\mathbb{R}^3_+$;
3. $\{(x,t,r) : r^{\frac{1}{p}} t^{\frac{1}{q}} \geq x, r \geq 0, t \geq 0\}$ with $1/p + 1/q = 1$, $p \in [2, \infty)$;
4. epi f_3^{π};
5. $\{(x,t,r) : r^{\frac{1}{p}} t^{\frac{1}{q}} \geq x \geq 0, r \geq 0, t \geq 0\}$ with $1/p + 1/q = 1$, $p \in [2, \infty)$.

How restrictive is our assumption?

According to Hildebrand '14, every 3D pointed, solid, and closed convex cone with automorphism group of dimension at least 2 is linearly isomorphic to exactly one of the five cones below :

1. $K_{\exp}$;
2. $\mathbb{R}^3_+$;
3. $\{(x,t,r) : r^{\frac{1}{p}} t^{\frac{1}{q}} \geq x, r \geq 0, t \geq 0\}$ with $1/p + 1/q = 1$, $p \in [2, \infty)$;
4. epi f_3^{π};
5. $\{(x,t,r) : r^{\frac{1}{p}} t^{\frac{1}{q}} \geq x \geq 0, r \geq 0, t \geq 0\}$ with $1/p + 1/q = 1$, $p \in [2, \infty)$.

$\therefore$ half of the isomorphism classes of nonpolyhedral cones with automorphism group of dimension at least 2 are covered by our assumption.

Exposed faces and $\mathbb{1}$-FRFs

Here shows all exposed faces of epi f^π and the dominating exponents in their $\mathbb{1}$-FRFs. (Wang, Lourenço, P. '25)

$z^* \in \mathrm{bdry}((\mathrm{epi}\, f^\pi)^*) \setminus \{0\}$	the exposed face $\mathrm{epi}\, f^\pi \cap \{z^*\}^\perp$	dominating exponent in $\mathbb{1}$-FRF
$z^* = (-x^*, r^*, t^*)$, $t^* > 0$, $\beta := x^*/t^* \in \mathrm{int}(\mathrm{dom}\, f^*)$, $r^* = t^* f^*(\beta)$	$\mathcal{F}^e_\beta = \{t(\bar\beta, 1, f(\bar\beta)) : t \geq 0\}$ with $\bar\beta := (f^*)'(\beta)$	1/2 if $\beta \notin f'(W)$? otherwise
$z^* = (-x^*, r^*, t^*)$, $t^* > 0$, $\beta := x^*/t^*$, $r^* \geq t^* f^*(\beta)$ $\beta \in \mathrm{bdry}(\mathrm{dom}\, f^*) \cap \mathrm{dom}\, f^*$	$\mathcal{F}^e_{\mathrm{bd}} = \{(x, 0, f_\infty(x)) : f_\infty(x) = \beta x\}$	1 if $r^* > t^* f^*(\beta)$? otherwise
$z^* = (-x^*, s^*, 0)$, $s^* > (f^*)_\infty(x^*)$, $x^* = 0$	$\mathcal{F}^e_\theta = \{(x, 0, s) : (x, s) \in \mathrm{epi}\, f_\infty\}$	?
$z^* = (-x^*, s^*, 0)$, $s^* > (f^*)_\infty(x^*)$, $x^* \neq 0$	$\mathcal{F}^e_{\neq\theta} = \{(0, 0, s) : s \geq 0\}$	1
$z^* = (-x^*, s^*, 0)$, $s^* = (f^*)_\infty(x^*)$, $x^* \neq 0$	$\mathcal{F}^e_{\mathbb{1}} = \{(x, t, r) : r \geq t f(\alpha), x = \alpha t, t \geq 0\}$ with α uniquely defined via $x^* \in \mathcal{N}_{\mathrm{dom}\, f}(\alpha)$	?

