Error bound for conic feasibility problems: case studies on the exponential cone

Ting Kei Pong
Department of Applied Mathematics
The Hong Kong Polytechnic University
Hong Kong

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(Joint work with Scott B. Lindstrom and Bruno F. Lourenço)

Conic programming problem

Conic program: Let $\mathcal{K}$ be a closed convex cone in a Euclidean space $\mathcal{E}$, $c \in \mathcal{E}$, $\mathcal{A}$ be a linear map on $\mathcal{E}$ and $b \in \text{Range}(\mathcal{A})$.

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$$K_{\exp} := \{x \in \mathbb{R}^3 \mid x_2 > 0, x_3 \geq x_2 e^{x_1/x_2}\} \cup \{(x_1, 0, x_3) \mid x_1 \leq 0, x_3 \geq 0\}.$$

  - ⋆ Epigraph of (the closure of) the perspective function of $z \mapsto \exp(z)$.
  - ⋆ Recent addition to MOSEK and other conic solvers.
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- Product cones of the above...

Why "exotic" cones?

- Richer modeling power. For example,

$$t \geq \ln\left(\sum_{i=1}^{n} e^{x_i}\right) \iff \sum_{i=1}^{n} z_i \leq 1 \text{ and } (z_i, 1, x_i - t) \in K_{\text{exp}} \;\; \forall i.$$

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- These and many other exotic cones have nice logarithmically homogeneous self-concordant barrier functions for the adaptation of existing IPM routines. (Coey, Kapelevich, Vielma '21)

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$$\mathcal{L}+a = \{x : \ \mathcal{A}x = b, \langle c, x \rangle = v\}.$$

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In this case, typically, $(\mathcal{L} + a) \cap \operatorname{ri} \mathcal{K} = \emptyset$.

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- Typically, $\mathrm{d}(x, \mathcal{K})$ and $\mathrm{d}(x, \mathcal{L} + a)$ are relatively easier to compute.
- Is x "a good soln." when $\max\{\mathrm{d}(x, \mathcal{K}), \mathrm{d}(x, \mathcal{L} + a)\}$ is small?

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Key: Compare the orders of magnitude of $\mathrm{d}(x, \mathcal{K} \cap (\mathcal{L} + a))$ and $\max\{\mathrm{d}(x, \mathcal{K}), \mathrm{d}(x, \mathcal{L} + a)\}$.

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Note: Typically, $(\mathcal{L} + a) \cap \mathrm{ri}\, \mathcal{K} = \emptyset$.

Error bounds

Definition: Let $\theta \in (0, 1]$. We say that $\{\mathcal{K}, \mathcal{L} + a\}$ satisfies a (uniform) Hölderian error bound with exponent θ if for every bounded set B, there exists $c_B > 0$ such that

$$\mathrm{d}(x, \mathcal{K} \cap (\mathcal{L} + a)) \leq c_B \left(\max\{\mathrm{d}(x, \mathcal{K}), \mathrm{d}(x, \mathcal{L} + a)\}\right)^{\theta} \quad \forall x \in B.$$

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Examples:

- If $\mathcal{K}$ is polyhedral, Lipschitz error bound holds. (Hoffman '57)
- If $(\mathcal{L} + a) \cap \mathrm{ri}\,\mathcal{K} \neq \emptyset$, Lipschitz error bound holds. (Bauschke, Borwein '96)

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- If $\mathcal{K} = \mathcal{S}^n_+$, Hölderian error bound with exponent $2^{-(\ell-1)}$ holds (Sturm '00); here ℓ has to do with facial reduction. (Borwein, Wolkowicz '81)

Faces and facial reduction

Definition: A sub-cone $\mathcal{F} \subseteq \mathcal{K}$ is called

- a face if $x,\ y \in \mathcal{K}$ and $x + y \in \mathcal{F}$ implies $x,\ y \in \mathcal{F}$;
- an exposed face if $\exists\ z \in \mathcal{K}^*$ such that $\mathcal{F} = \mathcal{K} \cap \{z\}^\perp$.

Note: Recall that $\mathcal{K}^* := \{x \mid \langle x, y \rangle \geq 0 \ \ \forall y \in \mathcal{K}\}$.

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Theorem 1. (Borwein, Wolkowicz '81, Lourenço, Muramatsu, Tsuchiya '18)
Suppose $\mathcal{K} \cap (\mathcal{L} + a) \neq \emptyset$. Then there exists a chain of faces

$$\mathcal{F}_\ell \subsetneq \cdots \subsetneq \mathcal{F}_1 = \mathcal{K}$$

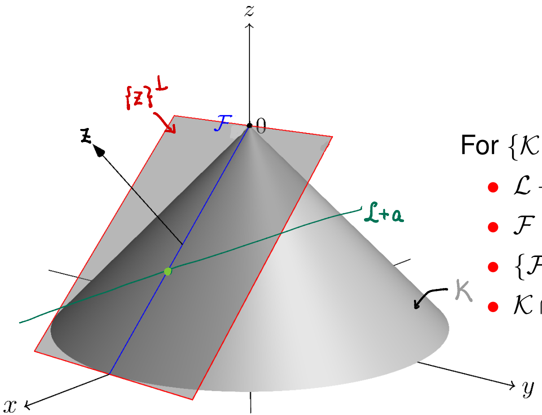
and vectors $\{z_1, \ldots, z_{\ell-1}\}$ satisfying

- For all $i \in \{1, \ldots, \ell - 1\}$,

$$z_i \in \mathcal{F}_i^* \cap \mathcal{L}^\perp \cap \{a\}^\perp \quad \text{and} \quad \mathcal{F}_{i+1} = \mathcal{F}_i \cap \{z_i\}^\perp.$$

- $\mathcal{F}_\ell \cap (\mathcal{L} + a) = \mathcal{K} \cap (\mathcal{L} + a)$ and $\{\mathcal{F}_\ell, \mathcal{L} + a\}$ satisfies a Lipschitz error bound.

Facial reduction: Illustration



The picture is provided by B. F. Lourenço.

For $\{\mathcal{K}, \mathcal{L} + a\}$:

- $\mathcal{L} + a \subseteq \{z\}^\perp$;
- $\mathcal{F} = \mathcal{K} \cap \{z\}^\perp$;
- $\{\mathcal{F}, \mathcal{L} + a\}$ has Lip. err. bd.
- $\mathcal{K} \cap (\mathcal{L} + a) = \mathcal{F} \cap (\mathcal{L} + a)$.

Sturm's error bounds and facial reduction

Key observation: (Sturm '00, Lourenço '21)
Let $\mathcal{F} \trianglelefteq \mathcal{S}^n_+$ and $z \in \mathcal{F}^*$. Then $\exists\, \kappa > 0$ such that for all x,

$$\mathrm{d}(x, \mathcal{F} \cap \{z\}^\perp) \leq \kappa\epsilon + \kappa\sqrt{\epsilon\|x\|},$$

where $\epsilon = \max\{\mathrm{d}(x, \mathcal{F}), \mathrm{d}(x, \{z\}^\perp)\}$.

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Error bound for $\{\mathcal{S}^n_+, (\mathcal{L} + a)\}$ follows from induction: For $x \in B$,

$$\mathrm{d}(x, \mathcal{K} \cap (\mathcal{L} + a)) = \mathrm{d}(x, \mathcal{F}_\ell \cap (\mathcal{L} + a)) \leq c_\ell \max\{\mathrm{d}(x, \mathcal{F}_\ell), \mathrm{d}(x, \mathcal{L} + a)\}$$

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&\leq c_\ell[\mathrm{d}(x, \mathcal{L} + a) + \mathrm{d}(x, \mathcal{F}_{\ell-1} \cap \{z_{\ell-1}\}^\perp)] \\
&\leq c_\ell[\mathrm{d}(x, \mathcal{L} + a) + c_{\ell-1}\big(\max\{\mathrm{d}(x, \mathcal{F}_{\ell-1}), \mathrm{d}(x, \{z_{\ell-1}\}^\perp)\} \\
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&= O([\max\{\mathrm{d}(x,\mathcal{F}_{\ell-1}), \mathrm{d}(x,\mathcal{L}+a)\}]^{\frac{1}{2}})
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&= O([\max\{\mathrm{d}(x, \mathcal{F}_{\ell-1}), \mathrm{d}(x, \mathcal{L} + a)\}]^{\frac{1}{2}}) = O([\max\{\mathrm{d}(x, \mathcal{K}), \mathrm{d}(x, \mathcal{L} + a)\}]^{\frac{1}{2^{\ell-1}}}).
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Facial residual function

Definition: (Lourenço '21, Lindstrom, Lourenço, P. '22)
Let $\mathcal{F} \trianglelefteq \mathcal{K}$ and $z \in \mathcal{F}^*$. Suppose $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

- ψ is nondecreasing in each argument and $\psi(0, t) = 0 \ \forall\, t \in \mathbb{R}_+$;
- It holds that

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Then ψ is called a 1-step facial residual function for $\mathcal{F}$ and z.

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Remarks:

- For $\mathcal{K} = \mathcal{S}^n_+$, we have $\psi(s, t) = \kappa \cdot (s + \sqrt{st})$ for some $\kappa > 0$.
- Induction arguments show that error bound can be derived given the face chain from facial reduction and by composing 1-step facial residual functions. (Lindstrom, Lourenço, P. '22)

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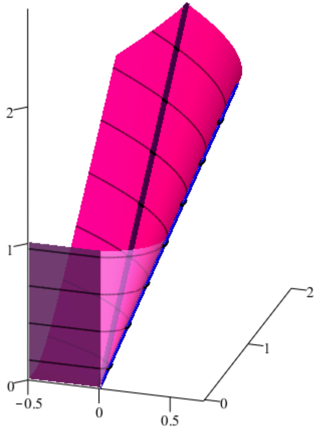
- Two key ingredients: For each nonpolyhedral $\mathcal{F}$,
  - ⋆ Identify all its exposed faces.
  - ⋆ Obtain all 1-step facial residual functions: Depends on $\mathcal{F}$ and z.