Exposed faces and $\mathbb{1}$-FRFs

Here shows all exposed faces of epi f^π and the dominating exponents in their $\mathbb{1}$-FRFs. (Wang, Lourenço, P. '25)

$z^* \in \mathrm{bdry}((\mathrm{epi}\, f^\pi)^*) \setminus \{0\}$	the exposed face epi $f^\pi \cap \{z^*\}^\perp$	dominating exponent in $\mathbb{1}$-FRF
$z^* = (-x^*, r^*, t^*)$, $t^* > 0$, $\beta := x^*/t^* \in \mathrm{int}(\mathrm{dom}\, f^*)$, $r^* = t^* f^*(\beta)$	$\mathcal{F}^e_\beta = \{t(\bar\beta, 1, f(\bar\beta)) : t \geq 0\}$ with $\bar\beta := (f^*)'(\beta)$	1/2 if $\beta \notin f'(W)$ **?** otherwise
$z^* = (-x^*, r^*, t^*)$, $t^* > 0$, $\beta := x^*/t^*$, $r^* \geq t^* f^*(\beta)$ $\beta \in \mathrm{bdry}(\mathrm{dom}\, f^*) \cap \mathrm{dom}\, f^*$	$\mathcal{F}^e_{\mathrm{bd}} = \{(x, 0, f_\infty(x)) : f_\infty(x) = \beta x\}$	1 if $r^* > t^* f^*(\beta)$ **?** otherwise
$z^* = (-x^*, s^*, 0)$, $s^* > (f^*)_\infty(x^*)$, $x^* = 0$	$\mathcal{F}^e_\theta = \{(x, 0, s) : (x, s) \in \mathrm{epi}\, f_\infty\}$	**?**
$z^* = (-x^*, s^*, 0)$, $s^* > (f^*)_\infty(x^*)$, $x^* \neq 0$	$\mathcal{F}^e_{\neq\theta} = \{(0, 0, s) : s \geq 0\}$	1
$z^* = (-x^*, s^*, 0)$, $s^* = (f^*)_\infty(x^*)$, $x^* \neq 0$	$\mathcal{F}^e_{\mathbb{1}} = \{(x, t, r) : r \geq t f(\alpha), x = \alpha t, t \geq 0\}$ with α uniquely defined via $x^* \in \mathcal{N}_{\mathrm{dom}\, f}(\alpha)$	**?**

Theorem 2. (Wang, Lourenço, P. '25)
The set of $z^* \in \mathrm{bdry}((\mathrm{epi}\, f^\pi)^*)$ corresponding to the "**?**" cases is of Hausdorff dimension 1.

Computing $\mathbb{1}$-FRFs

Difficulties:

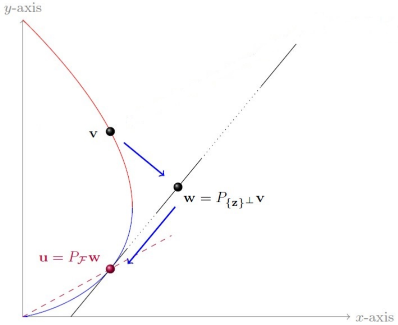
- Need to compare $\mathrm{d}(x, \mathcal{K})$, $\mathrm{d}(x, \{z\}^\perp)$, and $\mathrm{d}(x, \mathcal{K} \cap \{z\}^\perp)$.
- Projection onto $\mathcal{K}$ (and hence $\mathrm{d}(x, \mathcal{K})$) does not have an easy-to-analyze analytic form.

Computing $\mathbb{1}$-FRFs

Difficulties:

- Need to compare $\mathrm{d}(x, \mathcal{K})$, $\mathrm{d}(x, \{z\}^\perp)$, and $\mathrm{d}(x, \mathcal{K} \cap \{z\}^\perp)$.
- Projection onto $\mathcal{K}$ (and hence $\mathrm{d}(x, \mathcal{K})$) does not have an easy-to-analyze analytic form.

The $v - w - u$ approach:



Computing $\mathbb{1}$-FRFs cont.

Theorem 3. (Lindstrom, Lourenço, P. '23)
Let $\mathcal{K}$ be a closed convex cone and $z \in \mathcal{K}^*$ (with $\|z\| = 1$) be such that $\{z\}^\perp \cap \mathcal{K}$ is a nontrivial exposed face of $\mathcal{K}$.
Let $\eta > 0$, $\alpha \in (0, 1]$ and $\mathrm{g} : \mathbb{R}_+ \to \mathbb{R}_+$ be nondecreasing and satisfy $\mathrm{g}(0) = 0$ and $\mathrm{g} \geq |\cdot|^\alpha$. Consider

$$\gamma_{z,\eta} := \inf_v \left\{ \frac{\mathrm{g}(\|w - v\|)}{\|w - u\|} \;\middle|\; \begin{array}{c} v \in \mathrm{bdry}(\mathcal{K}) \cap B(\eta) \setminus (\{z\}^\perp \cap \mathcal{K}) \\ w = \mathrm{Proj}_{\{z\}^\perp} v, \;\; u = \mathrm{Proj}_{\{z\}^\perp \cap \mathcal{K}} w, \;\; w \neq u \end{array} \right\}.$$

Suppose that $\gamma_{z,\eta} \in (0, \infty]$. Then

$$\mathrm{d}(x, \{z\}^\perp \cap \mathcal{K}) \leq \kappa_{z,\eta} \mathrm{g}(\mathrm{d}(x, \mathcal{K})) \quad \forall\, x \in \{z\}^\perp \cap B(0, \eta),$$

where $\kappa_{z,\eta} := \max\{2\eta^{1-\alpha}, 2\gamma_{z,\eta}^{-1}\}$. Moreover,

$$\psi(s, t) := s + \kappa_{z,t} \mathrm{g}(2s)$$

is a $\mathbb{1}$-FRF of $\mathcal{K}$ and z.

Main idea

Consider those (infinitely many) 1-D face $\mathcal{F}^e_\beta$ exposed by a unique (up to scaling) $z^* \in \mathrm{bdry}((\mathrm{epi}\, f^\pi)^*) \setminus \{0\}$:

Let $\|z^*\| = 1$ and $\mathcal{F}^e_\beta = \mathrm{cone}\{(\bar{\beta}, 1, f(\bar{\beta}))\}$ with $\bar{\beta} = (f^*)'(\beta)$, $\beta \notin f'(W)$.

Main idea

Consider those (infinitely many) 1-D face $\mathcal{F}_\beta^e$ exposed by a unique (up to scaling) $z^* \in \text{bdry}((\text{epi}\, f^\pi)^*) \setminus \{0\}$:

Let $\|z^*\| = 1$ and $\mathcal{F}_\beta^e = \text{cone}\{(\bar{\beta}, 1, f(\bar{\beta}))\}$ with $\bar{\beta} = (f^*)'(\beta)$, $\beta \notin f'(W)$.

- If $v = (v_x, v_t, v_t f(v_x/v_t))$ with $v_t > 0$, then

  $$\|w - v\| = \Omega(v_t \cdot |h_1(v_x/v_t)|) \;\text{ and }\; \|w - u\| = O(v_t \cdot |h_2(v_x/v_t)|),$$

  where
  $$h_1(\zeta) := f(\zeta) + f^*(\beta) - \beta\zeta,$$
  $$h_2(\zeta) := |\zeta - \bar{\beta}| + |f(\zeta) - f(\bar{\beta})|.$$

  For those v_x/v_t close to $\bar{\beta}$, the exponent of $1/2$ pops up upon invoking local strong convexity of h_1 at $\zeta = \bar{\beta}$.
- Other cases can be dealt with similarly.

Conclusion

For a large class of perspective cones of nonnegative Legendre functions on $\mathbb{R}$:

- the set of $z^* \in \mathrm{bdry}(\mathcal{K}^*)\backslash\{0\}$ that does not admit a $\mathbb{1}$-FRF with exponent $1/2$ is of Hausdorff dimension 1.
- for $\mathcal{H}^2$-a.e. $z^* \in \mathrm{bdry}(\mathcal{K}^*)\backslash\{0\}$, they admit a $\mathbb{1}$-FRF with exponent $1/2$, as a consequence of differentiability and local strong convexity.

Reference:

- Xiaozhou Wang, Bruno F. Lourenço and Ting Kei Pong.
  Error bounds for perspective cones of a class of nonnegative Legendre functions
  Preprint. Available at https://arxiv.org/abs/2509.26289.

Thanks for coming!