Exponential cone

$$K_{\exp} = \{x \in \mathbb{R}^3 \mid x_2 > 0, x_3 \geq x_2 e^{x_1/x_2}\} \cup \{(x_1, 0, x_3) \mid x_1 \leq 0, x_3 \geq 0\}.$$



Faces of exponential cone

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Nontrivial exposed faces: (Lindstrom, Lourenço, P. '22)

- (Infinitely many) 1-D face exposed by a unique (up to scaling) $z \in \partial K^*_{\exp} \backslash \{0\}$:

$$\mathcal{F}_\beta := \{(t(1-\beta), t, te^{1-\beta} \mid t \geq 0\} \text{ for each } \beta \in \mathbb{R}.$$

- An exceptional extreme ray exposed by any $z \in \{(0, z_2, z_3) \mid z_2 \geq 0, z_3 > 0\} \subset \partial K^*_{\exp} \backslash \{0\}$:

$$\mathcal{F}_\infty := \{(x_1, 0, 0) \mid x_1 \leq 0\}.$$

- A unique 2-D face exposed uniquely (up to scaling) by $(0, 1, 0)$:

$$\mathcal{F}_{-\infty} := \{(x_1, 0, x_3) \mid x_1 \leq 0, x_3 \geq 0\}.$$

$\mathbb{1}$-FRFs for exponential cone

$\mathbb{1}$-FRFs: (Lindstrom, Lourenço, P. '22) Assume $\|z\| = 1$.

- (Infinitely many) 1-D face exposed by a unique $z \in \partial K_{\exp}^* \backslash \{0\}$:

$$\psi(s,t) = s + \kappa(t) \cdot \sqrt{2s},$$

  where $\kappa(\cdot)$ is nonnegative nondecreasing.

- The exceptional extreme ray $\mathcal{F}_\infty$:

$$\psi(s,t) = \begin{cases} s + 2\kappa(t)s & \text{if } z_2 > 0, \\ s + \kappa(t) \cdot \mathfrak{g}_\infty(2s) & \text{if } z_2 = 0, \end{cases}$$

  where $\kappa(\cdot)$ is nonnegative nondecreasing, and $\mathfrak{g}_\infty : \mathbb{R}_+ \to \mathbb{R}_+$ is

$$\mathfrak{g}_\infty(s) := \begin{cases} 0 & \text{if } s = 0, \\ -(\ln s)^{-1} & \text{if } 0 < s \leq e^{-2}, \\ 0.25(1 + e^2 s) & \text{else.} \end{cases}$$

$\mathbb{1}$-FRFs for exponential cone cont.

$\mathbb{1}$-FRFs cont.: Assume $\|z\| = 1$.

- The unique 2-D face exposed uniquely by $(0,1,0)$:

$$\psi(s,t) = s + \kappa(t) \cdot \mathfrak{g}_{-\infty}(2s),$$

where $\kappa(\cdot)$ is nonnegative nondecreasing, $\mathfrak{g}_{-\infty} : \mathbb{R}_+ \to \mathbb{R}_+$ is

$$\mathfrak{g}_{-\infty}(s) := \begin{cases} 0 & \text{if } s = 0, \\ -s \ln s & \text{if } 0 < s \leq e^{-2}, \\ s + e^{-2} & \text{else.} \end{cases}$$

$\mathbb{1}$-FRFs for exponential cone cont.

$\mathbb{1}$-FRFs cont.: Assume $\|z\| = 1$.

- The unique 2-D face exposed uniquely by $(0, 1, 0)$:

$$\psi(s, t) = s + \kappa(t) \cdot \mathfrak{g}_{-\infty}(2s),$$

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$$\mathfrak{g}_{-\infty}(s) := \begin{cases} 0 & \text{if } s = 0, \\ -s \ln s & \text{if } 0 < s \le e^{-2}, \\ s + e^{-2} & \text{else.} \end{cases}$$

Remark:

- As all nontrivial exposed faces are polyhedral, $\ell \le 2$. The final error bound has the same "order of magnitude" as the $\mathbb{1}$-FRFs.
- As $s \downarrow 0$: $-(\ln s)^{-1} \to 0$ slower than s^α for any $\alpha \in (0, 1]$; $-s \ln s \to 0$ faster than s^α for any $\alpha \in (0, 1)$, but is slower than s.
- These $\mathbb{1}$-FRFs are asymptotically the "best". (Lindstrom, Lourenço, P. '22)

Conclusion

Conclusion:

- Facial reduction and $\mathbb{1}$-FRFs are two key ingredients for deriving error bounds.
- Error bounds for $K_{\exp}$ feasibility problem.

References:

- S. B. Lindstrom, B. F. Lourenço and T. K. Pong.
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Thanks for coming! :)